Math Table: Polishing Euler’s Gem

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Abstract. We give a combinatorial proof of the Euler gem formula telling that a $d$-sphere has Euler characteristic $1 + (-1)^d$ and classify Platonic $d$-spheres.

1. The Euler characteristic of spheres

1.1. A finite simple graph $G = (V, E)$ consists of two finite sets, the vertex set $V$ and the edge set $E$ which is a subset of all sets $e = \{a, b\} \subset V$ with cardinality two. A graph is also called a network, the vertices are the nodes and the edges are the connections. A subset $W$ of $V$ generates a subgraph $(W, F)$ of $G$, where $F = \{\{a, b\} \in E \mid a, b \in W\}$. Given $G$ and $x \in V$, its unit sphere is the sub graph generated by $S(x) = \{y \in V \mid \{x, y\} \in E\}$. The unit ball is the sub graph generated by $B(x) = \{x\} \cup S(x)$. Given a vertex $x \in V$, the graph $G - x$ with $x$ removed is generated by $V \setminus \{x\}$. We can identify $W \subset V$ with the subgraph it generates in $G$.

1.2. The empty graph $0 = (\emptyset, \emptyset)$ is called the $(-1)$-sphere. The 1-point graph $1 = (\{1\}, \emptyset) = K_1$ is the smallest contractible graph. Inductively, a graph $G$ is contractible, if it is either 0 or if there exists $x \in V$ such that both $G - x$ and $S(x)$ are contractible. As seen by induction, all complete graphs $K_n$ and all trees are contractible. Complete subgraphs are also called simplices. Inductively a graph $G$ is called a $d$-sphere, if it is either 0 or if every $S(x)$ is a $(d - 1)$-sphere and if there exists a vertex $x$ such that $G - x$ is contractible.

1.3. Let $v_k$ denote the number of complete subgraphs $K_{k+1}$ of $G$. The vector $(v_0, v_1, \ldots)$ is the $f$-vector of $G$ and $\chi(G) = v_0 - v_1 + v_2 - \ldots$ is the Euler characteristic of $G$.

Theorem 1. If $G$ is a $d$-sphere, then $\chi(G) = 1 + (-1)^d$.

1.4. To prove this, we formulate two lemmas and a proposition. Given two subgraphs $A, B$ of $G$, the intersection $A \cap B$ as well as the union $A \cup B$ are sub graphs of $G$.

Lemma 1 (Valuation formula). $\chi(A) + \chi(B) = \chi(A \cap B) + \chi(A \cup B)$ for any two subgraphs $A, B$ of $G$.

Proof. Each of the functions $v_k(A)$ counting the number of $k$-dimensional simplices in a subgraph $A$ satisfies the identity. The Euler characteristic is a linear combination of such valuations and therefore satisfies the identity. □
1.5. A graph $G$ is a **unit ball**, if there exists a vertex $x$ in $G$ such that $B(x) = G$. We also say that $G$ is a **cone extension** of $S(x)$.

**Lemma 2** (Unit ball lemma). Every unit ball $B$ is contractible and has $\chi(B) = 1$.

**Proof.** Use induction with respect to the number of vertices in $B = B(x)$. It is true for $G = K_1$. Induction step: given a unit ball $B(x)$. Pick $y \in S(x)$ for which $B(y)$ is not equal to $B(x)$ (if there is none, then $B(x) = K_n$ for some $n$ and $B(x)$ is contractible with Euler characteristic 1). Now, both $B(y), B(y) - x$ and $S(x)$ are smaller balls so that by induction, all are contractible with Euler characteristic 1. As both $B(x) \setminus y$ and $S(y)$ are contractible, also $B(x)$ is contractible. By the valuation formula, $\chi(B(x)) = \chi(B(x) - y) + \chi(B(y)) - \chi(S(x)) = 1 + 1 - 1 = 1$.

**Proposition 1** (Contractibility). If $G$ is contractible then $\chi(G) = 1$.

**Proof.** Pick $x \in V$ for which $S(x)$ and $G - x$ are both contractible. By induction, $\chi(G - x) = 1$ and $\chi(S(x)) = 1$. By the unit ball lemma, $\chi(B(x)) = 1$. By the valuation lemma, $\chi(G) = \chi(B(x)) + \chi(G - x) - \chi(S(x)) = 1 + 1 - 1 = 1$.

1.6. Lets now prove the theorem.

**Proof.** For $G = 0$ we have $\chi(G) = 0$. This is the induction assumption. Assume the formula holds for all $d$-spheres. Take a $(d+1)$-sphere $G$ and pick a vertex $x$ for which $G - x$ is contractible. Now, since $S(x)$ is a $d$-sphere, $\chi(G) = \chi(G - x) + \chi(B(x)) - \chi(S(x)) = 1 + 1 - (1 + (-1)^d) = 1 + (-1)^{d+1}$.

2. **Platonic spheres**

2.1. A **Platonic $d$-sphere** is a $d$-sphere for which all unit spheres are isomorphic to a fixed Platonic $(d - 1)$-sphere $H$. The **curvature** $K(x)$ of a vertex is defined as $K(x) = \sum_{k=0} (-1)^k v_k(x)/(k+1)$, where $v_k(x)$ is the number of $k$-dimensional simplices $z$ in $G$ which contain $x$.

**Lemma 3** (Discrete Gauss-Bonnet). $\sum_{x \in V} K(x) = \chi(G)$.

**Proof.** By definition, $\chi(G) = \sum_{x \in G} \omega(x)$. Now distribute the charge $\omega(x)$ from $x$ equally to all zero dimensional parts of $x$.

2.2. For a triangle-free graph, $K(x) = v_0 - v_1/2 = 1 - \deg(x)/2$. For a 2-sphere we have $K(x) = v_0 - v_1/2 + v_2/3 = 1 - \deg(x)/6$, where $\deg(x)$ is the **vertex degree**.

**Theorem 2.** There exists exactly one $d$-sphere except for $d = 1, d = 2$ and $d = 3$. For $d = 1$ there are $\infty$ many, for $d = 2$ and $d = 3$ there are 2.

**Proof.** $d = -1, 0, 1$ are clear. For $d = 2$, the curvature $K(x) = 1 - V_0/2 + V_1/3 - V_2/3$ is constant adding up to 2. It is either 1/3 or 1/6. For $d = 3$, where each $S(x)$ must be either the octahedron or icosahedron, $G$ is the 16 cell or 600 cell. For $d = 4$, by Gauss-Bonnet, $K(x)$ add up to 2 and be of the form $L/12$. For $L = 1$, there exists the 4-dimensional cross polytope with $f$-vector $(10, 40, 80, 80, 32)$. There is no 4-sphere, for which $S(x)$ is the 600-cell as the $f$-vector of it is $(120, 720, 1200, 600)$. We would get $K(x) = 1 - 120/2 + 720/3 - 1200/4 + 600/5 = 1$ requiring $|V| = 2$ and $\dim(G) \leq 1$.