MAXIMAL AREA OF RECTANGLE. We want to extremize the area of a rectangle for which the length of the boundary is fixed 4. If the sides are \( x \) and \( y \), then we want to extremize \( f(x, y) = xy \) under the constraint \( g(x, y) = 2x + 2y = 4 \). The Lagrange equations \( y = 2\lambda, x = 2\lambda \) show that \( x = y \) and so \( x = y = 1 \).

The last problem could also been solved by substituting \( y = 2 - x \) into the area formula \( A = xy = x(2 - x) \) leading to a one-dimensional extremal problem: maximize \( f(x) = x(2 - x) \) on the interval \([0, 2]\). To do so, we have to find the extrema inside the interval and then consider also the boundary points \( x = 0, x = 2 \). Again, we get \( x = 1 \).

VOLUME OF CUBE. Extremize the volume \( f(x, y, z) = xyz \) of a box with fixed surface area \( xy + yz + xz = 3 \). To solve \( yz = \lambda(y + z), xz = \lambda(x + z), xy = \lambda(x + y), xy + yz + xz = 1 \), take quotients: \( z/x = (y + z)/(y + x), z/y = (z + x)/(y + x) \) which gives \( z(y + x) = x(y + z), z(y + x) = y(z + x) \) so that either \( xz = yz \) or \( xz = 0 \). Similarly, we get \( y = z \) or \( y = 0 \). The solution is \( x = y = z = 1 \).

ANOTHER SOLUTION. For a solution without Lagrange multipliers, we would plug in \( z = (1 - xy)/(y + x) \) and try to find the maximum of \( f(x, y) = xy(1 - xy)/(y + x) \) on the domain \( D = \{ x > 0, y > 0, xy \leq 1 \} \).

We first would have to find critical points inside the region \( D \): 
\[
    f_x(x, y) = -y(1 - 2xy)/(x + y) - xy(1 - xy)/(x + y)^2 = 0
\]
\[
    f_y(x, y) = -x(1 - 2xy)/(x + y) - xy(1 - xy)/(x + y)^2 = 0
\]
The difference of these two equations gives \( (x - y)(1 - 2xy) = 0 \) so that either \( x = y \) or \( xy = 1/2 \). The second case can not give us \( f_x = f_y = 0 \). The first condition \( x = y \) gives \( x = y = 1 \) which is not inside the region. However, on the boundary \( g(x, y) = xy = 1 \), the Lagrange equations \( \nabla f = \lambda \nabla g \) have a solution with \( (x, y) = (1, 1) \).

The example illustrates the power of Lagrange multipliers. The substitution method is more complicated.

TWO CONSTRAINTS. (informal) The calculation with Lagrange multipliers can be generalized: if the goal is to optimize a function \( f(x, y, z) \) under the constraints \( g(x, y, z) = c, h(x, y, z) = d \), take the Lagrange equations 
\[
\nabla f = \lambda \nabla g + \mu \nabla h, g = c, h = d
\]
which are 5 equations for the 5 unknowns \( x, y, z, \lambda, \mu \). Geometrically the gradient of \( f \) is in the plane spanned by the gradients of \( g \) and \( h \).
(This is the plane orthogonal to the curve \( \{ g = c, h = d \} \).)

GENERAL PROBLEM. Given a region \( G \) whose boundary is given by \( g(x, y) = c \). The task to maximize or minimize \( f(x, y) \) on \( G \) has the following steps:
I) Find extrema inside the region: compute critical points \( \nabla f = (0, 0) \) and classify them using the second derivative test.
II) Find extrema on the boundary using Lagrange: \( \nabla f = \lambda \nabla g, g = c \).
III) Compare the values of the functions obtained in I) and II) to find the maximum or minimum.
EXAMPLE. Extremize \( f(x, y) = 3x^2 - 4x - y^2 \) on the disc \( x^2 + y^2 \leq 1 \).

I) Inside the disc. There is only one critical point \((2/3, 0)\). The discriminant \( D = -6 \) so that \((2/3, 0)\) is a saddle point.

II) On the boundary solve \( 6x - 4 = 2Ax, -2y = 2\lambda y \). There are four solutions: \((1/2, -\sqrt{3}/2), (1/2, +\sqrt{3}/2), (1,0), (-1,0)\).

III) A list of all candidates:

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>(f(x,y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((2/3, 0))</td>
<td>(-4/3)</td>
</tr>
<tr>
<td>((1/2, -\sqrt{3}/2))</td>
<td>(-2)</td>
</tr>
<tr>
<td>((1/2, +\sqrt{3}/2))</td>
<td>(-2)</td>
</tr>
<tr>
<td>((1,0))</td>
<td>(-1)</td>
</tr>
<tr>
<td>((-1,0))</td>
<td>(7)</td>
</tr>
</tbody>
</table>

reveals that \((-1,0)\) is the maximum and \((1/2, -\sqrt{3}/2)\) are minima.

IN MATHEMATICA.

Here is how a machine solves the above problem. After defining the functions \( f \) and \( g \), the machine solves first the equations leading to critical points, and then the Lagrange equations (we put \( L = \lambda \)).

\[
\begin{align*}
  f[x_, y_] &:= 3x^2 - 4x - y^2 \\
  g[x_, y_] &:= x^2 + y^2 - 1 \\
  \text{Solve}\{D[f[x, y], x] == 0, D[f[x, y], y] == 0, \{x, y\}\} \\
  \text{Solve}\{D[f[x, y], x] == L \cdot D[g[x, y], x], D[f[x, y], y] == L \cdot D[g[x, y], y], g[x, y] == 0\}, \{x, y, L\}\end{align*}
\]

TRICKY LAGRANGE PROBLEM. Let \( p \) and \( q \) be positive constants such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Use the method of Lagrange Multipliers to prove that for any \( x > 0, y > 0 \), the following inequality is true:

\[ xy \leq \frac{x^p}{p} + \frac{y^q}{q} . \]

SOLUTION.

We have to find the maximum of \( f(x, y) = xy \) \((x > 0, y > 0)\) under the constraint \( \frac{x^p}{p} + \frac{y^q}{q} = c \).

The Lagrange equations \( x = \lambda x^{p-1}, y = \lambda y^{q-1} \) gives \( y/x = x^{p-1}/y^{q-1} \) so that \( y^q = x^p \).

Plugging this into \( x^p/q + y^q = c \) gives \( x^p(1/p + 1/q) = c \) or \( x = c^{1/p} \) and so \( y = c^{1/q} \). The maximal value of \( f(x, y) = xy \) is \( c^{1/p}c^{1/q} = c \). Therefore, everywhere

\[ xy = f(x, y) \leq c = x^p/p + y^q/q . \]

SNELLS LAW of refraction is the problem to determine the fastest path between two points, if the path crosses a border of two media and the media have different indices of refraction. The law can be derived from Lagrange:

PROBLEM. A light ray travels from \( A = (-1,1) \) to the point \( B = (1,-1) \) crossing a boundary between two media (air and water). In the air \((y > 0)\) the speed of the ray is \( v_1 = 1 \) (in units of speed of light). In the second medium \((y < 0)\) the speed of light is \( v_2 = 0.9 \). The light ray travels on a straight line from \( A \) to a point \( P = (x, 0) \) on the boundary and on a straight line from \( P \) to \( B \). Verify Snell’s law of refraction \( \sin(\theta_1)/\sin(\theta_2) = v_1/v_2 \), where \( \theta_1 \) is the angle the ray makes in air with the \( y \) axis and where \( \theta_2 \) is the angle, the ray makes in water with the \( y \) axis.

SOLUTION. Minimize \( f(x,y) = \sqrt{(-1-x)^2 + y^2/v_1} + \sqrt{(1-x)^2 + y^2/v_2} = l_1/v_1 + l_2/v_2 \) under the constraint \( G(x,y) = y = 0 \). The Lagrange equations show that \( f_x(x,y) = 0 \). This is already Snells law because \( f_x = v_12(x+1)/(2l_1) + v_22(1-x)/(2l_2) = 0 \) means \( v_1/v_2 = \sin(\theta_1)/\sin(\theta_2) \). If \( v_1 \) is larger, then \( \theta_1 \) is larger.