If \( f(x, y) \) is a function of two variables, then \( \frac{\partial}{\partial x} f(x, y) \) is defined as the derivative of the function \( g(x) = f(x, y) \), where \( y \) is considered a constant. It is called \textbf{partial derivative} of \( f \) with respect to \( x \). The partial derivative with respect to \( y \) is defined similarly.

We also write \( f_x(x, y) = \frac{\partial}{\partial x} f(x, y) \) and \( f_{yx} = \frac{\partial}{\partial x} \frac{\partial}{\partial y} f \).

For \( f(x, y) = x^4 - 6x^2y^2 + y^4 \), we have \( f_x(x, y) = 4x^3 - 12xy^2 \), \( f_{xx} = 12x^2 - 12y^2 \), \( f_y(x, y) = -12x^2y + 4y^3 \), \( f_{yy} = -12x^2 + 12y^2 \) and see that \( f_{xx} + f_{yy} = 0 \). A function which satisfies this equation is also called \textbf{harmonic}. The equation \( f_{xx} + f_{yy} = 0 \) is an example of a \textbf{partial differential equation} for the unknown function \( f(x, y) \) involving partial derivatives. The vector \( [f_x, f_y] \) is called the \textbf{gradient}.

\textbf{Clairaut’s theorem} If \( f_{xy} \) and \( f_{yx} \) are both continuous, then \( f_{xy} = f_{yx} \).

Proof: we look at the equations without taking limits first. We extend the definition and say that a background Planck constant \( h \) is positive, then \( f_x(x, y) = [f(x + h, y) - f(x, y)]/h \). For \( h = 0 \) we define \( f_x \) as before. Compare the two sides for fixed \( h > 0 \):

\[
\begin{align*}
hf_x(x, y) &= f(x + h, y) - f(x, y) \\
ha f_{xy}(x, y) &= f(x + h, y + h) - f(x + h, y + h) - (f(x + h, y) - f(x, y)) \\
h^2 f_{xy}(x, y) &= f(x + h, y + h) - f(x + h, y + h) - (f(x + h, y + h) - f(x + h, y + h)) - (f(x + h, y) - f(x, y))
\end{align*}
\]

No limits were taken. We established an identity which holds for all \( h > 0 \), the discrete derivatives \( f_x, f_y \) satisfy \( f_{xy} = f_{yx} \). It is a ”quantum Clairaut” theorem. If the classical derivatives \( f_{xy}, f_{yx} \) are both continuous, the limit \( h \to 0 \) leads to the classical Clairaut’s theorem. The quantum Clairaut theorem holds for \textbf{any} functions \( f(x, y) \) of two variables. Not even continuity is needed.

2 Find \( f_{xxxxxyy} \) for \( f(x) = \sin(x) + x^6y^{10}\cos(y) \). Hint: No need not compute, just think.

3 The continuity assumption for \( f_{xy} \) is necessary. The example \( f(x, y) = \frac{x^3y - xy^3}{x^2+y^2} \) contradicts Clairaut’s theorem:

\[ \partial_x f, \partial_y f \] were introduced by Carl Gustav Jacobi. Josef Lagrange had used the term ”partial differences”.
\[
f_x(x, y) = \frac{(3x^2y - y^3)/(x^2 + y^2) - 2x(x^3y - xy^3)/(x^2 + y^2)^2}{(x^2 + y^2)^2}, f_x(0, y) = -y, f_{xy}(0, 0) = -1,
\]
\[
f_y(x, y) = \frac{(x^3 - 3xy^2)/(x^2 + y^2) - 2y(x^3y - xy^3)/(x^2 + y^2)^2}{(x^2 + y^2)^2}, f_y(x, 0) = x, f_{y,x}(0, 0) = 1.
\]

\(f_x(x_0, y_0)\) measures the slope when slicing the graph \(z = f(x, y)\) in the \(x\)-direction.
\(f_{xx}\) measures the concavity when slicing the graph in the \(x\)-direction.
\(f_{xy}\) measures how the \(x\) slope changes when you move in the \(y\) direction.

An equation for an unknown function \(f(x, y)\) which involves partial derivatives with respect to at least two different variables is called a **partial differential equation**. If only the derivative with respect to one variable appears, it is called an **ordinary differential equation**.

Here are two examples of partial differential equations. We will look at them in more detail next time and try to make sense what they mean.

4. The **wave equation** \(f_{tt}(t, x) = f_{xx}(t, x)\) governs the motion of light or sound. The function \(f(t, x) = \sin(x - t) + \sin(x + t)\) satisfies the wave equation.

5. The **heat equation** \(f_t(t, x) = f_{xx}(t, x)\) describes diffusion of heat or spread of an epidemic. The function \(f(t, x) = \frac{1}{\sqrt{t}}e^{-x^2/(4t)}\) satisfies the heat equation.

6. The **wave equation** \(f_{tt}(t, x) = f_{xx}(t, x)\) governs the motion of light or sound. The function \(f(t, x) = \sin(x - t) + \sin(x + t)\) satisfies the wave equation.

7. The **heat equation** \(f_t(t, x) = f_{xx}(t, x)\) describes diffusion of heat or spread of an epidemic. The function \(f(t, x) = \frac{1}{\sqrt{t}}e^{-x^2/(4t)}\) satisfies the heat equation.

8. The **Laplace equation** \(f_{xx} + f_{yy} = 0\) determines the shape of a membrane. The function \(f(x, y) = x^3 - 3xy^2\) is an example satisfying the Laplace equation. Such functions are called **harmonic**.

9. The **advection equation** \(f_t = f_x\) is used to model transport in a wire. The function \(f(t, x) = e^{-(x+t)^2}\) satisfy the advection equation.

10. The **Burgers equation** \(f_t + ff_x = f_{xx}\) describes waves at the beach which break. The function \(f(t, x) = \frac{x}{t} \frac{\sqrt{t}e^{-x^2/(4t)}}{1 + \sqrt{t}e^{-x^2/(4t)}}\) satisfies the Burgers equation.