Lecture 10: The chain rule

How do we take the derivative of a composition of functions? It is the chain rule which will allow us to compute derivatives like for \( f(x) = \sin(x^2) \) which is a composition of two functions \( f(x) = x^2 \) and \( g(x) = \sin(x) \). The product rule does not work here. The functions are "chained", we evaluate first \( x^2 \) then apply \( \sin \) to it. In order to differentiate, we the derivative of the first function we evaluate \( x^2 \) then multiply this with the derivative of the function \( \sin \) at \( x^2 \). The answer is \( 7x^2 \cos(x^2) \).

\[
\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).
\]

The chain rule follows from the identity

\[
\frac{f(g(x) + h) - f(g(x))}{h} = \frac{[f(g(x) + (g(x) + h) - g(x))] - f(g(x))}{h} \cdot \frac{g(x + h) - g(x)}{h}.
\]

Write \( H(x) = g(x + h) - g(x) \) in the first part on the right hand side

\[
\frac{f(g(x + h)) - f(g(x))}{h} = \frac{[f(g(x) + H) - f(g(x))]}{H} \cdot \frac{g(x + h) - g(x)}{h}.
\]

As \( h \to 0 \), we also have \( H \to 0 \) and the first part goes to \( f'(g(x)) \) and the second factor has \( g'(x) \) as a limit. The chain rule is one reason why classical calculus is so elegant: \( D[f(g(x))] = (Df)(g(x))D(g(x)) \). The \( h \) has changed.

**1** Find the derivative of \( f(x) = (4x - 1)^7 \). **Solution** The inner function is \( g(x) = 4x - 1 \). It has the derivative 4. We get therefore \( f'(x) = 17(4x - 1)^6 \cdot 4 = 68(x - 1)^6 \). Remark. We could have expanded out the power \( (4x - 1)^7 \) first and avoided the chain rule. Avoiding the chain rule is called the **pain rule**.

**2** Find the derivative of \( f(x) = \sin(\pi \cos(x)) \) at \( x = 0 \). **Solution**: applying the chain rule gives \( \cos(\pi \cos(x)) \cdot (-\pi \sin(x)) \).

**3** For linear functions \( f(x) = ax + b, g(x) = cx + d \), the chain rule can readily be checked. We have \( f(g(x)) = a(cx + d) + b = acx + ad + b \) which has the derivative \( ac \). Indeed this is the definition of \( f \) times the derivative of \( g \). You can convince you that the chain rule is true also from this example since if you look closely at a point, then the function is close to linear.

One of the cool applications of the chain rule is that we can compute derivatives of inverse functions:

**4** Find the derivative of the natural logarithm function \( \log(x) \) \(^1\). **Solution** Differentiate the identity \( \exp(\log(x)) = x \). On the right hand side we have 1. On the left hand side the chain rule gives \( \exp(\log(x))^\prime \log(x) = x \log'(x) = 1 \). Therefore \( \log'(x) = 1/x \).

\(^1\)We always write \( \log(x) \) for the natural log. The In notation is old fashioned and only used in obscure places like calculus books and calculators from the last millennium.

\[
\frac{d}{dx} \log(x) = 1/x.
\]

Denote by \( \arccos(x) \) the inverse of \( \cos(x) \) on \([0, \pi]\) and with \( \arcsin(x) \) the inverse of \( \sin(x) \) on \([-\pi/2, \pi/2]\).

**5** Find the derivative of \( \arcsin(x) \). **Solution**: we write \( x = \sin(\arcsin(x)) \) and differentiate.

\[
\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1 - x^2}}.
\]

**6** Find the derivative of \( \arccos(x) \). **Solution**: we write \( x = \cos(\arccos(x)) \) and differentiate.

\[
\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1 - x^2}}.
\]

**7** \( f(x) = \sin(x^2 + 3) \). Then \( f'(x) = \cos(x^2 + 3)2x \).

**8** \( f(x) = \sin(\sin(x)) \). Then \( f'(x) = \cos(\sin(x))\cos(x) \).

Why is the chain rule called "chain rule"? The reason is that we can chain even more functions together.

**9** Lets compute the derivative of \( \sin(\sqrt{x^2 - 1}) \) for example. **Solution**: This is a composition of three functions \( f(g(h(x))) \), where \( h(x) = x^2 - 1, g(x) = \sqrt{x} \) and \( f(x) = \sin(x) \). The chain rule applied to the function \( \sin(x) \) and \( \sqrt{x^2 - 1} \) gives \( \cos(x) \sqrt{x^2 - 1} \frac{x}{2} \). Apply now the chain rule again for the derivative on the right hand side.

Here is the famous **falling ladder problem**. A stick of length 1 slides down a wall. How fast does it hit the floor if it slides horizontally on the floor with constant speed? The ladder connects the point \((0, y)\) on the wall with \((x, 0)\) on the floor. We want to express \( y \) as a function of \( x \). We have \( y = f(x) = \sqrt{1 - x^2} \). Taking the derivative, assuming \( x' = 1 \) gives \( f'(x) = -2x/\sqrt{1 - x^2} \).

In reality, the ladder breaks away from the wall. One can calculate the force of the ladder to the wall. The force becomes zero at the **break-away angle** \( \theta = \arcsin((2v^2/(3g))^{1/3}) \), where \( g \) is the gravitational acceleration and \( v = x' \) is the velocity.

**10** For the brave: find the derivative of \( f(x) = \cos(\cos(\cos(\cos(\cos(\cos(x))))))) \).
Homework

1. Find the derivatives of the following functions
   a) \( f(x) = \sin(\sqrt{x}) \)
   b) \( f(x) = \tan(1/x^2) \)
   c) \( f(x) = \exp(1/(1 + x)) \)
   d) \( (2 + \sin(x))^{-5} \)

2. Find the derivatives of the following functions at \( x = 1 \).
   a) \( f(x) = x^8 \log(x) \) (\( \log \) is natural log)
   b) \( \sqrt{x^5 + 1} \)

3. Find the derivative of \( f(x) = 1/x \) by differentiating the identity \( xf(x) = 1 \).
   b) Find the derivative of \( f(x) = \arccot(x) \) by differentiating \( \cot(\arccot(x)) = x \).

4. Find the derivative of \( f(x) = \sqrt{x} \) by differentiating the identity \( f(x)^2 = x \).
   b) Find the derivative of \( f(x) = x^{m/n} \) by differentiating the identity \( f(x)^n = x^m \).

   The function \( f(x) = (\exp(x) + \exp(-x))/2 \) is called \( \cosh(x) \).
   The function \( f(x) = (\exp(x) - \exp(-x))/2 \) is called \( \sinh(x) \).
   They are called **hyperbolic cosine** and **hyperbolic sine**. The first is even, the second is odd. You can see directly using \( \exp'(x) = \exp(x) \) and \( \exp'(-x) = -\exp(-x) \)
   that \( \sinh'(x) = \cosh(x) \) and \( \cosh'(x) = \sinh(x) \). Furthermore \( \exp = \cosh + \sinh \)
   writes \( \exp \) as a sum of an even and odd function.

5. Find the derivative of the inverse \( \arccosh(x) \) of \( \cosh(x) \).
   b) Find the derivative of the inverse \( \arcsinh(x) \) of \( \sinh(x) \).

   The \( \cosh \) function is the shape of a chain hanging at two points. The shape is the hyperbolic cosine.