6: Arc Length and Curvature

If \( t \in [a, b] \mapsto \vec{r}(t) \) is a curve with velocity \( \vec{r}'(t) \) and speed \( |\vec{r}'(t)| \), then \( L = \int_a^b |\vec{r}'(t)| \, dt \) is called the **arc length of the curve**. Written out in three dimensions, this is \( L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt \).

1. The arc length of the **circle** of radius \( R \) given by \( \vec{r}(t) = [R \cos(t), R \sin(t)] \) parameterized by \( 0 \leq t \leq 2\pi \) is \( 2\pi R \).

2. The **helix** \( \vec{r}(t) = [\cos(t), \sin(t), t] \) has velocity \( \vec{r}'(t) = [-\sin(t), \cos(t), 1] \) and constant speed \( |\vec{r}'(t)| = [\sin(t), \cos(t), 1] = \sqrt{2} \).

3. What is the arc length of the curve \( \vec{r}(t) = [t, \log(t), t^2/2] \) for \( 1 \leq t \leq 2 \)? **Answer:** Because \( \vec{r}'(t) = [1, 1/t, t] \), we have \( \vec{r}'(t) = \sqrt{1 + 1/t^2 + t^2} = \frac{1}{t} + t \) and \( L = \int_1^2 \frac{1}{t} + t \, dt = \log(t) + \frac{t^2}{2} \bigg|_1^2 = \log(2) + 2 - 1/2 \).

4. Find the arc length of the curve \( \vec{r}(t) = [3t^2, 6t, t^3] \) from \( t = 1 \) to \( t = 3 \).

5. What is the arc length of the curve \( \vec{r}(t) = [\cos^3(t), \sin^3(t)], 0 \leq t \leq 2\pi \)? **Answer:** We have \( |\vec{r}'(t)| = 3\sqrt{\sin^2(t) \cos^4(t) + \cos^2(t) \sin^4(t)} = \sqrt{(3/2)^2 (\sin(2t))} \). The absolute value forces us to split the integral into 4 intervals. Since \( \int_0^{\pi/2} \sin(2t) \, dt = 1 \), we have \( \int_0^{2\pi} (3/2) |\sin(2t)| \, dt = (3/2)4 = 6 \).

6. Find the arc length of \( \vec{r}(t) = [t^2/2, t^3/3] \) for \(-1 \leq t \leq 1 \). This cubic curve satisfies \( y^2 = x^38/9 \) and is an example of an **elliptic curve**. The speed is \( |\vec{r}'(t)| = \sqrt{t^2 + t^4} \). Because \( \int x\sqrt{1 + x^2} \, dx = (1 + x^2)^{3/2}/3 \), the arc length integral can be evaluated using substitution by as \( \int_{-1}^1 |t|\sqrt{1 + t^2} \, dx = 2 \int_0^1 t\sqrt{1 + t^2} \, dt = 2(1 + t^2)^{3/2}/3|_0^1 = 2(2\sqrt{2} - 1)/3 \).

7. The arc length of an **epicycle** \( \vec{r}(t) = [t + \sin(t), \cos(t)] \) parameterized by \( 0 \leq t \leq 2\pi \). We have \( |\vec{r}'(t)| = \sqrt{2 + 2 \cos(t)} \). so that \( L = \int_0^{2\pi} \sqrt{2 + 2 \cos(t)} \, dt \). A substitution \( t = 2u \) gives \( L = \int_0^\pi \sqrt{2 + 2 \cos(2u)} \, du = \int_0^\pi \sqrt{2 + 2 \cos^2(u) - 2 \sin^2(u)} \, du = \int_0^\pi \sqrt{4 \cos^2(u)} \, du = 4 \int_0^\pi \cos(u) \, du = 8 \).

8. Compute the arc length of the **catenary** \( \vec{r}(t) = [t, e^t + e^{-t}] \) on an interval \([a, b]\) can be computed as \( e^b - e^a - e^{-b} + e^{-a} \). By the way, \((e^t + e^{-t})/2\) is called the hyperbolic cosine and denoted by \( \cosh(t) \).

Because a parameter change \( t = t(s) \) corresponds to a **substitution** in the integration which does not change the integral, we immediately have

The arc length is independent of the parameterization of the curve.
The circle parameterized by \( \vec{r}(t) = [\cos(t^2), \sin(t^2)] \) on \( t = [0, \sqrt{2\pi}] \) has the velocity \( \vec{r}'(t) = 2t(-\sin(t), \cos(t)) \) and speed \( 2t \). The arc length is still \( \int_0^{\sqrt{2\pi}} 2t \, dt = t^2 |_0^{\sqrt{2\pi}} = 2\pi \).

We do not always have a closed formula for the arc length of a curve. The length of the Lissajous figure \( \vec{r}(t) = [\cos(3t), \sin(5t)] \) leads to \( \int_0^{2\pi} \sqrt{9 \sin^2(3t) + 25 \cos^2(5t)} \, dt \) which needs to be evaluated numerically.

Define the unit tangent vector \( \vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)| \) unit tangent vector.

The curvature if a curve at the point \( \vec{r}(t) \) is defined as \( \kappa(t) = |\vec{T}'(t)|/|\vec{T}(t)| \).

The curvature does not depend on the parametrization.

Proof. If \( s(t) \) be another parametrization, then by the chain rule \( d/dtT'(s(t)) = T'(s(t))s'(t) \) and \( d/dtr(s(t)) = r'(s(t))s'(t) \). We see that the \( s' \) cancels in \( T'/r' \).

Especially, if the curve is parametrized by arc length, meaning that the velocity vector \( r'(t) \) has length 1, then \( \kappa(t) = |T'(t)| \). It measures the rate of change of the unit tangent vector.

The curve \( \vec{r}(t) = [t, f(t)] \), which is the graph of a function \( f \) has the velocity \( \vec{r}'(t) = (1, f'(t)) \) and the unit tangent vector \( \vec{T}(t) = (1, f'(t))/\sqrt{1 + f'(t)^2} \). After some simplification we get

\[
\kappa(t) = |\vec{T}'(t)|/|\vec{T}'(t)| = |f''(t)|/\sqrt{1 + f'(t)^2}^3
\]

For example, for \( f(t) = \sin(t) \), then \( \kappa(t) = |\sin(t)|/\sqrt{1 + \cos^2(t)}^3 \).

If \( \vec{r}(t) \) is a curve which has nonzero speed at \( t \), then we can define \( \vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)| \), the unit tangent vector, \( \vec{N}(t) = \vec{r}'(t)/|\vec{r}'(t)| \), the normal vector and \( \vec{B}(t) = \vec{T}(t) \times \vec{N}(t) \) the bi-normal vector. The plane spanned by \( \vec{N} \) and \( \vec{B} \) is called the normal plane. It is perpendicular to the curve. The plane spanned by \( T \) and \( N \) is called the osculating plane.

If we differentiate \( \vec{T}(t) \cdot \vec{T}(t) = 1 \), we get \( \vec{T}'(t) \cdot \vec{T}(t) = 0 \) and see that \( \vec{N}(t) \) is perpendicular to \( \vec{T}(t) \). Because \( B \) is automatically normal to \( T \) and \( N \), we have shown:

The three vectors \( (\vec{T}(t), \vec{N}(t), \vec{B}(t)) \) are unit vectors orthogonal to each other.

A useful formula for curvature is

\[
\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}
\]

We prove this in class.