THE AMAZING WORLD OF SIMPLICIAL COMPLEXES

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Abstract. Defined by a single axiom, finite abstract simplicial complexes belong to the simplest constructs of mathematics. We look at a few theorems.

Theorems

1. Simplicial complexes

1.1. A finite abstract simplicial complex $G$ is a finite set of non-empty sets which is closed under the process of taking finite non-empty subsets. $G$ defines a finite simple graph $\Gamma = (V, E)$, where $V = G$ and $E$ is the set of un-ordered pairs, where one is contained in the other. The Barycentric refinement $G_1$ of $G$ is the simplicial complex formed by the subsets of $V$ which are vertex sets of complete sub-graphs of $\Gamma$. Other names for Whitney complexes are flag complexes or clique complexes.

Theorem: Barycentric refinements are Whitney complexes.

1.2. Examples of complexes not coming directly from graphs are buildings or matroids. Oriented matroids are examples of elements of the ring $\mathcal{R}$ generated by simplicial complexes. Still, the Barycentric refinement $G_1$ of $G$ always allows to study $G$ with the help of graph theory.

1.3. A subset $H$ of $G$ is called a sub-complex, if it is itself a simplicial complex. Any subset $H$ generates a sub-complex, the smallest simplicial complex in $G$ containing $H$. The set $\mathcal{G}$ of sub-complexes is a Boolean lattice because it is closed under intersection and union. The f-vector of $G$ is $f = (v_0, v_1, \ldots, v_r)$, where $v_k$ is the number of elements in $G$ with cardinality $k + 1$. The integer $r$ is the maximal dimension of $G$.

2. Poincaré-Hopf

2.1. A real-valued function $f : G \rightarrow \mathcal{R}$ is locally injective if $f(x) \neq f(y)$ for any $x \subset y$ or $y \subset x$. In other words, it is a coloring in the graph $\Gamma$ representing $G_1$. The unit sphere $S(x)$ of $x \in G$ is the set $\{y \in G | (x, y) \in E(\Gamma)\}$. It is the unit sphere in the metric space $G$, where the distance is the geodesic distance in the graph representing $G_1$. Define the stable unit sphere $S_f^-(x) = \{y \in S(x) | f(y) < f(x)\}$ and the index $i_f(x) = \chi(S_f^-(x))$. The Poincaré-Hopf theorem is

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2.2. Classically, for a smooth function with isolated critical points on a Riemannian manifold $M$, the same definitions and result apply for $i_f(x) = \lim_{r \to 0} \chi(S_{r,f}(x))$, where $S_r$ is the geodesic sphere of radius $r$ in $M$ centered at $x$.

2.3. If $f(x) = \dim(x)$, then $i_f(x) = \omega(x)$. Poincaré-Hopf tells then that $\chi(G) = \chi(G_1)$. If $f(x) = -\dim(x)$, then $i_f(x) = \omega(x)(1 - \chi(S(x)))$. For complexes for which every unit sphere is a 2$d$-sphere, we have $i_{\dim} = -i_{-\dim}$ implying $\chi(G) = 0$.

3. **Gauss-Bonnet**

3.1. Any probability space $(\Omega, \mathcal{A}, P)$ of locally injective functions defines a curvature $\kappa(x) = E[i_f(x)]$. As we have integrated over $f$, the curvature value $\kappa(x)$ only depends on $x$.

Theorem: $\sum_x \kappa(x) = \chi(G)$.

3.2. If $\Omega$ is the product space $\prod_{x \in G} [-1, 1]$ with product measure so that $f \to f(x)$ are independent identically distributed random variables, then $\kappa(x) = K(x)$ is the **Levitt curvature** $1 + \sum_{k=0} (-1)^k v_k(S(x))/(k+1)$. The same applies if the probability space consists of all colorings. If $f = 1 + v_0t + v_1t^2 + \ldots$ is the generating function of the $f$-vector of the unit sphere, with anti-derivative $F = t + v_0t^2/2 + v_1t^3/3\ldots$, then $\kappa = F(0) - F(-1)$. Compare $\chi(G) = f'(0) - f'(-1)$ and $\sum_{x} \chi(S(x)) = f'(0) - f'(-1)$.

3.3. If $P$ is the Dirac measure on the single function $f(x) = \dim(x)$, then the curvature is $\omega(x)$. If $P$ is the Dirac measure on $f(x) = -\dim(x)$, then the curvature is $\omega(x)(1 - \chi(S(x)))$.

4. **Valuations**

4.1. A real-valued function $X$ on $\mathcal{G}$ is called a **valuation** if $X(A \cap B) + \chi(A \cup B) = \chi(A) + \chi(B)$ for all $A, B \in \mathcal{G}$. It is called an **invariant valuation** if $X(A) = X(B)$ if $A$ and $B$ are isomorphic. Let $\mathcal{G}_r$ denote the set of complexes of dimension $r$. The **discrete Hadwiger theorem** assures:

Theorem: Invariant valuations on $\mathcal{G}_r$ have dimension $r + 1$.

4.2. A basis of the space of invariant valuations is given by $v_k : \mathcal{G} \to \mathbb{R}$. Every vector $X = (x_0, \ldots, x_r)$ defines a valuation $X(G) = X \cdot f(G)$ on $\mathcal{G}_r$.

5. **The Stirling Formula**

5.1. The $f$-vectors transform linearly under Barycentric refinements. Let $\text{Stirling}(x, y)$ denote the **Stirling numbers** of the second kind. It is the number of times one can partition a set of $x$ elements into $y$ non-empty subsets. The map $f \to Sf$ is the **Barycentric refinement operator**

Theorem: $f(G_1) = Sf$, where $S(x, y) = \text{Stirling}(y, x)x!$. 

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**Theorem:** $\sum_x i_f(x) = \chi(G)$. 

**Theorem:** $\sum_x \kappa(x) = \chi(G)$. 

**Theorem:** Invariant valuations on $\mathcal{G}_r$ have dimension $r + 1$. 

**Theorem:** $f(G_1) = Sf$, where $S(x, y) = \text{Stirling}(y, x)x!$. 

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5.2. The matrix is upper triangular with diagonal entries $k!$ the factorial. If $X(G) = \langle X, f(G) \rangle = X(G_1) = \langle X, Sf(G) \rangle = \langle S^T X f(G) \rangle$, then $X = S^T X$ so that $X$ is an eigenvector to the eigenvalue 1 of $S^T$. The valuation with $X = (1, -1, 1, -1, \ldots)$ is the **Euler characteristic** $\chi(G)$. This shows that Euler characteristic is unique in the class of translation invariant valuations.

**Theorem:** If $X(1) = 1$ and $X(G) = X(G_1)$ for all $G$, then $X = \chi$.

6. THE UNIMODULARITY THEOREM

6.1. A finite abstract simplicial complex $G$ of $n$ sets defines the $n \times n$ **connection matrix** $L(x, y) = 1$ if $x \cap y \neq \emptyset$ and $L(x, y) = 0$ if $x \cap y = \emptyset$. The **unimodularity theorem** is:

**Theorem:** For all $G \in G$, the matrix $L$ is unimodular.

7. Wu characteristic

7.1. Using the notation $x \sim y$ if $x \cap y \neq \emptyset$, define the **Wu characteristic**

$$\omega(G) = \sum_{x \sim y} \omega(x) \omega(y).$$

For a complete complex $K_{d+1}$ we have $\omega(K^{d+1}) = (-1)^d$. As every $x \in G$ defines a simplicial complex generated by $\{x\}$, the notation $\omega(x)$ is justified.

7.2. A complex $G$ is a $d$-complex if every unit sphere is a $(d-1)$-sphere. A complex $G$ is a $d$-complex with boundary if every unit sphere $S(x)$ is either $(d-1)$ sphere or a $d-1$-ball. The sets for which $S(x)$ is a ball form the **boundary** of $G$. A complex without boundary is **closed**. $d$-complexes with or without boundary are pure: all maximal sets have the same dimension.

**Theorem:** For a $d$-complex $G$ with boundary, $\omega(G) = \chi(G) - \chi(\delta G)$.

7.3. For any $d$ one can define higher **Wu characteristic**

$$\omega_k(G) = \sum_{x_1 \sim \ldots \sim x_k} \omega(x_1) \cdots \omega(x_d)$$

summing over all simultaneously intersecting sets in $G$. As before $\omega(x) = (-1)^{\dim(x)} = (-1)^{|x| - 1}$. It can be generalized by assuming $x_k \in G_k$ to be in different complexes. Especially important is the **intersection number** between two complexes $G, H$

$$\omega(G, H) = \sum_{x \sim y} \omega(x) \omega(y).$$
8. The energy theorem

8.1. As $L$ has determinant 1 or $-1$, the inverse $g = L^{-1}$ is a matrix with integer entries. The entries $g(x,y)$ are the potential energy values between the simplices $x, y$.

**Theorem:** For any complex $G$, we have $\sum_x \sum_y g(x,y) = \chi(G)$.

8.2. This energy theorem assures that the total potential energy of a complex is the Euler characteristic.

9. Homotopy

9.1. The graph $1 = K_1$ is contractible. Inductively, a graph is contractible if there exists a vertex $x$ such that both $S(x)$ and $G - x$ are contractible. The step $G \rightarrow G - x$ is a homotopy step. Two graphs are homotopic if there exists a sequence of homotopy steps or inverse steps which brings one into the other. Contractible is not the same than homotopic to 1. A graph $G$ is a unit ball if there exists a vertex such that $B(x) = G$.

**Theorem:** If $G$ is a unit ball then it is contractible.

9.2. It is proved by induction. It is not totally obvious. A cone extension $G = D + x$ for the dunce hat $D$ obtained by attaching a vertex $x$ to $D$ is a ball but we can not take $x$ away. Any other point $y$ can however be taken away by induction as $G - y$ is a ball with less elements.

**Theorem:** Contractible graphs have Euler characteristic 1.

9.3. The proof is done by induction starting with $G = 1$. It is not true that the Wu characteristic $\sum_{x\sim y} \omega(x)\omega(y)$ is a homotopy invariant as $\omega(K_{n+1}) = (-1)^n$.

10. Spheres

10.1. The empty graph $0$ is the $(-1)$ sphere. A $d$-sphere $G$ is a $d$-complex for which all $S(x)$ are $(d - 1)$ spheres and for which there exists a vertex $x$ such that $G - x$ is contractible. The 1-skeleton graphs of the octahedron and the icosahedron are examples of 2-spheres. Circular graphs with more than 3 vertices are 1-spheres. A simplicial complex $G$ is a $d$-sphere, if the graph $G_1$ is a $d$-sphere. Here is the polished Euler Gem

**Theorem:** $\chi(G) = 1 + (-1)^d$ for a $d$-sphere $G$.

**Theorem:** The join of a $p$-sphere with a $q$-sphere is a $p + q + 1$-sphere.
10.2. The generating function of $G$ is $f_G(t) = 1 + \sum_{k=0}^{\infty} v_k(G)t^{k+1}$ with $v_k(G)$ being the number of $k$-dimensional sets in $G$. It satisfies

\textbf{Theorem:} $f_{G+H}(t) = f_G(t) + f_H(t) - 1$ and $f_{G\oplus H}(t) = f_G(t)f_H(t)$.

For example, for $P_2 \oplus P_2 = S_4$ we have $(1 + 2t)(1 + 2t) = 1 + 4t + 4t^2$.

10.3. Given a $d$-graph. The function $\dim(x)$ has every point a critical point and $S^-(x) = \{ y \in S(x) \mid f(y) < f(x) \}$ and $S^+(x) = \{ y \in S(x) \mid f(y) > f(x) \}$ then $S(x) = S^-(x) + S^+(x)$.

10.4. Since by definition, a sphere becomes contractible after removing one of its points:

\textbf{Theorem:} $d$-spheres admit functions with exactly two critical points.

Spheres are the $d$-graphs for which the minimal number of critical points is 2. There are no $d$-graphs for which the minimal number of critical points is 1.

11. Platonic complexes

11.1. A combinatorial CW complex is an empty or finite ordered sequence of spheres $G = \{c_1, \ldots, c_n\}$ such that $G_n = \{c_1, \ldots, c_n\}$ is obtained from $G_{n-1} = \{c_1, \ldots, c_{n-1}\}$ by selecting a sphere $c_n$ in $G_{n-1}$ such that $c_n$ is either empty or different from any $c_j$. We identify $c_j$ with the cell filling out the sphere. Its dimension is 1 plus the dimension of the sphere. The Barycentric refinement $G_1$ of $G$ is the Whitney complex of the graph with vertex set $G$ and where two vertices $a, b$ are connected if one is a sub sphere of the other.

11.2. $G$ is a $d$-sphere if $G_1$ is a $d$-sphere as a simplicial complex. A subset $H$ of $G$ is a sub-complex of $G$ if $H_1 \subset G_1$ for the refinements.

11.3. The Levitt curvature of a cell $c_j$ is $F(0) - F(-1)$, where $F$ is the anti-derivative of the $f$-generating function $f = 1 + tv_0 + t^2v_1 + \ldots$ of the sphere $S(c_j)$. The curvature of a cell $x$ in a 2-sphere is $1 - v_0(S(x))/2 + v_1(S(x))/3 = 1 - v_0(S(x))/6$. The curvature of a cell in a 3 sphere is 0. Gauss-Bonnet assures that the sum of the curvatures is the Euler characteristic.

11.4. A $d$-sphere $G$ is called a \textbf{Platonic $d$-polytope} if for every $0 \leq k \leq d$ and any cell dimension $k$, there exists a Platonic $(d-1)$-sphere $P_k$ such that for any cell $c$ of dimension $k$, the unit sphere $S(x)$ is isomorphic to $P_k$. The $-1$-dimensional sphere 0 is assumed to be Platonic. The 0-dimensional sphere consisting of two isolated points is Platonic too. The 1-dimensional complexes $C_k$ with $k \geq 3$ are the Platonic 1-spheres. With $C_3$ one denotes the 1-skeleton complex of $K_3$. Let $P = (p(-1), p(0), p(1), p(2), \ldots$ denote the number of Platonic $d$-polytopes. In the CW case, we have the familiar \textbf{Schläfli classification}

\textbf{Theorem:} Platonic$_{CW} = (1, 1, \infty, 5, 6, 3, 3, 3, \ldots)$. 


The classification of Platonic polytopes of dimension \( d \) which are simplicial complexes is easier. There is a unique Platonic solid in each dimension except in dimensions 1, 2, 3. In the 1-dimensional case there are infinitely many. In the two-dimensional case, only the octahedron and icosahedron are Platonic. In the three dimensional case, there is only the 600 cell and the 16 cell. After that the curvature condition and Gauss-Bonnet constraint brings it down to the cross polytopes.

\[ \text{Theorem:} \quad \text{Platonic}_{SC} = (1, 1, \infty, 2, 2, 1, 1, 1, \ldots) . \]

\[ \text{12. Dehn-Sommerville relations} \]

\[ \text{12.1.} \quad \text{Given a } d \text{-dimensional complex } G, \text{ any integer vector } (X_0, \ldots, X_d) \text{ in } \mathbb{Z}^{d+1} \text{ defines a valuation } X(G) = X_0v_0 + \ldots + X_dv_d. \text{ By distributing the values } X_k \text{ attached to each } k \text{-simplex in } G \text{ equally to its } k+1 \text{ vertices, we get the curvature } K(x) = \sum_{k=0}^{d} X_k v_{k-1}(S(x))/(k+1) \text{ for the valuation } X \text{ and graph } G \text{ at the vertex } x. \text{ The formula } \sum_{x \in V} K(x) = X(G) \text{ is the Gauss-Bonnet theorem} \text{ for } X. \]

\[ \text{12.2.} \quad \text{In the case } X(G) = v_1, \text{ the curvature is the vertex degree divided by 2 and the formula reduces to the “Euler handshake”. If } X = v_d \text{ is the volume of } G, \text{ then } K(x) \text{ is the number of } d \text{-simplices attached to } x \text{ divided by } d+1. \text{ In the case } X = (1, -1, 1, -1, \ldots), \text{ } X \text{ is the Euler characteristic and } K \text{ is the discrete analogue of the Euler form in differential geometry entering the Gauss-Bonnet-Chern theorem. For } d \text{-graphs, there are some valuations which are zero. Define the Dehn-Sommerville valuations } \]

\[ X_{k,d} = \sum_{j=k}^{d-1} (-1)^{j+k} \binom{j+1}{k+1} v_j(G) + v_k(G). \]

\[ \text{Theorem:} \quad \text{For } d \text{-graphs, the Dehn-Sommerville curvatures are zero.} \]

\[ \text{12.3.} \quad \text{The proof is by noticing that the curvature of } X_{k,d} \text{ is } K(x) = X_{k-1,d-1}(S(x)). \text{ This follows from the relation } \]

\[ X_{k+1,d+1}(l+1)/(l+1) = X(k,d)(l)/(k+2). \]

\[ \text{Use Gauss-Bonnet and induction using the fact that the unit sphere of a geometric graph is geometric and that for } d = 1, \text{ a geometric graph is a cyclic graph } C_n \text{ with } n \geq 4. \text{ For such a graph, the Dehn-Sommerville valuations are zero.} \]

\[ \text{13. Dual Connection matrix} \]

\[ \text{13.1.} \quad \text{Define the dual connection matrix } \overline{L}(x,y) = 1 - L(x,y) \text{ of a complex } G \text{ with } n \text{ sets. } \overline{L} \text{ is the adjacency matrix of a dual connection graph } (G, E), \text{ where two simplices are connected, if they do not intersect: } E = \{(a,b) \mid a \cap b = \emptyset\}. \]

\[ \text{Theorem:} \quad 1 - \chi(G) = \text{det}(-L\overline{L}). \]
13.2. Let $E$ be the constant matrix $E(x, y) = 1$. The result follows from unimodularity $\det(L) = \det(g)$ and the energy theorem telling that $Lg = (E - L)g = Eg - 1$ has the eigenvalues of $Eg$ minus 1 which are $\chi(G)$ and 0. Assume $G$ has $n$ sets:

**Theorem:** $-L\bar{L}$ has $n - 1$ eigenvalues 1 and one eigenvalue $1 - \chi(G)$.

13.3. The above formula is not the first one giving the Euler characteristic as a determinant of a Laplacian. [12] show, using a formula of Stanley, that if $A(x, y) = 1$ if $x$ is not a subset of $y$ and $A(x, y) = 0$ else, then $1 - \chi(G) = \det(A)$.

14. **Alexander Duality**

14.1. The **Alexander dual** of $G$ over the vertex set $V$ is the simplicial complex $G^* = \{x \subset V \mid x^c \notin G\}$. It is the complex generated by the sets $x$ for which the complements are not in $G$. For the complete complex $K_d$, the dual is the empty complex. In full generality if $G$ has $n$ elements, one has for the Betti numbers $b_k(G)$:

**Theorem:** $b_k(G) = b_{n-3-k}(G), k = 1, \ldots, n - 1$

14.2. In order for this statement to have content, one needs $n \geq 5$. It does not for $G = C_4$ but it works for $G = C_5$ already, where $G^*$ is the complement of a circle in a 3-sphere. The combinatorial Alexander duality is due to Kalai and Stanley.

15. **Sard**

15.1. Given a locally injective function $f$ on a graph $G = (V, E)$, define for $c \notin f(V)$ the **level surface** $\{f = c\}$ as the subgraph of the Barycentric refinement of $G$ generated by simplices $x$ on which $f$ changes sign. Remember that $G$ is a $d$-graph if every unit sphere $S(x)$ is a $(d-1)$-sphere. A discrete Sard theorem is:

**Theorem:** For a $d$-graph, every level surface is a $(d - 1)$-graph.

If $G$ is a finite abstract simplicial complex, then $f : G \to R$ defines a function on the Barycentric refinement $G_1$ and the level surface is defined like that. This result has practical value as we can define discrete versions of classical surfaces.

15.2. Given a finite set of functions $f_1, \ldots, f_k$ on the vertex sets of successive Barycentric refinements $G_1, \ldots, G_k$ of a simplicial complex, we can now look at the $(d-k)$-graph $\{f = c\} = \{f_1 = c_1, \ldots, f_k = c_k\}$. Unlike in the continuum case, where the result only holds for almost all $c$, this holds for all $c$ disjoint from the range.

**Theorem:** Given $f_j : G_j \to R, j \leq k$, then $\{f = c\}$ is a $(d-k)$-complex.

16. **Symmetric index**

16.1. A $d$-sphere is a complex for which all unit spheres are $d-1$-spheres and which when punctured becomes contractible. A $d$-graph is a $d$-complex for which all unit spheres are $(d-1)$-spheres. For a locally injective function $f$ one can now look at the symmetric index $j_f(x) = i_f(x) + i_{-f}(x)$ and relate it to the center manifold $B_f(x) = \{f(y) = f(x)\}$ in the sense of the previous discrete Sard result.
Theorem: \[ 1 - \chi(S(x))/2 - \chi(B_f(x))/2 = (i_f(x) + i_f(x))/2 = j_f(x). \]

16.2. For odd \( d \), \( B_f(x) \) is a \((d - 2)\) complex and \( j_f(x) = -\chi(B_f(x))/2 = 0 \) implying \( \chi(G) = 0 \) and zero curvature \( K(x) = 0 \) for all \( x \). For even \( d \), the formula becomes \( j_f(x) = 1 - \chi(B_f(x))/2 \) and allows with Poincaré-Hopf to write the Euler characteristic of \( G \) as a sum of the Euler characteristic of smaller dimensional graphs. For \( d = 4 \), the curvature at a point is an expectation of \( 1 - \chi(B_f(x))/2 \), where \( B_f(x) \) are the random 2-dimensional center surfaces defined by \( f \) on \( S(x) \).

17. Jordan-Schoenflies

17.1. A complex \( H \) is embedded in another complex \( G \), if the vertex set of \( H_1 \) in \( G_1 \) generates the complex \( H_1 \). The complex \( H = \{\{1, 2\}, \{2, 3\}\} \) in \( G = \{\{1, 2, 3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\} \) for example is not embedded as it generates the entire complex. The Barycentric refinement \( H_1 \) however as a complex is embedded in the complex \( G_1 \).

17.2. The notion of \( d\)-sphere has been defined combinatorially as a \( d\)-complex which has \((d - 1)\) spheres as unit spheres and when punctured becomes contractible. A \( d\)-ball is a \( d\)-complex with boundary for which the boundary is a \( d - 1 \) sphere. By definition, a punctured \( d\)-sphere is a \( d\)-ball.

Theorem: A \( d - 1\)-sphere embedded in a \( d\) sphere separates \( G \) into two complementary components \( A, B \), which are both \( d\)-balls.

17.3. The case \( d = 2 \) is the Jordan-Brouwer separation theorem. The discrete version follows also from the piecewise linear version. In the Jordan curve case, the original proof already first dealt with the polygonal case.

17.4. The classical Jordan-Brouwer theorem tells that a \( d - 1 \) sphere embedded in a \( d\)-sphere \( G \) separates \( G \) into two complementary components \( A, B \). It needs some regularity to assure that the two components are balls. Alexander gave an example of a topological embedding of \( S^2 \) into \( S^3 \) for which one of the domains \( A \) is not simply connected.

18. Bonnet and Synge

18.1. The definition of positive curvature complexes is analog to the continuum. We aim for entirely combinatorial proofs of the results in the continuum.

18.2. Let \( G \) be a \( d\)-complex so that every unit sphere is a \((d - 1)\) sphere. A geodesic 2-surface is a subcomplex if the embedded graph does not contain a 3-simplex. \( G \) has positive sectional curvature if every geodesic embedded wheel graph \( W(x) \) has interior curvature \( \geq 5/6 \). The geomag lemma is that any wheel graph in a positive curvature \( G \) can be extended to an embedded 2-sphere.

18.3. An elementary analog of the Bonnet theorem is:

Theorem: A positive curvature complex has diameter \( \leq 4 \).
18.4. The simplest analog of Synge theorem is

**Theorem:** A positive curvature complex is simply connected.

18.5. The reason for both statements is the geomag lemma stating that any closed geodesic curve can be extended to a 2-complex which is a sphere and so simply connected. The strict curvature assumption as we can not realize a projective plane yet with so few cells. With weaker assumptions getting closer to the continuum, the work is harder:

18.6. Define more generally the sectional curvature to be $\geq \kappa$ if there exists $M$ such that the total interior curvature of any geodesic embedded 2-disk with $M$ interior points is $\geq \delta \cdot M$ and such that every geodesic embedded wheel graph $W(x)$ has non-negative interior curvature. A complex has **positive curvature** if there exists $\kappa > 0$ such that $G$ has sectional curvature $\geq \kappa$. The maximal $\kappa$ which is possible is then the "sectional curvature bound".

18.7. An embedded 2-surface of positive sectional curvature $\kappa$ must then have surface area $\leq 2/\kappa$. The classical theorem of Bonnet assures that a Riemannian manifold of positive sectional curvature is compact and satisfies an upper diameter bound $\pi/\sqrt{\kappa}$. An analog bound $C/\sqrt{\kappa}$ should work in the discrete.

18.8. Having a notion of sectional curvature allows to define Ricci curvature of an edge $e$ as the average over all sectional curvatures over all wheel graphs passing through $e$. The scalar curvature at a vertex $x$ is the average Ricci curvatures over all edges $e$ containing $x$. The Hilbert functional is then the total scalar curvature.

19. **AN INVERSE SPECTRAL RESULT**

19.1. Let $p(G)$ denote the number of positive eigenvalues of the connection Laplacian $L$ and let $n(G)$ the number of negative eigenvalues of $L$. One can hear the Euler characteristic of $G$ because of

**Theorem:** For all $G \in \mathcal{G}$ we have $\chi(G) = p(G) - n(G)$.

19.2. The proof checks this by deforming $L$ when adding a new cell. This result implies that Euler characteristic is a logarithmic potential energy of the origin with respect to the spectrum of $iL$.

**Theorem:** $\chi(G) = tr(\log(iL))(2\pi/i)$.

19.3. The proof shows also that after a CW ordering of the sets in a finite abstract simplicial complex, one can assign to every simplex a specific eigenvalue and so eigenvector of $L$. Each set in $G$ has become a wave, a quantum mechanical object.
20. The Green star formula

20.1. Given a simplex \( x \in G \), the stable manifold of the dimension functional \( \dim(x) \) is \( W^-(x) = \{ y \in G \mid y \subset x \} \). The unstable manifold \( W^+(x) = \{ y \in G \mid x \subset y \} \) is known as the star of \( x \). Unlike \( W^-(x) \) which is always a simplicial complex, the star \( W^+(x) \) is in general not a sub complex of \( G \).

Theorem: \( g(x, y) = \omega(x) \omega(y) \chi(W^+(x) \cap W^+(y)) \).

20.2. In comparison, we have \( W^-(x) \cap W^+(x) = \omega(x) \) and \( L(x, y) = \chi(W^-(x) \cap W^-(y)) \). The to \( L \) similar matrix \( M(x, y) = \omega(x) \omega(y) \chi(W^-(x) \cap W^-(y)) \) satisfies \( \sum_x \sum_y M(x, y) = \omega(G) \), the Wu characteristic.

21. Wu characteristic

21.1. The Euler characteristic \( \chi(G) = \omega_1(G) = \sum_{x \in G} \omega(x) \) of \( G \) is the simplest of a sequence of combinatorial invariants \( \omega_k(G) \). The second one, \( \omega(G) = \sum_{x, y, L(x, y) = 1} \omega(x) \omega(y) \), is the Wu characteristic of \( G \). The valuation \( \chi \) is an example of a linear valuation, while \( \omega \) is a quadratic valuation. The Wu characteristic also defines an intersection number \( \omega(A, B) \) between sub-complexes.

21.2. All multi-linear valuations feature Gauss-Bonnet and Poincaré-Hopf theorems, where the curvature of Gauss-Bonnet is an index averaging. For example, with \( K(v) = \sum_{v \in x, x \sim y} \omega(x) \omega(y)/(|x| + 1) \) The Gauss-Bonnet theorem for Wu characteristic is

Theorem: \( \omega(G) = \sum_v K(v) \).

22. The boundary formula

22.1. We think of the internal energy \( E(G) = \chi(G) - \omega(G) \) as a sum of potential energy and kinetic energy. A \( d \)-complex is a simplicial complex \( G \) for which every \( S(x) \) is a \((d - 1)\)-sphere. A \( d \)-complex with boundary is a complex \( S(x) \) is either a \((d - 1)\)-sphere or a \((d - 1)\)-ball for every \( x \in G \).

22.2. The \( d \)-complexes are discrete \( d \)-manifolds and \( d \)-complexes with boundary is a discrete version of a \( d \)-manifold with boundary. We denote by \( \delta G \) the boundary of \( G \). It is the \((d - 1)\)-complex consisting of boundary points. By definition, \( \delta \delta G = 0 \), the empty complex. The reason is that the boundary of a complex is closed, has no boundary. We can reformulate the formula given below as

Theorem: If \( G \) is a \( d \)-complex with boundary then \( E(G) = \chi(\delta(G)) \).

22.3. If \( G \) is a \( d \)-ball, then \( \delta G \) is a \((d - 1)\)-sphere and \( E(G) = 1 + (-1)^{d-1} \), by the polished Euler gem formula.
23. Zeta function

23.1. For a one-dimensional complex $G$, there is a spectral symmetry which will lead to a functional equation for the spectral zeta function.

Theorem: If dim$(G) = 1$, then $\sigma(L^2) = \sigma(L^{-2})$.

23.2. If $H$ is a Laplacian operator with non-negative spectrum like the Hodge operator $H$ or connection operator $L$, one can look at its zeta function

$$\zeta_H(s) = \sum_{\lambda \neq 0} \lambda^{-s},$$

where the sum is over all non-zero eigenvalues of $H$ or $L^2$. In the connection case, we take $L^2$ to have all eigenvalues positive.

Theorem: If dim$(G) = 1$, then $\zeta(s) = \zeta(-s)$.

23.3. The case of the connection Laplacian is especially interesting because one does not have to exclude any zero eigenvalue. The connection zeta function of $G$ is defined as $\zeta(s) = \sum_\lambda \lambda^{-s}$, where the sum is over all eigenvalues $\lambda$ of $L^2$. It is an entire function in $s$.

Theorem: If dim$(G) = 1$, then $\zeta(s) = \zeta(-s)$.

23.4. When doing Barycentric refinement steps, the zeta function converges to an explicit function.

$$\zeta(it) = \int_0^1 \frac{2 \cos \left(2t \log \left(\sqrt{4v^2 + 1 + 2v}\right)\right)}{\pi \sqrt{1 - v} \sqrt{v}} dv.$$

It is a hypergeometric series $\zeta(2s) = \pi_4 F_3 \left(\frac{1}{4}, \frac{3}{4}, -s, s; \frac{1}{2}, \frac{1}{2}, 1; -4\right)$. 

24. The Hydrogen formula

24.1. Given a simplicial complex $G$, let $\Lambda_k(G)$ denote the set of real valued functions on $k$-dimensional simplices. It is a $v_k$-dimensional vector space. Define the $v_k \times v_{k+1}$ matrices $d_k(x, y) = 1$ if $x \subset y$ and $d_k(x, y) = 0$ else. It is the sign-less incidence matrix. It can be extended to a $n \times n$ matrix $d$ so that $d = d_0 + d_1 + \cdots + d_r$ and $D = d + d^*$ and $H = (d + d^*)^2$, the sign-less Dirac and sign-less Hodge operator. In the one-dimensional case, we have $H = d^*d + dd^*$. The Hydrogen relations are

Theorem: If dim$(G) = 1$, then $L - L^{-1} = H$.

24.2. The relation allows to relate the spectra of $L$ and $H$. It allows to estimate the spectral radius or give explicit formulas for the spectrum of the connection Laplacian in the circular case. The relation was also needed to get the explicit dyadic zeta function.
24.3. Let $S(x)$ denote the **unit sphere** of a simplex $x \in G$. While $S(x)$ is at first a subset of $G$, it generates a sub-complex in $G_1$. As $g(x, x) = 1 - \chi(S(x)) = \chi(W^+(x))$, we have a functional $\sum_x \chi(S(x))$ of Dehn-Sommerville type. With $f(t) = 1 + \sum_{k=1}^{\infty} v_k t^k = 1 + v_0 t + v_1 t^2 + v_2 t^3 + \cdots$, the Euler characteristic of $G_1$ can be written as $\chi(G) = f(0) - f(-1)$. The following result holds for any simplicial complex:

**Theorem:** $\text{tr}(L - L^{-1}) = \sum_x \chi(S(x)) = f'(0) - f'(-1)$.

24.4. Compare that the Levitt curvature at a point $x$ was $F(0) - F(-1) = \int_{-1}^{0} f(t) \, dt$, where $F$ is the anti-derivative of the generating function of $S(x)$.

25. **Hopf Umlaufsatz**

25.1. The discrete analogue of a region in the complex plane is a **hexagonal complex** which is defined as a 2-complex which is flat in the interior. To consider flat 2-complexes might look special at first but the importance of the complex plane in the continuum justifies to look at this case particularly well. In the continuum, we have the **Frobenius classification** of normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ selecting out $\mathbb{C}$ as the only commutative complete normed division algebra. In the continuum, special **packing properties** select out dimensions like 2, 4, 8, 24: the hexagonal lattice is the densest in two dimensions.

25.2. If $d(x)$ is the vertex degree, then the Levitt curvature is $K(x) = (6 - d)/6 = 0$ in the interior and $K(x) = (4 - d)/6$ at the boundary. The vertex degrees at the boundary can be $1 < d < 6$ because $d = 1$ would mean a one dimensional hair sticking out and $d = 6$ would mean that the point is an interior point. The boundary curvature corresponds to the signed curvature of a boundary curve.

25.3. Let $b_0(G)$ denote the number of connected components of the region and $b_1(G)$ the genus, the number of holes. Informally, it is the number of bounded connected components of the complement of $G$ in the flat infinite hexagonal plane. The number $b_1(G)$ can also be defined algebraically as the nullity of the Hodge Laplacian block $H_1(G)$. A consequence of Gauss-Bonnet and Euler-Poincaré is the **Hopf Umlaufsatz**

**Theorem:** For a planar region $\sum_{x \in \partial G} K(x) = b_0(G) - b_1(G)$.

25.4. Let $|S_r(x)|$ denote the number of edges in the sphere $S_r(x)$. The **Puiseux curvature** of a boundary point is defined as

$$K(x) = (2|S_1(x)| - |S_2(x)|)/12.$$ 

It is a second order curvature measuring the discrepancy from linear growth of the wave fronts emanating from $x$. 

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25.5. With the Puiseux curvature, the Hopf Umlaufsatz needs more regularity from the region. A region is called **smooth** if it satisfies the following additional conditions: (i) $G$ equal to its closure meaning that if $d(x, y) = 1$ in the ambient hexagonal plane then $(x, y)$ is a simplex and (ii): any two interior points with a common boundary point either have distance 1 or are both adjacent to a third interior point.

**Theorem:** For a smooth planar region $\sum_{x \in \delta G} K_2(x) = b_0(G) - b_1(G)$.

26. **Brouwer-Lefschetz**

26.1. The **exterior derivative** $d$ for $G$ defines the **Dirac operator** $D = d + d^*$ of $d$. The Hodge Laplacian $H = D^2$ splits into a direct sum $H_0 \oplus H_1 \cdots \oplus H_d$. The null space of $H_k$ is isomorphic to the $k$'th cohomology group $H^k(G) = \ker(d_k)/\text{im}(d_{k-1})$. Its dimension $b_k$ is the $k$'th Betti number. The **Euler-Poincaré** relation assures that the combinatorial and cohomological Euler characteristics are the same:

**Theorem:** $\chi(G) = \sum_k (-1)^k b_k$.

26.2. An **endomorphism** $T$ of $G$ is a map from $G$ to $G$ which preserves the order structure. It is an automorphism if it is bijective. An endomorphism $T$ induces a linear map on cohomology $H^k(G)$. The super trace of this map is the **Lefschetz number** $\chi(T, G)$ of $T$. Given a fixed point $x \in G$ of $T$, its **Brouwer index** is defined as $i_T(x) = \omega(x) \text{sign}(T|x)$. Now

**Theorem:** $\chi(T, G) = \sum_{x=T(x)} i_T(x)$.

26.3. A special case is $T = 1$, where $\chi(1, G) = \chi(G)$ and $i_T(x) = \omega(x)$. The Brouwer-Lefschetz fixed point theorem is then the Euler-Poincaré theorem.

27. **McKean-Singer**

27.1. The **super trace** $\text{str}(A)$ of a $n \times n$ matrix defined for a complex with $n$ sets is defined as $\sum_{x \in G} \omega(x)L(x,x)$. By definition, we have $\text{str}(1) = \text{str}(L)$. For the Hodge operator $H = D^2 = (d + d^*)^2$ we have the **McKean-Singer formula**:

**Theorem:** $\text{str}(\exp(-tH)) = \chi(G)$ for all $t$.

27.2. The reason is that $\text{str}(H^k) = 0$ for $k > 0$, implying $\text{str}(\exp(tH)) = \text{str}(1) = \chi(G)$. The McKean-Singer identity is important as it allows to give almost immediate **heat deformation proofs** of the Lefschetz formulas in any framework in which the identity holds. We proposed in [135] to define a discrete version of a **differential complex** as McKean-Singer enables Atiyah-Singer or Atiyah-Bott like extensions of Gauss-Bonnet or Lefschetz. They are caricatures of the heavy theorems in the continuum.
27.3. The Hodge operator $H = (d + d^*)^2$ and the connection operator $L$ live on the same finite dimensional Hilbert space. There is no cohomology associated to $L$. But for the connection operator $L$, there is still a localized version of McKean-Singer:

**Theorem:** $\text{str}(L^k) = \chi(G)$ for $k = -1, 0, 1$.

28. **Barycentric limit**

28.1. The matrix $L$ with eigenvalues $\lambda_0 \leq \lambda_2 \leq \ldots \leq \lambda_{n-1}$ defines the spectral function $F(x) = \lambda_{[x]}$ on $[0, 1]$, where $[t]$ is the floor function giving the largest integer smaller or equal than $t$. The inverse function $k(x) = F^{-1}(x)$ is called the integrated density of states of $L$ and $\mu = k'$ is the density of states. The sequence $G_k$ of Barycentric refinements of $G$ defines a sequence of operators $L_k$ and so a sequence of spectral functions $F_n(x)$. Let $\mathcal{G}_r$ denote the set of complexes of dimension $r$. The following spectral universality is a central limit theorem:

**Theorem:** $\exists F = F(r)$ such that $F_n(G) \rightarrow_{L^1} F$ for all $G \in \mathcal{G}_r$.

28.2. For $r = 1$, we know $F(x) = 4\sin^2(\pi x/2)$. The function is important as it conjugates the Ulam map $z \rightarrow 4x(1 - x)$ to a linear function $T(F(x)) = F(2x)$. The measure $\mu$ maximizes metric entropy of the Ulam map and is equal to the topological entropy which is $\log(2)$ for $T$.

28.3. We think of $G_n \rightarrow G_{n+1}$ as a renormalization step like adding and normalizing two independent random variables. The result can be seen as a central limit theorem.

29. **The join monoid**

29.1. The join $G \oplus H$ of two complexes $G, H$ is the complex $G \cup H \cup \{x \cup y, x \in H, y \in G\}$. For graphs it is known as the Zykov sum. Given graphs $G = (V, E), H = (W, F)$ then the sum is $(V \cup W, E \cup F \cup \{(a, b) \mid a \in V, b \in W\})$. If $\overline{G}$ denotes the complement graph and $+$ the disjoint union, then $\overline{G} \oplus H = \overline{G} + H$.

29.2. The join of two simplicial complexes $G, H$ is defined as the complex generated by $G \oplus H = G \cup H \cup \{x \cup y \mid x \in G, y \in H\}$. Let $f_G(t) = 1 + v_0t + v_1t^2 + \ldots$ denote the generating function of $G$: then we have the multiplication formula:

**Theorem:** $f_{G \oplus H}(t) = f_G(t)f_H(t)$.

29.3. This gives $1 - \chi(G) = f_G(-1)$. The dimension function on $G$ not only defines a coloring on $G_1$, it also defines a hyperbolic splitting of the unit spheres. Let $S^{-}(x) = \{y \in S(x), \dim(y) < \dim(x)\}$ and $S^{+}(x) = \{y \in S(x), \dim(y) > \dim(x)\}$. We call them the stable sphere and unstable sphere.

**Theorem:** $S(x) = S^{-}(x) \oplus S^{+}(x)$. 
29.4. It follows that $g(x, x) = 1 - \chi(S(x)) = (1 - \chi(S^-(x)))(1 - \chi(S^+(x))) = \omega(x)(1 - \chi(S^+(x)))$. This implies that $\text{str}(L^{-1}) = \sum_x (1 - \chi(S^+(x))) = \chi(G)$ because this is the sum over the Poincaré-Hopf indices of the function $-\dim$.

29.5. The join monoid is isomorphic to the additive monoid of disjoint union. The zero element is 0, the $-1$ sphere. One can show by induction that if $H$ is contractible and $K$ arbitrary then $H + K$ is contractible. This implies:

**Theorem:** The join $G$ of two spheres $H + K$ is a sphere.

29.6. For example, the join of two zero dimensional spheres $P_2$ is the circle $C_4$. The join of two circles a three sphere. It is not the dimension but the clique number $\dim(G) + 1$ which is additive. The clique number of the $-1$ sphere 0 is 0.

### 30. HOPF-RYNOV

30.1. An **Eulerian $d$-complex** is defined as a simplicial complex for which the graph $G_1$ has the property that every unit sphere $S(y)$ admits a natural fixed point free involution $\phi_y$ so that every $S(x)$ is a 2 : 1 cover of a discrete projective space. Barycentric refinements $G_1$ always have the minimal chromatic number $d + 1$ and have a bipartite dual graph. Every 2-complex is Eulerian which in this case also agrees with the property of being an Eulerian graph.

30.2. The existence of an involution not only allow us to define projective spaces associated to each unit ball $S(y)$, it also allows to define to continue an one-dimensional arc $(x, y)$ to $(x, y, \phi_y(x))$ and so define a global geodesic flow.

**Theorem:** An Eulerian $d$-complex admits a natural globally defined geodesic flow.

30.3. Characterizing geodesics by distance minimization fails. As the tangent space is in cardinality smaller than the graph, one can not connect any two points by a geodesic. One needs to go to the wave equation which is a Schrödinger equation in the discrete to achieve a unique flow with that property. For general complexes, the star graph with 4 vertices shows the difficulty to continue a geodesic through the singularity.

### 31. THE STRONG RING

31.1. The **addition** $A + B$ of two complexes is the disjoint union. The empty complex 0 is the zero element. The **Cartesian product** $G \times H$ is not a simplicial complex any more. We can look at the ring $\mathcal{R}$ generated by simplicial complexes. It has the one point complex 1 = $K_1$ as one element. Connected elements are the additive primes, simplicial complexes are multiplicative primes. The Hodge operator $H$ and the connection operator $L$ can both be extended to the ring $\mathcal{R}$.

**Theorem:** $\sigma(H(A \times B)) = \sigma(H(A)) + \sigma(H(B))$,

The set addition on the right hand side is $\{ \lambda + \mu \}$ which consists of $n^2$ elements if $G$ has $n$ elements.
31.2. Furthermore, because $L(A \times B)$ is the matrix tensor product of $L(A)$ and $L(B)$

**Theorem:** $\sigma(L(A \times B)) = \sigma(L(A)) \cdot \sigma(L(B))$

The set multiplication on the right hand side is $\{\lambda \cdot \mu\}$ which consists of $n^2$ elements if $G$ has $n$ elements.

32. **Kuenneth formula**

32.1. The Betti numbers of a signed complex $b_k(G)$ are now signed with $b_k(-G) = -b_k(G)$. The maps assigning to $G$ its Poincaré polynomial $p_G(t) = \sum_{k=0} b_k(G)t^k$ or Euler polynomial $e_G(t) = \sum_{k=0} v_k(G)t^k$ are ring homomorphisms from $R$ to $\mathbb{Z}[t]$. Also $G \rightarrow \chi(G) = p(-1) = e(-1) \in \mathbb{Z}$ is a ring homomorphism.

**Theorem:** $e_G$ and $p_G$ are ring homomorphisms $R \rightarrow \mathbb{Z}[t]$.

32.2. The Kuenneth formula for cohomology groups $H^k(G)$ is explicit via Hodge: a basis for $H^k(A \times B)$ is obtained from a basis of the factors. The product in $R$ produces the strong product for the connection graphs. These relations generalize to Wu characteristic. $R$ is a subring of the full Stanley-Reisner ring $S$, a subring of a quotient ring of the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots]$. An object $G \in R$ can be visualized by its Barycentric refinement $G_1$ and its connection graph $G'$.

32.3. Theorems like Gauss-Bonnet, Poincaré-Hopf or Brouwer-Lefschetz for Euler and Wu characteristic extend to the strong ring. The isomorphism $G \rightarrow G'$ to a subring of the strong Sabidussi ring shows that the multiplicative primes in $R$ are the simplicial complexes and that connected elements in $R$ have a unique prime factorization.

32.4. The Sabidussi ring is dual to the Zykov ring. The Zykov join was the addition which is a sphere-preserving operation. The Barycentric limit theorem implies that the connection Laplacian remains invertible in the limit.

33. **Dimension**

33.1. The inductive dimension of a graph is defined inductively as $\dim(G) = 1 + \sum_{x \in V} \dim(S(x))/|V|$. For a general complex $G$ we can define $\dim(G) = \dim(G_1)$, as $G_1$ is now the Whitney complex of a graph. We have $\dim(G) \leq \maxdim(G) = \max_{x \in G}(|x| - 1)$, where the right hand side is the maximal dimension.

**Theorem:** $\dim(A \times B) = \dim(A) + \dim(B)$.

33.2. Under Barycentric refinements, the inductive dimension can only increase.

**Theorem:** $\dim(G_1) \geq \dim(G)$

33.3. The reason is that higher dimensional complexes have more offsprings than smaller dimensional ones.
33.4. This implies an inequality which resembles the corresponding inequality for Hausdorff dimension in the continuum:

**Theorem:** \( \dim((A \times B)_1) \geq \dim(A) + \dim(B) \).

33.5. Added March 7, 2019. The next result has been proven by Betre and Salinger [16]:

**Theorem:** \( \dim((A \oplus B)) = \dim(A) + \dim(B) + 1 \).

If we extend the dimension from the Zykov monoid to the Zykov group, then the functional \( d(G) = \dim(G) + 1 \) is a group homomorphism to \((\mathbb{Q}, +)\) similarly as the genus functional \( 1 - \chi(G) \) is a group homomorphism to \((\mathbb{Z}, \cdot)\). These facts illustrate the importance of Euler characteristic and dimension as a functional. For more functionals, see [118].

### 34. Random complexes

34.1. Given a probability space of complexes, one can study the expectations of random variables. The simplest probability space is the Erdős-Rényi space \( E(n, p) \) of random graphs equipped with the Whitney complex. Define the polynomials \( d_n(p) \) of degree \( \binom{n}{2} \) as

\[
d_{n+1}(p) = 1 + \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} d_k(p),
\]

where \( d_0 = -1 \). We can now estimate the inductive dimension.

**Theorem:** \( E_{G(n,p)}[\dim] = d_n(p) \).

34.2. As the Euler characteristic is one of the most important functionals, we want to estimate its expectation:

**Theorem:**

\[
E_{G(n,p)}[\chi] = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} p^k \left( \binom{k}{2} \right).
\]

34.3. We don’t yet know the expectation value of the Wu characteristic on \( E(n, p) \).

### 35. Lusternik-Schnirelmann

35.1. A complex \( G \) is **contractible** if there exists \( x \in G \) such that both the unit sphere \( S(x) \) as well as the complex \( G \setminus x \) are contractible. A complex is **homotopic to \( K=1 \)** if there exists a complex \( H \) such that \( H \) is contractible to both \( G \) and \( K \). The **dunce hat** is an example of a complex homotopic to \( 1 \) which is not contractible. The minimal number of contractible subcomplexes of \( G \) covering \( G \) is called the Lusternik-Schnirelman category of \( G \).

35.2. A \( x \in G \) is called a **critical point** of a function \( f \) if \( S_f^{-}(x) \) is not contractible. The minimal number of critical points which a function \( f \) on \( G \) can have is denoted by \( \text{cri}(G) \).
35.3. There is a graded multiplication $H^k(G) \times H^l(G) \to H^{k+l}(G)$ called the cup product. If $m - 1$ is the maximal number of $p > 0$-forms $f_1, \ldots, f_{m-1}$ for which $f_1 \cup \cdots \cup f_{m-1}$ is not zero, then $m$ is called the cup length of $G$.

35.4. The following result, established with Josellis in 2012 is completely analog to the continuum.

**Theorem:** $\cup(G) \leq \text{cat}(G) \leq \text{cri}(G)$.

35.5. For any critical point $x_i$, we can form the maximal complex $G_i$ which does not contain an other critical point. Each $U_i$ is contractible and cover $G$. This proves $\text{cat}(G) \leq \text{cri}(G)$. If $\text{cat}(G) = n$, let $\{U_k\}_{k=1}^n$ be a Lusternik-Schnirelmann cover. Given a collection of $k_j \geq 1$-forms $f_j$ with $f_1 \wedge f_2 \cdots \wedge f_n \neq 0$. Using coboundaries we can achieve that for any simple $y_k \in U_k$, we can change $f$ in the same cohomology class $f$ so that $f(y_k) = 0$. Because $U_k$ are contractible in $G$, we can render $f$ zero in $U_k$. This shows that we can choose $f_k$ in the relative cohomology groups $H^k(G, U_k)$ meaning that we can find representatives $k_j$ forms $f_j$ which are zero on each $p_k$ simplices in the in $G$ contractible sets $U_k$. But now, taking these representatives, we see $f_1 \wedge \cdots \wedge f_n = 0$. This shows $\cup(G) \leq n$.

36. Morse inequality

36.1. A locally injective scalar function $f$ on the vertex set of a d-graph is called a Morse function, if $S_f(x)$ is a sphere for every $x$. The Morse index is $m(x) = 1 + \dim(S_f(x))$. The Poincaré-Hopf index is $(-1)^{m(x)}$. For example, if $d = 2$, and $S_f(x)$ is 0-dimensional, then $m(x) = 1$ and $i_f(x) = -1$. A function $f$ on an abstract simplicial d-complex $G$ is a Morse function if it is a Morse function on the graph $G_1$.

**Theorem:** Every d-complex admits a Morse function.

36.2. We can build up $G$ as a discrete CW-complex. The number at which a simplex $x$ has been added is a Morse function as $S(x)$ and $S^-(x)$ are both spheres. Also the function dim($x$) is a Morse function. For d-complexes, the stars of two simplices intersect in a simplex so that:

**Theorem:** For a d-complex, the Green function takes values $1, -1, 0$.

We have $g(x, y) = \omega(x)\omega(y)\chi(W^+(x) \cap W^+(y))$. We have $W^+(x) \cap W^+(y) = (1 - S^+(x))(1 - \chi(S^+(x)))$ which is in $\{ -1, 1 \}$ if there is an intersection and 0 if not. Let $b_k(G)$ denote the $k$'th Betti number. Let $c_k(G)$ denote the number of critical points of index $k$. Here are the weak Morse inequalities:

**Theorem:** $b_k(G) \leq c_k(G)$.

We even have the strong Morse inequalities

**Theorem:** $(-1)^m \sum_{k=0}^m (-1)^k(c_k - b_k) \geq 0$

By Euler-Poincaré, this is zero for the entire sum. It appears as if the Witten deformation proof (see e.g. [41]) works in the discrete too.
37. Isospectral deformation

37.1. If \( d \) is the exterior derivative, the operator \( D = d + d^* \) is the Dirac operator of \( G \). The Dirac operator \( D \) admits an isospectral Lax deformations \( D' = [B, D] = BD - DB \), where \( B = d - d^* + \gamma ib \), if \( D = d + d^* + b \). The parameter \( \gamma \) is a tuning parameter. For \( \gamma = 0 \) the deformation stays real. For \( \gamma \neq 0 \), it is allowed to become complex. The Dirac operator \( D(t) \) defines for every \( t \) an elliptic complex \( D : E \to F \) meaning that we have a splitting \( D(t) : E \to F \) such that McKean-Singer relation holds.

**Theorem:** The Lax system for the Dirac operator is integrable.

37.2. The spectrum of \( D(t) \) stays constant. Actually, \( L = D(t)^2 \) stays constant.

37.3. We have a deformation of the complex for which all classical geometry like the wave equation stays the same because \( L \) does not change. It is only the underlying \( d \) which changes. The Connes formula \( \sup_{|Df|_{\infty} = 1} |f(x) - f(y)| \) allows to re-interpret the isospectral deformation as a deformation of the metric. And as curvature is defined by a measure on locally injective function, any deformation of this measure (for example by isospectral deformation of \( d \)), gives a deformation of the curvature.

38. Trees and Forests

38.1. Given a finite simple graph \( G \), a rooted spanning tree is a subgraph \( H \) of \( G \) which is a tree with the same vertex set together with a base point \( x \). A rooted spanning forest is a subgraph \( H \) of \( G \) which is a forest with the same vertex set together with a base point \( x \). Let \( K \) be the Kirchhoff Laplacian of the graph and \( \det(K) \) the pseudo determinant, the product of the non-zero eigenvalues of \( K \). It is \( \exp(-\zeta'(0)) \) for the spectral zeta function \( \zeta(s) = \sum_{\lambda \neq 0} \lambda^{-s} \) of \( K \).

**Theorem:** \( \det(K) \) is the tree number. \( \det(K + 1) \) is the forest number.

38.2. The tree number of a graph \( G \) is the number of rooted spanning tree in \( G \). The forest number of a graph is the number of rooted spanning forests. The first part of the following theorem is the Kirchhoff matrix tree theorem. The second part of the theorem is the Chebotarev-Shamis forest theorem.

**Theorem:** \( \det(K) \) is the tree number. \( \det(K + 1) \) is the forest number.

38.3. By Baker-Norine theory, the tree number is also the order of the Picard group which appears in the context of discrete Riemann-Roch.

38.4. If \( F, G \) are arbitrary \( n \times m \) matrices. Write \( p(x) = p_0(-x)^m + p_1(-x)^{m-1} + \cdots + p_k(-x)^{m-k} + \cdots + p_m \) for the characteristic polynomial of the \( m \times m \) matrix \( F^TG \) with \( p_0 = 1 \). The generalized Cauchy-Binet theorem is

**Theorem:** \( p_k(F^TG) = \sum_{|P|=k} \det(F_P) \det(G_P) \)

The sum is taken over \( k \)-minors \( P \) with the understanding that if \( |P| = 0 \), then \( \det(F_P) = 1 \). Generalized Cauchy-Binet implies the polynomial identity \( \det(1 + zF^TG) = \sum_P z^{|P|} \det(F_P) \det(G_P) \) in which the sum is over all minors \( A_P \) including the empty one \( |P| = 0 \) for which \( \det(F_P) \det(G_P) = 1 \).
39. Wave equation

39.1. Because the Hodge Laplacian is a square $L = D^2 = (d + d^*)^2$, the wave equation $u_{tt} = Lu$ has an explicit d’Alembert solution. Let $D^{-1}$ be the pseudo inverse of $D$. It is defined as $\sum_{k, \lambda_k \neq 0} u_k u_k^T / \lambda_k$, where $Du_k = \lambda_k u_k$ with an orthonormal eigenbasis $\{u_k\}$ of $D$. The matrices $u_k u_k^T$ are orthogonal projections onto the lines spanned by $u_k$.

Theorem: $u(t) = \cos(Dt)u(0) + i \sin(Dt)D^{-1}u'(0)$

39.2. With the complex wave $\psi(t) = u(t) - i Du'(0)$, we can write the solution of the real wave equation of $u$ as a solution of the Schrödinger equation.

Theorem: $\psi(t) = e^{iDt}\psi(0)$.

39.3. Just use the Euler identity $e^{iDt} = \cos(Dt) + i \sin(Dt)$ and plug in $\psi(t) = u(t) - i Du'(0)$ to see that the relation holds.

40. Euler-Poincaré

40.1. Let $\Lambda^p(G)$ be the functions from $G_p = \{ x \in G \mid \dim(x) = k \}$ to $R$ which are anti-symmetric. The exterior derivatives $d_p f(x_0, x_1, \ldots, x_p) = \sum_{j} (-1)^j f(x_0, \ldots, \hat{x}_j, \ldots, x_p)$ define linear map $d : \Lambda(G) \rightarrow \Lambda(G)$, where $\Lambda(G)$ is the Hilbert space of dimension $n = |G|$. Since $d^2 = 0$, the cohomology groups $H^p(G) = \ker(d_p)/\im(d_{p-1})$ are defined. Their dimensions are the Betti numbers $b_k(G)$. The matrix $H = (d + d^*)^2$ decomposes into blocks $H_k(G)$. We have the Hodge relations:

Theorem: $\dim(\ker(H_k)) = \dim(H^k)$.

40.2. Define the Poincaré polynomial $p_G(t) = \sum_{k=0} b_k(G) t^k$. The cohomological Euler characteristic is $p_G(-1) = b_0(G) - b_1(G) + b_2(G) - \cdots$. If the $f$-vector of $G$ is $(v_0, v_1, v_2, \ldots)$, then the Euler polynomial is $e_G(t) = \sum_{k=0} v_k(G) t^k$. By definition, we have $d_G(-1) = \chi(G)$. The Euler-Poincaré theorem tells that the combinatorial and cohomological Euler characteristic agree.

Theorem: $\chi(G) = e_G(-1) = p_G(-1)$.

41. Interaction cohomology

41.1. Let $\Lambda^p_2(G)$ be the functions from $G_p = \{ (x, y) \mid x \cap y \neq \emptyset, \dim(x) + \dim(y) = p \}$ which are anti-symmetric. Like Stokes theorem $df(x) = f(\delta x)$ for simplicial cohomology, we define the exterior derivative $df((x, y)) = f(\delta x, y) + (-1)^{\dim(x)} f(x, \delta y)$ with the understanding that $f(\delta x, y) = 0$ if $\delta x \cap y = \emptyset$ or $f(x, \delta y) = 0$ if $x \cap \delta y = \emptyset$. It defines a linear map $d : \Lambda_2(G) \rightarrow \Lambda_2(G)$, where $\Lambda_2(G)$ has as dimension the number of intersecting simplices $(x, y)$ in $G$. Again, we can define the Dirac operator $D = d + d^*$ and the Hodge operator $H = D^2$ and decompose the later into blocks $H_k$. As before:
Theorem: \( \dim(\ker(H_k)) = \dim(H^k) \).

41.2. The quadratic Poincaré polynomial \( p_G(t) = \sum_{k=0} b_k(G) t^k \) and quadratic Euler polynomial \( e_G(t) = \sum_{k=0} v_k(G) t^k \) are defined in the same way. By definition, we have \( d_G(-1) = \chi(G) \). The Euler-Poincaré theorem tells that the combinatorial and cohomological Wu characteristic agree.

Theorem: \( \omega(G) = e_G(-1) = p_G(-1) \).

42. Stokes theorem

42.1. Examples of orientation oblivious measurements are valuations \( F \) like \( F(A) = v_k(A) \) measuring the \( k \) dimensional volume of a subcomplex \( A \) of \( G \) or \( \chi(A) \) giving the Euler characteristic of a subcomplex. The length of a subcomplex \( A \) for example is \( v_1(A) \). In the continuum, such quantities are accessible via integral geometry, like Crofton type formulas. In the discrete one refers to it also as geometric probability theory.

42.2. If valuations are done after an orientation has been chosen on the elements of \( G \), we get a calculus which features a fundamental theorem. Given an arbitrary choice of orientation of the sets in \( G \), the boundary \( \delta A \) of a subcomplex is in general no more a subcomplex, it becomes a chain. Given a form \( F \in \Lambda \), we can still compute \( F(\delta A) \).

If \( G \) is orientable \( d \)-complex and \( A \) is a \( k \)-subcomplex with boundary \( \delta A \), then \( \delta A \) is a complex. Stokes theorem tells that for any \( k \)-subcomplex \( A \) with boundary \( \delta A \), and any \( k \)-form \( F \)

Theorem: \( dF(A) = F(\delta A) \).

42.3. For \( k = 1 \), we talk about the fundamental theorem of line integrals, for \( k = 2 \) we have Stokes theorem and \( k = 3 \) goes under the name divergence theorem. The derivative \( d_0 : \Lambda^0 \rightarrow \Lambda^1 \) is the gradient, the derivative \( d_1 : \Lambda^1 \rightarrow \Lambda^2 \) is the curl and \( d_2 : \Lambda^2 \rightarrow \Lambda^3 \) is the divergence (often just identified with the dual \( d^*_0 : \Lambda^1 \rightarrow \Lambda^0 \), as 2-forms and 1-forms in three dimensions are dual to each other). This Stokes theorem holds both for the familiar simplicial calculus related to Euler characteristic \( \chi(G) \) as well as the connection calculus related to the Wu characteristics \( \omega_k(G) \).

43. Quadratic Lefschetz fixed point

43.1. Given an automorphism \( T \), define the quadratic Lefschetz number \( \chi_T(G) \), the super trace of the induced map on cohomology.

Theorem: \( \chi_T(G) = \sum_{x \sim y, (x,y) = (T(x),T(y))} i_T(x,y) \)

43.2. We can especially look at the case when \( G \) is a ball. This is cohomologically non-trivial.

Theorem: An endomorphism of a ball \( G \) has a fixed \((x,y), x \cap y \neq \emptyset \).
44. Eulerian spheres

44.1. Let $G_d$ be the class of $d$-graphs, $S_d$ the class of $d$-spheres, $B_d$ the class of $d$-balls, and $C_k$ the class of graphs with chromatic number $k$. Note that all Barycentric refinements of a complex are Eulerian. We call the class $S_d \cap C_{d+1}$ the class of Eulerian spheres and $B_d \cap C_{d+1}$ the class of Eulerian disks. The 0-sphere $2 = 1 + 1$ is Eulerian. Eulerian 1-spheres are cyclic graphs with an even number of vertices.

Theorem: Every unit sphere of an Eulerian sphere is Eulerian.

44.2. The dual graph $\hat{G}$ of a $d$-sphere $G$ is the graph in which the $d$-simplices are the vertices and where two simplices are connected, if one is contained in the other. A graph $(V, E)$ is bipartite if $V = (A \cup B$ with disjoint $A, B$ such $E = \{ (a, b) \mid a \in A, b \in B \}$. Every Barycentric refinement of a complex is a bipartite graph as we can take $A = \{ x \in G \mid \dim(x) \text{ even} \}$ and $B = \{ x \in G \mid \dim(x) \text{ odd} \}$.

Theorem: For $G \in S_d$, then $\hat{G}$ is bipartite if and only if $G$ is Eulerian.

45. Hamiltonian property

45.1. A finite simple graph $G$ is called Hamiltonian, if there exists a Hamiltonian cycle, a path visiting each vertex of $G$ exactly once. Hassler Whitney proved in 1931 a theorem which can be reformulated as 2-spheres being Hamiltonian. One can generalize this to arbitrary dimension.

Theorem: Every $d$-graph is a Hamiltonian graph.

45.2. The proof of this theorem is constructive. The path can be constructed in a time polynomial of the number of vertices. We can cut out part of the cycles, find Hamiltonian cycles in both parts, then combine the cycles. See [142].

46. Riemann-Hurwitz

46.1. The automorphism group $\text{Aut}(G)$ of a simplicial complex is the group of all automorphisms of $G$. An endomorphism $T$ is a simplicial map $G \to G$. If an endomorphism $T$ is restricted to the attractor $\bigcap_k T^k(G)$ is an automorphism. An automorphism $T$ of $G$ induces automorphisms on Barycentric refinements and so graph automorphisms. The equivalence classes $G_2/A$ are complexes again.

Theorem: If $A \subset \text{Aut}(G)$, then $G_1/A$ is a simplicial complex.

46.2. We need a refinements. The automorphism $T(a) = b, T(b) = a$ on $G = \{ \{a, b\}, \{a\}, \{b\} \}$ has no complexes $G/A, G_1/A$ but $G_1/A$ is a circle. The circle is a 2:1 cover of the interval, ramified at the two boundary points of the interval.

46.3. We can see $G_1$ as a branched cover $G_1/A$, ramified over some points. If $G$ was a $d$-graph, then $G_1/A$ is a discrete orbifold. If there are no ramification points, then the cover $G \to G/A$ is a fibre bundle with structure group $A$. 

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46.4. Given an automorphism $T$, define the ramification index $e(x) = 1 - \sum_{T \neq 1, T(x) = x} \omega(x)$ of $X$. The following remark was obtained with Tom Tucker. It is a discrete Riemann-Hurwitz result:

**Theorem:** $\chi(G) = |A| \chi(G/A) - \sum_{x \in G} (e(x) - 1)$

46.5. For every subset $G_k$ of indices of fixed dimension $k$, we have by the Burnside lemma $\sum_{T \in A} \sum_{x \in G_k, T(x) = x} 1 = |A| |G_k|$. The super sum gives $\sum_{T \in A} \sum_{x, T(x) = x} \omega(x) = |A| \chi(H)$. This gives $\sum_{T \neq 1} \sum_{x \in G} \omega(x) + \sum_{x \in G} \omega(x) = |A| \chi(H)$.

46.6. Let $\chi(G, T)$ denote the Lefschetz number of $T$. From the Lefschetz fixed point formula we get

**Theorem:** $\chi(G/A) = \frac{1}{|A|} \sum_{T \in A} L(G, T)$

47. **RIEMANN-ROCH**

47.1. A divisor $X$ is an integer-valued function on $G$. The simplex Laplacian $L$ is defined as $L(x, y) = \omega(x) \omega(y) H_0(x, y)$, where $H_0$ is the Kirchhoff Laplacian of the simplex graph (Hasse diagram) in which $G$ is the vertex set and two $x, y$ are connected if one is contained in the other and the dimensions differ by 1. The simplex graph is one-dimensional as it has no triangles. A divisor $X$ is called principal if $X = Lf$ for some integer valued function $f$. Write $(f) = Lf$. We think of a divisor as a geometric object and define its Euler characteristic $\chi(G) = \sum_x \omega(x) X(x)$. A divisor is essential if $\omega(x) X(x) \geq 0$ for all $x$. The linear system $|X|$ of $X$ is the set of $f$ for which $X + (f)$ is essential. Its dimension $l(X)$ is the maximal $k \geq 0$ such that for every $m < k$ and every $Y$ of $\chi(Y) = m$, the divisor $X - Y$ is essential. With the canonical divisor $K(x) = 0$, the simplest Riemann-Roch theorem is

**Theorem:** $l(X) - l(K - X) = \chi(X)$.

47.2. This is Baker-Norine theory, slightly adapted to change the perspective: classically a divisors appear one a curve (Riemann surface or 1-dimensional graph) $G$ and $\deg(X) + \chi(G) = \chi(X)$. One usually centers at the geometric underlying object which gives the canonical divisor $K = -2$ (as a function) which is in the case when $G$ is one-dimensional is linearly equivalent to the negated curvature function $K(v) = -2 + \deg(v)$ on the vertices of $G$. In the discrete we can put an other coordinate system and reflect at 0 rather than the geometric object. This is not possible in the continuum. Riemann-Roch tells that the signed distance to the surface $\chi(G) = 0$ is $\chi(G)$.

47.3. Reflecting at 0 rather than at usual canonical divisor representing the curve $G$ allows to have a Riemann-Roch for arbitrary dimensions. Generalizing Baker-Norine naively to higher dimensional simplicial complexes does not work, as the curvature $\kappa$ of $\chi(G)$ has only in the one-dimensional case the property that $K = -2\kappa$ is a divisor. Classically $l(X)$ and $l(K - X)$ have cohomological interpretations. Also here, Riemann-Roch appears like a fancy Euler-Poincaré formula for a now signed cohomology, but...
it is deeper than Euler-Poincaré as the surface \( \ker(\chi) \) is bumpy: it contains both generic divisors as well as special divisors.

47.4. The image of \( L \) is a linear subspace of the set \( \ker(G) = \{ \chi(G) = 0 \} \). The quotient \( \ker(\chi)/\text{im}(L) \) is the Picard group or divisor class group. The equivalence classes of divisors can be represented by rooted spanning trees in the simplex graph. This defines a group structure on rooted spanning trees. That there is a bijective identification between divisor classes and spanning trees is the subject of:

**Theorem:** The Picard group is isomorphic to the tree group.

### References

47.5. For the history of topology[44, 85] and graph theory [168, 85, 62] and discrete geometry [23]. See [72, 198, 184] for notations in algebraic topology, [71, 18, 24] for graph theory.

47.6. Abstract simplicial complexes appeared in 1907 by Dehn and Heegaard [29, 165]. In [4] they appeared under the name unrestricted skeleton complex. In [211], J.H.C. Whitehead calls them symbolic complexes. For references on simplicial complexes, see [87, 201, 202].

47.7. The category of simplicial complexes is as an axiom system a complexity minimum in the landscape of mathematical constructs. Adding more conditions leads to constructs like buildings (which are simplicial complexes covered by apartments) or matroids (which is a variant of simplicial complex in which the empty set is included), removing conditions leads to larger axiom systems. Already the definition of a partially ordered set requires more input as it defines a binary relation and the specification of reflexivity, antisymmetry and transitivity.

47.8. The importance of the Dehn-Heegard definition can not be over estimated as it frees from the concept of Euclidean space and defining simplicial complexes in Euclidean spaces which is more complicated as it requires to specify conditions on intersections. From the computer science point of view, one has to store a data structure with Euclidean vectors and given such a data structure requires quite a bit of work to double check that it is a valid simplicial complex. Even in modern time of 3d printing, there can be frustrations in realizing a complex because the actual data have degenerate points [197, 150] meaning often they are actually not simplicial complexes.

47.9. Sometimes, in the definition of simplicial sets \( G \) the vertex set \( V \) with condition \( \{ x \} \in G \) added, but that is redundant as the set of points \( V = \bigcup_{x \in G} x \) is defined by \( G \) itself. To appreciate the elegance of the construct of Dehn and Heegaard, one might want to compare with the proposal of Euclid stating that a point is which has no part. The later is an extremely vague definition which would include “elementary particles”, a notion which is relative as things which “have no part” have often turned out to be made of smaller parts.
47.10. Some of the results generalize to ∆ sets, partially ordered sets or simplicial sets (all constructs which generalize simplicial complexes). In physics, locally finite posets are called causal sets. Some connection calculus however does not generalize. The unimodularity theorem for example does not hold for simplicial sets, at least for the approaches we tried so far. It does extend to the strong ring.

47.11. Homotopy theory as developed by [211] uses elementary expansions and contractions. Homotopic complexes are said to have the same “nucleus”. [211] uses “collapsible” for “homotopic to a point”. See also [210]. The notions appearing for simplices described by graph theory, see [84, 83, 31].

47.12. Dimension theory has a long history [39]. The inductive definition of graphs appeared first in [100]. We studied the average in [97].

47.13. Random graphs were first studied in [48]. The average Euler characteristic appears in [97].

47.14. The idea of seeing geometric quantities as expectations is central in integral geometry. The first time, that curvature was seen as an expectation of indices is Banchoff [9, 10]. Random methods in geometry is part of integral geometry as pioneered by Crofton and Blaschke [22, 167]. We have used in in [121, 104] and [103, 112]. Having curvature given as an expectation allows to deform it. Given a unitary flow $U_t$ on functions for example produces a deformation of the curvature.

47.15. Discrete curvature traces back to a combinatorial curvature appearing already in [46] and considered by Heesch [17] in the context of graph coloring and extended in [61]. The formula $K(p) = 1 - V_1(p)/6$ and for graphs on the sphere appears also in [176, 177], where it is also pointed out that $\sum_p K(p) = 2$ is Gauss-Bonnet formula. Discrete curvature was used in [76] and unpublished work of Ishida from 1990. Higushi uses $K(p) = 1 - \sum_{y \in S(p)} (1/2 - 1/d(y))$, where $d(y)$ are the cardinalities of the neighboring face degrees in the sphere $S(p)$. In [205], the combinatorial curvature $K(p) = 1 - d(p)/2 + \sum_q 1/n(q)$ for polyhedral graphs is considered, where $n(y)$ are the side-numbers of the polygons $q$ adjacent to $p$. For two dimensional graphs, where all faces are triangles, these curvatures simplify to $d_j = 3$ so that $K = 1 - |S|/6$, where $|S|$ is the cardinality of the sphere $S(p)$. In [100] second order curvatures were used.

47.16. The Levitt curvature in arbitrary dimension appears in [156]. We rediscovered it in [98] after tackling dimension by dimension separately, not aware of Levitt. We got into the topic while working on [100]. Chern’s proof is [33] followed [5, 50].

47.17. See [181, 41] for modern proofs of Gauss-Bonnet-Chern. Historical remarks are in [34].

47.18. The Erdős Rényi probability space were introduced in [48]. The formulas for the average dimension and Euler characteristic has been found in [97]. The recursive dimension was first used in [100]. We looked at more functionals in [118].
47.19. The discrete Hadwiger Theorem appears in [92]. The continuous version is [67]. For integral geometry and geometric probability, see [186]. The theory of valuations on distributive lattices has been pioneered by Klee [93] and Rota [183] who proved that there is a unique valuation such that $X(x) = 1$ for any join-irreducible element. See also [57].

47.20. Wu characteristic appeared in [208] and was discussed in [63]. We worked on it in [131] and announced cohomology in [147] and [148]. For the connection cohomology belonging to Wu characteristic, see [141].

47.21. For discrete Poincaré-Hopf see [102] and an attempt to popularize it in [106] or Mathematica demonstrations [99, 101]. It got pushed a bit more in [103]. For the classical Poincaré-Hopf, see [200]. For the classical case, Poincaré covered the 2-dimensional case in chapter VIII of [174] It got extended by Hopf in arbitrary dimensions [79]. It is pivotal in the proof of Gauss-Bonnet theorems for smooth Riemannian manifolds (i.e. [66, 199, 77, 73, 45, 15]).

47.22. Discrete McKean-Singer was covered in [105]. The best proof in the continuum is [41]. The classical result is [160]. In [135], the suggestion appeared to define elliptic discrete complexes using McKean-Singer.

47.23. The Zykov sum (join) was introduced in [214] to graph theory. The strong ring was covered in [137, 140]. The extension of the monoid to a group is what is familiar when constructing integers from natural numbers or fractions from integers and known as the Grothendieck group completion. We first found a multiplication making the Zykov group to a ring by trial and error, then later saw that this ring is dual to the Sabidussi multiplication [185]. See [70, 194, 196].

47.24. The Brouwer-Lefschetz theorem is [107]. It generalizes the 1-dimensional case [170]. The classical result is [155]. See also [80].

47.25. The classical Kuenenneth formula is [152]. The graph version [128], uses the Barycentric refinement $(A \times B)_1$ of the Cartesian product $A \times B$. In the strong ring things are almost immediate via Hodge.

47.26. About the history of discrete notions of manifolds, see [192]. The Evako definition of a sphere as a cell complex for which every unit sphere is a $n-1$ sphere and such that removing one point makes it contractible was predated by approaches of Vietoris or van Kampen. The later would have accepted homology spheres as unit spheres.

47.27. The classical Sard theorem is [187]. The discrete version appears in [129]. It lead us to look at geometric coloring questions.

47.28. Our proof in the discrete is [127]. For the Jordan-Brouwer separation theorem, see [88, 26] or [206, 2, 95, 69, 42, 182, 172]. The Mazur-Morse-Brown theorem [159, 163, 27] assures that the complementary domains $A, B$ are homeomorphic to Euclidean unit balls if the embedding of $H$ is locally flat. The Alexander sphere is [3]. The Schoenflies theme is introduced in [189, 190, 191]. Various discrete versions were studied [32, 49, 204, 207, 158, 203]. Discrete versions are also interesting for constructive proofs [13].
47.29. For the spectral universality, see [125, 130]. It uses a result of Lidskii-Last [195] which assures if $||\mu - \lambda||_1 \leq \sum_{i,j=1}^{n} |A - B|_{ij}$ for any two symmetric $n \times n$ matrices $A, B$ with eigenvalues $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$.

47.30. The discrete exterior derivative goes back to Betti and Poincaré and was already anticipated by Kirchhoff. As pointed out in [107], the discrete Hodge point is [47]. It appeared also in [81]. The discrete Dirac operator was stressed in [109].

47.31. The unimodularity theorem $|\det(L)| = 1$ was discovered in February 2016, announced in [149] and proven in [132]. An other proof was given in [164].

47.32. We have looked at the arithmetic of unit spheres in [139], especially in the context of the diagonal Green function entries. The other Green function entries are covered in [145].

47.33. The result $\chi(G) = p(G) - n(G)$ was proven in [138, 145]. The functional equation for the spectral zeta function of the connection Laplacian was proven in [143]. Earlier work in the Hodge Zeta case is [146]. The zeta function is called Dyadic because the Barycentric limit is in an ergodic setup a von Neumann-Kakutani system [96], which has the Prüfer group as the spectrum. The system is a group translation on the dyadic group of integers and also known as the adding machine.

47.34. The Hydrogen relation $H = L - L^{-1}$ for one-dimensional complexes was studied in [134, 136] and [144].

47.35. An earlier talk [119] summarizes things also. [110] is an earlier snapshot about the linear algebra part. [122, 106] summarize the calculus.

47.36. The matrix tree theorem is [90]. It is based on the Cauchy-Binet theorem [30, 19]. A generalization [117] gives the coefficients of the characteristic polynomial. The Chebotarev-Shamis theorem is [173, 178]. See also [108], where we initially were not aware of the work of Chebotarev and Shamis.

47.37. We used multi-linear algebra [60] for the proof of the generalized Cauchy-Binet. The closest proof of the classical theorem is in [82]. The classical theorem deals with the determinant of $F^T G$ which is one coefficient of the characteristic polynomial $p$. We first generalized it to the pseudo determinant which is the product of the non-zero eigenvalues and the first non-zero coefficient of $p$. A new proof of the generalized Cauchy-Binet formula avoiding multi-linear algebra is given in [78].

47.38. The Lax deformation of exterior derivatives was introduced in [114, 113] and was motivated by Witten deformation [212, 41]. Lax systems were introduced first to [154]. Commutation relations of that form have appeared earlier when describing free tops $L' = [B, L]$, where $B = I^{-1}L$ is the angular velocity and $L$ the angular velocity in $so(n)$, which are geodesics in $SO(n)$ [6].

47.39. The Connes formula [36] is elementary but crucial in the process of generalizing Riemannian geometry to non-commutative geometry.
47.40. After finding a multiplication completing the Zykov addition to a ring in [137], we realized it is the dual to the Sabidussi ring. In [140], we looked at the ring generated by the Cartesian product. It is a subring and consists of discrete CW complexes. Unlike for simplicial sets, the classical theorems like Gauss-Bonnet and energy theorem go over.

47.41. Riemann-Roch for graphs is [7]. See also [8]. We worked on Riemann-Hurwitz in [151]. The usual approach for Riemann-Hurwitz in graph theory is to see them as discrete analogues of algebraic curves or Riemann surfaces see [161].

47.42. [209] first looked for a combinatorial definition of spheres. Forman [54] defined spheres through the Reeb as objects admitting 2 critical points. See also [55]. More on discrete Morse theory in [56, 58].

47.43. We used data fitting to get first heuristically the Stirling formula then proved it. It is however considered "well known" [25]. It appears also in [201, 157, 75].

47.44. The history of polytopes is a "delicate task" [43]. The Euler polyhedron formula (Euler’s gem) was discussed in [180]. The early proofs of Schläfli and Staudt had still gaps according to [29]. See also [28, 68] for pointing out the importance of shellability. The difficulty is also explained in [153, 64].

47.45. The story of polyhedra is told in [180, 38]. Historically, it was developed in [188], [193], [175]. Coxeter [38] defines a polytope as a convex body with polygonal faces. [65] also works with convex polytopes in $\mathbb{R}^n$ where the dimension is the dimension of the affine span.

47.46. The perils of a general definition of a polytop were known since Poincaré (see [1, 180, 35, 153]). Polytop definitions are given in [188, 38, 65, 86]. Topologists started with new definitions [4, 53, 37, 198], and define first a simplicial complex and then polyhedra as topological spaces which admit a triangularization by a simplicial complex.

47.47. Dehn-Sommerville relations have traditionally been formulated for convex polytopes and then been generalized to situations where unit spheres can be realized as convex polytopes. See [94, 169, 166, 157, 25, 75, 91] or [14].

47.48. We started to think about graph coloring during the project [124]. The reports [120] and [126] explored this a bit more. It is related to Fisk theory [52, 51].

47.49. Some special graphs appearing when counting was considered in [133]. When writing this, we were not aware that the cell complex introduced already in [20] which goes much further than what we did. Other classes of complexes called orbital networks [111, 115, 116] were studied first with Montasser Ghachem.

47.50. For the Alexander duality, see [21]. Originally established by Alexander in 1922, it was formulated by Kalai and Stanley in combinatorial topology. We formulated it with cohomology rather than homology and cohomology. As such it is an identity where we have numbers on both sides.
Unlike in Regge calculus [179, 40, 59, 162], the definitions of Ricci and Hilbert action mentioned are combinatorial and do not depend on an embedding. There are other notions of curvature which are intrinsic and do not depend on an embedding into an ambient space: [171, 213, 89].

Questions

48. Inverse spectral questions

48.1. We have seen that the spectrum of $L$ does not determine the Betti numbers in general but that for a Barycentric refinement of $G$, the Betti numbers $b_0, b_1$ can be read off from the spectrum as the number of eigenvalues 1 and $-1$.

Question: Does the spectrum of $L$ determine $b_k$ for $k \geq 2$.

Question: Does the spectrum of $L$ determine the Wu characteristic $\omega(G)$?

49. Barycentric limit

We have seen that the limiting spectral measure can be computed in the case $d = 1$. It is a smooth measure. In higher dimensions, we see spectral gaps. These gaps have first been seen in the BeKeNePaPeTe paper [11].

Question: Prove spectral gaps in limiting spectral measure for $d \geq 2$.

50. Coloring

50.1. The four color theorem is equivalent to the statement that all 2-spheres are 4-colorable.

Question: Are all $d$-spheres $(d + 2)$-colorable?

This would imply that all $d$-spheres either have chromatic number $d + 1$ or $d + 2$. A generalized Heawood statement is that we have minimal chromatic number $d + 1$ if and only if every $d - 2$ dimensional simplex in $G$ has even degree. Every Barycentric refinement of a sphere is minimally colorable.

Question: Are all 2-graphs 5 colorable?

There are 2-complexes different from spheres which need 5 colors. One of the simplest is the projective plane. Note that for positive genus $g$, the coloring question of $d$-complexes is completely different than than the coloring question for embedded graphs. On the 2-torus, one can embed $K_7$ which needs 7 colors but $K_7$ is a 6 dimensional complex.
51. Connection Cohomology

51.1. While we know that connection cohomology is not a homotopy invariant, we have not yet proven that it is a topological invariant. A notion of homeomorphism appears in [123]. Alternatively, one could use “discrete homoeomorphic” in the sense that geometric realizations are homeomorphic to ask

**Question:** Is connection cohomology a topological invariant?

51.2. We would like to find more examples of triangulations of non-homeomorphic d-manifolds with different connection cohomology which can not be distinguished by other means:

**Question:** Can one distinguish homology spheres with Wu cohomology?

51.3. Something we have only started to look at is to extend Alexander duality to the Wu characteristic:

**Question:** Is there a duality for connection cohomology?

51.4. As connection cohomology is not a homotopy invariant, the naive generalization does not work.

52. Random Complexes

52.1. The probability spaces $E(n, p)$ of graphs define natural random spaces of simplicial complexes as we can take the Whitney complex of a graph. While we have a formula for the expectation of Euler characteristic, this is not yet available for Wu characteristic numbers $\omega_k$.

**Question:** What is the expected value of $\omega_k$ on $E(n, p)$?

52.2. We would also like to know the expectations of the Betti numbers:

**Question:** What is the expectation of $b_k(G)$ on $E(n, p)$?

53. Geodesic Flow

53.1. Having a global geodesic flow on d-complexes asks for dynamical properties like ergodicity. In a d-graph we always have a Hamiltonian path. The existence of geodesic flow is related to the question whether every unit sphere is Eulerian, admitting an Eulerian cycle. As we deal with finite complexes, every geodesic path is closed. Ergodicity means that the path visits every edge exactly once. This prompts the ergodicity question:

**Question:** Is there a d-complex with a geodesic Hamiltonian cycle?
53.2. For $d$-graphs with boundary the geodesic flow can be continued naturally using the billiard reflection condition which naturally exists for $d$-complexes. The geodesic flow is then a billiard. Again one can ask for the existence of an Eulerian geodesic path in the interior. Note that a billiard trajectory on the boundary is simply a geodesic on the boundary (which is a closed $(d - 1)$ complex.

**Question:** Is there an ergodic $d$-complex with boundary?

54. **Zeta function**

54.1. While various equivalent expressions exist for the connection zeta function in the Barycentric limit of a one-dimensional complex, we don’t yet have found a reference about where the roots of $\zeta$ are:

**Question:** The limiting zeta function $\zeta$ has roots on the imaginary axes.
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