Analytic

Let $D \subset \mathbb{C}$ be an open set. A continuous function $f : D \to \mathbb{C}$ is called analytic in $D$, if for all $z \in D$ the complex partial derivative

$$\frac{\partial f}{\partial z} := \lim_{|h| \to 0} \frac{1}{h} (f(z + h) - f(z))$$

exists and is finite. Analytic functions are also called holomorphic. Properties: the sum and the product of analytic functions are analytic. If $f_n$ is a sequence of analytic maps which converges uniformly on compact subsets of $D$ to a function $f$, then $f$ is analytic too.

Complex partial derivative

Define the complex partial derivative of a complex function $f(z) = f(x + iy)$ in the complex plane is defined as

$$\frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}) f.$$

Conformal map

A conformal map is a differentiable map from the complex plane to the complex plane which preserves angles.
- every conformal map which has continuous partial derivatives is analytic.
- An analytic function $f$ is conformal at every point where its derivative $f'(z)$ is different from 0.

Dirichlet problem

Solution of the Dirichlet problem. If $D$ is a regular domain in the complex plane and $f$ is a continuous function on the boundary of $D$, then there exists a unique harmonic function $f$ on $D$ such that $h(z) = f(z)$ for all boundary points of $D$.

Dirichlet problem

Let $K$ be a compact subset of the complex plane. Let $P(K)$ the set of all Borel probability measure on $K$. A measure $\nu$ maximizing the potential theoretical energy in $P(K)$ is called an equilibrium measure of $K$. Properties:
- every compact $K$ has an equilibrium measure.
- if $K$ is not polar then the equilibrium measure is unique.
The fine topology on the complex plane is defined as the coarsest topology on the plane which makes all subharmonic functions continuous.

Frostman’s theorem

[Frostman’s theorem]: If $\nu$ is the equilibrium measure on a compact set $K$, then the potential $p_\nu$ of $\nu$ is bounded below by $I(\nu)$ everywhere on $C$. Furthermore, $p_\nu = I(\nu)$ everywhere on $K$ except on a $F_{\sigma}$ polar subset $E$ of the boundary of $K$.

A function $h$ on the complex plane is called harmonic in a region $D$ if it satisfies the mean value property on every disc contained in $D$.

Frostman’s theorem

Harmonic measure

A harmonic measure $w_D$ on a domain $D$ is a function from $D$ to the set of Borel probability measures on the boundary of $D$. The measure for $z$ is defined as the functional $g \mapsto H_D(g)(z)$, where $H_D(g)$ is the Perron function of $g$ on $D$.

- if the boundary of $D$ is non-polar, there exists a unique harmonic measure for $D$.
- if $D = \text{Im}(z) < 0$, then $w_D(z, a, b) = \arg((z - b)/(z - a))/\pi$

Harnack inequality

The Harnack inequality assures that for any positive harmonic function $h$ on the disc $D(w, R)$ and for any $r < R$ and $0 < t < 2\pi$

$$h(w)(R - r)/(R + r) \leq h(w + re^{it}) \leq h(w)(R + r)/(R - r)$$

Extended Liouville theorem

The extended Liouville theorem: if $f$ is subharmonic on the complex plane $C - E$, where $E$ is a closed polar set and $f$ is bounded above then $f$ is constant.

Generalized Laplacian

The generalized Laplacian $\Delta(f)$ of a subharmonic function $f$ on a domain $D$ is the Radon measure $\mu$ on $D$ defined as the linear functional $g \mapsto \int_D u\Delta g \, dA$. The Laplacian of a subharmonic function is also called the Riesz measure. The Laplacian is known to exist and is unique. If $p_\mu$ is the potential associated to $\mu$, then $\Delta p_\mu = \mu$. 
Hadamard’s three circle theorem

Hadamard’s three circle theorem assures that for any subharmonic function \( f \) on the annulus \( \{ r < |z| < R \} \) the function \( M(f, r) = \sup_{|z|=r} f(z) \) is an increasing convex function of \( \log(r) \).

Jensen formula

Jensen formula If \( f \) is holomorphic in the disc \( D = B(0, R), r < R \) and \( a_1, \ldots, a_n \) are the zeros of \( f \) in the closure of \( D \) counted with multiplicity, then \( \int_0^{2\pi} \log |f(re^{it})| \, dt = \log |f(0)| + N \log(r) - \sum_{j=1}^{n} \log|a_j| \).

Jensen formula

If \( f \) is a subharmonic function in a neighborhood of a point \( z \) in the complex plane, then the limit \( \lim_{r \to 0} M(f, r)/\log(r) \) exists and is called the [Lelong number] of \( f \) at \( z \). Here \( M(f, r) = \sup_{|z|=r} f(z) \).

Hyperbolic domain

An open set in the extended complex plane is a [hyperbolic domain] if there is a subharmonic function on \( G \) that is bounded above and not constant on any component of \( G \). A domain which is not hyperbolic is called a parabolic domain. Known facts:

- every bounded region is a hyperbolic domain (take \( f(z) = \text{Re}(z) \)).
- an open not connected set is hyperbolic.
- the complex plane is not a hyperbolic domain

Perron function

The [Perron function] for a domain \( D \) is defined as the functional assigning to a continuous function \( g \) on the boundary of \( D \) the value \( H_D(g) \), which is the supremum of all subharmonic functions \( u \) satisfying \( \sup_{z \to w} u(z) \leq g(w) \).

Potential

A subharmonic function \( f \) is called a [potential] if \( f = p_\mu \), where \( \mu = \Delta f \) is the Laplacian of \( f \) and \( p_m u(z) = -\int_D \log |z - w|d_\mu(w)/(2\pi) \) is the potential defined by \( \mu \).
The \textit{logarithmic capacity} of a subset $E$ of the complex plane is defined as $c(E) = \sup_{\mu} \exp(-I(\mu))$, where $I(\mu)$ is the potential theoretical energy of $\mu$ and the supremum is taken over all Borel probability measures $\mu$ on $\mathbb{C}$ whose support is a compact subset of $E$. Known facts:

- $c(E) = 0$ if and only if $E$ is polar.
- A disc of radius $r$ has capacity $r$.
- A line segment of length $h$ has capacity $h/4$.
- If $K$ has diameter $d$, then $c(K) \leq d/2$.
- If $K$ has area $A$, then $c(A) \geq (A/\pi)^{1/2}$.

The \textit{mean value property} tells that if $h$ is a harmonic function in the disc $D(w, R)$ and $0 < r < R$, then $h(w) = \int_0^{2\pi} h(w + re^{it}) dt / (2\pi)$.

A subset $S$ of the complex plane is called a \textit{polar set} if the potential theoretical energy $I(\mu)$ is $-\infty$ for every finite Borel measure $\mu$ with compact support $\text{supp}(\mu)$ in $S$. Properties of polar sets:

- Every countable union of polar sets is polar.
- Every polar set has Lebesgue measure zero.

The \textit{Poisson integral formula}: if $h$ is harmonic on the disk $D(0, R')$, then for all $0 < r < R < R'$ and $0 < t < 2\pi$, $h(w + re^{it}) = \int_0^{2\pi} h(w + R'e^{is})(R^2 - r^2)/(R^2 - 2Rr\cos(s - t) + r^2) ds / (2\pi)$.

The \textit{potential theoretical energy} $I(\mu)$ of a finite Borel measure $\mu$ of compact support on the complex plane is defined as

$$I(\mu) = \int \int \int \log|z - w|d\mu(w)d\mu(z).$$
A function $f$ on an open subset $U$ of the complex plane is called [subharmonic] if it is upper semicontinuous and satisfies the local submean inequality. Examples:

- if $g$ is holomorphic then $f = \log |g|$ is subharmonic
- if $\mu$ is a Borel measure of compact support, then $f(z) = \int \log |z - w| \, d\mu(w)$ is subharmonic.
- any harmonic function is subharmonic.
- if $g$ is subharmonic, then $\exp(g)$ is subharmonic.

A boundary point $w$ of a domain $D$ is called [regular] if there exists a barrier at $w$. A barrier is a subharmonic function $f$ defined in a neighborhood $N$ of $w$ which is negative on $D \cap N$ and such that $\lim_{z \to w} f(z) = 0$. It is known that $z$ is a regular boundary point if and only if the complement of $D$ is non-thin at $z$.

A boundary point $w$ of a domain $D$ is called [irregular] if it is not regular. It is known that if $z$ has a neighborhood $N$ such that $N$ intersected with the boundary of $D$ is polar, then $z$ is irregular.

A domain $D$ for which every point is regular is called a [regular domain]. For example, a simply connected domain $D$ such that the complement of $D$ in the Riemann sphere contains at least two points, is regular.

The [Riemann mapping Theorem]: if $D$ is a simply connected proper subdomain of the complex plane, there exists a conformal map of $D$ onto the unit disc.

The [Riesz decomposition theorem] tells that every subharmonic function $f$ can be written as $f = p_\mu + h$, where $\mu = \Delta f$ is the Laplacian of $f$, $2\pi p_\mu$ is the potential of $\mu$ and where $h$ is harmonic.
The local [submean inequality] for a function in the complex plane tells that there exists $R > 0$ such that for all $0 < r < R$ one has
\[ f(w) \leq \int_0^{2\pi} f(w + re^{it}) \, dt/(2\pi) . \]

Let $f$ be a subharmonic function on a domain $D$. The [maximum principle] says that if $f$ attains a global maximum in the interior of $D$ then $f$ is constant.

A subset $S$ of the complex plane is called a [thin set] if for all $w$ in the closure of $S - w$ and all subharmonic functions $f$, \( \limsup_{z \to w} f(z) = f(w) \). Examples:

- every single point in the interior of $S$ is thin.
- $F_\sigma$ polar sets $S$ are thin at every point.
- connected sets of cardinality larger than 1 are non-thin at every point of their closure
- A domain $S$ is thin at a point $z \in S$ if and only if $z$ is regular.

The [Wiener criterion] gives a necessary and sufficient condition for a set $S$ to be thin at a point $w$. Let $S$ be a $F_\sigma$ subset of $C$ and let $w$ be in $S$. Let $a < 1$ and define $S_n = z \in S, a^n < |z - w| < a^{n-1}$ . The criterion says that $S$ is thin at $w$ if and only if \( \sum_{n \geq 1} n / \log(2/c(S_n)) < \infty \), where

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