LINE INTEGRALS.

2D: If \( F(x, y) \) is a vector field in the plane and \( \gamma : t \mapsto \vec{r}(t) \) is a curve, then \( \int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \) is called the line integral of \( F \) along the curve \( \gamma \).

3D: If \( F(x, y, z) \) is a vector field in space and \( \gamma : t \mapsto \vec{r}(t) \) is a curve, then \( \int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \) is called the line integral of \( F \) along the curve \( \gamma \).

NOTATION. The short-hand notation \( \int_a^b F \cdot dr \) is also used. In the literature, where curves are sometimes written as \( r(t) = (x(t), y(t), z(t)) \) or \( r(t) \), the notation \( \int_a^b F \cdot dr \) or \( \int_a^b F \cdot dr \) appears. For simplicity, we leave out below the arrows above the \( r(t) \) and \( F(r(t)) \) even so they are vectors.

WRITTEN OUT. If \( F = (M, N) \) and \( \vec{r} = (x(t), y(t)) \), we can write \( \int_a^b M(x(t), y(t))x'(t) + N(x(t), y(t))y'(t) \, dt \).

MORE NOTATION. One also can write \( \int_a^b M(x, y) \, dx + N(x, y) \, dy \). Warning: this later notation is only possible for certain type of curves and rather old fashioned. Avoid it.

EXAMPLE: Work. If \( F(x, y, z) \) is a force field, then the line integral \( \int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \) is called work.

EXAMPLE: Electric potential. If \( E(x, y, z) \) is an electric field, then the line integral \( \int_a^b E(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \) is called electric potential.

EXAMPLE: Gradient field. If \( F(x, y, z) = \nabla U(x, y, z) \) is a gradient field, then as we will see next hour \( \int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = U(r(b)) - U(r(a)) \). The gradient field has physical relevance. For example, if \( U(x, y, z) \) is the pressure distribution in the atmosphere, then \( \nabla U(x, y, z) \) is the pressure gradient roughly the wind velocity field.

EXAMPLE 1. Let \( \gamma : t \mapsto \vec{r}(t) = (\cos(t), \sin(t)) \) be a circle parametrized by \( t \in [0, 2\pi] \) and let \( F(x, y) = (-y, x) \). Calculate the line integral \( I = \int_\gamma F \cdot dr \).

ANSWER: We have \( I = \int_0^{2\pi} F(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_0^{2\pi} (-\cos(t), \sin(t)) \cdot (-\cos(t), \sin(t)) \, dt = \int_0^{2\pi} \cos^2(t) + \sin^2(t) \, dt = 2\pi \).

EXAMPLE 2. Let \( r(t) \) be a curve given in polar coordinates as \( r(t) = (\cos^2(t), \cos(t)\sin(t)) \). The velocity vector is then \( r'(t) = (-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t)) = (x(t), y(t)) \). The line integral is

\[
\int_0^\pi F(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_0^\pi (\cos^2(t)\sin(t), 0) \cdot (-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t)) \, dt \\
= -2 \int_0^\pi \sin^2(t)\cos^4(t) \, dt = -2(t/16 + \sin(2t)/64 - \sin(4t)/64 - \sin(6t)/192)|_0^\pi = -\pi/8
\]

WORK. If \( F \) is a force field and \( r(t) \) a path of a body, then \( F(\vec{r}(t)) \) is the force acting on the body. The component of that force in the velocity direction is \( G(t) = F(\vec{r}(t)) \cdot \vec{r}'(t)/|\vec{r}'(t)| \). For some small time \( dt \), the body will move a distance \(|\vec{r}'(t)|\, dt\). In physics, \( G(t) \, ds \) is the amount of work done when traveling this distance. Integrating up gives the total work or energy \( W = \int_a^b G(\vec{r}(t)) \, ds = \int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \).

\( W = \int_\gamma F \, ds \) is the energy gained by a body traveling along the path \( \gamma \) in a force field \( F \).
We could get the electric field strength \( E \) of the ions is parallel to the field, we know that \( E\Delta r \) is the voltage difference between the two plates and \( E \) is the energy difference of the particle. Because \( F = eE \), where \( e \) is the charge of the ion and the velocity \( r' \) of the ions is parallel to the field, we know that \( E\Delta r \) is the voltage difference and \( F\Delta r \) is the energy difference which is \( m\Delta v^2/2 \), where \( m \) is the mass and \( v \) the velocity of the ion, we could get the electric field strength \( E = mv^2/(2Lc) \).

### Adding and Subtracting Curves.

If \( \gamma_1, \gamma_2 \) are curves, then \( \gamma_1 + \gamma_2 \) denotes the curve obtained by traveling first along \( \gamma_1 \), then along \( \gamma_2 \). One writes \(-\gamma \) for the curve \( \gamma \) traveled backwards and \( \gamma_1 - \gamma_2 = \gamma_1 + (-\gamma_2) \).

**Examples.** If \( \gamma_1(t) = (t,0), \gamma_2(t) = (1,(t - 1)) \) for \( t \in [0,1] \), \( \gamma_3(t) = (1 - (t - 2),1) \) for \( t \in [2,3] \), \( \gamma_4(t) = (0,1 - (t - 3)) \) for \( t \in [3,4] \), then \( \gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \) for \( t \in [0,4] \) is the path which goes around a unit square. The path \(-\gamma\) travels around in the clockwise direction.

### Calculating with Line- Integrals.

- \( \int_{\gamma_1} F \cdot dr + \int_{\gamma_2} G \cdot dr = \int_{\gamma} (F + G) \cdot dr \).
- \( \int_{\gamma} cF \cdot dr = c \int_{\gamma} F \cdot dr \).
- \( \int_{\gamma_1 + \gamma_2} F \cdot dr = \int_{\gamma_1} F \cdot dr + \int_{\gamma_2} F \cdot dr \).
- \( \int_{-\gamma} F \cdot dr = -\int_{\gamma} F \cdot dr \).

### Volume-Pressure.

Processes involving gases or liquids can be described in a **Volume-pressure diagram**.

A periodic processes like a refrigerator defines a closed cycle \( \gamma : t \mapsto r(t) = (V(t),p(t)) \) in the \( V - p \) plane. The curve is parameterized by the time \( t \). At a given time the gas has a specific volume \( V(t) \) and a specific pressure \( p(t) \). Consider the vector field \( F(V,p) = (p,0) \) and a closed curve \( \gamma \) and the line integral \( \int_{\gamma} F \cdot ds \). Writing it out, we get \( \int_{0}^{2\pi} (p(t),0) \cdot (V'(t),p'(t)) \ dt = \int_{0}^{2\pi} p(t)V'(t) \ dt = \int_{0}^{2\pi} p dV. \)

If the volume of the gas changes under pressure \( p \), then the work on the system is \( pdV \). On the other hand, if the volume is kept constant, then for a gas, one does not do work on the system, when changing the pressure. Processes described by this approximation are called **adiabatic**.

For example, if the volume is decreased under high pressure and increased under low pressure then we do the work \( \int_{0}^{2\pi} p dV \). Let compute if \( r(t) = (2 + \cos(t), 2 + \sin(t)) \) for \( t \in [0,2\pi] \) and \( F(V,p) = (p,0) \).

\[
r'(t) = (-\sin(t), \cos(t)), \quad F(r(t)) = (2 + \sin(t),0).
\]

so that \( F(r(t)) \cdot r'(t) = -\sin(t)(2 + \sin(t)) \) and

\[
\int_{0}^{2\pi} F(r(t)) \cdot r'(t) \ dt = -\int_{0}^{2\pi} \sin^2(t) \ dt = \pi.
\]