5/14/2011: Second practice final exam

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points) No justifications are needed.

1) T F If a function $f(x)$ has a critical point 0 and $f''(0) = 0$ then 0 is neither a maximum nor minimum.

2) T F If $f' = g$ then $\int_0^x g(x) = f(x)$.

3) T F The function $f(x) = 1/x$ has the derivative $\log(x)$.

4) T F The function $f(x) = \arctan(x)$ has the derivative $1/\cos^2(x)$.

5) T F The fundamental theorem of calculus implies that $\int_a^b f'(x)\, dx = f(b) - f(a)$.

6) T F $\lim_{x\to 8} 1/(x-8) = \infty$ implies $\lim_{x\to 3} 1/(x-3) = \omega$.

7) T F A continuous function which satisfies $\lim_{x\to -\infty} f(x) = 3$ and $\lim_{x\to\infty} f(x) = 5$ has a root.

8) T F The function $f(x) = (x^7 - 1)/(x-1)$ has a limit at $x = 1$.

9) T F If $f_c(x)$ is an even function which depends on a parameter c and for $c < 3$ the function is concave up at $x = 0$ and for $c > 3$ the function is concave down at $x = 0$, then $c = 3$ is a catastrophe.

10) T F The function $f(x) = +\sqrt{x^2}$ has a continuous derivative 1 everywhere.

11) T F A rower rows on the Charles river leaving at 5 PM at the Harvard boat house and returning at 6 PM. If $f(t)$ is the distance of the rower at time t to the boat house, then there is a point where $f'(t) = 0$.

12) T F A global maximum of a function $f(x)$ on the interval $[0,1]$ is a critical point.

13) T F A continuous function on the interval $[2,3]$ has a global maximum and global minimum.

14) T F The intermediate value theorem assures that if f is continuous on $[a,b]$ then there is a root of f in (a,b).

15) T F On an arbitrary floor, a square table can be turned so that it does not wobble any more.

16) T F The derivative of $\log(x)$ is $1/x$.

17) T F If f is the marginal cost and $F = \int_0^x f(x)\, dx$ the total cost and $g(x) = F(x)/x$ the average cost, then points where $f = g$ are called "break even points".

18) T F At a function party, Log talks to Tan and the couple Sin and Cos, when she sees her friend Exp alone in a corner. Log: "What's wrong?" Exp: "I feel so lonely!" Log: "Go integrate yourself!" Exp sobbs: "Won't change anything." Log: "You are so right".

19) T F If a car's position at time t is $f(t) = t^3 - t$, then its acceleration at $t = 1$ is 6.

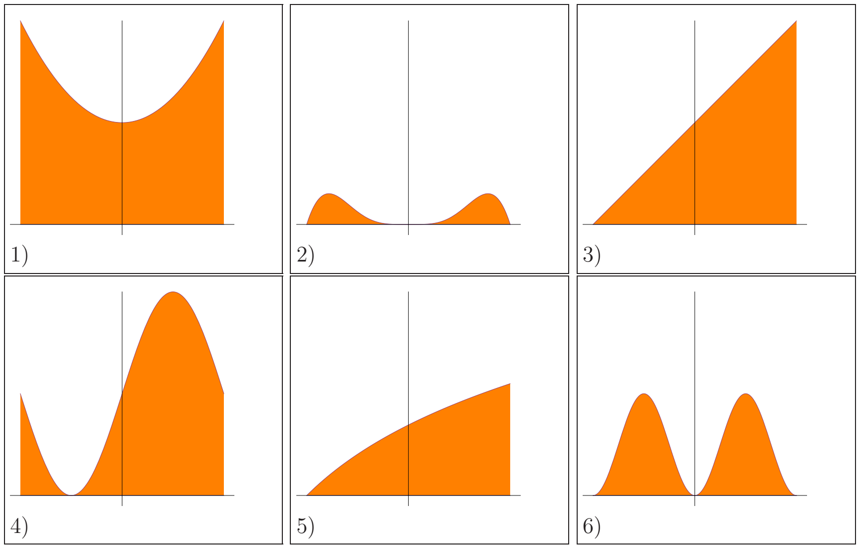
20) T F For trig substitution, the identities $u = \tan(x/2)$, $dx = \frac{2du}{(1+u^2)}$, $\sin(x) = \frac{2u}{1+u^2}$, $\cos(x) = \frac{1-u^2}{1+u^2}$ are useful.

Problem 2) Matching problem (10 points) No justifications are needed.

a) Match the following integrals with the graphs and (possibly signed) areas.

Integral	Enter 1-6
$\int_{-1}^{1} \sin(\pi x) x^3 \, dx.$	
$\int_{-1}^{1} \log(x+2) \, dx.$	
$\int_{-1}^{1} x + 1 \, dx.$	

Integral	Enter 1-6
$\int_{-1}^{1} (1 + \sin(\pi x)) \, dx.$	
$\int_{-1}^{1} \sin^2(x) \, dx.$	
$\int_{-1}^{1} x^2 + 1 \, dx.$	



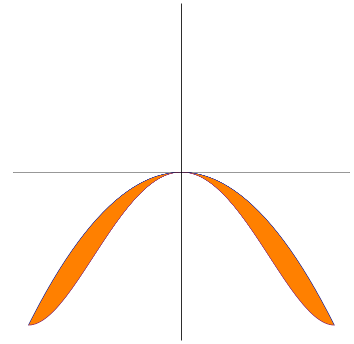
Problem 3) Matching problem (10 points) No justifications are needed.

Determine from each of the following functions, whether discontinuities appears at $x = 0$ and if, which of the three type of discontinuities it is at 0.

Function	Jump discontinuity	Infinity	Oscillation	No discontinuity
$f(x) = \log(\lvert x\rvert^5)$				
$f(x) = \cos(5/x))$				
$f(x) = \cot(1/x)$				
$f(x) = \sin(x^2)/x^3$				
$f(x) = \arctan(\tan(x - \pi/2))$				
$f(x) = 1/\tan(x)$				
$f(x) = 1/\sin(x)$				
$f(x) = 1/\sin(1/x)$				
$f(x) = \sin(\exp(x))/\cos(x)$				
$f(x) = 1/\log\lvert x\rvert$				

Problem 4) Area computation (10 points)

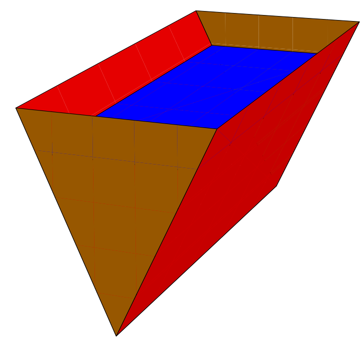
Find the area of the region enclosed by the graphs of the function $f(x) = x^4 - 2x^2$ and the function $g(x) = -x^2$.



Problem 5) Volume computation (10 points)

A farmer builds a bath tub for his warthog "Tuk". The bath has triangular shape of length 10 for which the width is $2z$ at height z. so that when filled with height z the surface area of the water is $20z$. If the bath has height 1, what is its volume?

P.S. Don't ask how comfortable it is to soak in a bath tub with that geometry. The answer most likely would be "Noink Muink".



Problem 6) Definite integrals (10 points)

Find the following definite integrals

a) (3 points) $\int_1^2 \sqrt{x} + x^2 - 1/\sqrt{x} + 1/x \, dx$.

b) (3 points) $\int_1^2 2x\sqrt{x^2-1} \, dx$

c) (4 points) $\int_1^2 2/(5x-1) \, dx$

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (3 points) $\int \frac{3}{1+x^2} + x^2 \, dx$

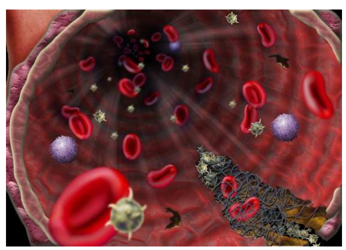
b) (3 points) $\int \frac{\tan^2(x)}{\cos^2(x)} \, dx$

c) (4 points) $\int \log(5x) \, dx$.

Problem 8) Implicit differentiation/Related rates (10 points)

A blood infusion bottle of cylindrical bag of volume $V = \pi r^2 h$ of changing radius r and fixed height $h = 2$ holds blood serum. Blood serum drips out at a constant rate $V'(t) = -1$. At which rate does the radius of the bag shrink when $r = 1/2$?

P.S: After experimenting with various bloody bag illustrations for this problem, Oliver fainted. He compromised to use a microscopic blood vessel picture and to make an other attempt to go for the "most gory exam problem ever written at Harvard" when writing the actual final exam.



Problem 9) Global extrema (10 points)

How about some sweets? We build a chocolate box which has 4 cubical containers of dimension $x \times x \times h$. The total material is $f(x,h) = 4x^2 + 12xh$ and the volume is $4x^2h$. Assume the volume is 4, what geometry produces the minimal cost?

Problem 10) Integration techniques (10 points)

Which integration technique works? It is enough to get the right technique and give the first step, not do the actual integration:

a) (2 points) $\int (x^2 + x + 1)\sin(x) \, dx$.

b) (2 points) $\int x/(1+x^2) \, dx$.

c) (2 points) $\int \sqrt{4-x^2} \, dx$.

d) (2 points) $\int \sin(\log(x))/x$.

e) (2 points) $\int \frac{1}{(x-6)(x-7)} \, dx$.

Problem 11) Hopital's rule (10 points)

Find the following limits as $x \to 0$ or state that the limit does not exist.

a) (2 points) $\frac{\tan(x)}{x}$

b) (2 points) $\frac{x}{\cos(x)-x}$.

c) (2 points) $x\log(1+x)/\sin(x)$.

d) (2 points) $x\log(x)$.

e) (2 points) $x/(1-\exp(x))$.

Problem 12) Applications (10 points)

The cumulative distribution function on $[0,1]$

$$F(x) = \frac{2}{\pi} \arcsin(\sqrt{x})$$

defines the **arc-sin** distribution.

a) Find the probability density function $f(x)$ on $[0,1]$.

b) Verify that $\int_0^1 f(x)\, dx = 1$.
Remark. The arc sin distribution is important chaos theory and probability theory. For the later see: [1]

[1]http://www.math.harvard.edu/ knill/oldtexas/Teaching/MA464/Text/33-onedimwalk2.ps