

THE STRONG RING OF SIMPLICIAL COMPLEXES

OLIVER KNILL

ABSTRACT. The strong ring R is a commutative ring generated by finite abstract simplicial complexes. To every $G \in R$ belongs a Hodge Laplacian $H = H(G) = D^2 = (d + d^*)^2$ determining the cohomology and a unimodular connection operator $L = L(G)$. The sum of the matrix entries of $g = L^{-1}$ is the Euler characteristic $\chi(G)$. For any $A, B \in R$ the spectra of H satisfy $\sigma(H(A \times B)) = \sigma(H(A)) + \sigma(H(B))$ and the spectra of L satisfy $\sigma(L(A \times B)) = \sigma(L(A)) \cdot \sigma(L(B))$ as $L(A \times B) = L(A) \otimes L(B)$ is the matrix tensor product. The inductive dimension of $A \times B$ is the sum of the inductive dimension of A and B . The dimensions of the kernels of the form Laplacians $H_k(G)$ in $H(G)$ are the Betti numbers $b_k(G)$ but as the additive disjoint union monoid is extended to a group, they are now signed with $b_k(-G) = -b_k(G)$. The maps assigning to G its Poincaré polynomial $p_G(t) = \sum_{k=0} b_k(G)t^k$ or Euler polynomials $e_G(t) = \sum_{k=0} v_k(G)t^k$ are ring homomorphisms from R to $\mathbb{Z}[t]$. Also $G \rightarrow \chi(G) = p(-1) = e(-1) \in \mathbb{Z}$ is a ring homomorphism. Kuenneth for cohomology groups $H^k(G)$ is explicit via Hodge: a basis for $H^k(A \times B)$ is obtained from a basis of the factors. The product in R produces the strong product for the connection graphs. These relations generalize to Wu characteristic. R is a subring of the full Stanley-Reisner ring S , a subring of a quotient ring of the polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$. An object $G \in R$ can be visualized by its Barycentric refinement G_1 and its connection graph G' . Theorems like Gauss-Bonnet, Poincaré-Hopf or Brouwer-Lefschetz for Euler and Wu characteristic extend to the strong ring. The isomorphism $G \rightarrow G'$ to a subring of the strong Sabidussi ring shows that the multiplicative primes in R are the simplicial complexes and that connected elements in R have a unique prime factorization. The Sabidussi ring is dual to the Zykov ring, in which the Zykov join is the addition, which is a sphere-preserving operation. The Barycentric limit theorem implies that the connection Laplacian of the lattice Z^d remains invertible in the infinite volume limit: there is a mass gap containing $[-1/5^d, 1/5^d]$ for any dimension d .

Date: Aug 5, 2017.

1991 Mathematics Subject Classification. 05C99, 13F55, 55U10, 68R05.

Key words and phrases. Topological arithmetic, Zykov ring, Sabidussi ring, Mass gap.

1. ENERGY THEOREM

1.1. A **finite abstract simplicial complex** is a finite set of non-empty sets which is closed under the operation of taking finite non-empty subsets. The **connection graph** G' of such a **simplicial complex** G has as the vertices the sets of G and as the edge set the pairs of sets which intersect. Given two simplicial complexes G and K , their **sum** $G \oplus K$ is the disjoint union and the **Cartesian product** $G \times K$ is the set of all set Cartesian products $x \times y$, where $x \in G, y \in K$. While $G \times K$ is no simplicial complex any more if both factors are different from the one-point complex K_1 , it still has a connection graph $(G \times K)'$, the graph for which the vertices are the sets $x \times y$ and where two sets are connected if they intersect. The **Barycentric refinement** $(G \times K)_1$ of $G \times K$ is the Whitney complex of the graph with the same vertex set as G' but where two sets are connected if and only if one is contained in the other. The matrix $L = L(G) = 1 + A$, where A is the adjacency matrix of G' is the **connection Laplacian** of G .

1.2. All geometric objects considered here are finite and combinatorial and all operators are finite matrices. Only in the last section when we look at the Barycentric limit of the discrete lattice Z^d , the operators become almost periodic on a profinite group, but there will be universal bounds on the norm of the inverse. One of the goals of his note is to extend the following theorem to the strong ring generated by $G_{i_1} \times \dots \times G_{i_k}$.

Theorem 1 (Unimodularity theorem). *For every simplicial complex G , the connection Laplacian $L(G)$ is unimodular.*

We have proven this in [21] inductively by building up the simplicial complex G as a discrete CW-complex starting with the zero-dimensional skeleton, then adding one-dimensional cells, reaching the one-dimensional skeleton of G , then adding triangles etc. continuing until the entire complex is built up. In every step, if a cell x is added, this means that a ball $B(x)$ is glued in along a sphere $S(x)$, the Fredholm determinant $\det(L) = \det(1 + A)$ of the adjacency matrix A of G' is multiplied by $\omega(x) = 1 - \chi(S(x)) = (-1)^{\dim(x)} \in \{-1, 1\}$, where $S(x)$ is the unit sphere in the Barycentric refinement G_1 of G , which is the Whitney complex of a graph. The determinant of $L(G)$ is now equal to the **Fermi characteristic** $\prod_x \omega(x) \in \{-1, 1\}$, a multiplicative analogue of the **Euler characteristic** $\sum_x \omega(x)$ of G . Having determinant 1 or -1 , the matrix is unimodular.

1.3. As a consequence of the unimodularity theorem, the inverse $g = L^{-1}$ of L produces **Green functions** in the form of integer entries $g(x, y)$ which can be seen as the **potential energy** between the two simplices x, y . The sum $V(x) = \sum_y g(x, y)$ is the **potential** at x as it adds up the potential energy $g(x, y)$ induced from the other simplices. It can also be interpreted as a **curvature** because the Euler characteristic $\chi(G) = \sum_{x \in G} \omega(x)$ is the sum over the Green function entries:

Theorem 2 (Energy theorem). *For every simplicial complex G , we have $\sum_x V(x) = \sum_{x,y} g(x, y) = \chi(G)$.*

1.4. This formula is a Gauss-Bonnet formula, when the row sum is interpreted as a curvature. It can also be seen as a **Poincaré-Hopf formula** because $V(x) = (-1)^{\dim(x)}(1 - \chi(S(x))) = (1 - \chi(S_f^-(x)))$ is a **Poincaré-Hopf index** for the **Morse function** $f(x) = -\dim(x)$ on the Barycentric refinement G_1 of G . A function on a graph is a **Morse functional** if every $S_f^-(x)$ is a discrete sphere, where $S_f^-(x) = \{y \in S(x) \mid f(y) < f(x)\}$. The proof reduces the energy theorem to the Poincaré-Hopf formula which is dual to the definition of Euler characteristic: the Poincaré-Hopf index of $-f = \dim$ is $\omega(x)$ and $\sum_x \omega(x)$ is the definition of Euler characteristic as we sum over simplices. In the Barycentric refinement, where we sum over vertices then $\omega(x)$ can be seen as a curvature. In order to reduce the energy theorem to Poincaré-Hopf, one has to show that $V(x) = \sum_y g(x, y)$ agrees with $(-1)^{\dim(x)}g(x, x)$. See also [26] for an interpretation of the diagonal elements.

1.5. The **Barycentric refined complex** G_1 is a set of subsets of the power set 2^G of G . It consists of set of sets in G which pairwise are contained in each other. It is the **Whitney complex** of the graph $G_1 = (V, E)$, where $V = G$ and E is the set of (a, b) with $a \subset b$ or $b \subset a$. Since G_1 is the Whitney complex of a graph, we can then use a more intuitive picture of the complex, as graphs can be drawn and visualized and are hardwired already as structures in computer algebra systems. Also definitions become easier. To define the inductive dimension for example, its convenient to define it for simplicial complexes as the corresponding number for the Barycentric refined complex, where one deals with the Whitney complex of a graph.

2. THE SABIDUSSI RING

2.1. The **strong product** of two finite simple graphs $G = (V, E)$ and $H = (W, F)$ is the graph $G \boxtimes H = (V \times W, \{((a, b), (c, d)) \mid a =$

$c, (b, d) \in F\} \cup \{(a, b), (c, d) \mid b = d, (a, c) \in E\} \cup \{(a, b), (c, d) \mid (a, c) \in E \text{ and } (b, d) \in F\}$. It is an associative product introduced by Sabidussi [31]. Together with the disjoint union \oplus as addition on signed complexes, it defines the **strong Sabidussi ring of graphs**. We have started to look at the arithmetic in [25]. We will relate it in a moment to the strong ring of simplicial complexes. In the Sabidussi ring of graphs, the additive monoid (\mathcal{G}_0, \oplus) of finite simple graphs with zero element (\emptyset, \emptyset) is first extended to a larger class (\mathcal{G}, \oplus) which is a Grothendieck group. The elements of this group are then **signed graphs**, where each connected component can have either a positive or negative sign. The **additive primes** in the strong ring are the connected components. An element in the group, in which both additive primes A and $-A$ appear, is equivalent to a signed graph in which both components are deleted. The **complement** of a graph $G = (V, E)$ is denoted by $\overline{G} = (V, \overline{E})$, where \overline{E} is the set of pairs (a, b) not in E .

2.2. The **join** $G+H = (V \cup W, E \cup F \cup \{(a, b) \mid a \in V, b \in W\})$ is an addition introduced by Zykov [34]. It is dual to the disjoint union $G+H = \overline{\overline{G} \oplus \overline{H}}$. The **large product** $G \star H = (V \times W, \{(a, b), (c, d) \mid (a, c) \in E \text{ or } (b, d) \in E\})$ is dual to the strong product. In other words, the dual to the strong ring $(\mathcal{G}, \oplus, \boxtimes, 0, 1)$ is the **large ring** $(\mathcal{G}, +, \star, 0, 1)$. Because the complement operation $G \rightarrow \overline{G}$ is invertible and compatible with the ring operations, the two rings are isomorphic. The additive primes in the strong ring are the connected sets. Sabidussi has shown that every connected set has a unique multiplicative prime factorization. It follows that the strong ring of graphs is an integral domain. It is not a unique factorization domain however. There are disconnected graphs which can be written in two different ways as a product of two graphs.

2.3. The Sabidussi and Zykov operations could also be defined for simplicial complexes but we don't need this as it is better to look at the Cartesian product and relate it to the strong product of connection graphs. But here is a definition: the disjoint union $G \oplus H$ is just $G \cup H$ assuming that the simplices are disjoint. The Zykov sum, or join is $G + H = G \cup H \cup \{x \cup y \mid x \in G, y \in H\}$. If π_k denote the projections from the **set theoretical Cartesian product** $X \times Y$ to X or Y , the Zykov product can be defined as $G \star H = \{A \subset X \times Y \mid \pi_1(A) \in G \text{ or } \pi_2(A) \in H\}$. It can be written as $G \star H = G \times 2^{V(H)} \cup 2^{V(G)} \times H$, where 2^X is the **power set** of X and $V(G) = \bigcup_{A \in G} A$.

3. THE STANLEY-REISNER RING

3.1. The **Stanley-Reisner ring** S is a subring $\bigcup_n \mathbb{Z}[x_1, x_2, \dots, x_n]/I_n$, where I_n is the ideal generated x_i^2 and $f_G - f_H$, where f_G, f_H are ring elements representing the isomorphic finite complexes G, H . The ring S contains all elements for which $f(0, 0, 0, \dots, 0) = 0$. It is a quotient ring of the **ring of chains** $C \subset \bigcup_n \mathbb{Z}[x_1, x_2, \dots, x_n]/J_n$ with J_n is the ideal generated by squares. A signed version of ring of chains is used in algebraic topology. It is the **free Abelian group** generated by finite simplices represented by monoids in the ring. The ring of chains C is larger than the Stanley-Reisner ring as for example, the ring element $f = x - y$ is zero in S . The Stanley-Reisner ring is helpful as every simplicial complex G and especially every Whitney complex of a graph can be described algebraically with a polynomial: if $V = \cup_{A \in G} A = \{x_1, \dots, x_n\}$ is the base set of the finite abstract simplicial complex G , define the monoid $x_A = \prod_{x \in A} x$ and then $f_G = \sum_{A \in G} x_A$. We initially have computed the product using this algebraic picture in [17].

3.2. The circular graph $G = C_4$ for example gives $f_G = x + y + z + w + xy + yz + zw + wx$ and the triangle $H = K_3$ is described by $f_H = a + b + c + ab + ac + bc + abc$. The Stanley-Reisner ring also contains elements like $7xy - 3x + 5y$ which are only chains and not simplicial complexes. The addition $f_G + f_H$ is the disjoint union of the two complexes, where different variables are used when adding two complexes. So, for $G = K_2 = x + y + xy$ we have $G + G = x + y + xy + a + b + ab$ and $G + G + G = x + y + xy + a + b + ab + u + v + uv$. The negative complex $-G$ is $-x - y - xy$. The complex $G - G = -x - y - xy + a + b + ab$ is in the ideal divided out so that it becomes 0 in the quotient ring. The Stanley-Reisner ring is large. It contains elements like xyz which can not be represented as linear combinations $\sum_i a_i G_i$ of simplicial complexes G_i . But we like to see the strong ring R embedded in the full Stanley-Reisner ring S .

3.3. If G and H are simplicial complexes represented by polynomials f_g, f_H , then the product $f_G f_H = f_{G \times H}$ is not a simplicial complex any more in general. Take $f_{K_2} f_{K_2} = (a + b + ab)(c + d + cd)$ for example which is $ac + bc + abc + ad + bd + abd + acd + bcd + abcd$. We can not interpret ac as a new single variable as bc is also there and their intersection $ac \cap bc = c$ could not be represented in the complex. A simplicial complex is by definition closed for non-empty intersections as such an intersection is a subset of both sets. We can still form the subring S in R generated by the simplicial complexes and call it the **strong ring**. The reason for the name is that on the connection

graph level it leads to the strong product of graphs. The strong ring of simplicial complexes will be isomorphic to a subring of the Sabidussi ring of graphs.

3.4. The Stanley-Reisner picture allows for a concrete implementation of the additive Grothendieck group which extends the monoid given by the disjoint union as addition. The Stanley-Reisner ring is usually a ring attached to a single geometric object. The full Stanley-Reisner ring allows to represent any element in the strong ring. It is however too large for many concepts in combinatorial topology, where finite dimensional rings are used to describe a simple complex. One can not see each individual element in S as a geometric object on its own, as cohomology and the unimodularity theorem fail on such an object. The chain $f = xy + yz + y$ for example is no simplicial complex. Its connection graph is a complete graph for which the Fredholm determinant $\det(1 + A)$ is zero. Its boundary is not a subset of f , its connection graph is K_3 as all ingredient, the two edges and the single vertex all intersect with each other. The elements of the Stanley-Reisner picture behave like measurable sets in a σ -algebra. One can attach to every f in the full Stanley-Reisner ring an **Euler characteristic** $\chi(f) = -f(-1, -1, \dots, -1)$ which satisfies $\chi(f + g) = \chi(f) + \chi(g)$, $\chi(fg) = \chi(f)\chi(g)$ but this in general does not have a cohomological analog via Euler-Poincaré. This only holds in the strong ring.

3.5. While the Cartesian product $G \times H$ of two simplicial complexes is not a simplicial complex any more, the product can be represented by an element $f_G f_H$ in the Stanley-Reisner ring. The **strong ring** is defined as the subring S of the full Stanley-Reisner ring R which is generated by simplicial complexes. Every element in the strong ring S is a sum $\sum_I a_I f_{G_I}$, where $a_I \in \mathbb{Z}$ and for every finite subset $I \subset \mathbb{N}$, the notation $f_{G_I} = \prod_{i \in I} f_{G_i}$ is used. The ring of chain contains $\sum_I a_I x_I$, with $x_I = x_{i_1} \cdot x_{i_k}$, where $A_I = \{x_{i_1}, \dots, x_{i_k}\}$ are finite subsets of $\{x_1, x_2, \dots\}$. Most of the elements are not simplicial complexes any more. The ring of chains has been used since the beginnings of combinatorial topology. But its elements can also be described by graphs, connection graphs.

3.6. The strong ring has the empty complex as the zero element and the one-point complex K_1 as the one element. The element $-K_1$ is the -1 element. The strong ring contains \mathbb{Z} by identifying the zero dimensional complexes P_n with n . It contains elements like $x + y + xy - (a + b + c + ab + bc + ac)$ which is a sum $K_2 + C_3$ of a Whitney

complex K_2 a non-Whitney complex C_3 . The triangle K_3 is represented by $(a + b + c + ab + bc + ac + abc)$. The strong ring does not contain elements like $x + y + xy - (y + z + yz)$ as in the later case, we don't have a linear combination of simplicial complexes as the sum is not a disjoint union. The element $G = x + y + xy - (a + b + ab)$ is identified with the zero element 0 as $G = K_2 - K_2$ and $G = x + y + xy + (a + b + ab) = K_2 + K_2$ can be written as $2K_2 = P_2 \times K_2$, where $P_2 = u + v$ is the zero-dimensional complex representing 2. The multiplication honors signs so that for example $G \times (-H) = -G \times H$ and more generally $a(G \times H) = (aG) \times H = G \times (aH)$ for any zero dimensional signed complex $a = P_a$ which follows from the distributivity $H \times (G_1 + G_2) = H \times G_1 + H \times G_2$.

4. THE CONNECTION LEMMA

4.1. The following lemma shows that the **strong ring of simplicial complexes** is isomorphic to a subring of the **Sabidussi ring**. First of all, we can extend the notion of **connection graph** from simplicial complexes to products $G = G_{i_1} \times \cdots \times G_{i_n}$ of simplicial complexes and so to the strong ring. The vertex set of G is the **Cartesian product** $V' = V_1 \times \cdots \times V_n$ of the base sets $V_i = V(G_i) = \bigcup_{A \in G_i} A$. Two different elements in V are connected in the connection graph G' if they intersect as sets. This defines a finite simple graph $G' = (V', E')$. Also a multiple λG of a complex is just mapped into the multiple $\lambda G'$ of the connection graph, if λ is an integer.

Lemma 1 (Connection lemma). $(G \times H)' = G' \boxtimes H'$.

Proof. In $G \times H$, two simplices $(a \times b), (c \times d)$ in the vertex set $V = \{(x, y) \mid x \in G, y \in H\}$ of $(G \times H)'$ are connected in $(G \times H)'$ if $(a \times b) \cap (c \times d)$ is not empty. But that means that either $a \cap c$ is not empty or then that $b \times d$ is not empty or then that $a = c$ and $b \cap d$ is not empty or then that $b = d$ and $a \cap c$ is not empty. \square

4.2. The strong connection ring is a sub ring of the **Sabidussi ring** $(\mathcal{G}, \oplus, \boxtimes, 0, 1)$, in which objects are signed graphs. While the later contains all graphs, the former only contains ring elements of the form G' , where G is a simplicial complex the **strong connection ring**. The lemma allows us to avoid seeing the elements as a subspace of abstract finite CW complexes for which the Cartesian product is problematic. Both the full Stanley-Reisner ring and the Sabidussi rings are too large. The energy theorem does not hold in the full Stanley-Reisner ring, as the example $G = xy + yz + y$ shows, where $G' = K_3$ is the complete graph for which the Fredholm determinant is zero. We would need to

complete it to a simplicial complex like $G = xy + yz + x + y + z$ for which the connection Laplacian $L(G)$ has a non-zero Fredholm determinant.

4.3. Given a simplicial complex G , let $\sigma(G)$ denote the **connection spectrum** of G . It is the spectrum of the connection Laplacian $L(G)$. The trace $\text{tr}(L(G))$ is the number of cells in the complex. It is a measure for the **total spectral energy** of the complex. Like the potential theoretic total energy $\chi(G)$, the connection spectrum and total energy are compatible with arithmetic. The trace $\text{tr}(G)$ is the total number of cells and can be written in the Stanley-Reisner picture as $f_G(1, 1, 1, \dots, 1)$.

Corollary 1 (Spectral compatibility). $\sigma(G \times H) = \sigma(G)\sigma(H)$.

Proof. It is a general fact that the Fredholm adjacency matrices tensor under the strong ring multiplication. This implies that spectra multiply. \square

The trace is therefore a ring homomorphism from the strong ring S to the integers but since the trace $\text{tr}(L(G))$ is the number of cells, this is obvious.

4.4. We see that from a spectral point of view, it is good to look at the Fredholm adjacency matrix $1 + A(L)$ of the connection graph. The operator $L(G)$ is an operator on the same Hilbert space than the Hodge Laplacian H of G which is used to describe cohomology algebraically. We will look at the Hodge Laplacian later.

4.5. As a consequence of the tensor property of the connection Laplacians, we also know that both the unimodularity theorem as well as the energy theorem extend to the strong connection ring.

Corollary 2 (Energy theorem for connection ring). *Every connection Laplacian of a strong ring element is unimodular and has the property that the total energy is the Euler characteristic.*

Proof. Linear algebra tells that if L is a $n \times n$ matrix and M is a $m \times m$ matrix, then $\det(L \otimes M) = \det(L)^m \det(M)^n$. If L, M are connection Laplacians, then $|\det(L)| = |\det(M)| = 1$ and the product shares the property of having determinant 1 or -1 . \square

4.6. In order to fix the additive part, we have to define the Fredholm determinant of $-G$, the negative complex to G . Since $\chi(-G) = -\chi(G)$, we have $\omega(-x) = -\omega(x)$ for simplices and if we want to preserve the property $\psi(G) = \prod_x \omega(x)$, we see that that defining $\psi(-G) = \det(-L(G))$ is the right thing. The **connection Laplacian** of $-G$

is therefore defined as $-L(G)$. This also extends the energy theorem correctly. The sum over all matrix entries of the inverse of L is the Euler characteristic.

5. THE SABIDUSSI THEOREM

5.1. An element G in the strong ring S is called an **additive prime** if it can not be decomposed as $G = G_1 \oplus G_2$ with both G_i being non-empty. The additive prime factorization is braking G into **connected components**. A **multiplicative prime** in the strong ring is an element which can not be written as $G = G_1 \times G_2$, where both G_i are not the one-element K_1 .

Theorem 3 (Sabidussi theorem). *Every additive prime in the Sabidussi ring has a unique multiplicative prime factorization.*

See [31] and also [3, 2], where also counter examples appear, if the connectivity assumption is dropped. The reason for the non-uniqueness is that $\mathbb{N}[x]$ has no unique prime factorization: $(1 + x + x^2)(1 + x^3) = (1 + x^2 + x^4)(1 + x)$.

5.2. Can there be primes in the strong ring that are not primes in the Sabidussi ring? The factors need not necessarily have to be simplicial complexes. Is it possible that a simplicial complex can be factored into smaller components in the Stanley-Reisner ring? The answer is no, because a simplicial complex G is described by a polynomial f_G has linear parts. A product does not have linear parts. We therefore also have a unique prime factorization for connected components in the strong ring. The Sabidussi theorem goes over to the strong ring.

Corollary 3. *Every additive prime in the strong ring has a unique multiplicative prime factorization. The connected multiplicative primes in the strong ring are the connected simplicial complexes.*

5.3. If we think of an element G in the ring as a **particle** and of $G \cup H$ as a **pair of particles**, then the **total spectral energy** is the sum as the eigenvalues adds up. As the individual L spectra multiply, this provokes comparisons with the **Fock space** of particles: when taking the disjoint union of spaces is that the Hilbert space of the particles is the product space. If we look at the product of two spaces, then the Hilbert space is the tensor product. In some sense, "particles" are elements in the strong ring. They are generated by prime pieces of "space". These are the elements in that ring which belong to simplicial complexes.

5.4. Whether this picture painting particles as objects generated by space has merit as a model in physics is here not important. Our point of view is purely mathematical: the geometric “parts” have **topological properties** like cohomology or energy or **spectral properties** like spectral energy attached to them. More importantly, the geometric objects are elements in a ring for which the algebraic operations are compatible with the topological properties. The additive primes in the ring are the connected components, the “particles” where each is composed of smaller parts thanks to a unique prime factorization $G_1 \times \cdots \times G_n$ into **multiplicative primes**. These **elementary parts or particles** are just the connected **simplicial complexes**.

6. SIMPLICIAL COHOMOLOGY

6.1. The **Whitney complex** of a finite simple graph $G = (V, E)$ is the simplicial complex in which the sets are the vertex sets of complete subgraphs of G . If d denotes the exterior derivative of the Whitney complex, then the **Hodge Laplacian** $H = (d+d^*)^2$ decomposes into blocks $H_k(G)$ for which $b_k(G) = \dim(\ker(H_k))$ are the **Betti numbers**, the dimensions of the **cohomology groups** $H^k(G) = \ker(d_k)/\text{im}(d_{k-1})$. The **Poincaré-polynomial** of G is defined as $p_G(x) = \sum_{k=0}^{\infty} b_k(G)x^k$. For every connected complex G , define $p_{-G}(x) = -p_G(x)$. Complexes can now have negative Betti numbers. There are also non-empty complexes with $p_G(x) = 0$ like $G = C_4 - C_5$.

6.2. The exterior derivative on a signed simplicial complex G is defined as $df(x) = f(\delta x)$, where δ is the boundary operation on simplices. The incidence matrix can also be defined as $d(x, y) = 1$ if $x \subset y$ and the orientation matches. Note that this depends on the choice of the orientation of the simplices as we do not require any compatibility. It is a choice of basis in the Hilbert space on which the Laplacian will work. The **exterior derivative** d of a product $G_1 \times G_2$ of two complexes each having the boundary operation δ_i is then given as

$$df(x, y) = f(\delta_1 x, y) + (-1)^{\dim(x)} f(x, \delta_2 y) .$$

The **Dirac operator** $D = d + d^*$ defines then the Hodge Laplacian $H = D^2$. Both the connection Laplacian and Hodge Laplacian live on the same space.

6.3. The Hodge theorem directly goes over from simplicial complexes to elements in the strong ring. Let $H = \oplus_{k=0}^n H_k$ be the block diagonal decomposition of the Hodge Laplacian and $G = \sum_{i=1}^n a_i G_i$ the additive decomposition into connected components of products $G_I = G_{i_1} \times \cdots \times G_{i_n}$ of simplicial complexes. For every product G_I , we can write down

a concrete exterior derivative d_I , Dirac operator $D(G_I) = d_I + d_I^*$ and Hodge Laplacian $H(G_I) = D(G_I)^2$.

Proposition 1 (Hodge relation). *We have $b_k(G_I) = \dim(\ker(H_k(G_I)))$.*

The proof is the same as in the case of simplicial complexes as the cohomology of G_I is based on a concrete exterior derivative. When adding connected components we **define** now

$$b_k(\sum_I a_I G_I) = \sum_I a_I b_k(G_I) .$$

6.4. The strong ring has a remarkable compatibility with cohomology. In order to see the following result, one could use discrete homotopy notions or then refer to the classical notions and call two complexes homotopic if their geometric realizations are homotopic. It is better however to stay in a combinatorial realm and ignore geometric realizations.

Theorem 4 (Kuenneth). *The map $G \rightarrow p_G(x)$ is a ring homomorphism from the strong connection ring to $\mathbb{Z}[x]$.*

Proof. If d_i are the exterior derivatives on G_i , we can write them as partial exterior derivatives on the product space. We get from $df(x, y) = d_1 f(x, y) + (-1)^{\dim(x)} d_2(f(x, y))$

$$d^* df = d_1^* d_1 f + (-1)^{\dim(x)} d_1^* d_2 f + (-1)^{\dim(x)} d_2^* d_1 f + d_2^* d_2 f ,$$

$$dd^* f = d_1 d_1^* f + (-1)^{\dim(x)} d_1 d_2^* f + (-1)^{\dim(x)} d_2 d_1^* f + d_2 d_2^* f .$$

Therefore $Hf = H_1 f + H_2 f + (-1)^{\dim(x)} (d_1^* d_2 + d_1 d_2^* + d_2^* d_1 + d_2 d_1^*) f(x, y)$. Since Hodge gives an orthogonal decomposition

$$\text{im}(d_i), \text{im}(d_i^*), \ker(H_i) = \ker(d_i) \cap \ker(d_i^*) ,$$

there is a basis in which $H(v, w) = (H(G_1)(v), H(G_2)(w))$. Every kernel element can be written as (v, w) , where v is in the kernel of H_1 and w is in the kernel of H_2 . \square

The Kuenneth formula follows also from [17] because the product is $(G \times H)_1$ and the cohomology of the Barycentric refinement is the same. It follows that the **Euler-Poincaré formula** holds in general for elements in the ring: the cohomological Euler characteristic $\sum_{k=0}^{\infty} b_k(G)(-1)^k$ is equal to the combinatorial Euler characteristic $\sum_{k=0}^{\infty} v_k(G)(-1)^k$, where (v_0, v_1, \dots) is the f -vector of G .

There is also a cohomology for the higher Wu characteristic $\omega_k(G)$.

7. GAUSS-BONNET, POINCARÉ-HOPF

7.1. The definition $\sum_x \omega(x) = \chi(G)$ of Euler characteristic, with curvature $\omega(x) = (-1)^{\dim(x)}$ can be interpreted as a **Gauss-Bonnet** result in G_1 , the Barycentric refinement of a simplicial complex G . If G is the Whitney complex of a graph, then we have a small set of vertices V , the zero dimensional simplices in G . Pushing the curvature from the simplices to the vertices v , then produces the curvature

$$K(v) = \sum_{k=0}^{\infty} \frac{(-1)^k V_{k-1}}{(k+1)} = 1 - \frac{V_0}{2} + \frac{V_1}{3} - \frac{V_2}{4} + \dots ,$$

where $V_k(v)$ is the number of $k+1$ -dimensional simplices containing v . See [5]. The formula appeared already in [28] but without seeing it as a Gauss-Bonnet result. Gauss-Bonnet makes sense for any simplicial complex G . If v is a 0-dimensional element in G and $V(v)$ is the number of simplices containing v , then the same curvature works. It can be formulated more generally for any element in strong ring. The curvatures just multiply in the product:

Theorem 5 (Gauss-Bonnet). *Given a ring element G . The curvature function K supported on the zero-dimensional part V of G satisfies $\sum_v K(v) = \chi(G)$. If $G = A \times B$, and $v = (a, b)$ is a 0-dimensional point in G , then $K_G(v) = K_A(a)K_B(b)$.*

Proof. The proof is the same. Lets take the product $A \times B$ of two simplicial complexes. We have $\sigma(A \times B) = \sum_{x,y} \omega(x)\omega(y)$, where the sum is over all pairs $(x, y) \in A \times B$ (the set theoretical Cartesian product). The sum does not change, if we distribute every value $\omega(x)$ equally to zero-dimensional subparts. This gives the curvature. \square

7.2. For Poincaré-Hopf [7], we are given a locally injective function f on G . Define the **Poincaré-Hopf index** $i_f(v)$ at a 0-dimensional simplex v in G as $1 - \chi(S_f^-(x))$, where $S_f^-(v) = \{x \in G \mid f(v) < f(x) \text{ and } v \subset x\}$ and the Euler characteristic is the usual sum of the $\omega(y)$, where y runs over the set $S_f^-(v)$. This can now be generalized to products:

Theorem 6 (Poincaré-Hopf). *Given a ring element G and a locally injective function f on G . The index function i_f supported on the zero-dimensional part V of G satisfies $\sum_v i_f(v) = \chi(G)$. If $G = A \times B$ and $v = (a, b)$ is a 0-dimensional point in G then $i_f(v) = i_f(a)i_f(b)$.*

Proof. Also here, the proof is the same. Instead of distributing the original curvature values $\omega(x)\omega(y)$ equally to all zero dimensional parts,

it is only thrown to the zero dimensional simplex (a, b) for which the function is minimal on (x, y) . \square

7.3. Also index averaging generalizes. Given any probability measure P on locally injective functions f , one can look at the expectation $K_P(v) = E[i_f(v)]$ which can now be interpreted as a curvature as it does not depend on an individual function f any more. There are various natural measures which produce the Gauss-Bonnet curvature $K(x)$. One is to look at the product measure $[-1, 1]$ indexed by G [8]. An other is the set of all colorings, locally injective functions on G [14]. Lets formulate it for colorings

Theorem 7 (Index averaging). *Averaging $i_f(x)$ over all locally injective functions on G with uniform measure gives curvature $K(x)$.*

7.4. For Brouwer-Lefschetz [11], we look at an endomorphisms T of an element G in the strong ring. The definition of the **Brouwer index** is the same as in the graph case: first of all, one can restrict to the attractor of T and get an automorphism T . For a simplex $x \in G$, define $i_T(x) = \text{sign}(T|x)\omega(x)$. Because T induces a permutation on the simplex x , the signature of $T|x$ is defined. Also the definition of the **Lefschetz number** $\chi_T(G)$ is the same. It is the super trace on cohomology

$$\chi_T(G) = \sum_{k=0} (-1)^k \text{tr}(T|H^k(G)) .$$

Theorem 8 (Brouwer-Lefschetz). $\sum_{x, T(x)=x} i_T(x) = \chi_T(G)$.

Proof. The fastest proof uses the heat flow $e^{-tH(G)}$ for the Hodge Laplacian. The super trace $\text{str}(H^k)$ is zero for $k > 0$ by McKean-Singer [9]. Define $l(t) = \text{str}(\exp(-tL)U_T)$, where $U_T f = f(T)$ is the Koopman operator associated to T . The function $f(t)$ is constant. This heat flow argument proves Lefschetz because $l(0) = \text{str}(U_T)$ is $\sum_{T(x)=x} i_T(x)$ and $\lim_{t \rightarrow \infty} l(t) = \chi_T(G)$ by Hodge. \square

7.5. There are more automorphisms T in $A \times B$ than product automorphisms $T_1 \times T_2$. An example is if $A = B$ and $T((x, y)) = (y, x)$. One could have the impression at first that such an involution does not have a fixed point, but it does. Lets for example take $A = B = K_2$. The product $A \times B$ has 9 elements and can be written as $(a+b+ab)(c+d+cd)$. The space is contractible so that only $H^0(G)$ has positive dimension and we are in the special case of the Brouwer fixed point case. The Lefschetz number $\chi_T(G)$ is equal to 1. There must be a fixed point. Indeed, it is the two dimensional simplex $((a, b) \times (c, d))$ represented in the Stanley-Reisner picture as $abcd$.

8. WU CHARACTERISTIC

8.1. Euler characteristic $\chi(G) = \omega_1(G)$ is the first of a sequence ω_k of **Wu characteristic**. The **Wu characteristic** $\omega(G) = \omega_2(G)$ is defined for a simplicial complex G as

$$\sum_{x \sim y} \omega(x)\omega(y),$$

where $\omega(x) = (-1)^{\dim(x)}$ and where the sum is taken over all intersecting simplices. The notation fits as $\omega(K_n) = (-1)^{n-1}$ which we have proven the fact that the Barycentric refinement of the complete graph is a ball, a discrete manifold with sphere boundary of dimension $n - 1$. A general formula for discrete manifolds with boundary then use the formula $\omega(G) = \chi(G) - \chi(\delta G)$. Higher order versions $\omega_k(G)$ are defined similarly than $\omega(G)$. We just have to sum over all k -tuples of simultaneously intersecting simplices in the complex. While we have seen $\omega_k((G \times H)_1) = \omega_k(G)\omega_k(H)$ and of course $\omega_k(G \oplus H) = \omega_k(G) + \omega_k(H)$, this insight was done for the Cartesian product $(G \times H)_1$ which was again a simplicial complex, the Whitney complex of a graph. The product property especially implies that the Barycentric refinement G_1 has the same Wu characteristics $\omega_k(G) = \omega_k(G_1)$. In other words, like Euler characteristic $\chi = \omega_1$, also the Wu characteristic $\omega = \omega_2$ and higher Wu characteristics $\omega_k(G)$ are **combinatorial invariants**.

8.2. The Wu characteristic can be extended to the strong ring. For simplicity, lets restrict to $\omega = \omega_2$. The notation $\omega(x) = (-1)^{\dim(x)}$ is extended to pairs of simplices as $\omega((x, y)) = \omega(x)\omega(y)$. So, ω is defined as a function on the elements (x, y) in the Cartesian product $G \times H$ of two simplicial complexes G and H . We can not use the original definition of Wu characteristic for the product as the product of two simplicial complexes is not a simplicial complex any more as the multiplicative primes in the ring are the simplicial complexes. Lets write $(x, y) \sim (a, b)$ if both $x \cap a \neq \emptyset$ and $y \cap b \neq \emptyset$. Now define

$$\omega(G \times H) = \sum_{(x,y) \sim (a,b)} \omega((x, y))\omega((a, b)) .$$

As this is equal to $\sum_{(x,y) \sim (a,b)} \omega(x)\omega(y)\omega(a)\omega(b)$ which is $\sum_{x \sim a} \sum_{y \sim b} \omega(x)\omega(a)\omega(y)\omega(b)$ or $(\sum_{x \sim a} \omega(x)\omega(a)) \sum_{y \sim b} \omega(y)\omega(b)$, which is $\omega(G)\omega(H)$, the product property is evident. We can also define $\omega_k(-G) = -\omega_k(G)$ so that

Proposition 2. *All Wu characteristics ω_k are ring homomorphisms from the strong ring to \mathbb{Z} .*

8.3. The just seen nice compatibility of Wu characteristic with the ring arithmetic structure renders the Wu characteristic quite unique among multi-linear valuations [20]. We have seen in that paper that for geometric graphs, there are analogue **Dehn-Sommerville relations** which are other valuations which are zero, answering a previously unresolved question of [1] from 1970. But this requires the simplicial complexes to be discrete manifolds in the sense that every unit sphere has to be a sphere. Dehn-Sommerville invariants are exciting that they defeat somehow the fate of exploding in the Barycentric limit as they are zero from the beginning and remain zero in the continuum limit. The local versions, the Dehn-Sommerville invariants of the unit spheres are local quantities which are **zero curvature** conditions. One might wonder why Euler curvature is not defined for odd dimensional manifolds for example. Indeed, Gauss-Bonnet-Chern is formulated only for even dimensional manifolds and the definition of curvature involves a Pfaffian, which only makes sense in the even dimensional case. But what really happens is that there are curvatures also in the odd dimensional case, they are just zero due to Dehn-Sommerville. When writing [5], we were not aware of the Dehn-Sommerville connection and had only conjectured that for odd dimensional geometric graphs the curvature is zero. It was proven in [8] using discrete integral geometry seeing curvature as an average of Poincaré-Hopf indices.

8.4. As a general rule, any result for Euler characteristic χ appears to generalize to Wu characteristic. For Gauss-Bonnet, Poncaré-Hopf and index expectation linking the two also the proofs go over. Start with the definition of Wu characteristic as a Gauss-Bonnet type result where $\omega_k(x)$ is seen as a curvature on simplices. Then push that curvature down to the zero dimensional parts. Either equally, leading to a **curvature**, or then directed along a gradient field of a function f , leading to **Poincaré-Hopf indices**. Averaging over all functions, then essentially averages over all possible “distribution channels” f leading for a nice measure on functions to a uniform distribution and so to curvature. The results and proofs generalize to products.

8.5. Lets look at Gauss-Bonnet first for Wu characteristic:

Theorem 9 (Gauss-Bonnet). *Given a ring element G . The curvature function K_k supported on the zero-dimensional part V of G satisfies $\sum_v K_k(v) = \omega_k(G)$. If $G = A \times B$, and $v = (a, b)$ is a 0-dimensional point in G , then $K_k(v) = K_A(a)K_B(b)$.*

8.6. For formulating Poincaré-Hopf for Wu characteristic, we define for zero-dimensional entries $v = (a, b)$ the stable sphere $S_f^-(a, b) = \{(x, y) \in G \times H \mid f((a, b)) < f((x, y)) \text{ and } a \subset x, b \subset y\}$. This stable sphere is the join of the stable spheres. The definition $i_{f,k}(v) = 1 - \omega_k(S_f^-(v))$ leads now to $i_{f,k}((a, b)) = i_{f,k}(a)i_{f,k}(b)$ and

Theorem 10 (Poincaré-Hopf). *Let f be a Morse function, then $\omega_k(G) = \sum_v i_{f,k}(v)$, where the sum is over all zero dimensional v in G .*

8.7. When looking at the index expectation results $K(x) = E[i_f(x)]$, one could either directly prove the result or then note that if we look at a direct product $G \times H$ and take probability measures P and Q on functions of G and H , then the random variables $f \rightarrow i_f(x)$ on the two probability spaces $(\Omega(G), P)$ and $(\Omega(H), Q)$ are independent. This implies $E[i_f(x)i_f(y)] = E[i_f(x)]E[i_f(y)]$ and so index expectation in the product:

Theorem 11 (Index expectation). *If the probability measure is the uniform measure on all colorings, then curvature $K_{G \times H, k}$ is the expectation of Poincaré-Hopf indices $i_{G \times H, f, k}$.*

8.8. The theorems of Gauss-Bonnet, Poincaré-Hopf and index expectation are not restricted to Wu characteristic. They hold for any **multi-linear valuation**. By the multi-linear version of the discrete Hadwiger theorem [4], a basis of the space of valuations is given. Quadratic valuations for example can be written as

$$X(G) = \sum_{x \sim y} X_{ij} V_{ij}(G) ,$$

where X is a symmetric matrix and where the **f -matrix** V_{ij} counts the number of pairs x, y of i -dimensional simplices x and j dimensional simplices y for which $x \cap y \neq \emptyset$.

8.9. To see how the f -vectors, the f -matrices and more generally the f -tensors behave when we take products in the ring, its best to look at their **generating functions**. Given a simplicial complex G with f -vector $f(G) = (v_0(G), v_1(G), \dots)$, define the **Euler polynomial**

$$e_G(t) = \sum_{k=0}^{\infty} v_k(G) t^k$$

or the multi-variate polynomials like in the quadratic case

$$V_G(t, s) = \sum_{k, l} V_{k, l}(G) t^k s^l ,$$

which encodes the cardinalities $V_{k,l}(G)$ of intersecting k and l simplices in the complex G . The convolution of the f -vectors becomes the product of Euler polynomials

$$e_{G \times H} = e_G e_H .$$

8.10. If we define the f -vector of $-G$ as $-f(G)$ and so the Euler polynomial of $-G$ as $-p_G$, we can therefore say that

Proposition 3. *The Euler polynomial extends to a ring homomorphism from the strong ring to the polynomial ring $\mathbb{Z}[t]$.*

8.11. This also generalizes to the multivariate versions. To the f -tensors counting k -tuple intersections in G , we can associate polynomials in $\mathbb{Z}[t_1, \dots, t_k]$ which encode the f -tensor and then have

Proposition 4. *For every k , the multivariate f -polynomial construction extends to ring homomorphisms from the strong ring to $\mathbb{Z}[t_1, \dots, t_k]$.*

8.12. We see that like in probability theory, where moment generating functions or characteristic functions are convenient as they “diagonalize” the combinatorial structure of random variables on product spaces (independent spaces), the use of f -polynomials helps to deal with the combinatorics of the f -tensors in the strong ring.

8.13. In order to formulate results which involve cohomology like the Lefschetz fixed point formula, which reduces if T is the identity to the Euler-Poincaré formula relating combinatorial and cohomological Euler characteristic, one has to define a cohomology. Because the name intersection cohomology is taken, we called it **interaction cohomology**. It turns out that this cohomology is finer than simplicial cohomology. Its Betti numbers are combinatorial invariants which allow to distinguish spaces which simplicial cohomology can not, the prototype example being the cylinder and Moebius strip [27]. The structure and proofs of the theorems however remain. The heat deformation proof of Lefschetz is so simple that it extends to the ring and also from Euler characteristic to Wu characteristic.

8.14. The definition of interaction cohomology involves explicit matrices d as exterior derivatives. The **quadratic interaction cohomology** for example is defined through the **exterior derivative** $dF(x, y) = F(\delta x, y) + (-1)^{\dim(x)} F(x, \delta y)$ on functions F on ordered pairs (x, y) of intersecting simplices in G . This generalizes the exterior derivative $dF(x) = F(\delta x)$ of simplicial cohomology. This definition resembles the de-Rham type definition of exterior derivative for the product of complexes, but there is a difference: in the interaction cohomology we only

look at pairs of simplices which interact (intersect). In some sense, it restricts to the simplices in the “diagonal” of the product.

8.15. It is obvious how to extend the definition of interaction cohomology to the product of simplicial complexes and so to the strong ring. It is still important to point out that at any stage we deal with finite dimensional matrices. The **quadratic interaction exterior derivative** d is defined as $dF(x, y) = d_1F + (-1)^{\dim(x)}d_2F$, where d_1 is the **partial exterior derivative** with respect to the first variable (represented by simplices in G) and d_2 the partial exterior derivative with respect to the second variable (represented by simplices in H). These partial derivatives are given by the intersection exterior derivatives defined above.

8.16. Lets restrict for simplicity to quadratic interaction cohomology in which the Wu characteristic $\omega = \omega_2$ plays the role of the Euler characteristic $\chi = \omega_1$. Let G first be a simplicial complex. If $b_p(G)$ are the Betti numbers of these interaction cohomology groups of G , then the **Euler-Poincaré** formula $\omega(G) = \sum_p (-1)^p b_p(G)$ holds. More generally, the **Lefschetz formula** $\chi_T(G) = \sum_{(x,y)=(T(x),T(y))} i_T(x, y)$ generalizes, where $\chi_T(G)$ is the **Lefschetz number**, the super trace of the Koopman operator U_T on cohomology and where $i_T(x, y) = (-1)^{\dim(x)+\dim(y)} \text{sign}(T|x) \text{sign}(T|y)$ is the **Brouwer index**. The heat proof generalizes.

8.17. The interaction cohomology groups are defined similarly for the product of simplicial complexes and so for general elements in the strong ring. The Künneth formula holds too. We can define the Betti numbers $b_k(-G)$ as $-b_k(G)$. The **interaction Poincaré polynomial** $p_G(x) = \sum_{k=0} \dim(H^k(G))x^k$ again satisfies $p_{G \times H}(x) = p_G(x)p_H(x)$ so that

Theorem 12. *For any k , the interaction cohomology polynomial extends to a ring homomorphism from the strong ring to $\mathbb{Z}[t]$.*

8.18. While we hope to be able to explore this more elsewhere we note for now just that all these higher order interaction cohomologies associated to the Wu characteristic generalize from simplicial complexes to the strong ring. Being able to work on the ring is practical as the interaction exterior derivatives are large matrices. So far we had worked with the Barycentric refinements of the products, where the matrices are bulky if we work with a full triangulation of the space. Similarly as de-Rham cohomology significantly cuts the complexity of cohomology computations, this is also here the case in the discrete. For

a triangulation of the cylinder G , the full Hodge Laplacian $(d + d^*)^2$ of quadratic interaction cohomology is a 416×416 matrix as there were $416 = \sum_{i,j} V_{ij}(G)$ pairs of intersecting simplices. When working in the ring, we can compute a basis of the cohomology group from the basis of the circles and be done much faster.

8.19. For the Möbius strip G however, where the Hodge Laplacian of a triangulation leads to 364×364 matrices, we can not make that reduction as G is not a product space. The smallest ring element which represents G is a simplicial complex. Unlike the cylinder which is a “composite particle”, G is an “elementary particle”. In order to deal with concrete spaces, one would have to use patches of Euclidean pieces and compute the cohomology using Mayer-Vietoris.

8.20. What could be more efficient is to patch the space with contractible sets and look at the cohomology of a nerve graph which is just Čech cohomology. But as in the Möbius case, we already worked with the smallest nerve at hand, this does not help in the computation in that case. It would only reduce the complexity if we had started with a fine mesh representing the geometric object G . We still have to explore what happens to the k -harmonic functions in the kernel of the Hodge blocks H_k and the spectra of H_k in the interaction cohomology case, if we cut a product space and glue it with reverse orientation as in G .

8.21. There is other geometry which can be pushed over from complexes to graphs. See [10, 12, 13] for snapshots for results formulated for Whitney complexes of graphs from 2012, 2013 and 2014. The **Jordan-Brouwer theorem** for example, formulated in [16] for Whitney complexes of graphs generalizes to simplicial complexes and more generally to products as we anyway refer to the Barycentric refinement there which is always a Whitney complex.

8.22. Also promising are subjects close to calculus like the discrete **Sard theorem** [18] which allow to define new spaces in given spaces by looking at the zero locus of functions. Also this result was formulated in graph theory but holds for any simplicial complex. The **zero locus** $\{f = c\}$ of a function f on a ring element G can be defined as the complex obtained from the cells where f changes sign. We have noticed that if G is a discrete d -manifold in the sense that every unit sphere in the Barycentric refinement is a $(d - 1)$ -sphere, then for any locally injective function f and any c different from the range of f the zero locus $f = c$ is a discrete manifold again.

9. TWO SIBLINGS: THE DIRAC AND CONNECTION OPERATOR

9.1. To every G in the strong ring belong two graphs G_1 and G' , the Barycentric refinement and the connection graph. The addition of ring elements produces disjoint unions of graphs. The multiplication naturally leads to products of the Barycentric refinements $A_1 \times B_1 = (A \times B)_1$ as well as connection graphs $A' \boxtimes B' = (A \times B)'$, where \boxtimes is the strong product.

9.2. An illustrative example is given in [22]. Both the **prime graph** G_1 as well as the **prime connection graph** have as the vertex set the set of square free integers in $\{2, 3, \dots, n\}$. In G_1 , two numbers are connected if one is a factor of the other. It is part of the Barycentric refinement of spectrum of the integers. In the **prime connection graph** G' two integers are connected if they have a common factor larger than 1. It has first appeared in [23]. This picture sees square free integers as simplices in a simplicial complex. The number theoretical Möbius function $\mu(k)$ has the property that $-\mu(k)$ is the Poincaré-Hopf index of the counting function $f(x) = x$. The Poincaré-Hopf theorem is then $\chi(G) = 1 - M(n)$, where $M(n)$ is the Mertens function. As the Euler characteristic $\sum_x \omega(x)$ can also be expressed through Betti numbers, there is a relation between the Mertens function and the kernels of Hodge operators D^2 . On the other hand, the Fermi characteristic $\prod_x \omega(x)$ is equal to the determinant of the connection Laplacian. This was just an example. Lets look at it in general.

9.3. To every $G \in R$ belong two operators D and L , the Dirac operator and connection operator. They are both symmetric matrices acting on the same Hilbert space. While $D(-G) = -D(G)$ does not change anything in the spectrum of G , a sign change of G changes the spectrum as $L(-G) = -L(G)$ is a different operator due to lack of symmetry between positive and negative spectrum. There are higher order Dirac operators D which belong to the exterior derivative d which is used in interaction cohomology. The operator D which belongs to the Wu characteristic for example does not seem to have any obvious algebraic relation to the Dirac operator belonging to Euler characteristic. Indeed, as we have seen, the nullity of the Wu Dirac operator is not homotopy invariant in general. On the other hand, the interaction Laplacian L belonging to pairs of interacting simplices is nothing else than the tensor product $L \otimes L$, which is the interaction Laplacian of $G \times G$.

9.4. There are various indications that the Dirac operator $D(G)$ is of **additive nature** while the connection operator $L(G)$ is of **multiplicative nature**. One indication is that $L(G)$ is always invertible and that

the product of ring elements produces a tensor product of connection operators. Let str denote the super trace of a matrix A defined as $\text{str}(A) = \sum_x \omega(x) A_{xx}$, where $\omega(x) = (-1)^{\dim(x)}$. The discrete version of the **Mc-Kean Singer formula** [29] can be formulated in a way which makes the additive and multiplicative nature of D and L clear:

Theorem 13 (Mc Kean Singer). $\text{str}(e^{-D}) = \chi(G)$ and $\text{str}(L^{-1}) = \chi(G)$.

Proof. The left identity follows from the fact that $\text{str}(D^{2k}) = 0$ for every even k different from 0, that for $k = 0$, we have the definition of $\chi(G)$ and that for odd $2k + 1$ the diagonal entries of D^k are zero. The right equation is a Gauss-Bonnet formula as the diagonal entries L_{ii}^{-1} are the Euler characteristics $\chi(S(x))$ of unit spheres so that $\omega(x)\chi(S(x)) = 1 - \chi(S_f^-(x))$ is a Poincaré-Hof index for the Morse function $f(x) = -\dim(x)$. \square

These identities generalize to a general ring element G in the strong ring.

9.5. A second indication for the multiplicative behavior of quantities related to the connection graph is the Poincaré-Hopf formula. The multiplicative analogue of the Euler characteristic $\chi(G) = \sum_x \omega(x)$ is the **Fermi characteristic** $\phi(G) = \prod_x \omega(x)$. Lets call a function f on a simplicial complex a **Morse function** if $S_f^-(x) = \{y \in S(x) \mid f(y) < f(x)\}$ is a discrete sphere. Since spheres have Euler characteristic in $\{0, 2\}$ the Euler-Poincaré index $i_f(x) = 1 - \chi(S_f^-(x))$ is in $\{-1, 1\}$.

Theorem 14 (Poincaré-Hopf). *Let f be a Morse function, then $\chi(G) = \sum_x i_f(x)$ and $\phi(G) = \prod_x i_f(x)$.*

Since for any simplicial complex G , we can find a Morse function, the multiplicative part can be used to show that $\psi(G) = \det(L(G))$ is equal to $\phi(G)$. This is the **unimodularity theorem**. As we have seen, this theorem generalize to the strong ring.

9.6. The most striking additive-multiplicative comparison of D and L comes through spectra. The spectra of the Dirac operator D behave additively, while the spectra of connection operators multiply. The additive behavior of Hodge Laplacians happens also classically when looking at Cartesian products of manifolds:

Theorem 15 (Spectral Pythagoras). *If λ_i are eigenvalues of $D(G_i)$, there is an eigenvalue λ_{ij} of $D(G_i \times G_j)$ such that $\lambda_i^2 + \lambda_j^2 = \lambda_{ij}^2$.*

9.7. We should compare that with the multiplicative behavior of the connection operators:

Theorem 16 (Spectral multiplicity). *If λ_i are eigenvalues of $L(G_i)$, then there is an eigenvalue λ_{ij} of $L(G_i \times G_j)$ such that $\lambda_i \lambda_j = \lambda_{ij}$.*

9.8. In some sense, the operator L describes energies which are multiplicative and so can become much larger than the energies of the Hodge operator H . Whether this has any significance in physics is not clear. So far this is pure geometry and spectral theory. The operator L does not have this block structure like D . When looking at Schrödinger evolutions e^{iDt} or e^{iLt} , we expect different behavior. Evolving with L mixes parts of the Hilbert space which are separated in the Hodge case.

10. DIMENSION

10.1. The **maximal dimension** $\dim_{\max}(G)$ of a simplicial complex G is defined as $|x| - 1$, where $|x|$ is the **cardinality** of a simplex x . The **inductive dimension** of G is defined as the inductive dimension of its Barycentric refinement graph G_1 (or rather the Whitney complex of that graph) [6]. The definition of inductive dimension for graphs is recursive:

$$\dim(x) = 1 + |S(x)|^{-1} \sum_{y \in S(x)} \dim(y)$$

starting with the assumption that the dimension of the empty graph is -1 . The unit sphere graph $S(x)$ is the subgraph of G generated by all vertices directly connected to x .

Proposition 5. *Both the maximal and inductive dimension can be extended so that additivity holds in full generality for non-zero elements in the strong ring.*

The zero graph \emptyset has to be excluded as when we multiply a ring element with the zero element, we get the zero element and $\dim(0) = -1$ would imply $\dim(G \times 0) = \dim(0) = -1$ for any G . The property $0 \times G = 0$ has to hold if we want the ring axioms to hold. We will be able to extend the clique number to the dual ring of the strong connection ring but not the dimension.

10.2. Since the inductive dimension satisfies

$$\dim((G \times H)_1) = \dim(G_1) + \dim(H_1) ,$$

(see [17]), the definition

$$\dim(G \times H) = \dim(G) + \dim(H)$$

is natural. We have also shown $\dim(G)_1 \geq \dim(G)$ so that for nonzero elements G and H : $\dim(G \times H)_1 \geq \dim(G) + \dim(H)$, an inequality which looks like the one for Hausdorff dimension in the continuum. Again, also this does not hold if one of the factors is the zero element.

10.3. In comparison, we have seen that the Zykov sum has the property that the clique number $\dim_{\max}(G) + 1$ is additive. and that the large Zykov product, which is the dual of the strong product has the property that the clique number is multiplicative. If we define the clique number of $-G$ as minus the clique number of G , then we have

Proposition 6. *The clique number from the dual R^* of the strong ring R to the integers \mathbb{Z} is a ring homomorphism.*

Proof. By taking complement, the identity $P_n \star P_m = P_n \boxtimes P_m = P_{nm}$ gives $K_n \boxtimes K_m = K_m \boxtimes K_n = K_{nm}$. We have justified before how to extend the clique number to $c(-G) = -c(G)$. \square

10.4. The clique number is a ring homomorphism, making the clique number to become negative for negative elements a natural choice. For dimension, this is not good: the dimension of the empty graph 0 is -1 and since $0 = -0$ this is not good. If we would take a one dimensional graph G , we would have to define $\dim(-G) = -1$ but that does not go well with the fact that the zero graph has dimension -1 . We want the inductive dimension to be the average of the dimensions of the unit spheres plus 1 which makes the choice $\dim(-G) = \dim(G)$ natural.

10.5. The **maximal dimension** $\dim_{\max}(G)$ of a complex is the dimension of the maximal simplex in G . The **clique number** $c(G) = \dim_{\max}(G) + 1$ is the largest cardinality which appears for sets in G . For graphs, it is the largest n for which K_n is a subgraph of G . The empty complex or empty graph has clique number 0. Extend the clique number to the entire ring by defining $c(-G) = -c(G)$. The strong ring multiplication satisfies $c(G \boxtimes H) = c(G)c(H)$ and $c(G \oplus H) = \max(c(G), c(H))$.

11. DYNAMICAL SYSTEMS

11.1. There is an isospectral **Lax deformation** $D' = [B, D]$ with $B = (d - d^*)$ of the Dirac operator $D = (d + d^*)$ of a simplicial complex. This works now also for any ring elements. The deformed Dirac operator is then of the form $d + d^* + b$ meaning that D has additionally to the geometric exterior derivative part also a diagonal part which is geometrically not visible. What happens is that the off diagonal part $d(t)$ decreases, leading to an expansion of space if we use the

exterior derivative to measure distances. The deformation of exterior derivatives is also defined for Riemannian manifolds but the deformed operators are pseudo differential operators.

11.2. A complex generalization of the system is obtained by defining $B = d - d^* + i\beta b$, where β is a parameter. This is similar to [33] who modified the Toda flow by adding $i\beta$ to B . The case above was the situation $\beta = 0$, where the flow has a situation for scattering theory. The case $\beta = 1$ leads asymptotically to a linear wave equation. The exterior derivative has become complex however. All this dynamics is invisible classically for the Hodge operator H as the Hodge operator D^2 does not change under the evolution.

11.3. As any Lax pair does, the eigenvalues are integrals of motion. The equation $U'(t) = B(t)U(t)$ produces a unitary curve $U(t)$ which has the property that if $D_1 = VD V^T$ corresponds to an other choice of simplex orientation then $D_1(t) = VD(t)V^T$ carries on. The evolution does not depend on the gauge (the choice of signs used to make the simplicial complexes signed) but the energy for $L(t) = U(t)LU(t)^*$ does.

Theorem 17. *For every G in the strong ring, we have nonlinear integrable Hamiltonian systems based on deformation of the Dirac operator.*

This is very concrete as for every ring element we can write down a concrete matrix $D(G)$ and evolve the differential equation. As for simplicial complexes, also when we deform a D from the strong ring, the Hodge Laplacian $H(G) = D(G)^2$ does not move under the deformation. Most classical physics therefore is unaffected by the expansion. For more details see [15, 19].

11.4. Any type of Laplacian produces Schrödinger type evolutions. Examples of Laplacians are the **Kirchhoff matrix** H_0 of the graph which has G as the Whitney complex or the Dirac matrix $D = (d + d^*)$ or the Hodge Laplacian D^2 or the connection graph L of G_n . One can also look at non-linear evolutions. An example is the non-linear Schrödinger equation obtained from a functional like $F(u) = \langle u, gu \rangle - V(|u|^2)$ on the Hilbert space. The Hamiltonian system is then $u' = \partial_{\bar{u}} F'(u)$. This **Helmholtz evolution** has both the energy $F(u)$ as well as $|u|^2$ as integrals of motion. We can therefore restrict to **states**, vectors of length 1. A natural choice compatible with the product structure is the **Shannon entropy** $V(|u|^2) = \beta S(u) = -\beta \sum_x p(x) \log(p(x))$ where $p(x) = |u(x)|^2$. Summands for which $p(x) = 0$ are assumed to be zero as $\lim_{p \rightarrow 0} p \log(p) = 0$.

11.5. We like the Helmholtz system because the Shannon entropy is essentially unique in the property that it is additive with respect to products. As $\langle 1, g1 \rangle = \chi(G)$ is the Euler characteristic, which is compatible with the arithmetic, the **Helmholtz free energy** $F(u)$ leads to a natural Hamiltonian system. The **inverse temperature** β can be used as a perturbation parameter. If $\beta = 0$, we have a Schrödinger evolution. If $\beta > 0$, it becomes a nonlinear Schrödinger evolution. So far, this system is pretty much unexplored. The choice of both the energy and entropy part was done due to arithmetic compatibility: Euler characteristic is known to be the unique valuation up to scale which is compatible with the product and entropy is known to be a unique quantity (again up to scale) compatible with the product by a theorem of Shannon. See [24] where we look at bifurcations of the minima under temperature changes. There are **catastrophes** already for the simplest simplicial complexes.

12. BARYCENTRAL CENTRAL LIMIT

12.1. We can use a Barycentric central limit theorem to show that in the **van Hove limit** of the strong nearest neighbor lattice $Z \times \cdots \times Z$, the spectral measure of the connection Laplacian L has a **mass gap**: not only L , but also the Green function, the inverse $g = L^{-1}$ has a bounded almost periodic infinite volume limit. The potential theory of the Laplacian remains bounded and nonlinear time-dependent partial difference equations like $Lu + cV(u) = W$ which are Euler equations of **Frenkel-Kontorova** type variational problems have unique solutions u for small c .

12.2. A **Fock space** analogy is to see an additive prime in the ring (a connected space) as a **particle state**. The sum of spaces is then a collection of independent particles and the product is an entangled **multi-particle system**. Every independent particle has a unique decomposition into **multiplicative primes** which are the “elementary particles”. But the union of two particles can decay into different type of elementary particles. For an entangled multi-particle system $G \times H$, the spectrum of that configuration consists of all values $\lambda_i \mu_j$, where λ_i are the eigenvalues of $L(G)$ and μ_j are the eigenvalues of $L(H)$.

12.3. While $G \boxtimes H = H \boxtimes G$, we don't have $L(G) \otimes L(H) = L(H) \otimes L(G)$ as the tensor product of matrices is not commutative. But these two products are unitary equivalent. We can still define an **anti-commutator** $L(H) \otimes L(G) - L(G) \otimes L(H)$. We observed experimentally that the kernel of this anti-commutator is non-trivial only if

both complexes have an odd number of vertices. While $L(G)$ can have non-simple spectrum, we have so far only seen simple spectrum for operators $L(G) \times L(H) - L(H) \otimes L(G)$.

12.4. When we take successive Barycentric refinements G_n of complex G we see a universal feature [15, 19]. This generalizes readily to the ring.

Theorem 18 (Barycentric central limit). *For any G in the ring and for each of the operators $A = L$ or $A = H$ or $A = H_k$, the density of states of $A(G_n)$ converges weakly to a measure which only depends on the maximal dimension of G .*

12.5. The reason why this is true is that the $(k + 1)$ 'th Barycentric refinement of a maximal d -simplex consists of $(d - 1)!$ smaller disks glued along d -dimensional parts which have a cardinality growing slower. The gluing process changes a negligible amount of matrix entries. This can be estimated using a result of Lidskii-Last [32]. Examples of Laplacians for which the result holds are the Kirchhoff matrix of the graph which has G as the Whitney complex or the Dirac matrix $D = (d + d^*)$ or the Hodge Laplacian D^2 or the connection graph L of G_n .

12.6. In the one-dimensional case with Kirchhoff Laplacian, the limiting measure is derivative of the inverse of $F(x) = 4 \sin^2(\pi x/2)$.

12.7. For $G = C_n$ we look at the eigenvalues of the ν -torus $C_n^{\text{nu}} = C_n \boxtimes C_n \cdots \boxtimes C_n$. The spectrum defines a discrete measure dk_n which has a weak limit dk .

Theorem 19 (Mass gap). *The weak limit dk_n exists, is absolutely continuous and has support away from 0. The limiting operator is almost periodic, bounded and invertible.*

Proof. For $\nu = 1$, the density of states has a support which contains two intervals. The interval $[-1/5, 1/5]$ is excluded as we can give a bound on $g = L^{-1}$. For general ν , the gap size estimate follows from the fact that under products, the eigenvalues of L multiply. \square

13. STABILITY IN THE INFINITE VOLUME LIMIT

13.1. To the lattice Z^ν belongs the Hodge Laplacian $H_0 u(n) = \sum_{i=1}^\nu u(n + e_i) - u(n)$. In the one-dimensional case, where $H_0 u(n) = u(n + 1) - 2u(n) - u(n - 1)$ corresponds after a Fourier transform to the multiplication by $2 \cos(x) - 2$ we see that the spectrum of H_0 is $[0, 4]$. The non-invertibility leads to “small divisor problems”. Also, the trivial

linear solutions $u(n) = \theta + \alpha n$ to $Lu = 0$ are minima. Its this minimality which allows the problem to be continued to nonlinear situations like $u(n+1) - 2u(n) - u(n-1) = c \sin(u(n))$ for Diophantine α . For $\alpha = p/q$, Birkhoff periodic points and for general α , minimizers in the form of Aubry-Mather sets survive. For the connection Laplacian L 0 is in a gap of the spectrum.

13.2. Unlike for the Hodge Laplacians in Z^d which naturally are expressed on the Pontryagin dual of T^d , for which the Laplacian has spectrum containing 0, we deal with the dual of is \mathbb{D}_2^ν , where \mathbb{D}_2 is the **dyadic group**. The limiting operator of L is almost periodic operator and has a bounded inverse. Whereas in the Hodge case without mass gap, a **strong implicit function theorem** is required to continue solutions of nonlinear Frenkel-Kontorova type Hamiltonian systems $Lu + \epsilon V(u) = 0$, the connection Laplacian is an invertible kinetic part and perturbation theory requires only the weak implicit function theorem. Solutions of the Poisson equation $Lu = \rho$ for example can be continued to nonlinear theories $(L + V)v = \rho$. Since Poisson equations have unique solutions, also the discrete Dirichlet has unique solutions. The classical Standard map model $L_0 u + c \sin(u) = 0$ for example which is the Chirikov map in 1D, requires KAM theory for solutions u to exist. Weak solutions also continue to exist by **Aubry-Mather theory** [30].

13.3. If the Hodge Laplacian is replaced by the connection Laplacian, almost periodic solutions to driven systems continue to exist for small c similarly as the **Aubry anti-integrable limit** does classically for large c . But these continuations are in general not interesting. Any Hamiltonian system with Hamiltonian $L + V$, where the kinetic energy is the connection Laplacian remains simple. If we look at $Lu = W$ where W is obtained from an continuous function on the dyadic integers, then the solution u is given by a continuous function on the dyadic integers. This could be of some interest but there is no interpretation of this solution as an orbit of a Hamiltonian system.

Corollary 4. *A nonlinear discrete difference equation $Lu + \epsilon V(u) = g$ has a unique almost periodic solution if $g(n) = g(T^n x)$ is obtained from Z^ν action on \mathbb{D}_2^ν .*

Proof. For $\epsilon = 0$, we have the Poisson equation $Lu = g$ which has the solution $u = L^{-1}g$, where $u(n) = h(T^n x)$ is an almost periodic process with $h \in C(\mathbb{D}_2^\nu)$. Since L is invertible, the standard implicit function theorem allows a continuation for small ϵ . \square

13.4. It could be useful to continue a Hamiltonian to the infinite limit. An example is the Helmholtz Hamiltonian like $H(\psi) = (\psi, g\psi)$, where $g = L^{-1}$. Since the Hessian g is invertible, we can continue a minimal solution to $H(\psi) + \beta V(\psi)$ for small β if V is smooth. We suspect that we can continue almost periodic solutions to the above defined Helmholtz system $V(\psi) = \beta S(|\psi|^2)$ with entropy S for small β but there is a technical difficulty as $p \rightarrow p \log |p|$ is not smooth at $p = 0$, only continuous. One could apply the weak implicit function theorem to continuous almost periodic functions on $X = D_2^\nu$ if $f \rightarrow S(|f|)$ was Fréchet differentiable on the Banach space X but we have not proved that. It might require to smooth out S_ϵ first then show that the solution survives in the limit $\epsilon \rightarrow 0$.

14. A PSEUDO RIEMANNIAN CASE

14.1. When implementing a pseudo Riemannian metric signature like $(+, +, +, -)$ on the lattice \mathbb{Z}^4 , this changes the sign of the corresponding Hodge dual d^* . The Dirac operator of that coordinate axes is now $d_i - d_i^*$ and the corresponding Hodge operator is $D^*D = -H$ has just changed sign. The connection Laplacian L is not affected by the change of Riemannian metric. Having a Pseudo Riemannian metric on \mathbb{Z}^4 , we can look at kernel elements $Hu = 0$ as solutions to a discrete wave equation. Global solutions on a compact space like the product of circular graphs $C_n^4 = C_n \times C_n \times C_n \times C_n$ are not that interesting. On an infinite lattice, we could prescribe solutions on the space hypersurface $t = 0$ and then continue it. Technically this leads to a **coupled map lattice**.

14.2. If H is the operator of the 4-torus C_n^4 with Lorentzian metric signature, then the eigenvalues are all of the form $\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4$, where λ_i are eigenvalues of C_n . This just shifts the eigenvalues. This completely answers also the question what the limiting density of states.

Proposition 7. *The spectrum of the Hodge operator of the ν -torus \mathbb{T}_n^ν with Lorentz signature (m, k) agrees with the spectrum of the Hodge operator for the signature $(\nu, 0)$ shifted by $-4k$.*

Proof. For $\nu = 1$, replacing H with $-H$ has the effect that that the spectrum changes sign. But this is $\sigma(H) - 4$. When taking the product, we get a convolution of spectra which commutes with the translation. As for C_n , the spectrum satisfies $\sigma(-H) = \sigma(H) - 4$, the switch to a Lorentz metric goes over to the higher products. \square

14.3. As the Lorentz space appears in physics, the geometry of the \mathbb{Z}^4 lattice with Lorentz signature metric $(+, +, +, -)$ is of interest. The Barycentric limit leads to more symmetry in this discrete lattice case. The limiting operators D and L are almost periodic on G_2^4 , the compact group of dyadic integers. Besides of the group translations, there are also **scaling symmetries**. By allowing both scaling transformations and group translations, we can implement symmetries which approximate Euclidean symmetries in the continuum. The obstacle of poor symmetry properties in a discrete lattice appears to disappear in the Barycentric limit.

15. ILLUSTRATIONS

15.1. Here is an illustration of an element $G = C_4 - 2K_3 + (L_2 \times L_3)$ in the strong ring. In the Stanley-Reisner picture, we can write the ring element as

$$f_G = a + ab + b + bc + c + cd + d + ad - 2(x + y + z + xy + xz + yz) + (u + uv + v)(p + pq + q + qr + r).$$

The Euler characteristic is $\chi(G) = -f_G(-1, -1, \dots) = -1$.

15.2. The ring element $G = C_4 - 2K_3 + (L_2 \times L_3)$ in the strong ring is a sum of three parts, where the first is a Whitney complex of a graph, the second is -2 times a Whitney complex and the third is a product of two Whitney complexes L_2 and L_3 .

15.3. In Figure (15.4), we drew the weak Cartesian product to visualize $L_2 \times L_3$. In reality, $L_2 \times L_3$ is not a simplicial complex. There are 6 two-dimensional **square cells** present, one for each of the 6 holes present in the weak product. A convenient way to fill the hole is to look at the Barycentric refinement $(L_2 \times L_3)_1$ as done in [17]. This is then a **triangulation** of $L_2 \times L_3$.

15.4. In Figure (2) we see the connection graph G' to the ring element G . It is a disjoint union of graphs, where the connection graphs to K_3 are counted negatively. The connection Laplacian is $L(G) = L(C_4) \oplus (-L(K_3)) \oplus (-L(K_3)) \oplus [L(L_2) \otimes L(L_3)]$. Figure (3) shows the Barycentric refinement graph G_1 .

15.5. Figure (4) shows $G = S^2 \times S^3$. The 2-sphere S^2 is implemented as the Octahedron graph with f -vector $(6, 12, 8)$, the smallest 2-sphere, which already Descartes has super-summed to $6 - 12 + 8 = 2$. The number of cells is 26. The complex S^3 is the suspension of S^2 . It has f -vector $(8, 24, 32, 16)$ which super-sums to $\chi(S^3) = 8 - 24 + 32 - 16 = 0$

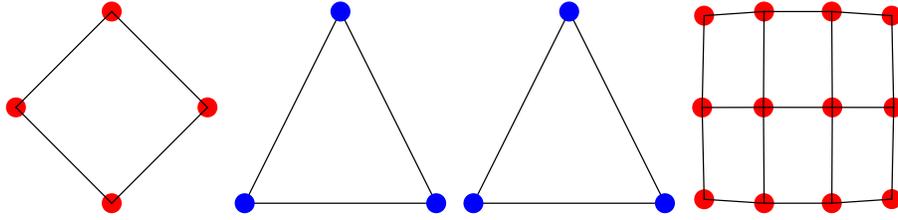


FIGURE 1. $G = C_4 - 2K_3 + (L_2 \times L_3)$ in the strong ring.

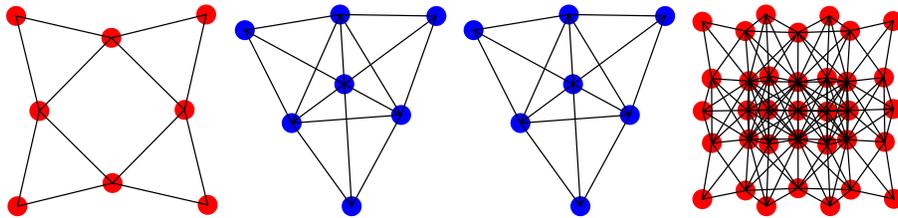


FIGURE 2. The connection graph of G .

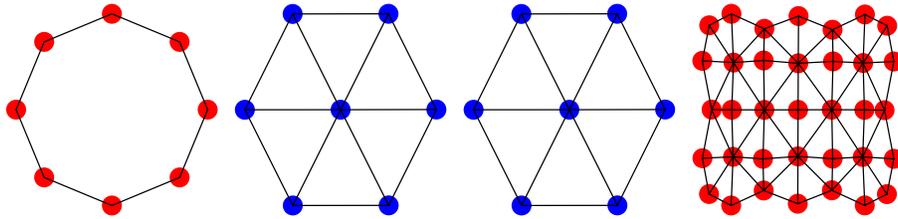


FIGURE 3. The Barycentric refinement G_1 of G .

as any 3-manifold does. The number of cells is 80. The product has $26 * 80 = 2080$ cells. It is a 3-sphere, a cross polytop. The Poincaré polynomial of G is $(1 + x^2)(1 + x^3) = 1 + x^2 + x^3 + x^5$. Indeed, the Betti vector of this 5-manifold is $(1, 0, 1, 1, 0, 1)$.

15.6. We see how useful the ring is. We did not have to build a triangulation of the 5-manifold as we had done in [17] where we defined the product to be G_1 in order to have a Whitney complex of a graph G_1 with 2080 vertices and 51232 edges. The Dirac and Hodge operator for $G = S^2 \times S^3$ is seen in Figure (S2S3). They are both 2080×2080 matrices. As for any 5-dimensional complex, the Hodge operator has 6 blocks. Blocks H_1, H_3, H_4, H_6 have a one-dimensional kernel. McKean-Singer super symmetry shows that for the union of the non-zero spectra H_1, H_3, H_5 is the union of the non-zero spectra of H_2, H_4, H_6 .

15.7. The ring element $G = C_4 - 2K_3 + (L_2 \times L_3)$ in the strong ring is a sum of three parts, where the first is a Whitney complex of a graph, the second is -2 times a Whitney complex and the third is a product of two Whitney complexes L_2 and L_3 . We drew the weak Cartesian product to visualize $L_2 \times L_3$. In reality, $L_2 \times L_3$ is not a simplicial complex. There are 6 additional two dimensional cells present in the CW complex representing it, one for each of the 6 holes present in the weak product.

15.8. Figure (5) illustrates non-unique factorization in the strong ring. It is adapted from [2] in weak ring. The two ring elements $G_1 \times G_2$ and $H_1 \times H_2$ are the same. It is the decomposition $(1 + x + x^2)(1 + x^3) = (1 + x^2 + x^4)(1 + x)$, where 1 is a point K_1 x is an interval K_2 . This gives the square $x^2 = K_2 \times K_2$, the cube $x^3 = K_2 \times K_2 \times K_2$ and hyper cube $x^4 = K_2 \times K_2 \times K_2 \times K_2$.

ARITHMETIC OF GRAPHS

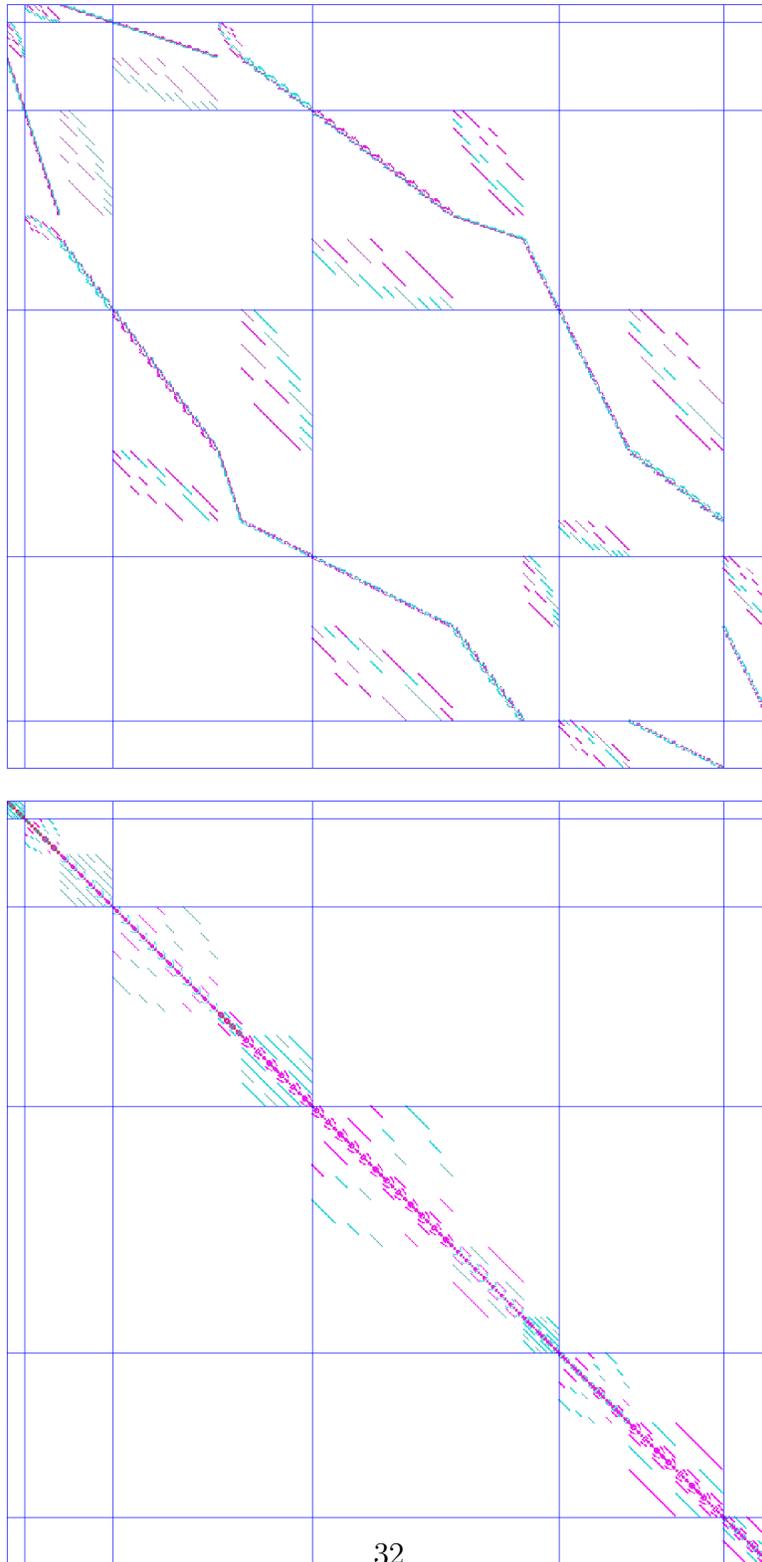


FIGURE 4. The Dirac and Hodge operator for $G = S^2 \times S^3$.

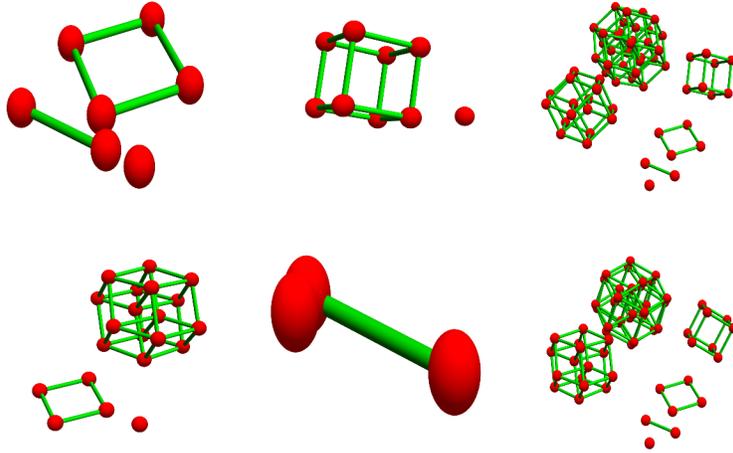


FIGURE 5. Non-unique prime factorization. The products AB and CD of the two ring elements produces the same G . This is only possible if G is not connected.

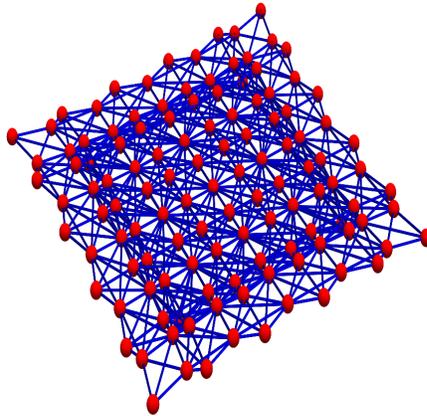


FIGURE 6. The connection graph of $G = L_5 \times L_5$ where L_n is the linear graph of length n is part of the connection graph of the discrete lattice $Z^1 \times Z^1$. The adjacency matrix A of the graph G seen here has the property that $L = 1 + A$ is the connection graph which is invertible.

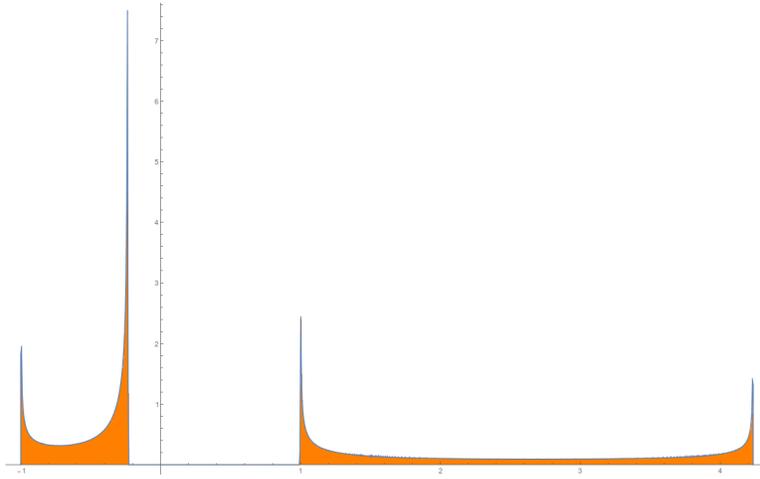


FIGURE 7. The density of states of the connection Laplacian of \mathbb{Z} has a mass gap at 0. We actually computed the eigenvalues of the connection Laplacian of C_{10000} which has a density of states close to the density of states of the connection Laplacian of \mathbb{Z} . The mass gap contains $[-1/5, 1/5]$.

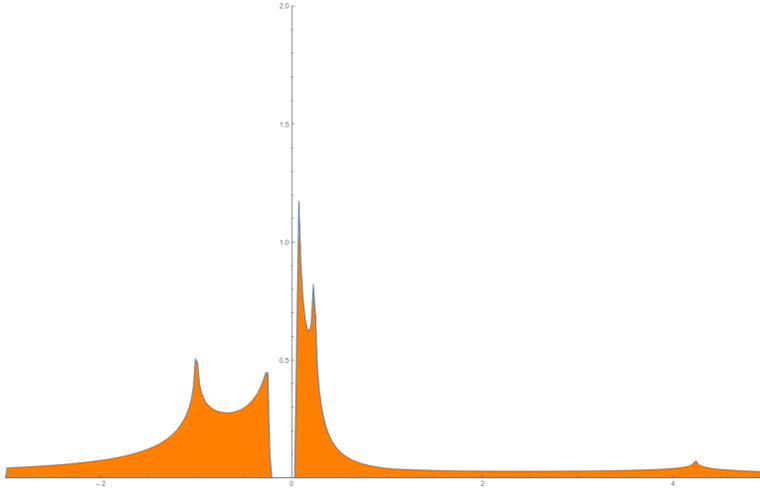


FIGURE 8. Part of the density of states of \mathbb{Z}^2 for finite dimensional approximations like $G = C_n \times C_n = C_{1000} \times C_{1000} = G_1 \times G_2$. The eigenvalues of $L(G)$ are the products $\lambda_i \lambda_j$ of the eigenvalues λ_i of $L(G_i)$. The mass gap contains $[-1/25, 1/25]$ which is independent of n .

CARTESIAN CLOSED CATEGORY

15.9. The goal of this appendix is to see the strong ring of simplicial complexes as a **cartesian closed category** and to ask whether it is a **topos**. As we have already finite products, the first requires to show the existence of exponentials. Cartesian closed categories are important in computer science as they have **simply typed lambda calculus** as language. Also here, we are close to computer science as we deal with a category of objects which (if they are small enough) can be realized in a computer. The elements can be represented as polynomials in a ring for example. We deal with a combinatorial category which can be explored without the need of finite dimensional approximations. It is part of combinatorics as all objects are finite.

15.10. In order to realize the ring as a category we need to define the **morphisms**, identifying an initial and terminal object (here $0 = \emptyset$ and $1 = K_1$) and show that **currying** works: there is an exponential object K^H in the ring such that the set of morphisms $C(G \times H, K)$ from $G \times H$ to K corresponds to the morphisms $C(G, K^H)$ via a **Curry bijection** seeing a graph $z = f(x, y)$ of a function of two variables as a graph of the function $x \rightarrow g_x(y) = f(x, y)$ from G to functions from H to K .

15.11. The existence of a product does not guarantee that a category is cartesian closed. Topological spaces or smooth manifolds are not cartesian closed but compactly generated Hausdorff spaces are. In our case, we are close to the category of finite sets which is cartesian closed. Like for finite sets we are close to computer science as procedures in computer programming languages are using the Curry bijection. Since the object K^H is in general very large, it is as for now more of theoretical interest.

15.12. The strong ring R resembles much the category of sets but there are negative elements in R . We can look at the category of **finite signed sets** which is the subcategory of zero dimensional signed simplicial complexes. Also this is a ring. It is isomorphic to \mathbb{Z} as a ring but the set of morphisms produces a category which has more structure than the ring \mathbb{Z} . This category of signed 0-dimensional simplicial complexes is Cartesian closed in the same way than the category of sets is. It is illustrative to see the exponential element 2^G is the set of all subsets of G . But this shows how exponential elements can become large.

15.13. A simplicial complex G as a finite set of non-empty sets, which is closed under the operation of taking non-empty subsets. [The usual definition is to looking at the **base set** $V = \bigcup_{x \in G} x$ and insisting that G is a set of subset of V with the property that if $y \subset x$ and $x \in G$ then $y \in G$ and also asking that $\{v\} \in G$ for every $v \in V$. Obviously the first given **point-free definition** is equivalent.] There is more structure than just the set of sets as the elements in G are partially ordered. A morphism between two simplicial complexes is not just a map between the sets but an **order preserving map**. This implies that the simplices are mapped into each other. We could rephrase this that a morphism induces a graph homomorphism between the barycentric refinements G_1 and H_1 but there are more graph homomorphisms in general on G_1 .

15.14. The class \mathcal{C} of simplicial complexes is a category for which the objects are the simplicial complexes and the morphisms are **simplicial functions**, functions which preserve simplices. In the point-free definition this means to look at functions from G to H which preserve the partial order. In order to be close to the definition of continuous functions (the morphisms in topological spaces) or measurable functions (the morphisms in measure spaces) one could ask that $f^{-1}(A)$ is a simplicial complex for every simplicial complex. As $f^{-1}(A)$ can be the empty complex this is fine. [If for some set $y \in H$, the inverse $f^{-1}(y)$ can be empty would not be good since simplicial complexes never contain the empty set. This is fine for the empty complex, which does not contain the empty set neither. But if we look at x as a complex by itself, then $f^{-1}(x)$ can be the empty complex. It is in general important to distinguish the elements x in the simplicial complex from the subcomplex x it represents, even so this is often not done.]

15.15. The category of simplicial complexes is close to the category of finite sets as the morphisms are just a subclass of all functions. There is an other essential difference: the product $G \times H$ of two simplicial complexes is **not** a simplicial complex any more in general, while the product $G \times H$ as sets is a set. This is also different from **geometric realizations** of simplicial complexes (called “polyhedra” in algebraic topology), where the product is a simplicial complex, the geometric realization of the Barycentric refinement of $G \times H$ will do.

15.16. [To compare with topologies O , where continuous maps have the property that $f^{-1}(x) \in O$ for every $x \in O$, morphisms of simplicial complexes have the property $f(x) \in H$ for $x \in G$. But only surjective morphisms also have the property that $f^{-1}(y) \in G$ for every $y \in H$,

the reason being that $f^{-1}(x)$ can be empty. A continuous map on topological spaces which is also open has the property that f and f^{-1} preserve the topology. A constant map for example is in general not open. We see that not only the object of simplicial complex is simpler but also that the morphisms are simpler.] It is better to therefore of a morphism f between simplicial complexes as a map for which both f and f^{-1} preserve sub simplicial complexes.

15.17. In order to work within the class of simplicial complexes (actually the special case of Whitney complexes of graphs), we had looked in [17] at the Barycentric refinements of Cartesian products and called this the Cartesian product. We had to live however with the consequence that the product $(G, H) \rightarrow (G \times H)_1$ is **not associative**: already $(G \times K_1)_1 = G_1$ is the Barycentric refinement of G . While the geometric realization of the Barycentric refinement G_1 is topologically equivalent to G , there is a problem with products as in the topological realization $|K_2 \times K_2| = |K_3|$ meaning that the arithmetic is not compatible. The geometric realization destroys the arithmetic. In the strong ring K_3 is a **multiplicative prime**, in the geometric realization, it is not; it decays as $K_2 \times K_2$.

15.18. Having enlarged the category to the strong ring, we have not only to deal with morphisms for simplicial complexes, we also have to say what the morphisms in the ring are. The definition is recursive with respect to the **degree** of a ring element, where the degree is the degree in the Stanley-Reisner polynomial representation. A map $G \rightarrow H$ is a morphism, if it is a morphism of simplicial complexes if G, H are simplicial complexes and if for every pair G, H in the ring, there is a pair A, B in the ring and morphisms $g : G \rightarrow A, h : H \rightarrow B$ such that $f(G \times H) = g(G) \times h(H)$. [By the way, the degree of a monomial in the Stanley-Reisner representation f_G only relates to the dimension if we G is prime, that is if G is a simplicial complex. In an product $A \times B$, the degree of a monomial is $c(A) + c(B)$, where c is the clique number. It is the clique number which is additive and not the dimension. The monomial $abcd$ in $(a + b + ab)(c + d + cd)$ for example belongs to a two-dimensional cell.]

15.19. The strong ring S is a ring and a category. But it is itself an element in the category of rings. The image of the map $\phi : G \rightarrow G'$ is a subring R of the Sabidussi ring of all graphs. The map ϕ is a ring isomorphism. If we think of S as a category, then R can be thought so too and ϕ is now a functor. This is nothing strange. Category is a universal language where objects of categories can be categories

themselves. A directed graph for example is a category, where the objects are the vertices and the morphisms are the directed edges.

15.20. The strong ring is also a **cartesian monoidal category**, a category with a notion of tensor product. The unit is the unit in the ring. It is also a **finitely complete category**, which is a category in which **pullbacks** exist: given any two ring elements G, H and two morphisms $g : G \rightarrow K, h : H \rightarrow K$, there is a subcomplex K of $G \times H$ such that for all $(x, y) \in K$, the equation $g(x) = h(y)$ holds. The subcomplex K is called a pullback.

15.21. The strong ring appears also to be a **topos** but we have not yet checked that. A topos is a cartesian closed category with a sub-object classifier. Examples of topoi are sets or the G -dynamical systems for a group G or the category of sheaves on a topological space. Topoi enjoy stability properties: the fundamental theorem of topos theory tells that a topos is stable under slicing, i.e that it is locally cartesian closed.

REFERENCES

- [1] B. Grünbaum. Polytopes, graphs, and complexes. *Bull. Amer. Math. Soc.*, 76:1131–1201, 1970.
- [2] R. Hammack, W. Imrich, and S. Klavžar. *Handbook of product graphs*. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, second edition, 2011. With a foreword by Peter Winkler.
- [3] W. Imrich and S. Klavzar. *Product graphs, Structure and recognition*. John Wiley and Sons, Inc. New York, 2000.
- [4] D.A. Klain and G-C. Rota. *Introduction to geometric probability*. Lezioni Lincee. Accademia nazionale dei lincei, 1997.
- [5] O. Knill. A graph theoretical Gauss-Bonnet-Chern theorem. <http://arxiv.org/abs/1111.5395>, 2011.
- [6] O. Knill. A discrete Gauss-Bonnet type theorem. *Elemente der Mathematik*, 67:1–17, 2012.
- [7] O. Knill. A graph theoretical Poincaré-Hopf theorem. <http://arxiv.org/abs/1201.1162>, 2012.
- [8] O. Knill. On index expectation and curvature for networks. <http://arxiv.org/abs/1202.4514>, 2012.
- [9] O. Knill. The McKean-Singer Formula in Graph Theory. <http://arxiv.org/abs/1301.1408>, 2012.
- [10] O. Knill. The theorems of Green-Stokes, Gauss-Bonnet and Poincare-Hopf in Graph Theory. <http://arxiv.org/abs/1201.6049>, 2012.
- [11] O. Knill. A Brouwer fixed point theorem for graph endomorphisms. *Fixed Point Theory and Applications*, 85, 2013.
- [12] O. Knill. The Dirac operator of a graph. <http://arxiv.org/abs/1306.2166>, 2013.
- [13] O. Knill. Classical mathematical structures within topological graph theory. <http://arxiv.org/abs/1402.2029>, 2014.
- [14] O. Knill. Curvature from graph colorings. <http://arxiv.org/abs/1410.1217>, 2014.
- [15] O. Knill. The graph spectrum of barycentric refinements. <http://arxiv.org/abs/1508.02027>, 2015.
- [16] O. Knill. The Jordan-Brouwer theorem for graphs. <http://arxiv.org/abs/1506.06440>, 2015.
- [17] O. Knill. The Künneth formula for graphs. <http://arxiv.org/abs/1505.07518>, 2015.
- [18] O. Knill. A Sard theorem for graph theory. <http://arxiv.org/abs/1508.05657>, 2015.
- [19] O. Knill. Universality for barycentric subdivision. <http://arxiv.org/abs/1509.06092>, 2015.
- [20] O. Knill. Gauss-Bonnet for multi-linear valuations. <http://arxiv.org/abs/1601.04533>, 2016.
- [21] O. Knill. On Fredholm determinants in topology. <https://arxiv.org/abs/1612.08229>, 2016.
- [22] O. Knill. On primes, graphs and cohomology. <https://arxiv.org/abs/1608.06877>, 2016.

ARITHMETIC OF GRAPHS

- [23] O. Knill. Some experiments in number theory.
<https://arxiv.org/abs/1606.05971>, 2016.
- [24] O. Knill. On Helmholtz free energy for finite abstract simplicial complexes.
<https://arxiv.org/abs/1703.06549>, 2017.
- [25] O. Knill. On the arithmetic of graphs.
<https://arxiv.org/abs/1706.05767>, 2017.
- [26] O. Knill. Sphere geometry and invariants.
<https://arxiv.org/abs/1702.03606>, 2017.
- [27] O. Knill. A case study in interaction cohomology.
<http://www.math.harvard.edu/knill/graphgeometry/papers/interactioncohomology.pdf>,
March, 18, 2016.
- [28] N. Levitt. The Euler characteristic is the unique locally determined numerical homotopy invariant of finite complexes. *Discrete Comput. Geom.*, 7:59–67, 1992.
- [29] H.P. McKean and I.M. Singer. Curvature and the eigenvalues of the Laplacian. *J. Differential Geometry*, 1(1):43–69, 1967.
- [30] J. Moser. *Selected chapters in the calculus of variations*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2003. Lecture notes by Oliver Knill.
- [31] G. Sabidussi. Graph multiplication. *Math. Z.*, 72:446–457, 1959/1960.
- [32] B. Simon. *Trace Ideals and their Applications*. AMS, 2. edition, 2010.
- [33] H. Toda. *Theory of nonlinear lattices*. Springer-Verlag, Berlin, 1981.
- [34] A.A. Zykov. On some properties of linear complexes. (russian). *Mat. Sbornik N.S.*, 24(66):163–188, 1949.

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA, 02138