COORDINATES. In this section we considered the question "What are coordinates?" Cartesian geometry was introduced by Fermat and Descartes (1596-1660) around 1636; it had a large influence on mathematics and introduced algebraic methods into geometry. The beginning of the vector concept came only later at the beginning of the 19th Century with the work of Bolzano (1783-1848). The full power of coordinates however is possible only if we allow to choose our coordinate system to adapt to the situation.

From Descartes biography. (A part which shows how far can dedication to teaching of mathematics go...

(...) In 1669 Queen Christina of Sweden persuaded Descartes to go to Stockholm. However the Queen wanted to draw tangents at 5 am, in the morning and Descartes broke the habit of his lifetime of getting up at 11 o'clock. After only a few months in the cold northern climate, walking to the palace at 5 o'clock every morning, he died of pneumonia.

VECTOR IN NEW COORDINATES. If a vector \( \mathbf{v} \) is written in a linear combination \( \mathbf{v} = \sum z_i \mathbf{v}_i \) of vectors \( \mathbf{v}_i \) in a basis \( \mathbf{B} = \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \} \), define \( S = [ \mathbf{v}_1 \ldots \mathbf{v}_n ] \). The transformation \( \mathbf{v} = \sum z_i \mathbf{v}_i \) is of course \( \mathbf{v} = S^{-1} \mathbf{v} \). We write \( \mathbf{v}_B = v = \left[ \begin{array}{c} z_1 \\ \vdots \\ z_n \end{array} \right] \) and call \( z_i \) the coordinates of \( \mathbf{v} \) with respect to the basis \( \mathbf{B} \).

EXAMPLE. Find the coordinates of \( \mathbf{v} = \left[ \begin{array}{c} 2 \\ 3 \end{array} \right] \) with respect to the basis \( \mathbf{B} = \{ \mathbf{v}_1 = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \} \).

We have \( S = \left[ \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right] \) and \( S^{-1} = \left[ \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right] \). Therefore \( \mathbf{v}_B = S^{-1} \mathbf{v} = \left[ \begin{array}{c} -1 \\ 3 \end{array} \right] \). Indeed \( -v_1 + 3v_2 = v \).

LINEAR TRANSFORMATION IN NEW COORDINATES. The transformation \( S^{-1} \) maps the coordinates from the standard basis \( \mathbf{B} \) into the coordinates of the new basis. In order to see what a transformation \( A \) does in the new coordinates, we map it back to the old coordinates, apply \( A \) and then map back again to the new coordinates: \( \mathbf{B} = S^{-1} AS \).

The transformation in "new" coordinates is

\[ A_{\mathbf{B}} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \]

A transformation in "good" coordinates.

IN OTHER WORDS. In a basis \( \mathbf{B} = \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \} \), the coordinates of a transformation \( T \) from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) is the matrix \( T_{\mathbf{B}} \) with columns \( T[\mathbf{v}_i]_{\mathbf{B}} \). In the standard basis, we have the matrix \( A \) with columns \( T[\mathbf{e}_i]_{\mathbf{B}} \).

BASIS IS EIGENBASIS. If \( \mathbf{v}_i \) are eigenvectors of \( A \) with eigenvalues \( \lambda_i \) then \( A \mathbf{v}_i = \lambda_i \mathbf{v}_i \). The matrix \( A \) is diagonal in that basis.