Lecture 2: Worksheet

In this lecture, we get acquainted with the most important functions.

Trigonometric functions

The cosine and sine functions can be defined geometrically by the coordinates \((\cos(x), \sin(x))\) of a point on the unit circle. The tangent function is defined as \(\tan(x) = \frac{\sin(x)}{\cos(x)}\).

\[
\begin{align*}
\cos(x) &= \text{adjacent side/hypothenuse} \\
\sin(x) &= \text{opposite side/hypothenuse} \\
\tan(x) &= \text{opposite side/adjacent side}
\end{align*}
\]

**Pythagoras theorem** gives us the important identity

\[
\cos^2(x) + \sin^2(x) = 1
\]

Define also \(\cot(x) = \frac{1}{\tan(x)}\). Less important but sometimes used are \(\sec(x) = \frac{1}{\cos(x)}\), \(\csc(x) = \frac{1}{\sin(x)}\).

1. Find \(\cos(\pi/3), \sin(\pi/3)\).
2. Where are the roots of \(\cos\) and \(\sin\)?
3. Find \(\tan(3\pi/2)\) and \(\cot(3\pi/2)\).
4. Find \(\cos(3\pi/2)\) and \(\sin(3\pi/2)\).
5. Find \(\tan(\pi/4)\) and \(\cot(\pi/4)\).
\[ \cos(x) \times L \times \sin(x) \times L \times 1 \]
\[ \cos(x) \times 2\pi \]
\[ \sin(x) \times 2\pi \]
\[ \tan(x) \times \frac{\pi}{2} \]
The exponential function

The function $f(x) = 2^x$ is first defined for positive integers like $2^{10} = 1024$, then for all integers with $f(0) = 1$, $f(-n) = 1/f(n)$. Using roots, it can be defined for rational numbers like $2^{3/2} = 8^{1/2} = \sqrt{8} = 2.828...$. Since the function $2^x$ is monotonone on the set of rationals, we can fill the gaps and define $f(x)$ for any real $x$. By taking square roots again and again for example, we see $2^{1/2}, 2^{1/4}, 2^{1/8}, ...$ we approach $2^0 = 1$.

There is nothing special about 2 and we can take any positive base $a$ and define the exponential $a^x$. It satisfies $a^0 = 1$ and the remarkable rule:
\[a^{x+y} = a^x \cdot a^y\]

It is spectacular because it provides a link between addition and multiplication.

We will especially consider the exponential \(\exp_h(x) = (1 + h)^{x/h}\), where \(h\) is a positive parameter. This is a super cool exponential because it satisfies \(\exp_h(x + h) = (1 + h)\exp_h(x)\) so that

\[
\frac{\exp_h(x + h) - \exp_h(x)}{h} = \exp_h(x).
\]

We will see this relation again. For cocktail party conversation say that ”the quantum derivative of the quantum exponential is the function itself for any Planck constant \(h\)”.

For \(h = 1\), we have the function \(2^x\) we have started with. In the limit \(h \to 0\), we get the important exponential function \(\exp(x)\) which we also call \(e^x\). For \(x = 1\), we get the **Euler number** \(e = e^1 = 2.71828\ldots\).

1. What is \(2^{-5}\)?
2. Find \(2^{1/2}\).
3. Find \(27^{1/3}\).
4. Why is \(A = 2^{3/4}\) smaller than \(B = 2^{4/5}\)? Take the 20th power!
5. Assume \(h = 2\) find \(\exp_h(4)\).