THE AMAZING WORLD OF SIMPLICIAL COMPLEXES

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Abstract. Defined by a single axiom, finite abstract simplicial complexes belong to the simplest constructs of mathematics. We look at a few theorems.

Theorems

1. Simplicial complexes

1.1. A **finite abstract simplicial complex** G is a finite set of non-empty sets which is closed under the process of taking finite non-empty subsets. The **Barycentric refinement** G_1 of G is the set of finite subsets of the power set of G which are pairwise contained into each other. The new complex G_1 defines a finite simple graph $\Gamma = (V, E)$, where $V = G$ and E are the pairs where one is contained in the other. G_1 agrees with the Whitney complex of Γ, the collection of vertex sets of complete sub graphs of Γ. Other names for Whitney complexes are **flag complexes** or **clique complexes**.

Theorem: Barycentric refinements are Whitney complexes.

1.2. Examples of complexes not coming directly from graphs are buildings or matroids. Oriented matroids are examples of elements of the ring $\mathcal{R}$ generated by simplicial complexes. Still, the Barycentric refinement G_1 of G always allows to study G with the help of graph theory.

1.3. A subset H of G is called a **sub-complex**, if it is itself a simplicial complex. Any subset H **generates** a sub-complex, the smallest simplicial complex in G containing H. The set $\mathcal{G}$ of sub-complexes is a Boolean lattice because it is closed under intersection and union. The **f-vector** of G is $f = (v_0, v_1, \dots, v_r)$, where v_k is the number of elements in G with cardinality $k + 1$. The integer r is the **maximal dimension** of G.

2. Poincaré-Hopf

2.1. A real-valued function $f : G \to \mathcal{R}$ is **locally injective** if $f(x) \neq f(y)$ for any $x \subset y$ or $y \subset x$. In other words, it is a **coloring** in the graph Γ representing G_1. The **unit sphere** $S(x)$ of $x \in G$ is the set $\{y \in G | (x, y) \in E(\Gamma)\}$. It is the unit sphere in the metric space G, where the distance is the geodesic distance in the graph representing G_1. Define the stable unit sphere $S_f^-(x) = \{y \in S(x) \mid f(y) < f(x)\}$ and the **index** $i_f(x) = \chi(S_f^-(x))$. The **Poincaré-Hopf theorem** is

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Theorem: $\sum_x i_f(x) = \chi(G)$.

2.2. Classically, for a smooth function with isolated critical points on a Riemannian manifold M, the same definitions and result apply for $i_f(x) = \lim_{r \to 0} \chi(S^-_{r,f}(x))$, where S_r is the geodesic sphere of radius r in M centered at x.

2.3. If $f(x) = \dim(x)$, then $i_f(x) = \omega(x)$. Poincaré-Hopf tells then that $\chi(G) = \chi(G_1)$. If $f(x) = -\dim(x)$, then $i_f(x) = \omega(x)(1 - \chi(S(x)))$. For complexes for which every unit sphere is a $2d$-sphere, we have $i_{\dim} = -i_{-\dim}$ implying $\chi(G) = 0$.

3. Gauss-Bonnet

3.1. Any probability space $(\Omega, \mathcal{A}, P)$ of locally injective functions defines a **curvature** $\kappa(x) = \mathrm{E}[i_f(x)]$. As we have integrated over f, the curvature value $\kappa(x)$ only depends on x.

Theorem: $\sum_x \kappa(x) = \chi(G)$.

3.2. If Ω is the product space $\prod_{x \in G}[-1,1]$ with product measure so that $f \to f(x)$ are independent identically distributed random variables, then $\kappa(x) = K(x)$ is the **Levitt curvature** $1 + \sum_{k=0}(-1)^k v_k(S(x))/(k+1)$. The same applies if the probability space consists of all colorings. If $f = 1 + v_0 t + v_1 t^2 + \dots$ is the generating function of the f-vector of the unit sphere, with anti-derivative $F = t + v_0 t^2/2 + v_1 t^3/3...$, then $\kappa = F(0) - F(-1)$. Compare $\chi(G) = f(0) - f(-1)$ and $\sum_x \chi(S(x)) = f'(0) - f'(-1)$.

3.3. If P is the Dirac measure on the single function $f(x) = \dim(x)$, then the curvature is $\omega(x)$. If P is the Dirac measure on $f(x) = -\dim(x)$, then the curvature is $\omega(x)(1 - \chi(S(x))$.

4. Valuations

4.1. A real-valued function X on $\mathcal{G}$ is called a **valuation** if $X(A \cap B) + \chi(A \cup B) = \chi(A) + \chi(B)$ for all $A, B \in \mathcal{G}$. It is called an **invariant valuation** if $X(A) = X(B)$ if A and B are isomorphic. Let $\mathcal{G}_r$ denote the set of complexes of dimension r. The **discrete Hadwiger theorem** assures:

Theorem: Invariant valuations on $\mathcal{G}_r$ have dimension $r + 1$.

4.2. A basis of the space of invariant valuations is given by $v_k : \mathcal{G} \to \mathbb{R}$. Every vector $X = (x_0, \dots, x_r)$ defines a valuation $X(G) = X \cdot f(G)$ on $\mathcal{G}_r$.

5. The Stirling formula

5.1. The f-vectors transform linearly under Barycentric refinements. Let $\mathrm{Stirling}(x, y)$ denote the **Stirling numbers** of the second kind. It is the number of times one can partition a set of x elements into y non-empty subsets. The map $f \to Sf$ is the **Barycentric refinement operator**

Theorem: $f(G_1) = Sf$, where $S(x, y) = \mathrm{Stirling}(y, x) x!$.

5.2. The matrix is upper triangular with diagonal entries $k!$ the factorial. If $X(G) = \langle X, f(G)\rangle = X(G_1) = \langle X, Sf(G)\rangle = \langle S^T X f(G)$, then $X = S^T X$ so that X is an eigenvector to the eigenvalue 1 of S^T. The valuation with $X = (1, -1, 1, -1, \dots)$ is the **Euler characteristic** $\chi(G)$. This shows that Euler characteristic is unique in the class of translation invariant valuations.

Theorem: If $X(1) = 1$ and $X(G) = X(G_1)$ for all G, then $X = \chi$.

6. The unimodularity theorem

6.1. A finite abstract simplicial complex G of n sets defines the $n \times n$ **connection matrix** $L(x, y) = 1$ if $x \cap y \neq \emptyset$ and $L(x, y) = 0$ if $x \cap y = \emptyset$. The **unimodularity theorem** is:

Theorem: For all $G \in \mathcal{G}$, the matrix L is unimodular.

7. Wu characteristic

7.1. Using the notation $x \sim y$ if $x \cap y \neq \emptyset$, define the **Wu characteristic**

$$\omega(G) = \sum_{x \sim y} \omega(x)\omega(y) .$$

For a complete complex K_{d+1} we have $\omega(K^{d+1}) = (-1)^d$. As every $x \in G$ defines a simplicial complex generated by $\{x\}$, the notation $\omega(x)$ is justified.

7.2. A complex G is a d**-complex** if every unit sphere is a $(d-1)$-sphere. A complex G is a d**-complex with boundary** if every unit sphere $S(x)$ is either $(d-1)$ sphere or a $d-1$-ball. The sets for which $S(x)$ is a ball form the **boundary** of G. A complex without boundary is **closed**. d-complexes with or without boundary are **pure**: all maximal sets have the same dimension.

Theorem: For a d-complex G with boundary, $\omega(G) = \chi(G) - \chi(\delta G)$.

7.3. For any d one can define higher **Wu characteristic**

$$\omega_k(G) = \sum_{x_1 \sim \dots x_k} \omega(x_1) \cdots \omega(x_d)$$

summing over all simultaneously intersecting sets in G. As before $\omega(x) = (-1)^{\dim(x)} = (-1)^{|x|-1}$. It can be generalized by assuming $x_k \in G_k$ to be in different complexes. Especially important is the **intersection number** between two complexes G, H

$$\omega(G, H) = \sum_{x \sim y} \omega(x)\omega(y) .$$

8. The energy theorem

8.1. As L has determinant 1 or -1, the inverse $g = L^{-1}$ is a matrix with integer entries. The entries $g(x, y)$ are the **potential energy values** between the simplices x, y.

Theorem: For any complex G, we have $\sum_x \sum_y g(x, y) = \chi(G)$.

8.2. This **energy theorem** assures that the total potential energy of a complex is the Euler characteristic.

9. Homotopy

9.1. The graph $1 = K_1$ is **contractible**. Inductively, a graph is **contractible** if there exists a vertex x such that both $S(x)$ and $G - x$ are contractible. The step $G \to G - x$ is a homotopy step. Two graphs are **homotopic** if there exists a sequence of homotopy steps or inverse steps which brings one into the other. Contractible is not the same than homotopic to 1. A graph G is a **unit ball** if there exists a vertex such that $B(x) = G$.

Theorem: If G is a unit ball then it is contractible.

9.2. It is proved by induction. It is not totally obvious. A **cone extension** $G = D + x$ for the **dunce hat** D obtained by attaching a vertex x to D is a ball but we can not take x away. Any other point y can however be taken away by induction as $G - y$ is a ball with less elements.

Theorem: Contractible graphs have Euler characteristic 1.

9.3. The proof is done by induction starting with $G = 1$. It is not true that the Wu characteristic $\sum_{x \sim y} \omega(x)\omega(y)$ is a homotopy invariant as $\omega(K_{n+1}) = (-1)^n$.

10. Spheres

10.1. The **empty graph** 0 is the (-1) sphere. A d**-sphere** G is a d-complex for which all $S(x)$ are $(d-1)$ spheres and for which there exists a vertex x such that $G - x$ is contractible. The **1-skeleton graphs** of the octahedron and the icosahedron are examples of 2-spheres. Circular graphs with more than 3 vertices are 1-spheres. A simplicial complex G is a d**-sphere**, if the graph G_1 is a d-sphere. Here is the **polished Euler Gem**

Theorem: $\chi(G) = 1 + (-1)^d$ for a d-sphere G.

Theorem: The join of a p-sphere with a q-sphere is a $p + q + 1$-sphere.

10.2. The **generating function** of G is $f_G(t) = 1 + \sum_{k=0}^{\infty} v_k(G) t^{k+1}$ with $v_k(G)$ being the number of k-dimensional sets in G. It satisfies

Theorem: $f_{G+H}(t) = f_G(t) + f_H(t) - 1$ and $f_{G \oplus H}(t) = f_G(t) f_H(t)$.

For example, for $P_2 \oplus P_2 = S_4$ we have $(1+2t)(1+2t) = 1 + 4t + 4t^2$.

10.3. Given a d-graph. The function $\dim(x)$ has every point a critical point and $S^-(x) = \{y \in S(x) \mid f(y) < f(x)\}$ and $S^+(x) = \{y \in S(x) \mid f(y) > f(x)\}$ then $S(x) = S^-(x) + S^+(x)$.

10.4. Since by definition, a sphere becomes contractible after removing one of its points:

Theorem: d-spheres admit functions with exactly two critical points.

Spheres are the d-graphs for which the minimal number of critical points is 2. There are no d-graphs for which the minimal number of critical points is 1.

11. Platonic complexes

11.1. A **combinatorial CW complex** is an empty or finite ordered sequence of spheres $G = \{c_1, \dots, c_n\}$ such that $G_n = \{c_1, \dots, c_n\}$ is obtained from $G_{n-1} = \{c_1, \dots, c_{n-1}\}$ by selecting a sphere c_n in G_{n-1} such that c_n is either empty or different from any c_j. We identify c_j with the cell filling out the sphere. Its dimension is 1 plus the dimension of the sphere. The Barycentric refinement G_1 of G is the Whitney complex of the graph with vertex set G and where two vertices a, b are connected if one is a sub sphere of the other.

11.2. G is a d-sphere if G_1 is a d-sphere as a simplicial complex. A subset H of G is a **sub-complex** of G if $H_1 \subset G_1$ for the refinements.

11.3. The **Levitt curvature** of a cell c_j is $F(0) - F(-1)$, where F is the anti-derivative of the f-generating function $f = 1 + tv_0 + t^2 v_1 + \dots$ of the sphere $S(c_j)$. The curvature of a cell x in a 2-sphere is $1 - v_0(S(x))/2 + v_1(S(x))/3 = 1 - v_0(S(x))/6$. The curvature of a cell in a 3 sphere is 0. Gauss-Bonnet assures that the sum of the curvatures is the Euler characteristic.

11.4. A d-sphere G is called a **Platonic d-polytope** if for every $0 \leq k \leq d$ and any cell dimension k, there exists a Platonic $(d-1)$-sphere P_k such that for any cell c of dimension k, the unit sphere $S(x)$ is isomorphic to P_k. The -1-dimensional sphere 0 is assumed to be Platonic. The 0-dimensional sphere consisting of two isolated points is Platonic too. The 1-dimensional complexes C_k with $k \geq 3$ are the Platonic 1-spheres. With C_3 one denotes the **1-skeleton complex** of K_3. Let $P = (p(-1), p(0), p(1), p(2), \dots$ denote the number of Platonic d-polytopes. In the CW case, we have the familiar **Schläfli classification**

Theorem: $\text{Platonic}_{CW} = (1, 1, \infty, 5, 6, 3, 3, 3, \dots)$.

11.5. The classification of Platonic polytopes of dimension d which are simplicial complexes is easier. There is a unique Platonic solid in each dimension except in dimensions $1, 2, 3$. In the 1-dimensional case there are infinitely many. In the two-dimensional case, only the **octahedron** and **icosahedron** are Platonic. In the three dimensional case, there is only the **600 cell** and the **16 cell**. After that the curvature condition and Gauss-Bonnet constraint brings it down to the cross polytopes.

Theorem: $\text{Platonic}_{SC} = (1, 1, \infty, 2, 2, 1, 1, 1, 1, \dots)$.

12. Dehn-Sommerville relations

12.1. Given a d-dimensional complex G, any integer vector $(X_0, \dots, X_d)$ in $\mathbb{Z}^{d+1}$ defines a **valuation** $X(G) = X_0 v_0 + \dots X_d v_d$. By distributing the values X_k attached to each k-simplex in G equally to its $k+1$ vertices, we get the **curvature** $K(x) = \sum_{k=0}^{d} X_k v_{k-1}(S(x))/(k+1)$ for the valuation X and graph G at the vertex x. The formula $\sum_{x \in V} K(x) = X(G)$ is the **Gauss-Bonnet theorem** for X.

12.2. In the case $X(G) = v_1$, the curvature is the vertex degree divided by 2 and the formula reduces to the "Euler handshake". If $X = v_d$ is the volume of G, then $K(x)$ is the number of d-simplices attached to x divided by $d+1$. In the case $X = (1, -1, 1, -1, \dots)$, X is the **Euler characteristic** and K is the discrete analogue of the Euler form in differential geometry entering the Gauss-Bonnet-Chern theorem. For d-graphs, there are some valuations which are zero. Define the **Dehn-Sommerville** valuations

$$X_{k,d} = \sum_{j=k}^{d-1} (-1)^{j+d} \binom{j+1}{k+1} v_j(G) + v_k(G) \ .$$

Theorem: For d-graphs, the Dehn-Sommerville curvatures are zero.

12.3. The proof is by noticing that the curvature of $X_{k,d}$ is $K(x) = X_{k-1,d-1}(S(x))$. This follows from the relation

$$X_{k+1,d+1}(l+1)/(l+1) = X(k,d)(l)/(k+2) \ .$$

Use Gauss-Bonnet and induction using the fact that the unit sphere of a geometric graph is geometric and that for $d = 1$, a geometric graph is a cyclic graph C_n with $n \geq 4$. For such a graph, the Dehn-Sommerville valuations are zero.

13. Dual Connection matrix

13.1. Define the **dual connection matrix** $\overline{L}(x,y) = 1 - L(x,y)$ of a complex G with n sets. $\overline{L}$ is the adjacency matrix of a **dual connection graph** (G, E), where two simplices are connected, if they do not intersect: $E = \{(a,b) \mid a \cap b = \emptyset\}$.

Theorem: $1 - \chi(G) = \det(-L\overline{L})$.

13.2. Let E be the constant matrix $E(x,y) = 1$. The result follows from unimodularity $\det(L) = \det(g)$ and the energy theorem telling that $\overline{L}g = (E-L)g = Eg - 1$ has the eigenvalues of Eg minus 1 which are $\chi(G)$ and 0. Assume G has n sets:

Theorem: $-L\overline{L}$ has $n-1$ eigenvalues 1 and one eigenvalue $1-\chi(G)$.

13.3. The above formula is not the first one giving the Euler characteristic as a determinant of a Laplacian. [12] show, using a formula of Stanley, that if $A(x,y) = 1$ if x is not a subset of y and $A(x,y) = 0$ else, then $1 - \chi(G) = \det(A)$.

14. Alexander Duality

14.1. The **Alexander dual** of G over the vertex set V is the simplicial complex $G^* = \{x \subset V \mid x^c \notin G\}$. It is the complex generated by the complements x^c of the sets x in G. For the **complete complex** K_d, the dual is the empty complex. In full generality if G has n elements, one has for the Betti numbers $b_k(G)$:

Theorem: $b_k(G) = b_{n-3-k}(G)$, $k = 1, \dots, n-1$

14.2. In order for this statement to have content, one needs $n \geq 5$. It does not for $G = C_4$ but it works for $G = C_5$ already, where G^* is the complement of a circle in a 3-sphere. The combinatorial Alexander duality is due to Kalai and Stanley.

15. Sard

15.1. Given a locally injective function f on a graph $G = (V, E)$, define for $c \notin f(V)$ the **level surface** $\{f = c\}$ as the subgraph of the Barycentric refinement of G generated by simplices x on which f changes sign. Remember that G is a d-graph if every unit sphere $S(x)$ is a $(d-1)$-sphere. A discrete Sard theorem is:

Theorem: For a d-graph, every level surface is a $(d-1)$-graph.

If G is a finite abstract simplicial complex, then $f : G \to R$ defines a function on the Barycentric refinement G_1 and the level surface is defined like that. This result has practical value as we can define discrete versions of classical surfaces.

15.2. Given a finite set of functions $f_1, \dots, f_k$ on the vertex sets of successive Barycentric refinements $G_1, \dots, G_k$ of a simplicial complex, we can now look at the $(d-k)$-graph $\{f = c\} = \{f_1 = c_1, \dots, f_k = c_k\}$. Unlike in the continuum case, where the result only holds for almost all c, this holds for all c disjoint from the range.

Theorem: Given $f_j : G_j \to \mathcal{R}, j \leq k$, then $\{f = c\}$ is a $(d-k)$-complex.

16. Jordan-Schoenflies

16.1. A complex H is **embedded** in an other complex G, if the vertex set of H_1 in G_1 generates the complex H_1. The complex $H = \{\{1,2\},\{2,3\}\}$ in $G = \{\{1,2,3\},\{1,2\},\{2,3\},\{1,3\}\}$ for example is not embedded as it generates the entire complex. The Barycentric refinement H_1 however as a complex is embedded in the complex G_1.

16.2. The notion of d-sphere has been defined combinatorially as a d-complex which when punctured becomes contractible. A d**-ball** is a d-complex with boundary for which the boundary is a $d-1$ sphere. By definition, a punctured d-sphere is a d-ball.

> **Theorem:** A $d-1$-sphere embedded in a d sphere separates G into two complementary components A, B, which are both d-balls.

16.3. The case $d = 2$ is the Jordan-Brouwer separation theorem. The discrete version follows also from the piecewise linear version. In the Jordan curve case, the original proof already first dealt with the polygonal case.

16.4. The classical Jordan-Brouwer theorem tells that a $d-1$ sphere embedded in a d-sphere G separates G into two complementary components A, B. It needs some regularity to assure that the two components are balls. Alexander gave an example of a topological embedding of S^2 into S^3 for which one of the domains A is not simply connected.

17. Bonnet and Synge

17.1. The definition of **positive curvature complexes** is analog to the continuum. We aim for entirely combinatorial proofs of the results in the continuum.

17.2. Let G be a d-complex so that every unit sphere is a $(d-1)$ sphere. A **geodesic** 2**-surface** is a subcomplex if the embedded graph does not contain a 3-simplex. G has positive sectional curvature if every geodesic embedded wheel graph $W(x)$ has interior curvature $\geq 5/6$. The **geomag lemma** is that any wheel graph in a positive curvature G can be extended to an embedded 2-sphere.

17.3. An elementary analog of the **Bonnet theorem** is:

> **Theorem:** A positive curvature complex has diameter ≤ 4.

17.4. The simplest analog of **Synge theorem** is

> **Theorem:** A positive curvature complex is simply connected.

17.5. The reason for both statements is the **geomag lemma** stating that any closed geodesic curve can be extended to a 2-complex which is a sphere and so simply connected. The strict curvature assumption as we can not realize a projective plane yet with so few cells. With weaker assumptions getting closer to the continuum, the work is harder:

17.6. Define more generally the **sectional curvature to be** $\geq \kappa$ if there exists M such that the total interior curvature of any geodesic embedded 2-disk with M interior points is $\geq \delta \cdot M$ and such that every geodesic embedded wheel graph $W(x)$ has non-negative interior curvature. A complex has **positive curvature** if there exists $\kappa > 0$ such that G has sectional curvature $\geq \kappa$. The maximal κ which is possible is then the "sectional curvature bound".

17.7. An embedded 2-surface of positive sectional curvature κ must then have surface area $\leq 2/\kappa$. The classical theorem of Bonnet assures that a Riemannian manifold of positive sectional curvature is compact and satisfies an upper diameter bound $\pi/\sqrt{k}$. An analog bound $C/\sqrt{k}$ should work in the discrete.

17.8. Having a notion of sectional curvature allows to define **Ricci curvature** of an edge e as the average over all sectional curvatures over all wheel graphs passing through e. The **scalar curvature** at a vertex x is the average Ricci curvatures over all edges e containing x. The **Hilbert functional** is then the total scalar curvature.

18. An inverse spectral result

18.1. Let $p(G)$ denote the number of positive eigenvalues of the connection Laplacian L and let $n(G)$ the number of negative eigenvalues of L. One can **hear** the Euler characteristic of G because of

Theorem: For all $G \in \mathcal{G}$ we have $\chi(G) = p(G) - n(G)$.

18.2. The proof checks this by deforming L when adding a new cell. This result implies that Euler characteristic is a logarithmic potential energy of the origin with respect to the spectrum of iL.

Theorem: $\chi(G) = \text{tr}(\log(iL))(2\pi/i)$.

18.3. The proof shows also that after a CW ordering of the sets in a finite abstract simplicial complex, one can assign to every simplex a specific eigenvalue and so eigenvector of L. Each set in G has become a wave, a quantum mechanical object.

19. The Green star formula

19.1. Given a simplex $x \in G$, the **stable manifold** of the dimension functional $\dim(x)$ is $W^-(x) = \{y \in G \mid y \subset x\}$. The **unstable manifold** $W^+(x) = \{y \in G \mid x \subset y\}$ is known as the **star** of x. Unlike $W^-(x)$ which is always a simplicial complex, the star $W^+(x)$ is in general not a sub complex of G.

Theorem: $g(x,y) = \omega(x)\omega(x)\chi(W^+(x) \cap W^+(y))$.

19.2. In comparison, we have $W^-(x) \cap W^+(x) = \omega(x)$ and $L(x,y) = \chi(W^-(x) \cap W^-(y))$. The to L similar matrix $M(x,y) = \omega(x)\omega(x)\chi(W^-(x) \cap W^-(y))$ satisfies $\sum_x \sum_y M(x,y) = \omega(G)$, the Wu characteristic.

20. Wu characteristic

20.1. The **Euler characteristic** $\chi(G) = \omega_1(G) = \sum_{x \in G} \omega(x)$ of G is the simplest of a sequence of combinatorial invariants $\omega_k(G)$. The second one, $\omega(G) = \sum_{x,y,L(x,y)=1} \omega(x)\omega(y)$, is the **Wu characteristic** of G. The valuation χ is an example of a linear valuation, while ω is a **quadratic valuation**. The Wu characteristic also defines an **intersection number** $\omega(A,B)$ between sub-complexes.

20.2. All multi-linear valuations feature Gauss-Bonnet and Poincaré-Hopf theorems, where the curvature of Gauss-Bonnet is an index averaging. For example, with $K(v) = \sum_{v \in x, x \sim y} \omega(x)\omega(y)/(|x|+1)$ The Gauss-Bonnet theorem for Wu characteristic is

Theorem: $\omega(G) = \sum_v K(v)$.

21. The boundary formula

21.1. We think of the **internal energy** $E(G) = \chi(G) - \omega(G)$ as a sum of **potential energy** and **kinetic energy**. A **d-complex** is a simplicial complex G for which every $S(x)$ is a $(d-1)$-sphere. A **d-complex with boundary** is a complex $S(x)$ is either a $(d-1)$-sphere or a $(d-1)$-ball for every $x \in G$.

21.2. The d-complexes are **discrete** d**-manifolds** and d-complexes with boundary is a discrete version of a d**-manifold with boundary**. We denote by δG the **boundary** of G. It is the $(d-1)$-complex consisting of boundary points. By definition, $\delta\delta G = 0$, the empty complex. The reason is that the boundary of a complex is closed, has no boundary. We can reformulate the formula given below as

Theorem: If G is a d-complex with boundary then $E(G) = \chi(\delta(G))$.

21.3. If G is a d-ball, then δG is a $(d-1)$-sphere and $E(G) = 1 + (-1)^{d-1}$, by the **polished Euler gem formula**.

22. Zeta function

22.1. For a one-dimensional complex G, there is a **spectral symmetry** which will lead to a **functional equation** for the zeta function.

Theorem: If $\dim(G) = 1$, then $\sigma(L^2) = \sigma(L^{-2})$.

22.2. If H is a Laplacian operator with non-negative spectrum like the **Hodge operator** H or **connection operator** L, one can look at its **zeta function**

$$\zeta_H(s) = \sum_{\lambda \neq 0} \lambda^{-s} ,$$

where the sum is over all non-zero eigenvalues of H or L^2. In the connection case, we take L^2 to have all eigenvalues positive.

22.3. The case of the connection Laplacian is especially interesting because one does not have to exclude any zero eigenvalue. The **connection zeta function** of G is defined as $\zeta(s) = \sum_\lambda \lambda^{-s}$, where the sum is over all eigenvalues λ of L^2. It is an entire function in s.

Theorem: If $\dim(G) = 1$, then $\zeta(s) = \zeta(-s)$.

22.4. When doing Barycentric refinement steps, the zeta function converges to an explicit function.

$$\zeta(it) = \int_0^1 \frac{2\cos\left(2t\log\left(\sqrt{4v^2+1}+2v\right)\right)}{\pi\sqrt{1-v}\sqrt{v}}\, dv \; .$$

It is a **hypergeometric series** $\zeta(2s) = \pi\, {}_4F_3\left(\frac{1}{4},\frac{3}{4},-s,s;\frac{1}{2},\frac{1}{2},1;-4\right)$.

23. The Hydrogen formula

23.1. Given a simplicial complex G, let $\Lambda_k(G)$ denote the set of real valued functions on k-dimensional simplices. It is a v_k-dimensional vector space. Define the $v_k \times v_{k+1}$ matrices $d_k(x,y) = 1$ if $x \subset y$ and $d_k(x,y) = 0$ else. It is the **sign-less incidence matrix**. It can be extended to a $n \times n$ matrix d so that $d = d_0 + d_1 + \cdots + d_r$ and $D = d + d^*$ and $H = (d+d^*)^2$, the **sign-less Dirac** and **sign-less Hodge operator**. In the one-dimensional case, we have $H = d^*d + dd^*$. The **Hydrogen relations** are

Theorem: If $\dim(G) = 1$, then $L - L^{-1} = H$.

23.2. The relation allows to relate the spectra of L and H. It allows to estimate the spectral radius or give explicit formulas for the spectrum of the connection Laplacian in the circular case. The relation was also needed to get the explicit **dyadic zeta function**.

23.3. Let $S(x)$ denote the **unit sphere** of a simplex $x \in G$. While $S(x)$ is at first a subset of G, it generates a sub-complex in G_1. As $g(x,x) = 1 - \chi(S(x)) = \chi(W^+(x))$, we have a functional $\sum_x \chi(S(x))$ of Dehn-Sommerville type. With $f(t) = 1 + \sum_{k=1}^\infty v_{k-1}t^k = 1 + v_0 t + v_1 t^2 + v_2 t^3 + \cdots$, the Euler characteristic of G_1 can be written as $\chi(G) = f(0) - f(-1)$. The following result holds for any simplicial complex:

Theorem: $\mathrm{tr}(L - L^{-1}) = \sum_x \chi(S(x)) = f'(0) - f'(-1)$.

23.4. Compare that the Levitt curvature at a point x was $F(0) - F(-1) = \int_{-1}^0 f(t)\, dt$, where F is the anti-derivative of the generating function of $S(x)$.

24. Hopf Umlaufsatz

24.1. The discrete analogue of a region in the complex plane is a **hexagonal complex** which is defined as a 2-complex which is flat in the interior. To consider flat 2-complexes might look special at first but the importance of the complex plane in the continuum justifies to look at this case particularly well. In the continuum, we have the **Frobenius classification** of normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ selecting out $\mathbb{C}$ as the only commutative complete normed division algebra. In the continuum, special **packing properties** select out dimensions like 2,4,8,24: the hexagonal lattice is the densest in two dimensions.

24.2. If $d(x)$ is the vertex degree, then the Levitt curvature is $K(x) = (6-d)/6 = 0$ in the interior and $K(x) = (4-d)/6$ at the boundary. The vertex degrees at the boundary can be $1 < d < 6$ because $d = 1$ would mean a one dimensional hair sticking out and $d = 6$ would mean that the point is an interior point. The boundary curvature corresponds to the signed curvature of a boundary curve.

24.3. Let $b_0(G)$ denote the number of connected components of the region and $b_1(G)$ the genus, the number of holes. Informally, it is the number of bounded connected components of the complement of G in the flat infinite hexagonal plane. The number $b_1(G)$ can also be defined algebraically as the nullity of the Hodge Laplacian block $H_1(G)$. A consequence of Gauss-Bonnet and Euler-Poincaré is the **Hopf Umlaufsatz**

Theorem: For a planar region $\sum_{x \in \delta G} K(x) = b_0(G) - b_1(G)$.

24.4. Let $|S_r(x)|$ denote the number of edges in the sphere $S_r(p)$. The **Puiseux curvature** of a boundary point is defined as

$$K(x) = (2|S_1(x)| - |S_2(x)|)/12 \; .$$

It is a second order curvature measuring the discrepancy from linear growth of the wave fronts emanating from x.

24.5. With the Puiseux curvature, the Hopf Umlaufsatz needs more regularity from the region. A region is called **smooth** if it satisfies the following additional conditions: (i) G equal to its closure meaning that if $d(x, y) = 1$ in the ambient hexagonal plane then (x, y) is a simplex and (ii): any two interior points with a common boundary point either have distance 1 or are both adjacent to a third interior point.

Theorem: For a smooth planar region $\sum_{x \in \delta G} K_2(x) = b_0(G) - b_1(G)$.

25. Brouwer-Lefschetz

25.1. The **exterior derivative** d for G defines the **Dirac operator** $D = d + d^*$ of d. The Hodge Laplacian $H = D^2$ splits into a direct sum $H_0 \oplus H_1 \cdots \oplus H_d$. The null space of H_k is isomorphic to the k'th cohomology group $H^k(G) = \ker(d_k)/\mathrm{im}(d_{k-1})$. Its dimension b_k is the k'th Betti number. The **Euler-Poincaré** relation assures that the combinatorial and cohomological Euler characteristics are the same:

Theorem: $\chi(G) = \sum_k (-1)^k b_k$.

25.2. An **endomorphism** T of G is a map from G to G which preserves the order structure. It is an automorphism if it is bijective. An endomorphism T induces a linear map on cohomology $H^k(G)$. The super trace of this map is the **Lefschetz number** $\chi(T, G)$ of T. Given a fixed point $x \in G$ of T, its **Brouwer index** is defined as $i_T(x) = \omega(x)\mathrm{sign}(T|x)$. Now

Theorem: $\chi(T, G) = \sum_{x=T(x)} i_T(x)$.

25.3. A special case is $T = 1$, where $\chi(1, G) = \chi(G)$ and $i_T(x) = \omega(x)$. The Brouwer-Lefschetz fixed point theorem is then the Euler-Poincaré theorem.

26. McKean-Singer

26.1. The **super trace** $\text{str}(A)$ of a $n \times n$ matrix defined for a complex with n sets is defined as $\sum_{x \in G} \omega(x) L(x, x)$. By definition, we have $\text{str}(1) = \text{str}(L)$. For the Hodge operator $H = D^2 = (d + d^*)^2$ we have the **McKean-Singer formula**:

Theorem: $\text{str}(\exp(-tH)) = \chi(G)$ for all t.

26.2. The reason is that $\text{str}(H^k) = 0$ for $k > 0$, implying $\text{str}(\exp(tH)) = \text{str}(1) = \chi(G)$. The McKean-Singer identity is important as it allows to give almost immediate **heat deformation proofs** of the Lefschetz formulas in any framework in which the identity holds. We proposed in [130] to define a discrete version of a **differential complex** as McKean-Singer enables Atiyah-Singer or Atiyah-Bott like extensions of Gauss-Bonnet or Lefschetz. They are caricatures of the heavy theorems in the continuum.

26.3. The Hodge operator $H = (d + d^*)^2$ and the connection operator L live on the same finite dimensional Hilbert space. There is no cohomology associated to L. But for the connection operator L, there is still a localized version of McKean-Singer:

Theorem: $\text{str}(L^k) = \chi(G)$ for $k = -1, 0, 1$.

27. Barycentric limit

27.1. The matrix L with eigenvalues $\lambda_0 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$ defines the **spectral function** $F(x) = \lambda_{[nx]}$ on $[0, 1)$, where $[t]$ is the **floor function** giving the largest integer smaller or equal than t. The inverse function $k(x) = F^{-1}(x)$ is called the **integrated density of states** of L and $\mu = k'$ is the **density of states**. The sequence G_k of Barycentric refinements of G defines a sequence of operators L_k and so a sequence of spectral functions $F_n(x)$. Let $\mathcal{G}_r$ denote the set of complexes of dimension r. The following **spectral universality** is a **central limit theorem**:

Theorem: $\exists F = F(r)$ such that $F_n(G) \to_{L^1} F$ for all $G \in \mathcal{G}_r$.

27.2. For $r = 1$, we know $F(x) = 4 \sin^2(\pi x/2)$. The function is important as it conjugates the **Ulam map** $z \to 4x(1 - x)$ to a linear function $T(F(x)) = F(2x)$. The measure μ maximizes **metric entropy** of the Ulam map and is equal to the **topological entropy** which is $\log(2)$ for T.

27.3. We think of $G_n \to G_{n+1}$ as a renormalization step like adding and normalizing two independent random variables. The result can be seen as a **central limit theorem**.

28. The join monoid

28.1. The **join** $G \oplus H$ of two complexes G, H is the complex $G \cup H \cup \{x \cup y, x \in H, y \in G\}$. For graphs it is known as the **Zykov sum**. Given graphs $G = (V, E), H = (W, F)$ then the sum is $(V \cup W, E \cup F \cup \{(a, b) \mid a \in V, b \in W\})$. If $\overline{G}$ denotes the **complement graph** and $+$ the disjoint union, then $\overline{G \oplus H} = \overline{G} + \overline{H}$.

28.2. The **join** of two simplicial complexes G, H is defined as the complex generated by $G \oplus H = G \cup H \cup \{x \cup y \mid x \in G, y \in H\}$. Let $f_G(t) = 1 + v_0 t + v_1 t^2 + \dots$ denote the **generating function** of G: then we have the multiplication formula:

> **Theorem:** $f_{G \oplus H}(t) = f_G(t) f_H(t)$.

28.3. This gives $1 - \chi(G) = f_G(-1)$. The dimension function on G not only defines a coloring on G_1, it also defines a hyperbolic splitting of the unit spheres. Let $S^-(x) = \{y \in S(x), \dim(y) < \dim(x)\}$ and $S^+(x) = \{y \in S(x), \dim(y) > \dim(x)\}$. We call them the **stable sphere** and **unstable sphere**.

> **Theorem:** $S(x) = S^-(x) \oplus S^+(x)$.

28.4. It follows that $g(x, x) = 1 - \chi(S(x)) = (1 - \chi(S^-(x)))(1 - \chi(S^+(x))) = \omega(x)(1 - \chi(S^+(x)))$. This implies that $\text{str}(L^{-1}) = \sum_x (1 - \chi(S^+(x))) = \chi(G)$ because this is the sum over the Poincaré-Hopf indices of the function $-\dim$.

28.5. The join monoid is isomorphic to the additive monoid of disjoint union. The **zero element** is 0, the -1 sphere. One can show by induction that if H is contractible and K arbitrary then $H + K$ is contractible. This implies:

> **Theorem:** The join G of two spheres $H + K$ is a sphere.

28.6. For example, the join of two zero dimensional spheres P_2 is the circle C_4. The join of two circles a three sphere. It is not the dimension but the **clique number** $\dim(G) + 1$ which is additive. The clique number of the -1 sphere 0 is 0.

29. Hopf-Rynov

29.1. An **Eulerian d-complex** is defined as a simplicial complex for which the graph G_1 has the property that every unit sphere $S(y)$ admits a natural fixed point free involution ϕ_y to that every $S(x)$ is a $2:1$ cover of a discrete projective space. Barycentric refinements G_1 have the minimal chromatic number $d + 1$ are Eulerian and have this property always in dimension 2. The existence of an involution not only allows to define **projective spaces** associated to each unit ball $S(y)$, it also allows to define to continue an one dimensional arc (x, y) to $(x, y, \phi_y(x))$ and so define a global **geodesic flow**.

> **Theorem:** An Eulerian d-complex admits a natural globally defined geodesic flow.

29.2. Characterizing geodesics by distance minimization fails. As the tangent space is in cardinality smaller than the graph, one can not connect any two points by a geodesic. One needs to go to the wave equation which is a Schrödinger equation in the discrete to achieve a unique flow with that property. For general complexes, the star graph with 4 vertices shows the difficulty to continue a geodesic through the singularity.

30. The strong ring

30.1. The **addition** $A+B$ of two complexes is the disjoint union. The empty complex 0 is the **zero element**. The **Cartesian product** $G \times H$ is not a simplicial complex any more. We can look at the ring $\mathcal{R}$ generated by simplicial complexes. It has the one point complex $1 = K_1$ as **one element**. Connected elements are the **additive primes**, simplicial complexes are **multiplicative primes**. The **Hodge operator** H and the **connection operator** L can both be extended to the ring $\mathcal{R}$.

Theorem: $\sigma(H(A \times B)) = \sigma(H(A)) + \sigma(H(B))$,

The **set addition** on the right hand side is $\{\lambda + \mu\}$ which consists of n^2 elements if G has n elements.

30.2. Furthermore, because $L(A \times B)$ is the **matrix tensor product** of $L(A)$ and $L(B)$

Theorem: $\sigma(L(A \times B)) = \sigma(L(A)) \cdot \sigma(L(B))$

The **set multiplication** on the right hand side is $\{\lambda \cdot \mu\}$ which consists of n^2 elements if G has n elements.

31. Kuenneth formula

31.1. The **Betti numbers** of a signed complex $b_k(G)$ are now signed with $b_k(-G) = -b_k(G)$. The maps assigning to G its Poincaré polynomial $p_G(t) = \sum_{k=0} b_k(G)t^k$ or **Euler polynomial** $e_G(t) = \sum_{k=0} v_k(G)t^k$ are ring homomorphisms from R to $\mathbb{Z}[t]$. Also $G \to \chi(G) = p(-1) = e(-1) \in \mathbb{Z}$ is a ring homomorphism.

Theorem: e_G and p_G are ring homomorphisms $\mathcal{R} \to \mathbb{Z}[t]$.

31.2. The **Kuenneth formula** for cohomology groups $H^k(G)$ is explicit via Hodge: a basis for $H^k(A \times B)$ is obtained from a basis of the factors. The product in R produces the strong product for the connection graphs. These relations generalize to Wu characteristic. R is a subring of the full **Stanley-Reisner ring** S, a subring of a quotient ring of the polynomial ring $Z[x_1, x_2, \dots]$. An object $G \in R$ can be visualized by ts Barycentric refinement G_1 and its connection graph G'.

31.3. Theorems like Gauss-Bonnet, Poincaré-Hopf or Brouwer-Lefschetz for Euler and Wu characteristic extend to the strong ring. The isomorphism $G \to G'$ to a subring of the strong Sabidussi ring shows that the multiplicative primes in R are the simplicial complexes and that connected elements in R have a unique prime factorization.

31.4. The **Sabidussi ring** is dual to the Zykov ring. The Zykov join was the addition which is a sphere-preserving operation. The Barycentric limit theorem implies that the connection Laplacian remains invertible in the limit.

32. Dimension

32.1. The **inductive dimension** of a graph is defined inductively as $\dim(G) = 1 + \sum_{v \in V} \dim(S(x))/|V|$. For a general complex G we can define $\dim(G) = \dim(G_1)$, as G_1 is now the Whitney complex of a graph. We have $\dim(G) \leq \text{maxdim}(G) = \max_{x \in G}(|x| - 1)$, where the right hand side is the **maximal dimension**.

Theorem: $\dim(A \times B) = \dim(A) + \dim(B)$.

32.2. Under Barycentric refinements, the inductive dimension can only increase.

Theorem: $\dim(G_1) \geq \dim(G)$

32.3. The reason is that higher dimensional complexes have more off-springs than smaller dimensional ones.

32.4. This implies a inequality which resembles the corresponding inequality for **Hausdorff dimension** in the continuum:

Theorem: $\dim((A \times B)_1) \geq \dim(A) + \dim(B)$.

33. Random complexes

33.1. Given a probability space of complexes, one can study the expectations of random variables. The simplest probability space is the **Erdös-Rényi space** $E(n,p)$ of random graphs equipped with the Whitney complex. Define the polynomials $d_n(p)$ of degree $\binom{n}{2}$ as

$$d_{n+1}(p) = 1 + \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} d_k(p) ,$$

where $d_0 = -1$. We can now estimate the inductive dimension.

Theorem: $\mathrm{E}_{G(n,p)}[\dim]) = d_n(p)$.

33.2. As the Euler characteristic is one of the most important functionals, we want to estimate its expectation:

Theorem:

$$\mathrm{E}_{G(n,p)}[\chi] = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} p^{\binom{k}{2}} .$$

33.3. We don't yet know the expectation value of the Wu characteristic on $E(n,p)$.

34. Lusternik-Schnirelmann

34.1. A complex G is **contractible** if there exists $x \in G$ such that both the unit sphere $S(x)$ as well as the complex $G \setminus x$ are contractible. A complex is **homotopic to K=1** if there there exists a complex H such that H is contractible to both G and K. The **dunce hat** is an example of a complex homotopic to 1 which is not contractible. The minimal number of contractible subcomplexes of G covering G is called the Lusternik-Schnirelman category of G.

34.2. A $x \in G$ is called a **critical point** of a function f if $S_f^-(x)$ is not contractible. The minimal number of critical points which a function f on G can have is denoted by $\text{cri}(G)$.

34.3. There is a graded multiplication $H^k(G) \times H^l(G) \to H^{k+l}(G)$ called the **cup product**. If $m-1$ is the maximal number of $p > 0$-forms $f_1, \dots, f_{m-1}$ for which $f_1 \cup \cdots \cup f_{m-1}$ is not zero, then m is called the **cup length** of G.

34.4. The following result, established with Josellis in 2012 is completely analog to the continuum.

Theorem: $\text{cup}(G) \leq \text{cat}(G) \leq \text{cri}(G)$.

34.5. For any critical point x_i, we can form the maximal complex G_i which does not contain an other critical point. Each U_i is contractible and cover G. This proves $\text{cat}(G) \leq \text{cri}(G)$. If $\text{cat}(G) = n$, let $\{U_k\}_{k=1}^n$ be a **Lusternik-Schnirelmann cover**. Given a collection of $k_j \geq 1$-forms f_j with $f_1 \wedge f_2 \cdots \wedge \dots f_n \neq 0$. Using coboundaries we can achieve that for any simplex $y_k \in U_k$, we can change f in the same cohomology class f so that $f(y_k) = 0$. Because U_k are contractible in G, we can render f zero in U_k. This shows that we can choose f_k in the relative cohomology groups $H^k(G, U_k)$ meaning that we can find representatives k_j forms f_j which are zero on each p_{k_j} simplices in the in G contractible sets U_k. But now, taking these representatives, we see $f_1 \wedge \cdots \wedge f_n = 0$. This shows $\text{cup}(G) \leq n$.

35. Morse inequality

35.1. A locally injective scalar function f on the vertex set of a d-graph is called a **Morse function**, if $S_f^-(x)$ is a sphere for every x. The **Morse index** is $m(x) = 1 + \dim(S_f^-(x))$. The **Poincaré-Hopf index** is $(-1)^{m(x)}$. For example, if $d = 2$, and $S_f^-(x)$ is 0-dimensional, then $m(x) = 1$ and $i_f(x) = -1$. A function f on an abstract simplicial d-complex G is a Morse function if it is a Morse function on the graph G_1.

Theorem: Every d-complex admits a Morse function.

35.2. We can build up G as a **discrete** CW**-complex**. The number at which a simplex x has been added is a Morse function as $S(x)$ and $S^-(x)$ are both spheres. Also the function $\dim(x)$ is a Morse function. For d-complexes, the stars of two simplices intersect in a simplex so that:

Theorem: For a d-complex, the Green function takes values $1, -1, 0$.

We have $g(x,y) = \omega(x)\omega(y)\chi(W^+(x) \cap W^+(y))$. We have $W^+(x) \cap W^+(y) = (1 - S^+(x))(1 - \chi(S^+(x))$ which is in $\{-1, 1\}$ if there is an intersection and 0 if not. Let $b_k(G)$ denote the k'th Betti number. Let $c_k(G)$ denote the number of critical points of index k. Here are the **weak Morse inequalities**:

Theorem: $b_k(G) \leq c_k(G)$.

We even have the **strong Morse inequalities**

Theorem: $(-1)^p \sum_{k=0}^{p} (-1)^k (c_k - b_k) \geq 0$

By Euler-Poincaré, this is zero for the entire sum. It appears as if the **Witten deformation** proof (see e.g. [39]) works in the discrete too.

36. Isospectral deformation

36.1. If d is the exterior derivative, the operator $D = d + d^*$ is the **Dirac operator** of G. The Dirac operator D admits an **isospectral Lax deformations** $D' = [B, D] = BD - DB$, where $B = d - d^* + \gamma ib$, if $D = d + d^* + b$. The parameter γ is a tuning parameter. For $\gamma = 0$ the deformation stays real. For $\gamma \neq 0$, it is allowed to become complex. The Dirac operator $D(t)$ defines for every t an **elliptic complex** $D : E \to F$ meaning that we have a splitting $D(t) : E \to F$ such that McKean-Singer relation holds.

Theorem: The Lax system for the Dirac operator is integrable.

36.2. The spectrum of $D(t)$ stays constant. Actually, $L = D(t)^2$ stays constant.

36.3. We have a deformation of the complex for which all classical geometry like the wave equation stays the same because L does not change. It is only the underlying d which changes. The **Connes formula** $\sup_{|Df|_\infty = 1} |f(x) - f(y)|$ allows to re-interpret the isospectral deformation as a deformation of the metric. And as curvature is defined by a measure on locally injective function, any deformation of this measure (for example by isospectral deformation of d), gives a deformation of the curvature.

37. Trees and Forests

37.1. Given a finite simple graph G, a **rooted spanning tree** is a subgraph H of G which is a tree with the same vertex set together with a base point x. A **rooted spanning forest** is a subgraph H of G which is a forest with the same vertex set together with a base point x. Let K be the **Kirchhoff Laplacian** of the graph and $\text{Det}(K)$ the **pseudo determinant**, the product of the non-zero eigenvalues of K. It is $\exp(-\zeta'(0))$ for the **spectral zeta function** $\zeta(s) = \sum_{\lambda \neq 0} \lambda^{-s}$ of K.

37.2. The **tree number** of a graph G is the number of rooted spanning tree in G. The **forest number** of a graph is the number of rooted spanning forests. The first part of the following theorem is the **Kirchhoff matrix tree theorem**. The second part of the theorem is the **Chebotarev-Shamis forest theorem**.

Theorem: $\text{Det}(K)$ is the tree number. $\det(K + 1)$ is the forest number.

37.3. By Baker-Norine theory, the tree number is also the order of the **Picard group** which appears in the context of discrete Riemann-Roch.

37.4. If F, G are arbitrary $n \times m$ matrices. Write $p(x) = p_0(-x)^m + p_1(-x)^{m-1} + \cdots + p_k(-x)^{m-k} + \cdots + p_m$ for the **characteristic polynomial** of the $m \times m$ matrix $F^T G$ with $p_0 = 1$. The **generalized Cauchy-Binet theorem** is

Theorem: $p_k(F^T G) = \sum_{|P|=k} \det(F_P) \det(G_P)$

The sum is taken over k-minors P with the understanding that if $|P| = 0$, then $\det(F_P) = 1$. Generalized Cauchy-Binet implies the polynomial identity $\det(1 + zF^T G) = \sum_P z^{|P|} \det(F_P) \det(G_P)$ in which the sum is over all minors A_P including the empty one $|P| = 0$ for which $\det(F_P) \det(G_P) = 1$.

38. Wave equation

38.1. Because the Hodge Laplacian is a square $L = D^2 = (d + d^*)^2$, the wave equation $u_{tt} = Lu$ has an explicit **d'Alembert solution**. Let D^{-1} be the **pseudo inverse** of D. It is defined as $\sum_{k, \lambda_k \neq 0} u_k u_k^T / \lambda_k$, where $Du_k = \lambda_k u_k$ with an orthonormal eigenbasis $\{u_k\}$ of D. The matrices $u_k u_k^T$ are **orthogonal projections** onto the lines spanned by u_k.

Theorem: $u(t) = \cos(Dt) u(0) + i \sin(Dt) D^{-1} u'(0)$

38.2. With the **complex wave** $\psi(t) = u(t) - iDu'(0)$, we can write the solution of the **real wave** equation of u as a solution of the **Schrödinger equation**.

Theorem: $\psi(t) = e^{iDt} \psi(0)$.

38.3. Just use the Euler identity $e^{iDt} = \cos(Dt) + i \sin(Dt)$ and plug in $\psi(t) = u(t) - iDu'(0)$ to see that the relation holds.

39. Euler-Poincaré

39.1. Let $\Lambda^p(G)$ be the functions from $G_p = \{x \in G \mid \dim(x) = k\}$ to R which are anti-symmetric. The **exterior derivatives**

$$d_p f(x_0, x_1, \ldots, x_p) = \sum_j (-1)^j f(x_0, \ldots, \hat{x_j}, \ldots, x_p)$$

define linear map $d : \Lambda(G) \to \Lambda(G)$, where $\Lambda(G)$ is the Hilbert space of dimension $n = |G|$. Since $d^2 = 0$, the **cohomology groups** $H^p(G) = \ker(d_p)/\mathrm{im}(d_{p-1})$ are defined. Their dimensions are the Betti numbers $b_p(G)$. The matrix $H = (d + d^*)^2$ decomposes into blocks $H_k(G)$. We have the **Hodge relations**:

Theorem: $\dim(\ker(H_k)) = \dim(H^k)$.

39.2. Define the **Poincaré polynomial** $p_G(t) = \sum_{k=0} b_k(G)t^k$. The **cohomological Euler characteristic** is $p_G(-1) = b_0(G) - b_1(G) + b_2(G) - \cdots$. If the f-vector of G is $(v_0, v_1, v_2, \dots)$, then the **Euler polynomial** is $e_G(t) = \sum_{k=0} v_k(G)t^k$. By definition, we have $d_G(-1) = \chi(G)$. The **Euler-Poincaré theorem** tells that the combinatorial and cohomological Euler characteristic agree.

Theorem: $\chi(G) = e_G(-1) = p_G(-1)$.

40. Interaction cohomology

40.1. Let $\Lambda_2^p(G)$ be the functions from $G_p = \{(x,y) \mid x \cap y \neq \emptyset, \dim(x) + \dim(y) = p\}$ which are anti-symmetric. Like **Stokes theorem** $df(x) = f(\delta x)$ for simplicial cohomology, we define the **exterior derivative** $df((x,y)) = f(\delta x, y) + (-1)^{\dim}(x) f(x, \delta y)$ with the understanding that $f(\delta x, y) = 0$ if $\delta x \cap y = \emptyset$ or $f(x, \delta y) = 0$ if $x \cap \delta y = \emptyset$. It defines a linear map $d : \Lambda_2(G) \to \Lambda_2(G)$, where $\Lambda_2(G)$ has as dimension the number of intersecting simplices (x,y) in G. Again, we can define the **Dirac operator** $D = d + d^*$ and the **Hodge operator** $H = D^2$ and decompose the later into blocks H_k. As before:

Theorem: $\dim(\ker(H_k)) = \dim(H^k)$.

40.2. The **quadratic Poincaré polynomial** $p_G(t) = \sum_{k=0} b_k(G)t^k$ and **quadratic Euler polynomial** $e_G(t) = \sum_{k=0} v_k(G)t^k$ are defined in the same way. By definition, we have $d_G(-1) = \chi(G)$. The **Euler-Poincaré theorem** tells that the combinatorial and cohomological Wu characteristic agree.

Theorem: $\omega(G) = e_G(-1) = p_G(-1)$.

41. Stokes theorem

41.1. Examples of **orientation oblivious** measurements are valuations F like $F(A) = v_k(A)$ measuring the k dimensional volume of a subcomplex A of G or $\chi(A)$ giving the Euler characteristic of a subcomplex. The **length** of a subcomplex A for example is $v_1(A)$. In the continuum, such quantities are accessible via **integral geometry**, like **Crofton type formulas**. In the discrete one refers to it also as **geometric probability theory**.

41.2. If valuations are done after an orientation has been chosen on the elements of G, we get a **calculus** which features a **fundamental theorem**. Given an arbitrary choice of orientation of the sets in G, the boundary δA of a subcomplex is in general no more a subcomplex, it becomes a **chain**. Given a form $F \in \Lambda$, we can still compute $F(\delta A)$. If G is **orientable** d-complex and A is a k-subcomplex with **boundary** δA, then δA is a complex. **Stokes theorem** tells that for any k-subcomplex A with boundary δA, and any k-form F

Theorem: $dF(A) = F(\delta A)$.

41.3. For $k = 1$, we talk about the **fundamental theorem of line integrals**, for $k = 2$ we have **Stokes theorem** and $k = 3$ goes under the name **divergence theorem**. The derivative $d_0 : \Lambda^0 \to \Lambda^1$ is the **gradient**, the derivative $d_1 : \Lambda^1 \to \Lambda^2$ is the **curl** and $d_2 : \Lambda^2 \to \Lambda^3$ is the divergence (often just identified with the dual $d_0^* : \Lambda^1 \to \Lambda^0$, as 2-forms and 1-forms in three dimensions are dual to each other). This Stokes theorem holds both for the familiar **simplicial calculus** related to Euler characteristic $\chi(G)$ as well as the **connection calculus** related to the Wu characteristics $\omega_k(G)$.

42. Quadratic Lefschetz fixed point

42.1. Given an automorphism T, define the **quadratic Lefschetz number** $\chi_T(G)$, the super trace of the induced map on cohomology.

Theorem: $\chi_T(G) = \sum_{x \sim y, (x,y)=(T(x),T(y))} i_T(x,y)$

42.2. We can especially look at the case when G is a ball. This is cohomologically non-trivial.

Theorem: An endomormorphism of a ball G has a fixed (x, y), $x \cap y \neq \emptyset$.

43. Eulerian spheres

43.1. Let $\mathcal{G}_d$ be the class of d**-graphs**, $\mathcal{S}_d$ the class of d**-spheres**, $\mathcal{B}_d$ the class of d**-balls**, and $\mathcal{C}_k$ the class of graphs with **chromatic number** k. Note that all Barycentric refinements of a complex are Eulerian. We call the class $\mathcal{S}_d \cap \mathcal{C}_{d+1}$ the class of **Eulerian spheres** and $\mathcal{B}_d \cap C_{d+1}$ the class of Eulerian disks. The 0-sphere $2 = 1 + 1$ is Eulerian. Eulerian 1-spheres are cyclic graphs with an even number of vertices.

Theorem: Every unit sphere of an Eulerian sphere is Eulerian.

43.2. The **dual graph** $\hat{G}$ of a d-sphere G is the graph in which the d-simplices are the vertices and where two simplices are connected, if one is contained in the other. A graph (V, E) is **bipartite** if $V = (A \cup B$ with disjoint A, B such $E = \{(a, b) \mid a \in A, b \in B\}$. Every Barycentric refinement of a complex is a bipartite graph as we can take $A = \{x \in G \dim(x) \text{ even}\}$ and $B = \{x \in G \dim(x) \text{ odd}\}$.

Theorem: For $G \in \mathcal{S}_d$, then $\hat{G}$ is bipartite if and only if G is Eulerian.

44. Riemann-Hurwitz

44.1. The **automorphism group** $\mathrm{Aut}(G)$ of a simplicial complex is the group of all **automorphisms** of G. An **endomorphism** T is a simplicial map $G \to G$. If an endomorphism T is restricted to the **attractor** $\bigcap_k T^k(G)$ is an automorphism. An automorphism T of G induces automorphisms on Barycentric refinements and so **graph automorphisms**. The equivalence classes G_2/A are complexes again.

Theorem: If $A \subset \mathrm{Aut}(G)$, then G_1/A is a simplicial complex.

44.2. We need a refinements. The automorphism $T(a) = b, T(b) = a$ on $G = \{\{a,b\},\{a\},\{b\}\}$ has no complexes $G/A, G_1/A$ but G_1/A is a circle. The circle is a 2:1 cover of the interval, ramified at the two boundary points of the interval.

44.3. We can see G_1 as a **branched cover** G_1/A, **ramified** over some points. If G was a d-graph, then G_1/A is a discrete **orbifold**. If there are no ramification points, then the cover $G \to G/A$ is a fibre bundle with structure group A.

44.4. Given an automorphism T, define the **ramification index** $e(x) = 1 - \sum_{T \neq 1, T(x)=x} \omega(x)$ of X. The following remark was obtained with Tom Tucker. It is a discrete **Riemann-Hurwitz** result:

Theorem: $\chi(G) = |A|\chi(G/A) - \sum_{x \in G}(e(x) - 1)$

44.5. For every subset $\mathcal{G}_k$ of indices of fixed dimension k, we have by the **Burnside lemma** $\sum_{T \in A} \sum_{x \in \mathcal{G}_k, T(x)=x} 1 = |A||\mathcal{G}_k|$. The super sum gives $\sum_{T \in A} \sum_{x, T(x)=x} \omega(x) = |A|\chi(H)$. This gives $\sum_{T \neq 1} \sum_{x \in G} \omega(x) + \sum_{x \in G} \omega(x) = |A|\chi(H)$.

44.6. Let $\chi(G,T)$ denote the **Lefschetz number** of T. From the Lefschetz fixed point formula we get

Theorem: $\chi(G/A) = \frac{1}{|A|} \sum_{T \in A} L(G,T)$

45. Riemann-Roch

45.1. A **divisor** X is an integer-valued function on G. The **simplex Laplacian** L is defined as $L(x,y) = \omega(x)\omega(y)H_0(x,y)$, where H_0 is the Kirchhoff Laplacian of the **simplex graph** (**Hasse diagram**) in which G is the vertex set and two x, y are connected if one is contained in the other and the dimensions differ by 1. The simplex graph is one-dimensional as it has no triangles. A divisor X is called **principal** if $X = Lf$ for some integer valued function f. Write $(f) = Lf$. We think of a divisor as a **geometric object** and define its Euler characteristic $\chi(G) = \sum_x \omega(x)X(x)$. A divisor is **essential** if $\omega(x)X(x) \geq 0$ for all x. The **linear system** $|X|$ of X is the set of f for which $X + (f)$ is essential. Its **dimension** $l(X)$ is the maximal $k \geq 0$ such that for every $m < k$ and every Y of $\chi(Y) = m$, the divisor $X - Y$ is essential. With the **canonical divisor** $K(x) = 0$, the simplest Riemann-Roch theorem is

Theorem: $l(X) - l(K - X) = \chi(X)$.

45.2. This is **Baker-Norine theory**, slightly adapted to change the perspective: classically a divisors appear one a **curve** (Riemann surface or 1-dimensional graph) G and $\deg(X) + \chi(G) = \chi(X)$. One usually centers at the geometric underlying object which gives the canonical divisor $K = -2$ (as a function) which is in the case when G is one-dimensional is linearly equivalent to the **negated curvature function** $K(v) = -2 + \deg(v)$ on the **vertices** of G. In the discrete we can put an other coordinate system and reflect at 0 rather than the geometric object. This is not possible in the continuum. Riemann-Roch tells that the signed distance to the surface $\chi(G) = 0$ is $\chi(G)$.

45.3. Reflecting at 0 rather than at usual canonical divisor representing the curve G allows to have a Riemann-Roch for arbitrary dimensions. Generalizing Baker-Norine naively to higher dimensional simplicial complexes does not work, as the curvature κ of $\chi(G)$ has only in the one-dimensional case the property that $K = -2\kappa$ is a divisor. Classically $l(X)$ and $l(K-X)$ have cohomological interpretations. Also here, Riemann-Roch appears like a fancy Euler-Poincaré formula for a now signed cohomology, but it is deeper than Euler-Poincaré as the surface $\ker(\chi)$ is **bumpy**: it contains both **generic divisors** as well as **special divisors**.

45.4. The image of L is a linear subspace of the set $\ker(G) = \{\chi(G) = 0\}$. The quotient $\ker(\chi)/\mathrm{im}(L)$ is the **Picard group** or **divisor class group**. The equivalence classes of divisors can be represented by rooted spanning trees in the simplex graph. This defines a group structure on **rooted spanning trees**. That there is a bijective identification between divisor classes and spanning trees is the subject of:

Theorem: The Picard group is isomorphic to the tree group.

References

45.5. For the history of topology[42, 81] and graph theory [162, 81, 59] and discrete geometry [22]. See [68, 192, 178] for notations in algebraic topology, [67, 17, 23] for graph theory.

45.6. Abstract simplicial complexes appeared in 1907 by Dehn and Heegaard [27, 159]. In [4] they appeared under the name **unrestricted skeleton complex**. In [205], J.H.C. Whitehead calls them **symbolic complexes**. For references on simplicial complexes, see [83, 195, 196].

45.7. The category of simplicial complexes is as an axiom system a complexity minimum in the landscape of mathematical constructs. Adding more conditions leads to constructs like buildings (which are simplicial complexes covered by apartments) or matroids (which is a variant of simplicial complex in which the empty set is included), removing conditions leads to larger axiom systems. Already the definition of a partially ordered set requires more input as it defines a **binary relation** and the specification of **reflexivity**, **antisymmetry** and **transitivity**.

45.8. The importance of the Dehn-Heegard definition can not be over estimated as it frees from the concept of Euclidean space and defining simplicial complexes in Euclidean spaces which is more complicated as it requires to specify conditions on intersections. From the computer science point of view, one has to store a data structure with Euclidean vectors and given such a data structure requires quite a bit of work to double check that it is a valid simplicial complex. Even in modern time of 3d printing, there can be frustrations in realizing a complex because the actual data have degenerate points [191, 144] meaning often they are actually not simplicial complexes.

45.9. Sometimes, in the definition of simplicial sets G the vertex set V with condition $\{x\} \in G$ added, but that is redundant as the **set of points** $V = \bigcup_{x \in G} x$ is defined by G itself. To appreciate the elegance of the construct of Dehn and Heegaard, one might want to compare with the proposal of Euclid stating that a **point** is which **has no part**. The later is an extremely vague definition which would include "elementary particles", a notion which is relative as things which "have no part" have often turned out to be made of smaller parts.

45.10. Some of the results generalize to Δ sets, **partially ordered sets** or **simplicial sets** (all constructs which generalize simplicial complexes). In physics, locally finite posets are called **causal sets**. Some connection calculus however does not generalize. The unimodularity theorem for example does not hold for simplicial sets, at least for the approaches we tried so far. It does extend to the strong ring.

45.11. Homotopy theory as developed by [205] uses elementary expansions and contractions. Homotoptic complexes are said to have the same "nucleus". [205] uses "collapsible" for "homotopic to a point". See also [204]. The notions appearing for simplices described by graph theory, see [80, 79, 29].

45.12. Dimension theory has a long history [37]. The inductive definition of graphs appeared first in [96]. We studied the average in [93].

45.13. Random graphs were first studied in [45]. The average Euler characteristic appears in [93].

45.14. The idea of seeing geometric quantities as expectations is central in **integral geometry**. The first time, that curvature was seen as an expectation of indices is Banchoff [9, 10]. Random methods in geometry is part of integral geometry as pioneered by Crofton and Blaschke [21, 161]. We have used in in [116, 100] and [99]. Having curvature given as an expectation allows to deform it. Given a unitary flow U_t on functions for example produces a deformation of the curvature.

45.15. Discrete curvature traces back to a combinatorial curvature considered by Heesch [16] in the context of graph coloring and extended in [58]. The formula $K(p) = 1 - V_1(p)/6$ and for graphs on the sphere appears also in [170, 171], where it is also pointed out that $\sum_p K(p) = 2$ is Gauss-Bonnet formula. Discrete curvature was used in [72] and unpublished work of Ishida from 1990. Higushi uses $K(p) = 1 - \sum_{y \in S(p)}(1/2 - 1/d(y))$, where $d(y)$ are the cardinalities of the neighboring face degrees in the sphere $S(p)$. In [199], the **combinatorial curvature** $K(p) = 1 - d(p)/2 + \sum_q 1/n(q)$ for polyhedral graphs is considered, where $n(y)$ are the side-numbers of the polygons q adjacent to p. For two dimensional graphs, where all faces are triangles, these curvatures simplify to $d_j = 3$ so that $K = 1 - |S|/6$, where $|S|$ is the cardinality of the sphere $S(p)$. In [96] second order curvatures were used.

45.16. The **Levitt curvature** in arbitrary dimension appears in [150]. We rediscovered it in [94] after tackling dimension by dimension separately, not aware of Levitt. We got into the topic while working on [96]. Chern's proof is [31] followed [5, 47].

45.17. See [175, 39] for modern proofs of Gauss-Bonnet-Chern. Historical remarks are in [32].

45.18. The Erdös Rényi probability space were introduced in [45]. The formulas for the average dimension and Euler characteristic has been found in [93]. The recursive dimension was first used in [96]. We looked at more functionals in [113].

45.19. The **discrete Hadwiger Theorem** appears in [88]. The continuous version is [64]. For integral geometry and geometric probability, see [180]. The theory of valuations on distributive lattices has been pioneered by Klee [89] and Rota [177] who proved that there is a unique valuation such that $X(x) = 1$ for any join-irreducible element. See also [54].

45.20. Wu characteristic appeared in [202] and was discussed in [60]. We worked on it in [126] and announced cohomology in [141] and [142]. For the **connection cohomology** belonging to Wu characteristic, see [136].

45.21. For discrete Poincaré-Hopf see [98] and an attempt to popularize it in [102] or Mathematica demonstrations [95, 97]. It got pushed a bit more in [99]. For the classical Poincaré-Hopf, see [194]. For the classical case, Poincaré covered the 2-dimensional case in chapter VIII of [168] It got extended by Hopf in arbitrary dimensions [75]. It is pivotal in the proof of Gauss-Bonnet theorems for smooth Riemannian manifolds (i.e. [63, 193, 73, 69, 43, 15]).

45.22. Discrete McKean-Singer was covered in [101]. The best proof in the continuum is [39]. The classical result is [154]. In [130], the suggestion appeared to define elliptic discrete complexes using McKean-Singer.

45.23. The Zykov sum (join) was introduced in [208] to graph theory. The **strong ring** was covered in [132, 135]. The extension of the monoid to a group is what is familiar when constructing integers from natural numbers or fractions from integers and known as the **Grothendieck group completion**. We first found a multiplication making the Zykov group to a ring by trial and error, then later saw that this ring is dual to the Sabidussi multiplication [179]. See [66, 188, 190].

45.24. The Brouwer-Lefschetz theorem is [103]. It generalizes the 1-dimensional case [164]. The classical result is [149]. See also [76].

45.25. The classical **Kuenneth formula** is [146]. The graph version [123], uses the Barycentric refinement $(A \times B)_1$ of the Cartesian product $A \times B$. In the strong ring things are almost immediate via Hodge.

45.26. About the history of discrete notions of manifolds, see [186]. The Evako definition of a sphere as a cell complex for which every unit sphere is a $n-1$ sphere and such that removing one point makes it contractible was predated by approaches of Vietoris or van Kampen. The later would have accepted homology spheres as unit spheres.

45.27. The classical Sard theorem is [181]. The discrete version appears in [124]. It lead us to look at geometric coloring questions.

45.28. Our proof in the discrete is [122]. For the Jordan-Brouwer separation theorem, see [84, 25] or [200, 2, 91, 65, 40, 176, 166]. The Mazur-Morse-Brown theorem [153, 157, 26] assures that the complementary domains A, B are homeomorphic to Euclidean unit balls if the embedding of H is locally flat. The Alexander sphere is [3]. The Schoenflies theme is introduced in [183, 184, 185]. Various discrete versions were studied [30, 70, 46, 198, 201, 152, 197]. Discrete versions are also interesting for constructive proofs [13].

45.29. For the spectral universality, see [120, 125]. It uses a result of Lidskii-Last [189] which assures if $||\mu - \lambda||_1 \leq \sum_{i,j=1}^n |A - B|_{ij}$ for any two symmetric $n \times n$ matrices A, B with eigenvalues $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$.

45.30. The discrete exterior derivative goes back to Betti and Poincaré and was already anticipated by Kirchhoff. As pointed out in [103], the discrete Hodge point is [44]. It appeared also in [77]. The **discrete Dirac operator** was stressed in [105].

45.31. The unimodularity theorem $|\det(L)| = 1$ was discovered in February 2016, announced in [143] and proven in [127]. An other proof was given in [158].

45.32. We have looked at the arithmetic of unit spheres in [134], especially in the context of the diagonal Green function entries. The other Green function entries are covered in [139].

45.33. The result $\chi(G) = p(G) - n(G)$ was proven in [133, 139]. The functional equation for the spectral zeta function of the connection Laplacian was proven in [137]. Earlier work in the Hodge Zeta case is [140]. The zeta function is called Dyadic because the Barycentric limit is in an ergodic setup a **von Neumann-Kakutani system** [92], which has the Prüfer group as the spectrum. The system is a group translation on the dyadic group of integers and also known as the **adding machine**.

45.34. The **Hydrogen relation** $H = L - L^{-1}$ for one-dimensional complexes was studied in [129, 131] and [138].

45.35. An earlier talk [114] summarizes things also. [106] is an earlier snapshot about the linear algebra part. [117, 102] summarize the calculus.

45.36. The matrix tree theorem is [86]. It is based on the **Cauchy-Binet theorem** [28, 18]. A generalization [112] gives the coefficients of the characteristic polynomial. The **Chebotarev-Shamis theorem** is [167, 172]. See also [104], where we initially were not aware of the work of Chebotarev and Shamis.

45.37. We used multi-linear algebra [57] for the proof of the generalized Cauchy-Binet. The closest proof of the classical theorem is in [78]. The classical theorem deals with the determinant of $F^T G$ which is one coefficient of the characteristic polynomial p We first generalized it to the pseudo determinant which is the product of the non-zero eigenvalues and the first non-zero coefficient of p. A new proof of the generalized Cauchy-Binet formula avoiding multi-linear algebra is given in [74].

45.38. The **Lax deformation** of exterior derivatives was introduced in [109, 108] and was motivated by **Witten deformation** [206, 39]. Lax systems were introduced first to [148]. Commutation relations of that form have appeared earlier when describing **free tops** $L' = [B, L]$, where $B = I^{-1}L$ is the angular velocity and L the angular velocity in $so(n)$, which are geodesics in $SO(n)$ [6].

45.39. The Connes formula [34] is elementary but crucial in the process of generalizing Riemannian geometry to **non-commutative geometry**.

45.40. After finding a multiplication completing the Zykov addition to a ring in [132], we realized it is the dual to the Sabidussi ring. In [135], we looked at the ring generated by the Cartesian product. It is a subring and consists of discrete CW complexes. Unlike for simplicial sets, the classical theorems like Gauss-Bonnet and energy theorem go over.

45.41. Riemann-Roch for graphs is [7]. See also [8]. We worked on Riemann-Hurwitz in [145]. The usual approach for Riemann-Hurwitz in graph theory is to see them as discrete analogues of algebraic curves or Riemann surfaces see [155].

45.42. [203] first looked for a combinatorial definition of spheres. Forman [51] defined spheres through the Reeb as objects admitting 2 critical points. See also [52]. More on discrete Morse theory in [53, 55].

45.43. We used data fitting to get first heuristically the Stirling formula then proved it. It is however considered ”well known” [24]. It appears also in [195, 151, 71].

45.44. The history of polytopes is a “delicate task” [41]. The **Euler polyhedron formula** (Euler's gem) was discussed in [174]. The early proofs of Schläfli and Staudt had still gaps according to [27]. The difficulty is also explained in [147, 61].

45.45. The story of polyhedra is told in [174, 36]. Historically, it was developed in [182], [187], [169]. Coxeter [36] defines a polytop as a convex body with polygonal faces. [62] also works with convex polytopes in R^n where the dimension is the dimension of the affine span.

45.46. The perils of a general definition of a polytop were known since Poincaré (see [1, 174, 33, 147]). Polytop definitions are given in [182, 36, 62, 82]. Topologists started with new definitions [4, 50, 35, 192], and define first a simplicial complex and then polyhedra as topological spaces which admit a **triangularization** by a simplicial complex.

45.47. Dehn-Sommerville relations have traditionally been formulated for convex polytopes and then been generalized to situations where unit spheres can be realized as convex polytopes. See [90, 163, 160, 151, 24, 71, 87] or [14].

45.48. We started to think about graph coloring during the project [119]. The reports [115] and [121] explored this a bit more. It is related to Fisk theory [49, 48].

45.49. Some special graphs appearing when counting was considered in [128]. When writing this, we were not aware that the cell complex introduced already in [19] which goes much further than what we did. Other classes of complexes called **orbital networks** [107, 110, 111] were studied first with Montasser Ghachem.

45.50. For the Alexander duality, see [20]. Originally established by Alexander in 1922, it was formulated by Kalai and Stanley in combinatorial topology. We formulated it with cohomology rather than homology and cohomology. As such it is an identity where we have numbers on both sides.

45.51. Unlike in **Regge calculus** [173, 38, 56, 156], the definitions of Ricci and Hilbert action mentioned are combinatorial and do not depend on an embedding. There are other notions of curvature which are intrinsic and do not depend on an embedding into an ambient space: [165, 207, 85].

Questions

46. Inverse spectral questions

46.1. We have seen that the spectrum of L does not determine the Betti numbers in general but that for a Barycentric refinement of G, the Betti numbers b_0, b_1 can be read of from the spectrum as the number of eigenvalues 1 and -1.

Question: Does the spectrum of L determine b_k for $k \geq 2$.

Question: Does the spectrum of L determine the Wu characteristic $\omega(G)$?

47. Barycentric limit

We have seen that the limiting spectral measure can be computed in the case $d = 1$. It is a smooth measure. In higher dimensions, we see spectral gaps. These gaps have first been seen in the **BeKeNePaPeTe paper** [11].

Question: Prove spectral gaps in limiting spectral measure for $d \geq 2$.

48. Coloring

48.1. The **four color theorem** is equivalent to the statement that all 2-spheres are 4-colorable.

Question: Are all d-spheres $(d+2)$-colorable?

This would imply that all d-spheres either have chromatic number $d+1$ or $d+2$. A generalized **Heawood statement** is that we have minimal chromatic number $d+1$ if and only if every $d-2$ dimensional simplex in G has even degree. Every Barycentric refinement of a sphere is minimally colorable.

Question: Are all 2-graphs 5 colorable?

There are 2-complexes different from spheres which need 5 colors. One of the simplest is the projective plane. Note that for positive genus g, the coloring question of d-complexes is completely different than than the coloring question for embedded graphs.

On the 2-torus, one can embed K_7 which needs 7 colors but K_7 is a 6 dimensional complex.

49. Connection Cohomology

49.1. While we know that connection cohomology is not a homotopy invariant, we have not yet proven that it is a topological invariant. A notion of homeomorphism appears in [118]. Alternatively, one could use "discrete homoemorphic" in the sense that geometric realizations are homeomorphic to ask

Question: Is connection cohomology a topological invariant?

49.2. We would like to find more examples of triangulations of non-homeomorphic d-manifolds with different connection cohomology which can not be distinguished by other means:

Question: Can one distinguish **homology spheres** with Wu cohomology?

49.3. Something we have only started to look at is to extend Alexander duality to the Wu characteristic:

Question: Is there a duality for connection cohomology?

49.4. As connection cohomology is not a homotopy invariant, the naive generalization does not work.

50. Random complexes

50.1. The probability spaces $E(n,p)$ of graphs define natural random spaces of simplicial complexes as we can take the Whitney complex of a graph. While we have a formula for the expectation of Euler characteristic, this is not yet available for Wu characteristic numbers ω_k.

Question: What is the expected value of ω_k on $E(n,p)$?

50.2. We would also like to know the expectations of the Betti numbers:

Question: What is the expectation of $b_k(G)$ on $E(n,p)$?

51. Geodesic flow

51.1. Having a global geodesic flow on d-complexes asks for dynamical properties like ergodicity. We know that (generalizing Tutte and Whitney) that every d-sphere is a Hamiltonian graph in the sense that it admits a Hamiltonian cycle. The existence of geodesic flow is related to the fact that every unit sphere is Eulerian, admitting an Eulerian cycle but that is easier as the Eulerian path existence is decided by local properties alone. As we deal with finite complexes, every geodesic path is closed. **Ergodicity** means that the path visits every edge exactly once. This prompts the **ergodicity question**:

Question: Is there a d-complex with a geodesic Hamiltonian cycle?

51.2. For d-graphs with boundary the geodesic flow can be continued naturally using the **billiard reflection condition** which naturally exists for d-complexes. The geodesic flow is then a **billiard**. Again one can ask for the existence of a Hamiltonian path in the interior. Note that a billiard trajectory on the boundary is simply a geodesic on the boundary (which is a closed $(d-1)$ complex. Lets simply call a region with boundary **ergodic** if there exists a Hamiltonian path in the interior (the graph without boundary edges)

Question: Is there an ergodic d-complex with boundary?

52. Zeta function

52.1. While various equivalent expressions exist for the **connection zeta function** in the Barycentric limit of a one-dimensional complex, we don't yet have found a reference about where the roots of ζ are:

Question: The limiting zeta function ζ has roots on the imaginary axes.

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