

ONE CAN HEAR THE EULER CHARACTERISTIC OF A SIMPLICIAL COMPLEX

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ABSTRACT. We prove that the number p of positive eigenvalues of the connection Laplacian L of a finite abstract simplicial complex G matches the number b of even dimensional simplices in G and that the number n of negative eigenvalues matches the number f of odd-dimensional simplices in G . The Euler characteristic $\chi(G)$ of G therefore can be spectrally described as $\chi(G) = p - n$. This is in contrast to the more classical Hodge Laplacian H which acts on the same Hilbert space, where $\chi(G)$ is not yet known to be accessible from the spectrum of H . Given an ordering of G coming from a build-up as a CW complex, every simplex $x \in G$ is now associated to a unique eigenvector of L and the correspondence is computable. The Euler characteristic is now not only the potential energy $\sum_{x \in G} \sum_{y \in G} g(x, y)$ with $g = L^{-1}$ but also agrees with a logarithmic energy $\text{tr}(\log(iL))/2/(i\pi)$ of the spectrum of L . We also give here examples of isospectral but non-isomorphic abstract finite simplicial complexes. One example shows that we can not hear the cohomology of the complex.

1. INTRODUCTION

1.1. The energy theorem [5] identifies the Euler characteristic $\chi(G)$ of a finite abstract simplicial complex G as a potential theoretical energy

$$E(G) = \sum_{x \in G} \sum_{y \in G} g(x, y) ,$$

where $g = L^{-1}$ is the Green's function, an integer matrix. Is there a spectral interpretation of Euler characteristic $\chi(G) = \sum_x (-1)^{\dim(x)}$ for a self-adjoint Laplacian of a simplicial complex? We answer this question with yes. We are not aware of any other self-adjoint Laplacian for a simplicial complex which determines $\chi(G)$ from its spectrum.

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1.2. There is a non-selfadjoint matrix which produces $\chi(G)$: if Γ is an acyclic directed graph with m vertices, define $A(i, j) = 1$ if and only if (i, j) is not an edge and 0 else. Define the diagonal $D = \text{Diag}(x_1, \dots, x_m)$. In [8], the determinant formula $\det(1 + DA) = \sum_x x_1 x_2 \dots x_{|x|}$ is proven, where the sum is over all paths x in the graph and where $|x|$ denotes the number of vertices in x . Applying this to the graph where G is the vertex set and (x, y) is an edge if $x \subset y$, and $x_j = -1$, then $A(i, j) = -1$ if and only if i is not a subset of j . and 0 else, Bera and Muherjee note that this implies $\det(A) = (1 - \chi(G))$ [1]. As $\det(A)$ is the product of the eigenvalues of a matrix A defined by G , this is a spectral description. But the matrix A is not symmetric and in general has complex eigenvalues so that it can not be conjugated to a symmetric matrix.

1.3. Besides pointing out that the spectrum of the connection Laplacian L determines the Euler characteristic we also show that there is a canonical bijection between simplices of G ordered as a CW complex and the eigenvalues of L . Every simplex has so a well defined energy value and explicitly defined eigenvector of L in the Hilbert space. The simplices, the basic building blocks of a simplicial complex therefore have a particle feature. They manifest as eigenstates of the energy matrix $g = L^{-1}$. Our result extends to the strong ring generated by simplicial complexes. For these more general objects, still the number of positive minus the number of negative eigenvalues is the Euler characteristic. We already knew that the total energy $\chi(G)$ is a ring homomorphism from the ring to the integers and that the eigenvectors of the connection Laplacian are multiplicative. Now, an eigenvalue to $\lambda\mu$ of $G \times H$ can be associated to a specific product simplex $x \times y$ with $x \in G, y \in H$.

1.4. If L is the connection Laplacian of a finite abstract simplicial complex G defined by $L(x, y) = \text{sign}(|x \cap y|)$ then by definition, the **super trace** $\sum_{x \in G} \omega(x)L(x, x)$ of L is the Euler characteristic $\chi(G) = \sum_{x \in G} \omega(x) = b - f$, where $\omega(x) = (-1)^{\dim(x)} = (-1)^{|x|-1}$, where $b = |\{x \in G \mid |x| \in 2\mathbb{N} - 1\}|$ is the number of even-dimensional simplices and $f = |\{x \in G \mid |x| \in 2\mathbb{N}\}|$ is the number odd-dimensional simplices in G . The trace of L is $b + f = |G|$ is a spectral invariant. The super trace however is not unitary invariant: it does not provide a spectral description of $\chi(G)$. We can not see $\chi(G)$ from the eigenvalues of the Hodge Laplacian $H(G) = (d + d^*)^2$ yet. We know that H decomposes into blocks $H_k(G)$ which have as nullity the Betti numbers $b_k(G)$. By

Euler-Poincaré, one has then $\chi(G) = \sum_{k=0}^{\infty} (-1)^k b_k(G)$. If we look at the spectrum of H as a set of real numbers without any labels, we can not see whether an eigenvalue comes from odd or even forms.

1.5. The Dirac operator $D = (d + d^*)$ defines a super-symmetry between the nonzero eigenvalues of H restricted to the even and odd dimensional part of the exterior bundle. The symmetry is $f \rightarrow Df$. This implies the celebrated McKean-Singer formula $\chi(G) = \text{str}(\exp(-tH))$ which followed from the fact that $\text{str}(H^k) = 0$ for all $k > 0$ and $\text{str}(1) = \chi(G)$ [7, 3]. Still, also here, as the super trace refers to the $\omega(x)$, the Euler characteristic is not yet a true spectral property. Applying a unitary base change which exchanges some odd and even dimensional parts can change the super trace. There is an analogue McKean-Singer formula

$$\chi(G) = \text{str}(L^{-1}) = \text{str}(L^1) = \text{str}(L^0)$$

for L . But the lack of super symmetry for L does not allow us to find identities $\chi(G) = \text{str}(L^{-t})$ for t different from 0, 1 or -1 .

2. A SPECTRAL FORMULA

2.1. We formulate now a spectral interpretation of the Euler characteristic $\chi = \chi(G)$. Let $p = p(G)$ be the number of positive eigenvalues of the connection Laplacian L and let $n = n(G)$ be the number of negative eigenvalues of L . Let $b(G)$ be the number of even-dimensional simplices in G and $f(G)$ denote the number of odd-dimensional simplices in G .

Theorem 1 (Spectral formula for Euler characteristic). *For any complex G we have $p(G) = b(G)$ and $n(G) = f(G)$ so that $\chi = p - n$.*

2.2. As the proof will show, the result holds more generally for discrete CW complexes, where cells are inductively added to spheres. It also holds for products of simplicial complexes which are no simplicial complexes any more. This can appear strange at first as we are familiar with the Hodge situation, which features a complete symmetry between positive and negative energy states. While for the Hodge operator H which lives on the same Hilbert space, the harmonic forms are topologically interesting, it appears that the other eigenvalues and eigenvectors change under coordinate changes. For the connection operator L - which has no kernel - all eigenvectors appear to be of topological interest.

2.3. We have seen that the Hodge Laplacian H is “spectrally additive” in the sense that the eigenvalues of $H(A \times B)$ are the sum of the eigenvalues of $H(A)$ and $H(B)$ (as for the Laplacian in the continuum) and that L is “spectrally multiplicative” in that the eigenvalues of $L(A \times B)$ are the product of the eigenvalues of $L(A)$ and $L(B)$, which has no continuum analogue yet [6]. We see from this that the spectral formula goes over to the strong ring generated by simplicial complexes. An eigenvector of $A \times B$ which corresponds to a positive energy state can either be decomposed into two positive energy eigenstates or two negative eigenstates.

2.4. Example 1: Let $G = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$. Then

$$L = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

which has the eigenvalues $(1 \pm \sqrt{5})/2$, $(3 \pm \sqrt{13})/2$ and 1. There are $p = 3$ positive eigenvalues and $n = 2$ negative ones. This matches the Euler characteristic $\chi(G) = b - f = 3 - 2$.

Example 2: The cube graph complex $G = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 7\}, \{4, 6\}, \{4, 7\}, \{5, 8\}, \{6, 8\}, \{7, 8\}\}$ has a Laplacian L with eigenvalues $\{3 \pm \sqrt{10}, (2 \pm \sqrt{5})^{(3)}, (1 \pm \sqrt{2})^{(3)}, (-1)^{(5)}, 1\}$. There are $p = 8$ positive and $n = 12$ negative eigenvalues. (There are 4 negative eigenvalues -1 which do not match up naturally). There are $b = 8$ zero dimensional simplices and $f = 12$ one dimensional simplices so that $\chi(G) = 8 - 12 = -4$.

Example 3: For the cyclic complex $G = C_4 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\}$, the eigenvalues are $\{2 \pm \sqrt{5}, (1 \pm \sqrt{2})^{(2)}, 1, -1\}$, featuring four positive and four negative eigenvalues, matching $\chi(G) = 0$.

3. PROOF

3.1. We have seen that $g(x, x) = 1 - \chi(S(x))$, where $S(x)$ is the unit sphere of x in the graph $G_1 = (V, E)$, where $V = G$ and where $(a, b) \in E$ if and only if $a \subset b$ or $b \subset a$. What are the other entries $g(x, y)$? An element x in G is called a **facet**, if it is not contained in any other simplex y of G .

Lemma 1. *If x is a facet and $y \subset x$, then $g(x, y) = \omega(y)$.*

Proof. We can rearrange the simplices so that the facet x is the last one. The vector $\{g(x, y) \mid y = 1, \dots, n\}$ is then the last column of g . The Cramer formula tells that $g(x, y)$ is $\det(L(y))/\det(L)$, where $L(y)$ is the matrix L in which column x has been replaced by $e(y)$. We prove the statement by induction with respect to the number of elements $|G|$ of G . Now make a Laplace expansion with respect to row x . This gives a sum, using induction as we now have a simplicial complex $G \setminus \{x\}$ and $\sum_{z \subset x, z \in S(y) \cup \{y\}} w(z) = 1 - \chi(S(y)) = \omega(y)$. \square

3.2. The proof of the theorem is done by induction on the number $|G|$ of elements in G . As in the proof of the unimodularity theorem [4], we prove the result more generally for discrete CW complexes, where we have a finer built-up and so a stronger induction assumption during the proof. Given a connection Laplacian L to a complex G ; if we add a new cell x and consider the complex $H = G +_A x$ in which x is attached to A . Let L be the connection Laplacian of G and K the connection Laplacian of H . Define $K(t)$ as the matrix defined by $K(t)(y, z) = K(y, z)$ if z is different from x and $K(t)(y, x) = tK(y, x)$ if $y \neq x$.

Proposition 1. $\det(K(t)) = (1 - t \cdot \chi(A))\det(L)$.

Proof. Make a Laplace expansion with respect to the last column. The formula

$$\det \begin{pmatrix} L_{11} & L_{12} & \cdot & \cdot & \cdot & L_{1n} & t_1 K_{1,x} \\ L_{21} & L_{22} & \cdot & \cdot & \cdot & \cdot & t_2 K_{2,x} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_3 K_{3,x} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ L_{n1} & \cdot & \cdot & \cdot & \cdot & L_{nn} & t_n K_{n,x} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \det(L) - \sum_{y \subset x} t_y \det(L) \omega(y)$$

immediately proves the statement. Just set all variables $t_k = t$. We have used the Lemma rephrased that the minor $\det(L[y, x])$ is $\omega(y)$. \square

3.3. We see that during the deformation $K(0)$ to $K(1)$, when adding an even-dimensional cell, all eigenvalues stay on the same side of 0. When adding an odd dimensional cell, the new eigenvalue 1 which is present in $K(0)$ but not present in L crossed from $t = 0$ to $t = 1$ from the positive to the negative side and no other eigenvalue crosses, as the characteristic polynomial has a simple root there. The formula also shows that if $\chi(A) = 0$, then $\det(K(t))$ is constant and that $\chi(K(t))$ changes linearly in all cases.

Corollary 1. *a) If $\chi(A) = 0$, then the matrix $K(t)$ is invertible for all t . The number of positive eigenvalues of K has increased by 1 and the number of negative eigenvalues has remained the same.*

b) If $\chi(A) = 2$, then the matrix $K(t)$ is not invertible exactly for $t = 1/2$ and has a one-dimensional kernel. The number of positive eigenvalues of K and L are the same. The number of negative eigenvalues has increased by 1.

3.4. The proof of the result does not directly follow from the **unimodularity theorem** $|\det(L)| = 1$ or the **energy theorem** $\sum_{x,y} g(x,y) = \chi(G)$, if $g = L^{-1}$. We have shown the unimodularity theorem for discrete CW complexes, which generalize simplicial complexes. Adding a cell x along a sphere $S(x)$ multiplies the determinant by $1 - \chi(S(x)) = (-1)^{\dim(x)}$. So, if we add an odd dimensional simplex, the determinant gets multiplied by -1 meaning that an other eigenvalue -1 is added. This does not prove the theorem yet as it could be possible for example that the addition renders two negative eigenvalues positive or even more complicated changes of the spectrum happen.

3.5. We still have not completely cleared up the meaning of all the matrix entries $g(x,y)$. We know $g(x,x) = 1 - \chi(S(x))$ as this was essential for the energy theorem. Let $W^+(x,y) = W^+(x) \cap W^+(y)$ denote the intersections of the stable manifolds of x and y . Here $W^+(x)$ is the set of simplices containing x but not being equal to x . And $W^-(x)$ is the set of simplices contained in x but not equal to x . We know therefore that $g(x,y) = (1 - \chi(W^+(x,y)))(1 - \chi(W^-(x,y)))$ holds if $x = y$ or $y \subset x$ and x is a facet. In the later case this is $\omega(y)$. We also know that if $g(x,x) = 0$ then $g(x,y) = 0$ if $y \subset x$. We have tried in various ways to modify the in general incorrect formula $g(x,y) = (1 - \chi(W^+(x,y)))(1 - \chi(W^-(x,y)))$ to get an identity which works for all x,y .

3.6. The modification $g(x,y) = (\chi(x \cap y) - \chi(W^+(x,y)))(\chi(x \cap y) - \chi(W^-(x,y)))$ comes already closer as it covers the cases we know, like when there is not connection at all between x and y . It still fails in general. More experimentation is needed. An other approach to the puzzle is to give meaning to the minors $\det(L[x,y])$ as $g(x,y) = (-1)^{i(x)+i(y)} \det(L[x,y]) / \det(L)$, where $i(x)$ is the position in the ordering of G defining the matrix L . The determinant $\det(L[x,y])$ is a signed sum over all one dimensional paths where one of the connectivity components goes from x to y . Representing Green function values as path integrals is a familiar theme in mathematical physics.

4. COROLLARIES

4.1. As the number of positive eigenvalues corresponds to the number of even-dimensional complete sub-complexes and the number of negative eigenvalues corresponds to the number of odd-dimensional complete sub-complexes, we know that L is never positive or negative definite if the dimension of the complex G is positive:

Corollary 2 (There is anti-matter). *For any CW complex of positive dimension, the connection Laplacian is never positive nor negative definite.*

Proof. A positive dimensional simplicial complex contains both even and odd dimensional simplices. \square

4.2. We can write the sign function $\text{sign}(x)$ as the imaginary part of $\log(ix)(2/(i\pi))$. From the unimodularity theorem follows that

$$\sum_k \log(i\lambda_k) = i\left(\frac{\pi}{2}\right)\chi(G)$$

if the branch of the log has been chosen so that $\arg(z) \in [-\pi/2, \pi/2]$ for $\text{Re}(z) \geq 0$ or equivalently

Corollary 3 (Potential theoretical energy).

$$\chi(G) = \frac{2}{i\pi} \text{tr}(\log(iL)) .$$

5. ISOSPECTRAL COMPLEXES

5.1. The question whether the spectrum of $L(G)$ determines G is equivalent to the question whether the adjacency spectrum of the connection matrix determines G . The topic of finding relations between a geometry space and the spectrum of its Laplacian is a classical one [2] and has been ported to finite geometries like graphs. We give in this section examples of non-isospectral complexes with respect to the connection Laplacian.

5.2. As the eigenvalues of L do not determine the complex G one can ask whether the eigenvectors do. We know that the eigenvalues and eigenvectors together determine L and so G . Now we can look for the class of unitary matrices U for which the operator UDU^T has an inverse g for which the sum $\sum_{x,y} g(x,y)$ is the Euler characteristic.

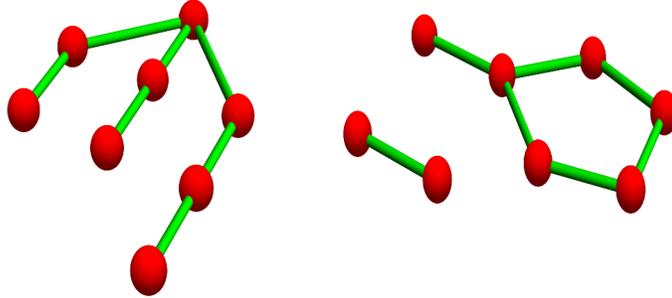


FIGURE 1. **Example 1:** The complex $G = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{1, 2\}, \{1, 3\}, \{2, 6\}, \{2, 7\}, \{4, 5\}, \{6, 8\}, \{7, 4\}\}$ and the complex $H = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{1, 2\}, \{1, 5\}, \{1, 7\}, \{2, 8\}, \{3, 4\}, \{5, 6\}, \{8, 6\}\}$ are isospectral.

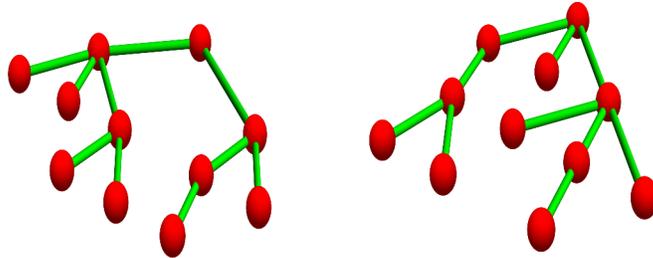


FIGURE 2. **Example 2:** The complex $G = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{1, 2\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{4, 6\}, \{5, 8\}, \{5, 11\}, \{6, 7\}, \{6, 10\}, \{7, 9\}\}$ and the complex $H = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{1, 2\}, \{1, 3\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 7\}, \{7, 8\}, \{7, 9\}, \{7, 11\}, \{9, 10\}\}$ are isospectral.

5.3. Remarkably, the Wu characteristic

$$\omega(G) = \omega_2(G) = \sum_{x, y, x \cap y \neq \emptyset} \omega(x)\omega(y)$$

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of the two complexes in these examples agree. In example 1), the Wu characteristic is $\omega_2 = 1$. Also the higher Wu characteristic agree: $\omega_3 = -5$ and $\omega_4 = 13$. In example 2), there are both the same 9. Also the higher Wu characteristic $\omega_3(G) = -35, \omega_4(G) = 105$ both do agree. This prompts the question:

Question: are the Wu characteristics ω_k determined by the spectrum? We remind that $\omega_k(G)$ is the sum over all products $\omega(x_1) \cdots \omega(x_k)$ of simultaneously intersecting simplices x_1, \dots, x_k in G . So far we only know the answer for the Euler characteristic $\omega_1(G) = \chi(G)$.

REFERENCES

- [1] S. Bera and S.K. Mukherjee. Combinatorial proofs of some determinantal identities. *Linear and Multilinear algebra*, 0(0):1–9, 2017.
- [2] M. Kac. Can one hear the shape of a drum? *Amer. Math. Monthly*, 73:1–23, 1966.
- [3] O. Knill. The McKean-Singer Formula in Graph Theory. <http://arxiv.org/abs/1301.1408>, 2012.
- [4] O. Knill. On Fredholm determinants in topology. <https://arxiv.org/abs/1612.08229>, 2016.
- [5] O. Knill. Sphere geometry and invariants. <https://arxiv.org/abs/1702.03606>, 2017.
- [6] O. Knill. The strong ring of simplicial complexes. <https://arxiv.org/abs/1708.01778>, 2017.
- [7] H.P. McKean and I.M. Singer. Curvature and the eigenvalues of the Laplacian. *J. Differential Geometry*, 1(1):43–69, 1967.
- [8] R. P. Stanley. A matrix for counting paths in acyclic digraphs. *Journal of Combinatorial Theory, Series A*, 74:169–172, 1996.

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