5: Curves

We have already seen lines like \( \vec{r}(t) = [x(t), y(t), z(t)] = [1+t, 2-3t, 4-3t] \). We can generalize this and replace the entries with general functions \( x(t), y(t), z(t) \). Depending on how many coordinates we use, we have either a curve in the plane or a curve in three dimensional space.

A parametrization of a planar curve is a map \( \vec{r}(t) = [x(t), y(t)] \) from a parameter interval \( R = [a, b] \) to the plane. The functions \( x(t), y(t) \) are called coordinate functions. The image of the parametrization is called a parametrized curve in the plane. The parametrization of a space curve is \( \vec{r}(t) = [x(t), y(t), z(t)] \). The image of \( r \) is a parametrized curve in space.

Here are some pictures of cool curves

We think of the parameter \( t \) as time. For a fixed \( t \), we have a vector \( [x(t), y(t), z(t)] \) in space. As \( t \) varies, the end point of this vector moves along a curve. The parametrization contains more information about the curve then the curve. It tells also how fast and in which direction we trace the curve.

1. The parametrization \( \vec{r}(t) = [2 + t, 3 + t, 1 + t] = [2, 3, 1] + t[1, 1, 1] \) is a line in space.

2. The parametrization \( \vec{r}(t) = [2 + 3\cos(t), 4 + 3\sin(t)] \) is a circle of radius 3 centered at \( (2, 4) \).

3. \( \vec{r}(t) = [\cos(3t), \sin(5t)] \) defines a Lissajous curve example.

4. If \( x(t) = t, y(t) = t^2 \), the curve \( \vec{r}(t) = [t, t^2] \) traces the graph of the function \( f(t) = t^2 \). It is a parabola.

5. With \( \vec{r}(t) = [2\cos(t), 5\sin(t)] = [x(t), y(t)] \) describes an ellipse \( x(t)^2/4 + y(t)^2/25 = 1 \).

6. The space curve \( \vec{r}(t) = [\cos(t), \sin(t), t] \) traces a helix.

7. If \( x(t) = \cos(2t), y(t) = \sin(2t), z(t) = 2t \) is the same curve as before but the parameterization has changed.

8. With \( x(t) = \cos(-t), y(t) = \sin(-t), z(t) = -t \) it is traced in the opposite direction.
With \( \vec{r}(t) = [\cos(t), \sin(t)] + 0.1[\cos(17t), \sin(17t)] \) we have an example of an epicycle, where a circle turns on a circle. It was used in the Ptolemaic geocentric system which predated the Copernican system still using circular orbits and then the modern Keplerian system, where planets move on ellipses and which can be derived from Newton’s laws.

If \( \vec{r}(t) = [x(t), y(t), z(t)] \) is a curve, then \( \vec{r}'(t) = [x'(t), y'(t), z'(t)] = [\dot{x}, \dot{y}, \dot{z}] \) is called the velocity at time \( t \). Its length \( |\vec{r}'(t)| \) is called speed and \( \vec{v}/|\vec{v}| \) is called direction of motion. The vector \( \vec{r}''(t) \) is called the acceleration. The third derivative \( \vec{r}'''(t) \) is called the jerk. Then come snap, crackle, pop and (if you like) the Harvard.

If \( \vec{r}'(t) \neq \vec{0} \), any vector parallel to \( \vec{r}'(t) \) is called tangent to the curve at \( \vec{r}(t) \).

The addition rule in one dimension \((f + g)' = f' + g'\), the scalar multiplication rule \((cf)' = cf'\) and the Leibniz rule \((fg)' = f'g + fg'\) and the chain rule \((f(g))' = f'(g)g'\) generalize to vector-valued functions because in each component, we have the single variable rule.

The process of differentiation of a curve can be reversed using the fundamental theorem of calculus. If \( \vec{r}'(t) \) and \( \vec{r}(0) \) is known, we can figure out \( \vec{r}(t) \) by integration \( \vec{r}(t) = \vec{r}(0) + \int_0^t \vec{r}'(s) \, ds \).

Assume we know the acceleration \( \vec{a}(t) = \vec{r}''(t) \) at all times as well as initial velocity and position \( \vec{r}'(0) \) and \( \vec{r}(0) \). Then \( \vec{r}(t) = \vec{r}(0) + t\vec{r}'(0) + \vec{R}(t) \), where \( \vec{R}(t) = \int_0^t \vec{v}(s) \, ds \) and \( \vec{v}(t) = \int_0^t \vec{a}(s) \, ds \).

The free fall is the case when acceleration is constant. In particular, if \( \vec{r}''(t) = [0, 0, -10] \), \( \vec{r}'(0) = [0, 1000, 2] \), \( \vec{r}(0) = [0, 0, h] \), then \( \vec{r}(t) = [0, 1000t, h + 2t - 10t^2/2] \).

If \( \vec{r}''(t) = \vec{F} \) is constant, then \( \vec{r}(t) = \vec{r}(0) + t\vec{r}'(0) - \vec{F}t^2/2 \).