KÜNNETH FORMULA
IN GRAPH THEORY

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http://arxiv.org/abs/1505.07518
CLASSICAL PRODUCT

H

G \times H

G

no Künneth formula

\begin{align*}
\text{dim}(H) &= 1 \\
\text{dim}(G \times H) &= 1 \\
\text{dim}(G) &= 1
\end{align*}

p(G,x) = 1 

p(H,x) = 1

p(G \times H, x) = 1 + x

not Euler Charact. multiplicative

X(G) = X(H) = 1

X(G \times H) = 0

not dimension additive

\text{Poincaré polynomial}
THE CHAIN RING

G finite simple graph

\[ g = x + y + z + w + xy + yz + zx + yw + xyz \]

in ring \( \mathbb{Z}[x,y,z,w] \)
$g = x + y + z + w + xy + yz + zx + yw + xyz$

Monoids become vertices connected if one divides the other.
THE PRODUCT

\[ g = x + y + xy \]
\[ h = v + w + vw \]

\[ gh = vx + wx + vwxy + vy + wy + vwy + vx\ y + wxy + vwxy \]

\[ G \times H \]
EXAMPLE

\[ G \times H = G \times H \]
DIMENSION

\[ G \times H = G \times H \]

dim=1.4666

dim=1.333

dim=2.702
COHOMOLOGY

$H^2(G \times H) = H^1(G) \oplus H^1(H)$
EULER
CHARACTERISTIC
**Euler and Poincaré Polynomials**

\[
q(G, x) = \sum v_k(G) x^k \quad q(G, x) = \sum \tilde{v}_k(G) x^k
\]

**Euler**

\[
p(G, x) = \sum \dim(H^k(G)) x^k
\]

**Poincaré**

\[
\chi(G) = q_{dR} (G, -1) = q(G, -1) = p(G, -1)
\]

G,H arbitrary finite simple graphs

\[
q(G \times H, x) = q_{dR}(G, x) \cdot q_{dR}(H, x)
\]

\[
p(G \times H, x) = p(G, x) \cdot p(H, x)
\]
DIMENSION
DIMENSION LEMMA

$\dim(G) \geq \dim(G)$

23/15 = 1.5333

71/44 = 1.6136

$G \times K_1 = G_1$

higher dim simplices spawn more new vertices
DIMENSION FORMULA

\[ \dim(G_1) + \dim(H_1) = \dim(G \times H) \]

holds even locally!
Super Additivity

$$\dim(G) + \dim(H) \leq \dim(G \times H)$$

Same formula as for Hausdorff dimension!
EXAMPLE

\[ G = H \]

\[ G \times H \]

 Exists graphs with dimension close to 0 whose product have dimension close to 1
\( \dim(G_1) - \dim(G) \)
EVOLUTION

$G = G_0$

$G_1$

$G_2$

$G_3$
CHROMATOLOGY
COLORING BY DIM

\[ c(G) = 3 \]

\[ c(G_1) = 3 \]

\[ c(G_1) \leq c(G) \]
GEOMETRIC GRAPHS
TORUS

geometric

$\chi = 0$

$b_0 = b_2 = 1$

$b_1 = 2$

$q(G_1, x) = 24 + 24x$

$q(H_1, x) = 10 + 10x$

$G \times H = C_{12} \times C_5$

not geometric

$\chi = 0$

$v_0 = 240$

$v_1 = 720$

$v_2 = 480$

$\bar{v}_0 = 240$

$\bar{v}_1 = 480$

$\bar{v}_2 = 240$

$v_0 = 60$

$v_1 = 120$

$C_{12} ' \times ' C_5 = G ' \times ' H$

$b_0 = 1, b_2 = 0$

$b_1 = -61$
HOMOTOPY GROUPS

Have different cohomology but same homotopy groups

$S^3 \times P^2$  

$P^3 \times S^2$
Question: \( c(G) \geq c(G_1) \)
JOIN

\[ * = *_n \]

use interval \( I = L_n \)

\[ G \ast H = G \times H \times I / \sim \]

\( (x, y_1, 0) \sim (x, y_2, 0) \)

\( (x_1', y, 1) \sim (x_2', y, 1) \)

\( C_9 \ast_4 S_0 \)

suspension
We can glue locally trivial bundles.
THE END