13: Linearization

The linear approximation of a function $f(x)$ at a point $a$ is the linear function

$$L(x) = f(a) + f'(a)(x - a).$$

The graph of the function $L$ is a line close to the graph of $f$ near $a$. We generalize this to higher dimensions:

The linear approximation of $f(x, y)$ at $(a, b)$ is the linear function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

In three dimensions, the linear approximation of a function $f(x, y, z)$ at $(a, b, c)$ is

$$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c).$$

Using the gradient $\nabla f(x, y) = [f_x, f_y]$ resp. $\nabla f(x, y, z) = [f_x, f_y, f_z]$, the linearization can be written as $L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$. By keeping the second variable $y = b$ fixed, we get to a one-dimensional situation, where the only variable is $x$. Now $f(a, b) + f_x(a, b)(x - a)$ is the linear approximation. Similarly, if $x = x_0$ is fixed, then $y$ is the single variable, then $f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$ is an approximation. Knowing the linear approximations in both the $x$ and $y$ variables, produces the linearization $L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

1. What is the linear approximation of the function $f(x, y) = \sin(\pi xy^2)$ at the point $(1, 1)$? We have $(f_x(x, y), y) = \left(\pi y^2 \cos(\pi xy^2), 2\pi x \cos(\pi xy^2)\right)$ which is at the point $(1, 1)$ equal to $\nabla f(1, 1) = [\pi \cos(\pi), 2\pi \cos(\pi)] = [-\pi, 2\pi]$.

2. Linearization can be used to estimate functions near a point. In the previous example,

$$-0.00943 = f(1+0.01, 1+0.01) \sim L(1+0.01, 1+0.01) = -\pi 0.01 - 2\pi 0.01 + 3\pi = -0.00942.$$  

3. Find the linear approximation to $f(x, y, z) = xy + yz + zx$ at the point $(1, 1, 1)$. Since $f(1, 1, 1) = 3$, and $\nabla f(x, y, z) = [y+z, x+z, y+x]$, $\nabla f(1, 1, 1) = [2, 2, 2]$. We have $L(x, y, z) = f(1, 1, 1) + [2, 2, 2] \cdot [x-1, y-1, z-1] = 3 + 2(x-1) + 2(y-1) + 2(z-1) = 2x + 2y + 2z - 3$.

4. Estimate $f(0.01, 24.8, 1.02)$ for $f(x, y, z) = e^x \sqrt{yz}$.

Solution: take $(x_0, y_0, z_0) = (0.25, 1)$, where $f(x_0, y_0, z_0) = 5$. Solution. The gradient is $\nabla f(x, y, z) = (e^x \sqrt{yz}, e^x z/(2\sqrt{y}), e^x \sqrt{y})$. At the point $(x_0, y_0, z_0) = (0.25, 1)$ the gradient is the vector $(5, 1/10, 5)$. The linear approximation is $L(x, y, z) = f(x_0, y_0, z_0) +$
\( \nabla f(x_0, y_0, z_0)(x-x_0, y-y_0, z-z_0) = 5 + (5, 1/10, 5)(x-0, y-25, z-1) = 5x + y/10 + 5z - 2.5. \)

We can approximate \( f(0.01, 24.8, 1.02) \) by \( 5 + [5, 1/10, 5] \cdot [0.01, -0.2, 0.02] = 5 + 0.05 - 0.02 + 0.10 = 5.13. \) The actual value is \( f(0.01, 24.8, 1.02) = 5.1306, \) very close to the estimate.

5. Find the tangent line to the graph of the function \( g(x) = x^2 \) at the point \((2, 4)\). Solution: the tangent line is the level curve of the linearization of \( L(x, y) \) of \( f(x, y) = y - x^2 = 0 \) which passes through the point. We compute the gradient \([a, b] = \nabla f(2, 4) = [-g'(2), 1] = [-4, 1]\) and forming \( ax + by = -4x + y = d \), where \( d = -4 \cdot 2 + 1 \cdot 4 = -4 \). The answer is \( -4x + y = -4 \).

6. The Barth surface is defined as the level surface \( f = 0 \) of

\[
\begin{align*}
&f(x, y, z) = (3 + 5t)(-1 + x^2 + y^2 + z^2)^2(-2 + t + x^2 + y^2 + z^2)^2 \\
&\quad + 8(x^2 - t^4y^2)(-t^4x^2 + z^2)(y^2 - t^4z^2)(x^4 - 2x^2y^2 + y^4 - 2x^2z^2 - 2y^2z^2 + z^4),
\end{align*}
\]

where \( t = (\sqrt{5} + 1)/2 \) is a constant called the golden ratio. If we replace \( t \) with \( 1/t = (\sqrt{5} - 1)/2 \) we see the surface to the middle. For \( t = 1 \), we see to the right the surface \( f(x, y, z) = 8 \). Find the tangent plane of the later surface at the point \((1, 1, 0)\). Solution: We find the level curve of the linearization by computing the gradient \( \nabla f(1, 1, 0) = [64, 64, 0] \). The surface is \( x + y = d \) for some constant \( d \). By plugging in the point \((1, 1, 0)\) we see that \( x + y = 2 \).

7. The quartic surface

\[
f(x, y, z) = x^4 - x^3 + y^2 + z^2 = 0
\]

is called the piriform. What is the equation for the tangent plane at the point \( P = (2, 2, 2) \) of this pair shaped surface? Solution. We get \([a, b, c] = [20, 4, 4]\) and so the equation of the plane \( 20x + 4y + 4z = 56 \), where we have obtained the constant to the right by plugging in the point \((x, y, z) = (2, 2, 2)\).