2: Vectors and Dot product

Two points \( P = (a, b, c) \) and \( Q = (x, y, z) \) in space define a vector \( \vec{PQ} = \vec{v} = [x-a, y-b-z-c] \) pointing from \( P \) to \( Q \). The real numbers \( v_1, v_2, v_3 \) in \( \vec{v} = [v_1, v_2, v_3] \) are the components of \( \vec{v} \).

Similar definitions hold in two dimensions, where vectors have two components. Vectors can be drawn everywhere in space. Two vectors with the same components are considered equal.  \(^1\)

The addition of two vectors is \( \vec{u} + \vec{v} = [u_1, u_2, u_3] + [v_1, v_2, v_3] = [u_1 + v_1, u_2 + v_2, u_3 + v_3] \). The scalar multiple \( \lambda \vec{u} = \lambda [u_1, u_2, u_3] = [\lambda u_1, \lambda u_2, \lambda u_3] \). The difference \( \vec{u} - \vec{v} \) can be seen as the addition of \( \vec{u} \) and \( (-1) \cdot \vec{v} \).

The addition and scalar multiplication of vectors satisfy the laws you know from arithmetic. commutativity \( \vec{u} + \vec{v} = \vec{v} + \vec{u} \), associativity \( \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} \) and \( r(s \cdot \vec{v}) = (r \cdot s) \cdot \vec{v} \) as well as distributivity \( (r+s)\vec{v} = r\vec{v} + s\vec{v} \) and \( r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w} \), where \( * \) is scalar multiplication.

The length or magnitude \( |\vec{v}| \) of a vector \( \vec{v} = \vec{PQ} \) is defined as the distance \( d(P,Q) \) from \( P \) to \( Q \). A vector of length 1 is called a unit vector. A synonym is direction. Nonzero vectors have length and magnitude.

\[ |[3, 4]| = 5 \text{ and } |[3, 4, 12]| = 13. \] Examples of unit vectors are \( |\vec{i}| = |\vec{j}| = |\vec{k}| = 1 \) and \( [3/5, 4/5] \) and \( [3/13, 4/13, 12/13] \). The only vector of length 0 is the zero vector \( |\vec{0}| = 0 \).

The dot product of two vectors \( \vec{v} = [a, b, c] \) and \( \vec{w} = [p, q, r] \) is defined as \( \vec{v} \cdot \vec{w} = ap + bq + cr \).

The dot product determines distance and distance determines the dot product.

Proof: Using the dot product one can express the length of \( \vec{v} \) as \( |\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}} \). On the other hand, \( (\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2(\vec{v} \cdot \vec{w}) \) allows to solve for \( \vec{v} \cdot \vec{w} \):
\[
\vec{v} \cdot \vec{w} = (|\vec{v} + \vec{w}|^2 - |\vec{v}|^2 - |\vec{w}|^2)/2.
\]

The Cauchy-Schwarz inequality tells \( |\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}| \).

Proof. We only need to show it in the case \( |\vec{w}| = 1 \). Define \( a = \vec{v} \cdot \vec{w} \) and estimate \( 0 \leq (\vec{v} - a\vec{w}) \cdot (\vec{v} - a\vec{w}) \) to get \( 0 \leq (\vec{v} - (\vec{v} \cdot \vec{w})\vec{w}) \cdot (\vec{v} - (\vec{v} \cdot \vec{w})\vec{w}) = |\vec{v}|^2 + (\vec{v} \cdot \vec{w})^2 - 2(\vec{v} \cdot \vec{w})^2 = |\vec{v}|^2 - (\vec{v} \cdot \vec{w})^2 \) which means \( (\vec{v} \cdot \vec{w})^2 \leq |\vec{v}|^2 \).

The angle between two nonzero vectors is defined as the unique \( \alpha \in [0, \pi] \) which satisfies \( \vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha) \).

\(^1\)We cover 2400 years of math from Pythagoras (500 BC), Al Kashi (1400), Cauchy (1800) to Hamilton (1850).
Al Kashi’s theorem: A triangle $ABC$ with side lengths $a, b, c$ and angle $\alpha$ opposite to $c$ satisfies $a^2 + b^2 = c^2 + 2ab \cos(\alpha)$.

Proof. Define $\vec{v} = \vec{AB}, \vec{w} = \vec{AC}$. Because $c^2 = |\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{v} \cdot \vec{w}$, we know $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$ so that $c^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}| \cdot |\vec{w}| \cos(\alpha) = a^2 + b^2 - 2ab \cos(\alpha)$.

The triangle inequality tells $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$

Proof: $|\vec{u} + \vec{v}|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u}^2 + \vec{v}^2 + 2\vec{u} \cdot \vec{v} \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u}| \cdot |\vec{v}| = (|\vec{u}| + |\vec{v}|)^2$.

Two vectors are called orthogonal or perpendicular if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. For example, $\vec{v} = [2, 3]$ is orthogonal to $\vec{w} = [-3, 2]$.

Pythagoras theorem: if $\vec{v}$ and $\vec{w}$ are orthogonal, then $|\vec{v} - \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$.

Proof: $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w}$. 

The vector $P(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w}$ is called the projection of $\vec{v}$ onto $\vec{w}$. The scalar projection $\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$ is plus or minus the length of the projection of $\vec{v}$ onto $\vec{w}$. The vector $\vec{b} = \vec{v} - P(\vec{v})$ is a vector orthogonal to $\vec{w}$.

2. Find the projection of $\vec{v} = [0, -1, 1]$ onto $\vec{w} = [1, -1, 0]$. Answer: $P(\vec{v}) = [1/2, -1/2, 0]$.

3. A wind force $\vec{F} = [2, 3, 1]$ is applied to a car which drives in the direction of the vector $\vec{w} = [1, 1, 0]$. Find the projection of $\vec{F}$ onto $\vec{w}$, the force which accelerates or slows down the car. Answer: $\vec{w}(\vec{F} \cdot \vec{w}/|\vec{w}|^2) = [5/2, 5/2, 0]$.

4. How can we visualize the dot product? Answer: the absolute value of the dot product is the length of the projection. Positive dot product means $\vec{v}$ and $\vec{w}$ form an acute angle, negative if that angle is obtuse.

5. Given $\vec{v} = [2, 1, 2]$ and $\vec{w} = [3, 4, 0]$. Find a vector which is in the plane defined by $\vec{v}$ and $\vec{w}$ and which bisects the angle between these two vectors. Answer. Normalize the two vectors to make them unit vectors then add them to get $[13, 17, 10]/15$.

6. Given two vectors $\vec{v}, \vec{w}$ which are perpendicular. Under which condition is $\vec{v} + \vec{w}$ perpendicular to $\vec{v} - \vec{w}$? Answer: Find the dot product of $\vec{v} + \vec{w}$ with $\vec{v} - \vec{w}$ and set it zero.

7. Is the angle between $[1, 2, 3]$ and $[-15, 2, 4]$ acute or obtuse? Answer: the dot product is 1. Ah! Cute!