MATH TABLE: POLISHING EULER'S GEM

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ABSTRACT. We give a combinatorial proof of the Euler gem formula telling that a d-sphere has Euler characteristic $1 + (-1)^d$ and classify Platonic d-spheres.

1. The Euler characteristic of spheres

1.1. A **finite simple graph** $G = (V, E)$ consists of two finite sets, the **vertex set** V and the **edge set** E which is a subset of all sets $e = \{a, b\} \subset V$ with cardinality two. A graph is also called a **network**, the vertices are the **nodes** and the edges are the **connections**. A subset W of V **generates** a subgraph (W, F) of G, where $F = \{\{a, b\} \in E \mid a, b \in W\}$. Given G and $x \in V$, its **unit sphere** is the sub graph generated by $S(x) = \{y \in V | \{x, v\} \in E\}$. The **unit ball** is the sub graph generated by $B(x) = \{x\} \cup S(x)$. Given a vertex $x \in V$, the graph $G - x$ **with** x **removed** is generated by $V \setminus \{x\}$. We can identify $W \subset V$ with the subgraph it generates in G.

1.2. The empty graph $0 = (\emptyset, \emptyset)$ is called the (-1)**-sphere**. The 1-point graph $1 = (\{1\}, \emptyset) = K_1$ is the smallest contractible graph. Inductively, a graph G is **contractible**, if it is either 1 or if there exists $x \in V$ such that both $G - x$ and $S(x)$ are contractible. As seen by induction, all **complete graphs** K_n and all trees are contractible. Complete subgraphs are also called **simplices**. Inductively a graph G is called a d**-sphere**, if it is either 0 or if every $S(x)$ is a $(d-1)$-sphere and if there exists a vertex x such that $G - x$ is contractible.

1.3. Let v_k denote the number of complete subgraphs K_{k+1} of G. The vector $(v_0, v_1, \dots)$ is the f**-vector** of G and $\chi(G) = v_0 - v_1 + v_2 - \dots$ is the **Euler characteristic** of G.

Theorem 1. *If G is a d-sphere, then $\chi(G) = 1 + (-1)^d$.*

1.4. To prove this, we formulate two lemmas and a proposition. Given two subgraphs A, B of G, the **intersection** $A \cap B$ as well as the **union** $A \cup B$ are sub graphs of G.

Lemma 1 (Valuation formula)**.** *$\chi(A) + \chi(B) = \chi(A \cap B) + \chi(A \cup B)$ for any two subgraphs A, B of G.*

Proof. Each of the functions $v_k(A)$ counting the number of k-dimensional simplices in a subgraph A satisfies the identity. The Euler characteristic is a linear combination of such valuations and therefore satisfies the identity. $\square$

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1.5. A graph G is a **unit ball**, if there exists a vertex x in G such that $B(x) = G$. We also say that G is a **cone extension** of $S(x)$.

Lemma 2 (Unit ball lemma)**.** *Every unit ball B is contractible and has $\chi(B) = 1$.*

Proof. Use induction with respect to the number of vertices in $B = B(x)$. It is true for $G = K_1$. Induction step: given a unit ball $B(x)$. Pick $y \in S(x)$ for which $B(y)$ is not equal to $B(x)$ (if there is none, then $B(x) = K_n$ for some n and $B(x)$ is contractible with Euler characteristic 1). Now, both $B(y), B(y) - x$ and $S(x)$ are smaller balls so that by induction, all are contractible with Euler characteristic 1. As both $B(x) \setminus y$ and $S(y)$ are contractible, also $B(x)$ is contractible. By the valuation formula, $\chi(B(x)) = \chi(B(x) - y) + \chi(B(y)) - \chi(S(x)) = 1 + 1 - 1 = 1$. □

Proposition 1 (Contractibility)**.** *If G is contractible then $\chi(G) = 1$.*

Proof. Pick $x \in V$ for which $S(x)$ and $G - x$ are both contractible. By induction, $\chi(G - x) = 1$ and $\chi(S(x)) = 1$. By the unit ball lemma, $\chi(B(x)) = 1$. By the valuation lemma, $\chi(G) = \chi(B(x)) + \chi(G - x) - \chi(S(x)) = 1 + 1 - 1$. □

1.6. Lets now prove the theorem.

Proof. For $G = 0$ we have $\chi(G) = 0$. This is the induction assumption. Assume the formula holds for all d-spheres. Take a $(d+1)$-sphere G and pick a vertex x for which both $S(x)$ and $G - x$ are contractible. Now, $\chi(G) = \chi(G - x) + \chi(B(x)) - \chi(S(x)) = 1 + 1 - (1 + (-1)^d) = 1 + (-1)^{d+1}$. □

2. Platonic spheres

2.1. A **Platonic d-sphere** is a d-sphere for which all unit spheres are isomorphic to a fixed Platonic $(d-1)$-sphere H. The **curvature** $K(x)$ of a vertex is defined as $K(x) = \sum_{k=0}(-1)^k v_k(x)/(k+1)$, where $v_k(x)$ is the number of k-dimensional simplices z in G which contain x.

Lemma 3 (Discrete Gauss-Bonnet)**.** $\sum_{x \in V} K(x) = \chi(G)$.

Proof. By definition, $\chi(G) = \sum_{x \in G} \omega(x)$. Now distribute the charge $\omega(x)$ from x equally to all zero dimensional parts of x. □

2.2. For a triangle-free graph, $K(x) = v_0 - v_1/2 = 1 - \deg(x)/2$. For a 2-sphere we have $K(x) = v_0 - v_1/2 + v_2/3 = 1 - \deg(x)/6$, where $\deg(x)$ is the **vertex degree**.

Theorem 2. *There exists exactly one d-sphere except for $d = 1, d = 2$ and $d = 3$. For $d = 1$ there are ∞ many, for $d = 2$ and $d = 3$ there are 2.*

Proof. $d = -1, 0, 1$ are clear. For $d = 2$, the curvature $K(x) = 1 - V_0/2 + V_1/3 - V_2/3$ is constant adding up to 2. It is either 1/3 or 1/6. For $d = 3$, where each $S(x)$ must be either the octahedron or icosahedron, G is the 16 cell or 600 cell. For $d = 4$, by Gauss-Bonnet, $K(x)$ add up to 2 and be of the form $L/12$. For $L = 1$, there exists the 4-dimensional cross polytope with f-vector $(10, 40, 80, 80, 32)$. There is no 4-sphere, for which $S(x)$ is the 600-cell as the f-vector of it is $(120, 720, 1200, 600)$. We would get $K(x) = 1 - 120/2 + 720/3 - 1200/4 + 600/5 = 1$ requiring $|V| = 2$ and $\dim(G) \leq 1$. □

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