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Prosthaphaeresis Revisited

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The trigonometric identities for the product of two sines and the product of two cosines were first published in 1588. The discovery of the two equations, however, clearly predates that publication. Credit for priority was debated at the end of the last century and tentatively assigned to Paul Wittich of Breslau. But at least some of the conclusions reached at that time were not completely sound, even with respect to the evidence then available; and in the meantime, new evidence has come to light. Reevaluation of the issue now suggests a very questionable role for Wittich in the discovery of either equation. The first one was certainly discovered by Johannes Werner in about 1510, and probably resurrected from his papers by Wittich, while the second one appears to have been discovered from a knowledge of the first by Joost Bürgi in about 1585. But if Wittich loses one aspect of his priority in this reevaluation, he gains in another aspect. For it is now clear that Wittich had developed the method of prosthaphaeresis—the idea of using a product formula to simplify calculations—well before he arrived on Hven in 1580. © 1988 Academic Press, Inc.

Die trigonometrischen Identitäten für das Produkt zweier Sinus und das Produkt zweier Cosinus wurden 1588 zum ersten Mal veröffentlicht. Ohne Zweifel jedoch ereignete sich die Entdeckung der zwei Gleichungen vor jener Veröffentlichung. Am Ende des letzten Jahrhunderts wurde das Verdienst der Priorität debattiert und vorerst Paul Wittich aus Breslau zugeschrieben. Auf jeden Fall waren einige der zu der Zeit gezogenen Schlüsse (selbst in Anbetracht der damals zugänglichen Beweisstücke) nicht sehr gründlich durchdacht. In der Zwischenzeit sind neue Befunde zum Vorschein gekommen. Eine neue Untersuchung der Streitfrage ergibt eine fragwürdige Rolle Wittichs in bezug auf die Entdeckung der beiden Gleichungen. Die erste, bestimmt von Johannes Werner ca. 1510 entdeckt, hat Wittich wahrscheinlich in Werners Nachlaß gefunden. Es scheint, als ob die zweite anhand der Kenntnis der ersten von Joost Bürgi ca. 1585 entdeckt wurde. Wenn aber Wittich einen Teil seiner Priorität dank diesen neuen Überlegungen verliert, so gewinnt er einen anderen. Es ist nämlich jetzt klar, daß Wittich die Methode der "Prosthaphaeresis" entwickelt hat, eine Methode, die das Produkt einer Formel verwendet, um Berechnungen zu vereinfachen. © 1988 Academic Press, Inc.

Les identités trigonométriques pour les produits de deux sinus et de deux cosinus ont été publiées pour la première fois en 1588. Il est pourtant clair que leur découverte est antérieure à cette publication. A la fin du XIX^e siècle, après avoir débattu la question, on a attribué avec hésitation cette découverte à Paul Wittich de Breslau. Toutefois, même en tenant compte uniquement des documents alors connus, certaines conclusions de cette époque reposent sur des assises chancelantes. Depuis, de nouveaux éléments sonts venus s'ajouter au dossier. Le rôle de Wittich dans la découverte de ces deux identités apparaît maintenant contestable. La première identité fut certainement découverte par Johannes Werner vers 1510, et aurait vraisemblablement été reprise des papiers de ce dernier par Wittich. Par ailleurs, la seconde identité semble avoir été déduite de la première par Joost Bürgi vers 1585. Même si Wittich perd ici la priorité de ces découvertes, il y gagne sur un autre plan. En effet, il devient maintenant évident que Wittich avait développé la méthode de prosthaphaeresis—l'idée d'employer une formule de produit pour simplifier les calculs—bien avant son arrivée à Hven en 1580. © 1988 Academic Press, Inc.

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In a number of articles published nearly a century ago, several pioneers of the history of mathematics—Rudolf Wolf, Moritz Cantor, Anton von Braunmühl, and Axel Björnbo—explored in some depth an episode in the history of mathematics that was, even then, largely forgotten—the development of the so-called "method of prosthaphaeresis." This method rested on the discovery in the 16th century of what are now termed the "product formulas" of trigonometry: the identities for $\sin A \sin B [= \frac{1}{2}\cos(A - B) - \frac{1}{2}\cos(A + B)]$, $\sin A \cos B$, and $\cos A \cos B$. When the method is mentioned at all, nowadays, it is to be cited as the forerunner of logarithms, for what it amounted to was a process for substituting addition/subtraction operations for multiplication/division ones. A person did not have to be computing to very many significant digits before such substitutions saved a great deal of work, and probably many mistakes as well.

The method was first published, with the formulas for the product of sines and the product of cosines, in 1588 in a small book entitled *Fundamentum astronomicum*, written by Nicolai Reymers Ursus. Publication normally carries scientific priority with it. But in this case there were a number of alternative claims clouding the issue. One was by the famous astronomer Tycho Brahe, issued in several letters [Brahe 1913–1927 VII, 58, 281; VIII, 201] dating from 1580 onward, on behalf of himself and Paul Wittich, who worked with him on the island of Hven in late 1580. It was buttressed by Tycho's publication in 1588 of an allusion to the method [Brahe 1913–1927 IV, 233], and a manuscript discovered in the 1880s which proved to be a manual of trigonometry used by Tycho's calculators on Hven [Studnicka 1886]. Some of Tycho's claims took the form of accusing Ursus of plagiarism. Ursus therefore issued a disclaimer in 1597 which amounted to a second claim. According to him, Wittich had, indeed, brought the *method* to the astronomical observatory of the Landgrave of Hessen-Cassel in 1584; but what he brought was only one prosthaphaeretic equation (for $\sin A \sin B$), and no *proof* for it! It had been the Landgrave's clock-maker, Joost Bürgi, Ursus said, who devised a geometrical proof for that identity. And from that proof the formula for the product of two cosines and its proof had more or less jumped out simultaneously [Ursus 1597, I3r].

In 1611, finally, a third claim appeared. In a book entitled *Theoria lunae*, Jacob Christmann asserted that the formula for the product of two sines had been discovered by Johannes Werner a century earlier [Braunmühl 1900, 136].

There were certainly many questions that the late 19th-century historians could have asked, because the story of the subsequent development of the third formula and the application of all three to the solution of every possible case of plane and spherical triangles has still not been completely described today. But the problem that exercised them was the immediate one of priority. By 1907 the various claims had been sorted out, and credit apportioned roughly as follows. First, there could be no doubt that Werner had really discovered, albeit in a rather strange form, the prosthaphaeretic equivalent of sin A sin B. The manuscript text in which Werner had presented it had fallen into Rheticus' hands in the middle of the 16th century, and passed from him to Valentin Otho in 1574. From Cracow (where Rheticus had held it) Otho took it to Wittenberg and then to Heidelberg, where it came under Christmann's scrutiny. All that could be questioned was whether the other known developments from 1580 onward were independent of Werner's work, or whether the developers were "transcriptors" as Christmann had charged [Björnbo 1907, 167; Braunmühl 1896, 107].

If there had been a "copyist," it was presumed that it would have been Tycho. After all, the "public" history of prosthaphaeresis began with Tycho and Wittich at Uraniborg, and Tycho was the senior member of the pair both in age and by the implication of some of his statements. At the same time, however, Wittich's arrival on the scene was clearly crucial in some respect. The working hypothesis that seems to have been shared by the analysts was that Tycho somehow got a start on the problem, and Wittich developed it into something worthy of writing up as a manual of trigonometry. In Braunmühl's view, the most accessible point of Werner's work was an allusion Werner made in his *De motu octavae sphaerae* to a special computation he could make [Braunmühl 1900, 195]. It was an attractive possibility, because Tycho actually cited the context of Werner's remark—even if for another purpose. But Braunmühl decided that it would have been nearly impossible to derive any real help from the reference anyway; thus it did not really take Björnbo's reading [1907, 167] of Tycho's letters (which showed that Tycho did not get Werner's pamphlet until the late 1580s) to terminate that line of inquiry.

The other alternative was that during his various travels Tycho had somehow obtained access to Werner's manuscript. The problem here was that Tycho's movements were well enough documented to make it very unlikely that he ever reached Cracow, where the manuscript was known to have been from the time Tycho was a small boy until he settled on Hven. All that remained, then, was the possibility that an astronomer named Praetorius might have conveyed knowledge of it to Tycho. He had been with Rheticus in Cracow in 1569, and had had enough contact with Werner's manuscript to be able to quote from it in 1599, many years after he had last seen it [Björnbo 1907, 170; Braunmühl 1900, 136–137]. Perhaps during Praetorius' tenure as Professor of Mathematics at Wittenberg from 1571 to 1576, he had given out some information which had been useful to Tycho. This was, however, rather unlikely, and Björnbo [1907, 168–169] was candid enough to register an opinion that the general approach used in Tycho's manual of trigonometry was really quite different from Werner's, anyway.

With the simple process of transmission apparently ruled out for the inauguration of the prosthaphaeretic method on Hven, the investigators moved on to the second stage. Here the apparently straightforward proposition that Wittich simply conveyed to Cassel the contents—or, at least, the basis—of the manual he and Tycho had constructed at Uraniborg, was clouded by Ursus' claim that Wittich had brought only the equation for the product of two sines. But here, for reasons that are far from clear, the cloud was resolved with no trouble at all. Because the manual of trigonometry preserved from Hven included the equation for the product of two cosines, Ursus' statement was summarily disregarded, if not completely ignored.

There were certainly good reasons for discounting Ursus' report. The pamphlet in which it was issued was, from beginning to end, an exercise in sophistry [1]. The display of literary cleverness that not merely infused, but dominated, the text was clearly at least as important to Ursus as the basic theme of his pamphlet (the protest of his innocence in the matter of plagiarizing Tycho's System of the World). And if there are strong reasons for doubting that he was always telling the truth about other issues, there are stronger reasons for doubting that he even knew the truth about this one. If Ursus ever met Wittich at all, it was not at Cassel. Ursus went to Cassel in early 1586, at least a year and a half after Wittich had left. Wittich was, in fact, dead by that time. So what Ursus knew could only have been related to him by Bürgi himself. To make matters more difficult, Ursus was a man of humble origins like Bürgi, and seems to have felt a certain sociological kinship with him, designating him already in his booklet of 1588 as his "preceptor." Yet, despite the possibility that Ursus may simply have been making mischief or deliberately trying to transfer to Bürgi and himself credit that he thought would otherwise accrue at least partially to his enemy Tycho, it seems worth considering the fact that Bürgi—a shy, retiring individual with little formal education, whose achievements became known almost in spite of himself—also mentioned specifically that Wittich brought the formula for the product of two sines, even if in a context that did not rule out Wittich's having brought the cosine-product formula as well [Wolf 1871, 10–11] [2]. What was clearly the sticking point, however, was that last century's analysts saw no way of crediting Ursus' statement without assuming that Wittich composed the manual with Tycho in 1580, and then somehow forgot—or neglected to reveal—the formula for the product of two cosines when he got to Cassel in 1584. There is, however, a much simpler way of doing it: untie the Gordian knot binding Tycho's surviving manual to the one started in 1580.

It seems to have been 1916 before anyone even thought of the possibility. And then it was the astronomer, historian of astronomy, and biographer of Tycho, J. L. E. Dreyer, who in a neutral context mentioned off-handedly that the dates heading Tycho's manual must surely mean "that in the form we have it the book was put together on the first of January 1591, while the only copy now existing was written in 1595" [1916, 127]. Unfortunately, even Dreyer was unable to accord these dates any nontrivial significance. So preconditioned was he by his (and all the mathematical analysts') presupposition that "at Uraniborg there is no sign of the *method* having received any further development" [Dreyer 1916, 129; my emphasis] that he was unable to conceive of any development in the *manual*—even though he himself noted in another context that the manual uses the word "tangens" which was only introduced by Thomas Fincke in 1583 [Dreyer 1916, 130]. What Dreyer did not seem to pick up was that the manual also gives rules for the formula tan $C = (c \sin B)/(a - c \cos B)$, which did not appear in print until Viète published it in 1593 (in a less commodious form, according to Braunmühl [1900, 201, 178]). Once we know that the original manual was revised in at least one respect, it is not hard to imagine that the date 1591 refers to such a revision, done perhaps by the then recently arrived Longomontanus. We could even speculate further that, since the manual is dated 1591 rather than simply labeled "revised" or "new," there may have been an earlier revision. In fact, we know there was. For in a letter of 1590 [Brahe 1913–1927 VII, 281], written to a student who left Hven in mid-1588, Tycho alludes to the "rule" containing the tangent formula as one known to the student. By the same logic, however, since Tycho also refers to rules that include the formula for the product of two cosines, we know that the second prosthaphaeretic formula was in Tycho's manual before Ursus' *Fundamentum astronomicum* appeared in the Fall of 1588.

If we are to suppose that the cosine-product formula was not in Tycho's manual when Wittich left Hven, then we must also suppose that there was an independent way for it to get into Tycho's manual before Ursus published. And, in fact, there was. Tycho's principal assistant, Peder Flemløse, who had already been with him 2½ years when Wittich came to Hven in 1580, went to Cassel on an errand for Tycho in 1586 [Brahe 1913–1927 VI, 53]. The burden of his trip was to show the Landgrave that the overhaul of his instruments directed by Wittich had been inspired from beginning to end by ideas Wittich had picked up from Tycho's instruments at Uraniborg. It is not at all unlikely that from this moral high ground Flemløse might have levered a *quid pro quo* out of the Landgrave's assistants. (After all, Ursus got his information there at about the same time.) So the probable presence of the cosine-product formula in Tycho's manual before 1588 does not contradict Ursus' story. And, interestingly enough, Tycho never did, either, although he lived 4 years after Ursus published his statement, and wrote a number of letters alluding to Ursus' book.

Of course, it would be nice to be able to confirm Ursus' statement directly: but probably nothing short of the discovery of a different (earlier) version of Tycho's manual will accomplish that. On the other hand—and here is the crux of this article—there is a relatively certain way to disprove the essence of Ursus' assertion: find in Wittich's papers or Tycho's logs a citation or use of the formula for the product of two cosines that unambiguously antedates 1584 or 1586, respectively. Dreyer presumed that he had one, a notation appended to the log for 10 December 1580, where "there is written in a different hand instructions how the place of Mars is to be found 'per VI . . . et deinde per IX dogma' " [Brahe 1913–1927 X, 85]. But if it had ever occurred to Dreyer that the original manual might have been substantially different from the later one, he would have realized that this statement, like most of the other additions and computations in Tycho's logs, probably dates from many years after the event. In fact, if Dreyer did not recognize the "other" hand as that of Tycho's sole assistant of that time (Flemløse), or of Wittich, whose hand on five associated pages was later attested to by Monavius, then the annotation was, indeed, made later—probably much later, in the 1590s.

Ideally, an investigator who was as thoroughly immersed in every case and nuance of spherical trigonometry as, say, Braunmühl, might scrutinize Tycho's various computations and spot places in which he failed to use the "dogmata" that he should have had available to him, if he had access to his mature manual—and draw appropriate conclusions. Dreyer noted a couple of these in Tycho's computations on the comet of 1577: unhappily, even though much of the work was done after 1580, there is no good reason for doubting that the computations cited by Dreyer [Brahe 1913–1927 IV, 41, 46, 72] were actually made before Wittich came to Hven. But, of the three prosthaphaeretic computations made in connection with the comet of 1580—one by Wittich cited by Dreyer, another by him not cited by Dreyer, and one by Tycho [Brahe 1913–1927 XIII, 316, 322, 332]—all seem to involve only the formula for sin A and sin B. And the example given in the "Liège" copy of Wittich's *De revolutionibus* (kindly called to my attention and photocopied for me by Owen Gingerich) seems also to utilize sin A sin B.

Once we take seriously Ursus' statement that Wittich came to Cassel with only one prosthaphaeretic identity and no proof, it seems all the more suspicious that that identity should have been the one that Werner had discovered. Is it possible that Braunmühl *et al.* missed something when they considered and rejected the possibility of a "transcription" from Werner? As it turns out, they did. They failed to consider the possibility that it may have been Wittich who was the transcriptor. It is only fair to say that they would have had little success in following up on this possibility, had they tried; for, even today, Wittich's life is still very obscure. But, thanks to the researches of Owen Gingerich and Robert Westman [Gingerich & Westman 1983], we now know that Wittich was *not* the young student of meager background that he was previously assumed to have been. We also know that he arrived on Hven with at least the one product formula, with a clear idea of how to use that formula to avoid tedious multiplications, and even with some kind of fairly formal exposition of the method. How he might have obtained access to the equation can only be a matter of the same kind of speculation that was devoted a century ago to Tycho's possible access to it. But Wittich fits the possibilities in a way that Tycho never did. He was acquainted with, and perhaps even a student of, Praetorius, who is known to have had some kind of familiarity with Werner's manuscript. Praetorius sent one of his students, Valentin Otho, to Cracow to work for Rheticus [Burmeister 1967–1968 I, 160]; perhaps he sent more than one. Rheticus himself testified that he utilized a number of calculators [Braunmühl 1900, 212], and there were neither very many jobs for aspiring mathematicians nor very many aspiring mathematicians. Wittich may even have had his own special access to Rheticus. According to Burmeister [1967–1968 I, 129, 137], Rheticus was acquainted with a certain Johannes Wittich, who was born in 1537, and was thus probably 10 to 15 years older than Paul. At any rate, it is worth pointing out that we now have *time* for Wittich to have circulated through Cracow, in a way that the previous investigators did not, for we now know that Wittich matriculated at Wittenberg in 1566, and therefore could have had a full decade of travels before his arrival on Hven.

If the priority of Wittich remains obscure, that of Tycho does not, at least from a mathematical standpoint. He may have been the only one whose computational activities give him a reason to appreciate the value of the prosthaphaeretic method. He may have directed the application of the method to the requirements of his data reduction scheme. He certainly provided the institutional support for the production of his manual of trigonometry, and probably made the subsequent revisions in it. But while all of this may have entitled him to some of the credit he claimed, it could not have been a credit that is of much interest to mathematicians. Ursus, the first to publish the method, is almost surely entitled to even less. According to him, Bürgi developed the diagram that proved Wittich's formula, and from that demonstration another one immediately emerged [Ursus 1597, I3r]. If this is a claim for Ursus himself, it is ever so subtle [3]. And it does not seem to have impressed Braunmühl [1900, 196], who clearly did not believe that anyone who developed the first proof could have failed to see the second, long before Ursus even arrived at Cassel. Thus, unless Ursus or Bürgi was guilty of the most blatant attempt imaginable to steal credit, *Bürgi*—the shy, humble, independent inventor of logarithms—is entitled to the credit for two proofs and a formula (and an ingenious substitution [Dreyer 1916, 128–129] to compensate for the lack of the third formula).

Now that we know exactly what the uncertainties are, it is entirely reasonable to imagine resolving them. Most dramatic would be the finding of either the exposition written by Wittich, or an earlier form of the manual Tycho and his assistants used on Hven. But answers could very well lie buried in Bürgi's papers at Cassel, or in the mass of computations that Tycho published in his *De mundi* . . . of 1588 [Brahe 1913–1927 IV]. And still other questions remain to be answered concerning the subsequent development of the method to maturity, at the hands of Curtius, Clavius, and Joestelius.

NOTES

1. For a different, and undoubtedly more philosophically informed, view, see Jardine [1983].

2. Apparently Bürgi's context is a discussion of how he developed his method of *computing* the values of sines from the formula brought by Wittich. I have seen only Wolf's description of the manuscript, not Bürgi's manuscript.

3. Since it has been inferred as a claim by most commentators, I quote the Latin: "ex qua visa Demonstratione foecundum quoddam lucrum pariter elucescebat: vz. cum casus alter, una cum sua etiam Demonstratione, tum ratio solvendi quaecunque Triangula. . . ."

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