Lecture 10: The chain rule

How do we take the derivative of a composition of functions like $f(x) = \sin(x^7)$? The product rule does not work here. The functions are "chained", we evaluate first x^7 then apply sin to it. In order to differentiate, we the derivative of the first function we evaluate x^7 then multiply this with the derivative of the function sin at x^7. The answer is $7x^6 \cos(x^7)$.

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) .$$

The chain rule follows from the identity

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + (g(x+h) - g(x))) - f(g(x))]}{[g(x+h) - g(x)]} \cdot \frac{[g(x+h) - g(x)]}{h} .$$

Write $H(x) =$ g(x+h)-g(x) in the first part on the right hand side

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + H) - f(g(x))]}{H} \cdot \frac{g(x+h) - g(x)}{h} .$$

As $h \to 0$, we also have $H \to 0$ and the first part goes to $f'(g(x))$ and the second factor has $g'(x)$ as a limit.

1 Find the derivative of $f(x) = (4x - 1)^{17}$. **Solution** The inner function is $g(x) = 4x - 1$. It has the derivative 4. We get therefore $f'(x) = 17(4x - 1)^6 \cdot 4 = 28(4x - 1)^6$. Remark. We could have expanded out the power $(4x - 1)^{17}$ first and avoided the chain rule. Avoiding the **chain rule** is called the **pain rule**.

2 Find the derivative of $f(x) = \sin(\pi \cos(x))$ at $x = 0$. **Solution**: applying the chain rule gives $\cos(\pi \cos(x)) \cdot (-\pi \sin(x))$.

3 For linear functions $f(x) = ax + b, g(x) = cx + d$, the chain rule can readily be checked: we have $f(g(x)) = a(cx + d) + b = acx + ad + b$ which has the derivative ac. This agrees with the definition of f times the derivative of g. You can convince you that the chain rule is true also from this example since if you look closely at a point, then the function is close to linear.

One of the cool applications of the chain rule is that we can compute derivatives of inverse functions:

4 Find the derivative of the natural logarithm function $\log(x)$ [1] **Solution** Differentiate the identity $\exp(\log(x)) = x$. On the right hand side we have 1. On the left hand side the chain rule gives $\exp(\log(x)) \log'(x) = x \log'(x) = 1$. Therefore $\log'(x) = 1/x$.

$\frac{d}{dx} \log(x) = 1/x.$

[1] We always write $\log(x)$ for the natural log. The ln notation is old fashioned and only used in obscure places like calculus books.

Denote by $\arccos(x)$ the inverse of $\cos(x)$ on $[0,\pi]$ and with $\arcsin(x)$ the inverse of $\sin(x)$ on $[-\pi/2,\pi/2]$.

5 Find the derivative of $\arcsin(x)$. **Solution**. We write $x = \sin(\arcsin(x))$ and differentiate.

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} \; .$$

6 Find the derivative of $\arccos(x)$. **Solution**. We write $x = \cos(\arccos(x))$ and differentiate.

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}} \; .$$

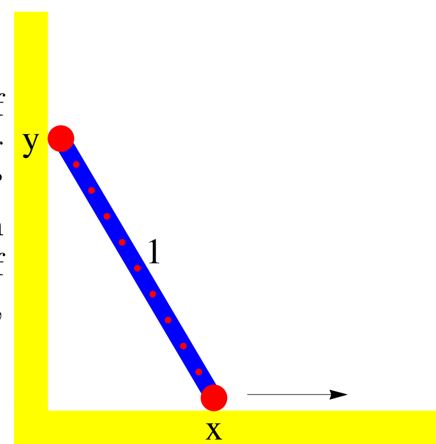
7 $f(x) = \sin(x^2+3)$. Then $f'(x) = \cos(x^2+3)2x$.

8 $f(x) = \sin(\sin(\sin(x)))$. Then $f'(x) = \cos(\sin(\sin(x)))\cos(\sin(x))\cos(x)$.

Why is the chain rule called "chain rule". The reason is that we can chain even more functions together.

9 Lets compute the derivative of $\sin(\sqrt{x^5-1})$ for example. **Solution:** This is a composition of three functions $f(g(h(x)))$, where $h(x) = x^5-1$, $g(x) = \sqrt{x}$ and $f(x) = \sin(x)$. The chain rule applied to the function $\sin(x)$ and $\sqrt{x^5-1}$ gives $\cos(\sqrt{x^5-1})\frac{d}{dx}\sqrt{x^5-1}$. Apply now the chain rule again for the derivative on the right hand side.

10 Here is a famous **falling ladder problem**. A stick of length 1 slides down a wall. How fast does it hit the floor if it slides horizontally on the floor with constant speed? The ladder connects the point $(0,y)$ on the wall with $(x,0)$ on the floor. We want to express y as a function of x. We have $y = f(x) = \sqrt{1-x^2}$. Taking the derivative, assuming $x' = 1$ gives $f'(x) = -2x/\sqrt{1-x^2}$.



In reality, the ladder breaks away from the wall. One can calculate the force of the ladder to the wall. The force becomes zero at the **break-away angle** $\theta = \arcsin((2v^2/(3g))^{2/3})$, where g is the gravitational acceleration and $v = x'$ is the velocity.

11 For the brave: find the derivative of $f(x) = \cos(\cos(\cos(\cos(\cos(\cos(\cos(x)))))))$.

Practice

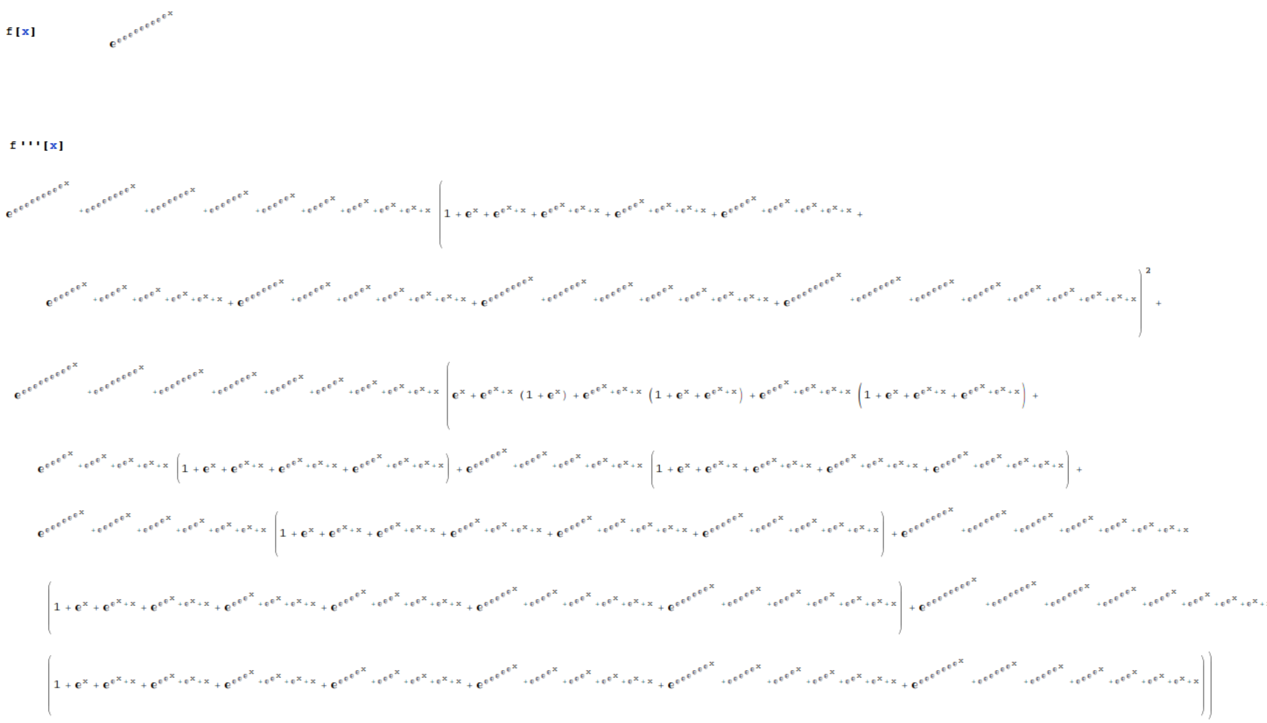
12 Take the derivative of $f_3(x) = e^{e^{e^x}}$.
Solution We can also write this as $\exp(\exp(\exp(x))$. The derivative is

$$\exp(\exp(\exp(x)))\exp(\exp(x))\exp(x) \ .$$

13 Lets push that to the extreme and differentiate

$$f(x) = \exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(x))))))))))$$

two times. Here is a picture of the formula obtained when running this in Mathematica:



14 Find the derivative of $1/\sin(x)$ using the quotient rule.
Solution $-\cos(x) \cdot 1/\sin^2(x)$.

15 Find the derivative of $f(x) = 1/\sin(x)$ using the chain rule.
Solution. The outer function is $f(x) = 1/x$. Therefore $f'(x) = -\cos(x)/\sin^2(x)$.

Homework

1 Find the derivatives of the following functions:

a) $f(x) = \sin(\log(x))$

b) $f(x) = \tan(x^9)$

c) $f(x) = \exp(1/(1+x^2))$

d) $(2+\sin(x))^{-5}$

2 Find the derivatives of the following functions at $x = 1$.

a) $f(x) = x^8 \log(x)$. (where log is natural log)

b) $\sqrt{x^5+1}$

c) $(1+x^2+x^4)^{100}$

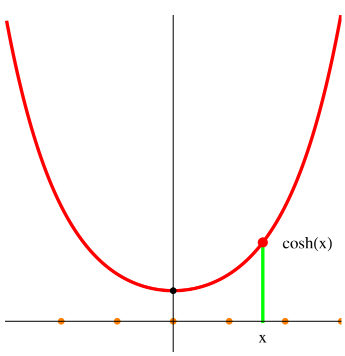
d) $\frac{5x^4}{2\sqrt{x^5+1}}$

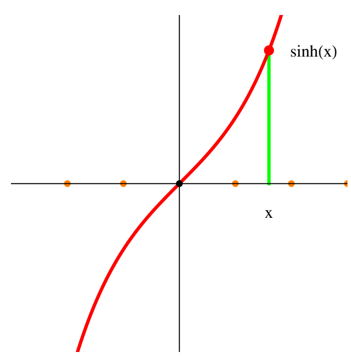
3 a) Find the derivative of $f(x) = 1/x$ by differentiating the identity $xf(x) = 1$ and using the product rule.
b) Find the derivative of $f(x) = \mathrm{arccot}(x)$ by differentiating $\cot(\mathrm{arccot}(x)) = x$ and using the chain rule.

4 a) Find the derivative of $\sqrt{x}$ by differentiating the identity $f(x)^2 = x$ leaving f as it is and solving for $f'(x)$.
b) Find the derivative of $x^{m/n}$ by differentiating the identity $f(x)^n = x^m$ leaving f as it is and solving for $f'(x)$.

The function $f(x) = [\exp(x) + \exp(-x)]/2$ is called $\cosh(x)$.
The function $f(x) = [\exp(x) - \exp(-x)]/2$ is called $\sinh(x)$.
They are called **hyperbolic cosine** and **hyperbolic sine**. The first is even, the second is odd. You can see directly using $\exp'(x) = \exp(x)$ and $\exp'(-x) = -\exp(-x)$ that $\sinh'(x) = \cosh(x)$ and $\cosh'(x) = \sinh(x)$. Furthermore $\exp = \cosh + \sinh$ writes exp as a sum of an even and odd function. You can easily check that $\cosh^2(x) - \sinh^2(x) = 1$.

5 a) Find the derivative of the inverse $\mathrm{arccosh}(x)$ of $\cosh(x)$ by using the chain rule.
b) Find the derivative of the inverse $\mathrm{arcsinh}(x)$ of $\sinh(x)$ by using the chain rule.





The cosh function is the shape of a chain hanging at two points. The shape is the hyperbolic cosine. You can use the identity $\cosh^2(x) - \sinh^2(x) = 1$.