Error correcting code I 2/23/2001	**Math21b, O.Knill**

We try out the error correcting code as in the book on pages 144-145.

I) Encoding.
To do so, we encode the letters of the alphabet by pairs of three vectors containing zeros and ones:

$$
\begin{array}{rrr}
A = (0,0,0,1),(0,0,0,1) & B = (0,0,0,1),(0,0,1,0) & C = (0,0,0,1),(0,0,1,1) \\
D = (0,0,0,1),(0,1,0,1) & E = (0,0,0,1),(0,1,1,0) & F = (0,0,0,1),(0,1,1,1) \\
G = (0,0,0,1),(1,0,0,1) & H = (0,0,0,1),(1,0,1,0) & I = (0,0,0,1),(1,0,1,1) \\
J = (0,0,0,1),(1,1,0,1) & K = (0,0,0,1),(1,1,1,0) & L = (0,0,0,1),(1,1,1,1) \\
M = (0,0,1,0),(0,0,0,1) & N = (0,0,1,0),(0,0,1,0) & O = (0,0,1,0),(0,0,1,1) \\
P = (0,0,1,0),(0,1,0,1) & Q = (0,0,1,0),(0,1,1,0) & R = (0,0,1,0),(0,1,1,1) \\
S = (0,0,1,0),(1,0,0,1) & T = (0,0,1,0),(1,0,1,0) & U = (0,0,1,0),(1,0,1,1) \\
V = (0,0,1,0),(1,1,0,1) & W = (0,0,1,0),(1,1,1,0) & X = (0,0,1,0),(1,1,1,1) \\
Y = (0,0,1,1),(1,0,0,1) & Z = (0,0,1,1),(1,0,1,0) & ? = (0,0,1,1),(1,0,1,1) \\
! = (0,0,1,1),(1,0,0,1) & . = (0,0,1,1),(1,0,1,0) & , = (0,0,1,1),(1,0,1,1)
\end{array}
$$

Choose a letter $\boxed{}$ and build (Mx, My), where M is the matrix $M = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$. After having chosen a letter in the form (x, y), we encode each block (using $1 + 1 = 0$) with:

$$
Mx = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} . \\ . \\ . \\ . \end{vmatrix} = \begin{vmatrix} . \\ . \\ . \\ . \\ . \\ . \\ . \end{vmatrix}
\qquad
My = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} . \\ . \\ . \\ . \end{vmatrix} = \begin{vmatrix} . \\ . \\ . \\ . \\ . \\ . \\ . \end{vmatrix}
$$

II) Transmission.
Now, give the lower part of the paper to your neighbor who will copy the two vectors (Mx, My) but with one error. (Switch one 1 to 0 or one 0 to 1 in the above vectors and put that down).

$$u = Mx + e = \begin{vmatrix} . \\ . \\ . \\ . \\ . \\ . \\ . \end{vmatrix} \qquad v = My + f = \begin{vmatrix} . \\ . \\ . \\ . \\ . \\ . \\ . \end{vmatrix}$$

III) Detect the error e and f.
Detect errors by forming

$$Hu = \begin{vmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} \begin{vmatrix} . \\ . \\ . \\ . \\ . \\ . \\ . \end{vmatrix} = \begin{bmatrix} . \\ . \\ . \end{bmatrix}$$

$$Hv = \begin{vmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} \begin{vmatrix} . \\ . \\ . \\ . \\ . \\ . \\ . \end{vmatrix} = \begin{bmatrix} . \\ . \\ . \end{bmatrix}.$$

From checking in which column Hu or Hv is, we obtain e and f: $e = \begin{bmatrix} . \\ . \\ . \\ . \\ . \\ . \\ . \end{bmatrix}$, $f = \begin{bmatrix} . \\ . \\ . \\ . \\ . \\ . \\ . \end{bmatrix}$.

IV) Decode the message.

Let $P \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \\ g \end{bmatrix}$. Determine $Pe = P \begin{bmatrix} . \\ . \\ . \\ . \\ . \\ . \\ . \end{bmatrix} = \begin{bmatrix} . \\ . \\ . \\ . \end{bmatrix}$, $Pf = P \begin{bmatrix} . \\ . \\ . \\ . \\ . \\ . \\ . \end{bmatrix} = \begin{bmatrix} . \\ . \\ . \\ . \end{bmatrix}$. In an error-free transmission (Pu, Pv) would give the right result back. Now

$$Pu = \begin{bmatrix} . \\ . \\ . \\ . \end{bmatrix}, \qquad Pv = \begin{bmatrix} . \\ . \\ . \\ . \end{bmatrix}$$

satisfy $Pu = x + Pe, Pv = y + Pf$. We recover the original message by subtracing Pe, Pf from that

$$x = \begin{bmatrix} . \\ . \\ . \\ . \end{bmatrix}, \qquad y = \begin{bmatrix} . \\ . \\ . \\ . \end{bmatrix}$$

The letter is $\boxed{}$.