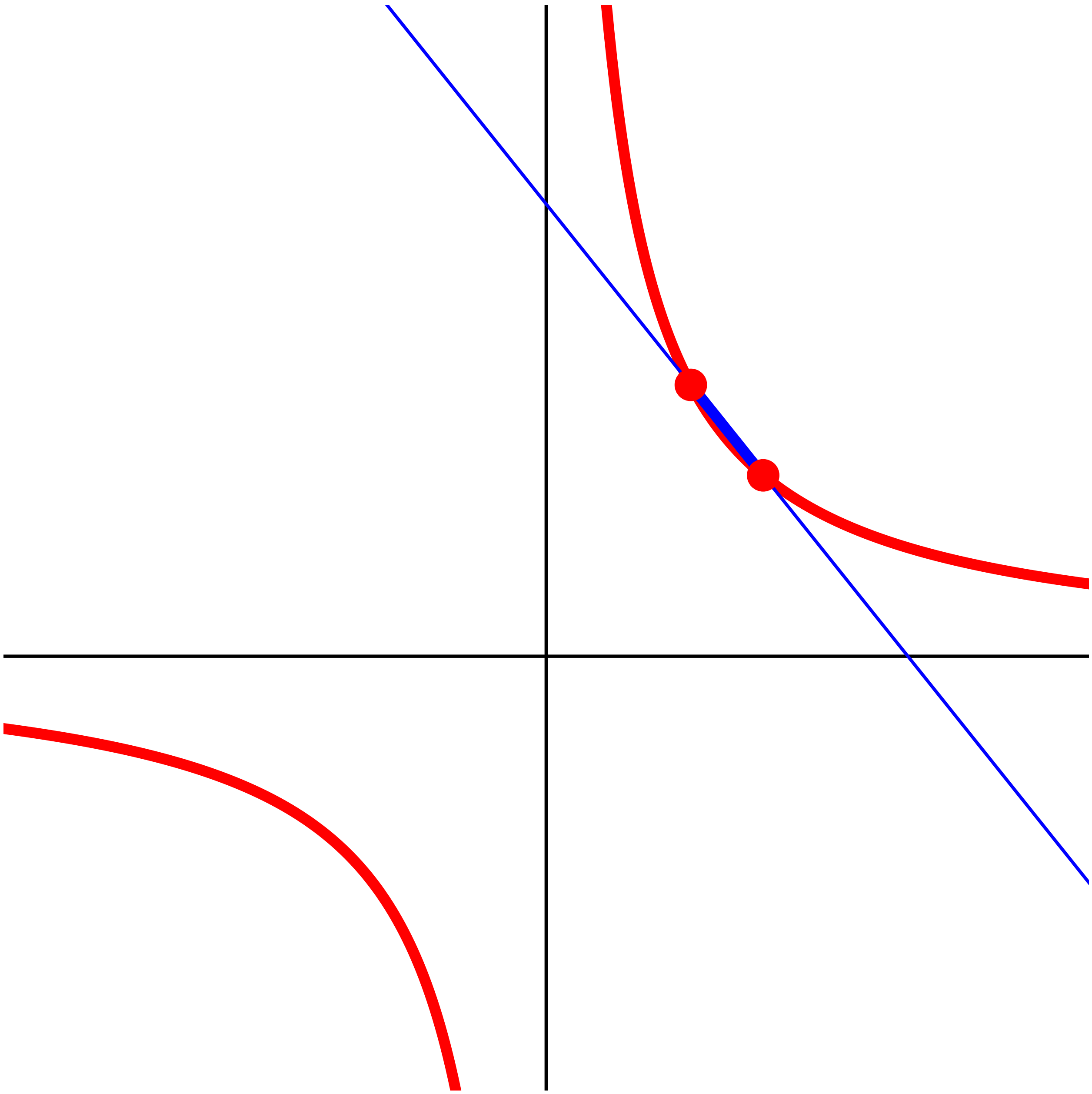
Lecture 7: Worksheet

Rate of change

We compute the derivative of $f(x) = 1/x$ by taking limits.

a) Simplify $\frac{1}{x+h} - \frac{1}{x}$.
b) Now take the limit $\frac{1}{h}[\frac{1}{x+h} - \frac{1}{x}]$ when $h \to 0$.
c) Is there any point where $f'(x) > 0$?



Derivatives

Differentiation rules

$$\frac{d}{dx}x^n = nx^{n-1}$$
$$e^{ax} = ae^{ax}$$
$$\frac{d}{dx}\cos(ax) = -a\sin(ax)$$
$$\frac{d}{dx}\sin(ax) = a\cos(ax)$$

1 Find the derivatives of the function $f(x) = \sin(3x) + x^5$

2 Find the derivative of $f(x) = \cos(7x) - 8x^4$.

3 Find the derivative of $f(x) = e^{5x} + \cos(2x)$.

Two trigonometric identities

During lecture, we need the identities

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

and

$$\sin(x+y) = \cos(x)\sin(y) + \sin(x)\sin(y) \; .$$

You might know these identities from pre-calculus.

We do not work with complex numbers in this course but the verification of these identities is so elegant with the **Euler** formula

$$e^{ix} = \cos(x) + i\sin(x)$$

that it is a crime to prove the identities differently. Just compare the real and imaginary components of

$$e^{i(x+y)} = \cos(x+y) + i\sin(x+y)$$

with the real and imaginary parts of

$$\begin{aligned} e^{ix}e^{iy} &= (\cos(x) + i\sin(x))(\cos(y) + i\sin(y)) \\ &= \cos(x)\cos(y) - \sin(x)\sin(y) + i(\cos(x)\sin(y) + \sin(x)\cos(y)) \; . \end{aligned}$$

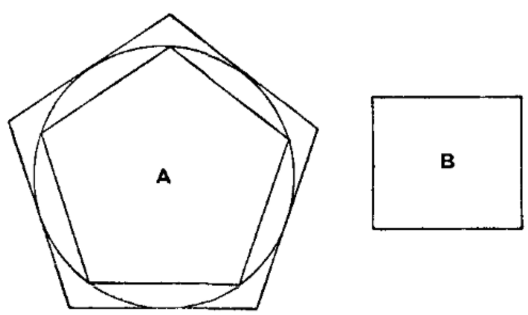
A page from Archimedes

Proposition 6.

"Similarly we can show that, *given two unequal magnitudes and a sector, it is possible to circumscribe a polygon about the sector and inscribe in it another similar one so that the circumscribed may have to the inscribed a ratio less than the greater magnitude has to the less.*

And it is likewise clear that, *if a circle or a sector, as well as a certain area, be given, it is possible, by inscribing regular polygons in the circle or sector, and by continually inscribing such in the remaining segments, to leave segments of the circle or sector which are [together] less than the given area.* For this is proved in the *Elements* [Eucl. XII. 2].

But it is yet to be proved that, *given a circle or sector and an area, it is possible to describe a polygon about the circle or sector, such that the area remaining between the circumference and the circumscribed figure is less than the given area.*"



The proof for the circle (which, as Archimedes says, can be equally applied to a sector) is as follows.

Let A be the given circle and B the given area.

Now, there being two unequal magnitudes $A + B$ and A, let a polygon (C) be circumscribed about the circle and a polygon (I) inscribed in it [as in Prop. 5], so that

$$C : I < A + B : A \quad \text{.....................(1).}$$

The circumscribed polygon (C) shall be that required.