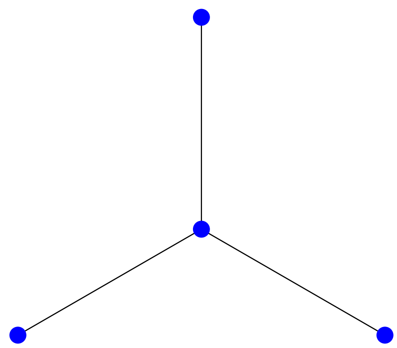
Networks and Matrices

Assume you have three friends who do not know each other. This defines a network containing 4 nodes in which you are the central node. One calls this network also a **graph**. In this case it has 4 vertices and 3 edges.

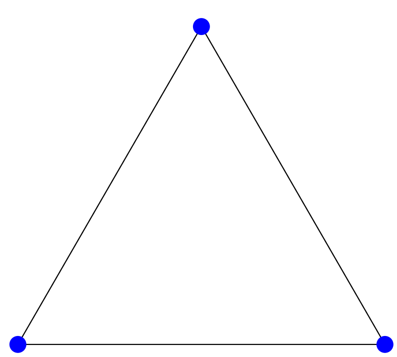


A graph can be encoded by its **adjacency matrix** A. The encoding is to put $A_{ij} = 1$, if person i is connected to person j. We do not connect a person to itself so that the matrix has zeros in the diagonal.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} .$$

You see that the rows appear several times. The matrix has zero determinant. It also follows from the fact that there is no closed loop going through all this network visiting each person exactly once and passing from one to the other if they are friends. In other words, in the Leibniz sum of the determinant, there is not even one single term. Note that one could also set up a network, where friendship relations can go also one way. This is called a **directed graph** or **digraph**. We deal here with undirected graphs and call them **finite simple graphs**.

Here is an other network, representing a network of three friends:



Its adjacency matrix is

$$A(G) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} .$$

Its determinant is 2. There are two "patterns", cyclic paths through the network visiting each node once.

Why are determinants important? You might have seen that they provide explicit formulas for solutions of systems of linear equations or that they are used to define eigenvalues as the roots of the characteristic polynomial.

But first of all, they came historically decades before matrices were defined. Indeed, the word "matrix" comes from **"mater"** which is latin for mother. The mother of what? They are the mother of determinants since they produce determinants!

The determinant of a matrix A is a natural function on the space of matrices. One main reason is the **Cauchy-Binet multiplication formula**

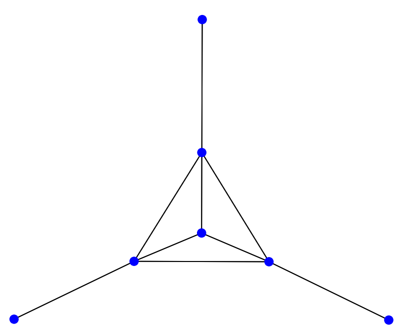
$$\det(AB) = \det(A)\det(A) ,$$

found in 1812 and which shows that det is compatible with multiplication. It is curious for example that in 1812, when the product formula was found independently by Cauchy and Binet, there were no matrices around yet.

Cliques and Connection Graphs

A subgraph of a graph G is called a **clique** or **simplex** or **complete sub graph** if each vertex is connected to each other in that graph. This corresponds to a group of people in which each one knows each other. A single person is a clique, but also a group of two friends as well as a group of three friends. We can now look at the individual cliques as vertices of a new graph and connect two different cliques if they have a mutual common friend. This new graph is called the **connection graph** of G.

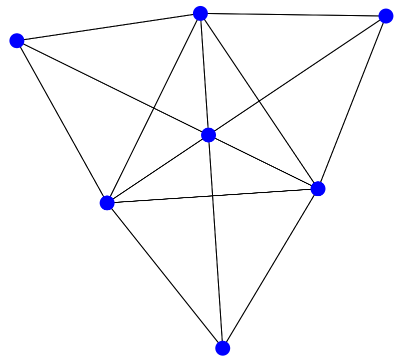
Here is the connection graph of the star graph:



Its adjacency matrix is

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

It has determinant 0. You can see this by looking at the zero 0×0 block.
And here is the connection graph G' of the triangle $G = K_3$. It also has 7 vertices.



Its adjacency matrix of G' is

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} .$$

It has determinant 6. The Leibniz definition of the determinant means here to sum over the 90 of the 7!=5040 nonzero possible patterns $A_{1,\pi(1)} \cdots A_{7,\pi(7)}$ and adding up their signatures. The result is $\det(A) = 6$. Of course, in linear algebra we learn to do that more effectively using row reduction. For combinatorics, it is useful to see determinants as a sum over permutations.

Now lets look at the Fredholm matrices. For the connection graph G' of the star graph G, it is

$$1 + A(G') = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

Its determinant is -1. For the connection graph G' of the triangle graph G, the Fredholm matrix is

$$1 + A(G') = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} .$$

Its determinant is again -1. Coincidence? If you see coincidences like that, it is good to have a computer and try many different cases. In February, 2016, I noticed first that the Fredholm determinant of connection graphs always either have determinant 1 or -1. This is surprising as it implies that the inverse of the Fredholm matrix of a connection graph always has integer entries. Lets look at that next.

Unimodular Matrices

The space of $n \times n$ integer matrices is also called $M(n, \mathbb{Z})$. It contains $GL(n, \mathbb{Z})$, the **general linear group** above the integers. The inverse of an integer $n \times n$ matrix is not an integer matrix any more in general since the entries can become fractions. Lets look at the 2×2 case, where a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has the inverse

$$A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} /(ad - bc)$$

We see here that the determinant matters. Indeed, there is a general **Laplace formula** for the inverse of a matrix which involves determinants of minors. The entry of $A^{-1}_{ij}\det(A)$ is $(-1)^{i+j}$ times the **minor** M_{ji} (the determinant of the matrix where row j and column i has been deleted). We see that if $\det(A) = \pm 1$, then the inverse matrix has integer entries. Related is **Cramer's rule**, found by the Swiss mathematician **Gabriel Cramer (1704-1752)**. It is an explicit formula for $\vec{x}$ if $A\vec{x} = \vec{b}$. The Laplace formula and Cramer rules are equivalent since $\vec{x} = A^{-1}\vec{b}$ and knowing solutions $A\vec{x} = e_j$ gives columns $\vec{x}$ of A^{-1}.


Gabriel Cramer and Pierre-Simon Laplace

The space of unimodular matrices $GL(n, \mathbb{Z})$ however form a group. And we can state a theorem! Here it is:

> The Fredholm matrix of the adjacency matrix of a connection graph is always unimodular.

A question is now of course, why this is so and what it means. The proof is done but is quite complicated still but hopefull appears in the next couple of weeks. We have two type of graphs now, one class for which the connection Fredholm determinant is equal to 1 and an other class, for which the connection Fredholm determinant is equal to -1. Lets find out how to characterize the classes. To do so, we experimented a lot more and correlated with other properties of graphs. After some search, (of course with the assistance of a computer) the following appeared (actually only in September 2016):

> The Fredholm determinant of a connection graph G' is 1 if there are an even number of cliques in G with even number of members and -1 if there is an odd number of cliques in G with an even number of members.

In other words, it is the cardinality of the number of odd dimensional cliques in G which matters. This implies that the log of the determinant is a valuation. The value for the union is the sum of the values of the individual graphs minus the value of the intersection.

Some Genealogy

The name Fredholm determinant was used by **John Archibald Wheeler** (1911-2008) in physics to determine scattering and cross sections [10]. Wheeler is given credit for spreading the name "black hole" in a lecture of 1967 (but it was used by a journalist **Ann Ewing** (1921-2010) in an article "Black Holes in space" earlier in 1964 [2].) Wheeler is the doctor father of **Richard Feynman** and **Arthur Wightman** who in turn was the doctor father of my doctor father **Oscar Lanford** as well as the doctor father of my postdoc mentors **Barry Simon** at Caltech and **Raphael de la Llave** at UT.

By the way, the book "Trace ideals" of Simon [7] is the best reference about Fredholm determinants.

John Wheeler and Ann Ewing

Bowen-Lanford Zeta function

The **Bowen-Lanford Zeta function** of a shift transformation T on the set of all sequences on a finite alphabet V is defined as

$$\zeta(z) = \exp(\sum_{n=1}^{\infty} z^n N_n/n) ,$$

where N_n is the number of fixed points of T^n. [1] The function therefore encodes the number of periodic points of the transformation. One can extend this in various way like looking at smooth maps on a compact manifold. Far reaching generalizations were given by Artin-Mazur and Ruelle.

One is often interested in closed invariant subsets of T obtained by restricting the possible words. This can be encoded by a finite graph. An edge (a, b) in that graph tells that the transition $a \to b$ and $b \to a$ is possible. Using $N_n = \text{tr}(A^n)$ we see that that $\sum_{n=1}^{\infty} z^n N_n/n = -\text{tr}(\log(1 - zA)) = -\log(\det(1 - zA)$ and $\zeta(z) = 1/\det(1 - zA)$.

It is more exciting! If we look at the set of all possible strings as a space and the shift T as a dynamical system on this space, then we can look at prime orbits. For example, the sequence ....[312312][312312]... is 6 periodic, but it is not prime as it is also ...[312][312][312].... If we sum over all possible prime orbits x of T and denote by $|x|$ its length, then $\zeta(z) = \prod_{x \text{ prime}}(1 - z^{|x|})^{-1}$. With a change of variables $p = e^{|x|}$ and $s = -\log(z)$ we can write this as an **Ihara** type zeta function

$$\prod_{x \text{ prime}} (1 - p(x)^{-s})^{-1}$$

which looks a lot now like the **Riemann zeta function** $\prod_p (1 - p^{-s})^{-1}$. See [4, 6]. The **prime number theorem** telling that the number of primes $\leq x$ is about $x/\log(x)$ has an analogue in graph theory: the number of prime orbits $\leq x$ is about λ_1^n/n where λ_1 is the largest Perron-Frobenius eigenvalue. Pollicott calls this the **Prime Graph Theorem**. By the way, there is a whole zoo of zeta functions. Several dozen of them [9]. See also the book [8].

Rufus Bowen and Oscar Lanford

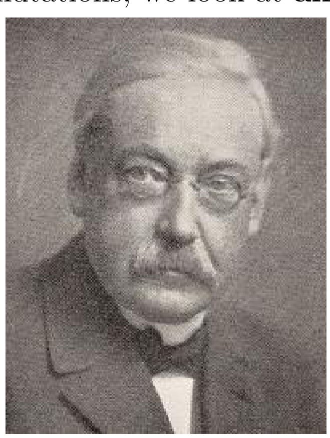
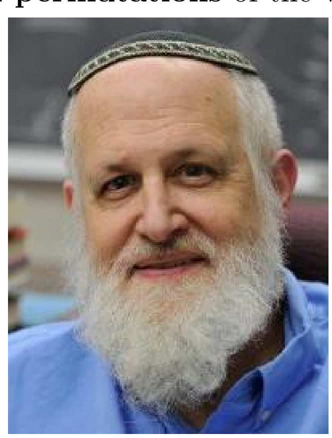
Audrey Terras and Mark Pollicott

Fredholm Determinants

The **Fredholm determinant** of a matrix A is defined as $\det(1 + A)$, where 1 is the identity matrix and det denotes the usual determinant.

Why are Fredholm determinants important? Lets look at a graph G and its adjacency matrix A. By definition, the determinant $\det(A)$ is a sum over all **cyclic permutations** of the vertex set of V which can occur.

Now, if we look at $\det(1 + A)$, then this allows for orbits to stay at some vertex. In other words, instead of all cyclic permutations, we look at **all permutations** of the vertex set of V.

Erik Fredholm and Barry Simon

Forests and Trees

Here is an other illustration what changes when we move from a matrix A to a matrix $1 + A$. The **Chebotarev-Shamis forest theorem** [5] tells that $\det(1 + L)$ is the number of rooted forests in a graph G if $L = B - A$ is the Laplacian. Here B is the vertex degree diagonal matrix. The pseudo determinant $\det(L)$ however counts the number of rooted spanning trees. This is the famous **Kirchhoff matrix tree theorem**.

The forest theorem follows immediately from a generalized Cauchy-Binet theorem [3], telling that

$$\det(1 + F^T G) = \sum_P \det(F_P)\det(G_P)$$

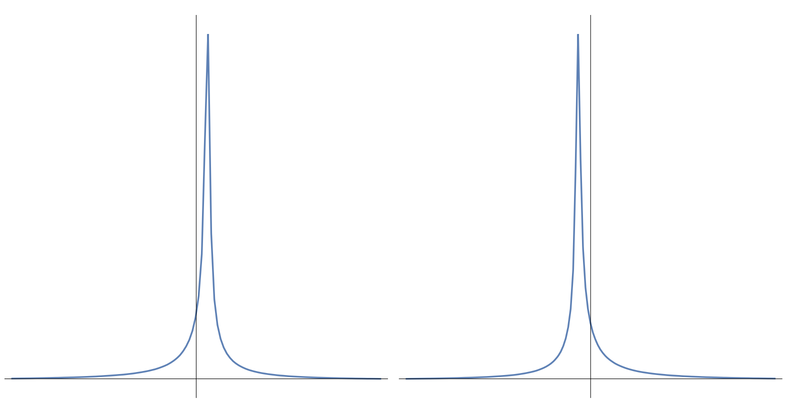
which holds for any pair F, G of $n \times m$ matrices and were the right hand side is a dot product of the minor giving all possible **minors** of the matrix, defined by the index set P which can be empty in which case $\det(A_P) = 1$.

Similarly, the Fredholm determinant $\det(1 + A)$ for the adjacency matrix A can be written as $\det(1 + A) = \sum_P \det(A_P)\det(1_P)$ which implies $\det(1 + A) = \sum_{k=0}^{\infty} \mathrm{tr}(\Lambda^k(A))$, where $\Lambda^k(A)$ is the k'th exterior power of A. This can now be defined if A is a trace class because $||\Lambda^k(A)||_1 \leq ||A^k||_1/k!$ where $|| \cdot ||_1 = \mathrm{tr}(|A|)$ is the trace class norm. This implies $\det(1 + A) \leq e^{||A||_1}$. This allows to extend the notion of determinant to compact operators which are trace class.

Pavel Chebotarev and Elena Shamis

Open Problems

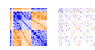
We believe that both random variables $G \to \det(1 + A(G))$ as well as $G \to \det(1 + A(G))$ have a continuous limiting distribution on random graphs when suitably normalized: here are the distributions obtained by computing it for one million graphs with 50 nodes and edge probability $p = 1/2$:



What is the structure of the inverse matrices of $1 + A(G')$, where $A(G')$ is the connection graph of the connection graph G' of G?

Here are some measurements. For many graphs, including the complete graph, wheel graphs, cross polytopes or cycle graphs, the inverse of the matrix $1 + A(G')$ only takes the values $-1, 0, 1$. For **star graphs** with n rays, we see that the values of the inverse of $1+A(G)$ is $-(n-1), -1, 0, 1$. For the utility graph, or the Petersen graph, the values are $-2, -1, 0, 1$. Among the 13 Archimedian solids, we see 6 for which the value takes values in $\{-2, -1, 0, 1\}$ and 7 where the value takes value in $\{-1, 0, 1\}$. On the 13 Catalan solids, the cases $\{-1, 0, 1\}$, $\{-3, -2, -1, 0, 1\}$, $\{-4, -2, -1, -1, 0, 1\}$ and $\{-4, -3, -2, -1, 0, 1\}$ occur.

We see that the values which the inverse matrix of $1+A(G')$ is the same for G and the **Barycentric refinement** of G which has the same vertex set like G' but for which two cliques are connected if one is contained in the other. We have no explanation for this yet. This would for example that the minimal value of $(1 + A(G)')^{-1}_{ij}$ is a **topological invariant** of G and could be used to get topological invariants of continuum spaces like manifolds or varieties.
Here we see the Fredholm matrix of a random connection graph. Most of its entries of $(1+A(G)')^{-1}_{ij}$ are $0, 1, -1$, but there are a few entries different from $0, 1, -1$.



A connection Fredholm matrix and its inverse

References

[1] R. Bowen and O. E. Lanford, III. Zeta functions of restrictions of the shift transformation. In *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*, pages 43–49. Amer. Math. Soc., 1970.

[2] E. Brown. Ann Ewing dies; science journalist turned nation's eyes to black holes. *Washington Post, August 1*, 2010.

[3] O. Knill. A Cauchy-Binet theorem for Pseudo determinants. *Linear Algebra and its Applications*, 459:522–547, 2014.

[4] M. Pollicott. Dynamical Zeta functions. Warwick, 2015.

[5] E.V. Shamis P.Yu, Chebotarev. A matrix forest theorem and the measurement of relations in small social groups. *Avtomat. i Telemekh.*, 9:125–137, 1997.

[6] D. Ruelle. *Dynamical Zeta Functions for Piecewise Monotone Maps of the Interval.* CRM Monograph Series. AMS, 1991.

[7] B. Simon. *Trace Ideals and their Applications.* AMS, 2. edition, 2010.

[8] A. Terras. *Zeta functions of Graphs*, volume 128 of *Cambridge studies in advanced mathematics.* Cambridge University Press, 2011.

[9] M.R. Watkins. List of known zeta functions. http://empslocal.ex.ac.uk/people/staff/mrwatkin//zeta/directoryofzetafunctions.htm, Retrieved: October 16, 2016.

[10] J. Wheeler. On the mathematical description of light nuclei by the method of resonating group structure. *Physical Review*, 52:1107, 1937.