10: Functions

A function $f(x,y)$ with domain $R$ is called **continuous at a point** $(a,b) \in R$ if $f(x,y) \rightarrow f(a,b)$ whenever $(x,y) \rightarrow (a,b)$. The function $f$ is **continuous on** $R$, if $f$ is continuous for every point $(a,b)$ on $R$. Sometimes, we can extend the domain $R$ to make it continuous there too. The function $f(x,y) = \sin(x) y/x$ for example can be assigned the value $y$ on the axes $x=0$ as l’Hopital shows.

1. The function $f(x,y) = x^2 + y^4 + xy + \sin(y + \sin \sin \sin(x)^2)$ is continuous on the entire plane. It is built up from functions which are continuous using addition, multiplication or composition of functions which are all continuous.

2. $f(x,y) = 1/(x^2 + y^2)$ is continuous everywhere except at the origin, where it is not defined. We can not extend the value as the value at $(0,0)$ would have to be arbitrarily large.

3. $f(x,y) = y + \sin(x)/|x|$ is continuous except at $x=0$. At every point $(0,y)$ it is discontinuous. $f(1/n,y) \rightarrow y + 1$ and $f(-1/n,y) \rightarrow y - 1$ for $n \rightarrow \infty$.

4. $f(x,y) = \sin(1/(x+y))$ is continuous except on the line $x+y=0$.

5. $f(x,y) = (x^4 - y^5)/(x^2 + y^2)$ is continuous at $(0,0)$. Use polar coordinates to see that it is $r^2 \cos^2(t)$.

6. The function $f(x) = e^{-1/x^2}$ is continuous everywhere. Actually, even all derivatives are continuous and are zero at 0. Still, the function is not constant zero.
There are three sources for discontinuous behavior: there can be jumps, there can be poles, or the function can oscillate. An example of a jump appears with \( f(x) = \frac{\sin(x)}{|x|} \), a pole example is \( g(x) = \frac{1}{x} \) leads to a vertical asymptote and the function going to infinity. An example of a function discontinuous due to oscillations is \( h(x) = \sin(\frac{1}{x}) \). Its graph is the devil’s comb.

There are two handy tools to discover a discontinuities:
1) Use polar coordinates with coordinate center at the point to analyze the function.
2) Restrict the function to one dimensional curves and check continuity on that curve, where one has a function of one variables.

Determine whether the function \( f(x, y) = \frac{\sin(x^2+y^2)}{x^2+y^2} \) is continuous at \((0, 0)\). **Solution** Use polar coordinates to write this as \( \sin(r^2)/r^2 \) which is continuous at 0 (apply l’Hopital twice if you want to verify this).

Is the function \( f(x, y) = \frac{x^2-y^2}{x^2+y^2} \) continuous at \((0, 0)\)? **Solution** Use polar coordinates to see that this \( \cos(2\theta) \). We see that the value depends on the angle only. Arbitrarily close to \((0, 0)\), the function takes any value from \(-1\) to \(1\).

Is the function \( f(x, y) = \frac{x^2y}{x^4+y^4} \) continuous?

**Solution.** Look on the parabola \( x^2 = y \) to get the function \( x^4/(2x^4) = 1/(2x^2) \) which is not continuous at 0. This example is shocking because it is continuous through each line through the origin: if \( y = ax \), then \( f(x, ax) = ax^3/(x^4 + a^2x^2) = ax/(x^2 + a^2) \). This converges to 0 for \( x \to 0 \) as long as \( a \neq 0 \). If \( a = 0 \) however, we have \( y = 0 \) and \( f = 0/x^4 \) which can be continuously extended to \( x = 0 \) too.

What about the function

\[ f(x, y) = \frac{xy^2 + y^3}{x^2 + y^2} \]

**Solution.** Use polar coordinates and write \( r^3 \sin^2(\theta) \cos(\theta) + \sin(\theta)/r^2 = r \sin^2(\theta) \cos(\theta) + \sin(\theta) \) which shows that the function converges to 0 as \( r \to 0 \).

Is the function \( f(x, y) = \frac{\sin(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} \) continuous everywhere? **Solution.** Use polar coordinates to see that this is \( \sin(r)/r \). This function is continuous at 0 by Hôpital’s theorem.