MATHEMATICA ROUTINES FOR THE LEFSCHETZ FORMULA AND ZETA FUNCTION FOR GRAPHS

OLIVER KNILL

ABSTRACT. Mathematica routines for the fixed point theorem [1].

The routine takes a graph $G = s$ and finds the Lefschetz number $L(T)$ (here $M(p)$) for all the automorphisms $T = p$ of G as well as the zeta function $Z(T)$. Code is compressed to ease communication. It can be helpful for the reader to expand and annotate the code. Take a graph s, delete the first line and explore what the individual routines do. Important first goal is the list "aut" of all automorphisms.

```
Lefschetz[s_]:=Module[{L,v,c,S,per,T,h,F,g,aut,M,A,IsAut},
  L=Length; Sig=Signature; Po=Position;
  v=VertexList[s];n=L[v];e=EdgeList[s];m=L[e];
  c=Table[{e[[k,1]],e[[k,2]]},{k,m}];
  S[k_]:=Module[{r,b,a=Subsets[v,{k+1,k+1}]},
    b[kk_]:=Binomial[kk+1,2];
    If[k==0,r=Table[{v[[j]]},{j,L[v]}]];
    If[k==1,r=Table[{e[[j,1]],e[[j,2]]},{j,L[e]}]];
    If[k>1,r={};Do[sj=a[[j]];sg=EdgeList[Subgraph[s,sj]];
    If[L[sg]==b[k], r=Append[r,sj]],{j,L[a]}]];r];
  per=Permutations[v];q=L[per];
  T[x_,p_]:=Table[p[[x[[j]]]],{j,L[x]}];
  iter[p_,k_]:=Module[{pp=p},Do[pp=T[pp,p],{k-1}];pp];
  h=Module[{r,j=2},r={S[0],S[1]};
    While[L[S[j]]>0,r=Append[r,S[j]];j++];r];
  F[p_]:=Module[{r={}}, Do[Do[
    If[L[Complement[T[h[[k,l]],p],h[[k,l]]]]==0,
          r=Append[r,h[[k,l]]]]],
    {l,L[h[[k]]]}],{k,L[h]}];r];
  g[q_,y_]:=Sig[Table[Po[y,q[[k]]][[1,1]],{k,L[q]}]];
  c=Union[Table[{e[[k,2]],e[[k,1]]},{k,m}],
          Table[{e[[k,1]],e[[k,2]]},{k,m}]];
  IsAut[p_]:=Module[{e1},
    e1=Table[{p[[e[[k,1]]]],p[[e[[k,2]]]]},{k,L[e]}];
    t=True;Do[t=And[t,MemberQ[c,e1[[k]]]],{k,L[e1]}];t];
  aut={}; Do[pe=per[[j]];
    If[IsAut[pe],aut=Append[aut,pe]],{j,q}];
```

Date: May 6, 2012.
1991 *Mathematics Subject Classification.* Primary: 05C80,05C82,05C10,90B15,57M15 .
Key words and phrases. Random Graph Theory, Complex Networks, Graph Dimension, Graph Curvature, Euler characteristic.

```
  M[p_]:=Module[{},f=F[p];
    u=Table[-(-1)^L[f[[k]]]*g[f[[k]],T[f[[k]],p]],{k,L[f]}];
    Sum[u[[k]],{k,L[u]}]];
  A=Table[M[aut[[k]]],{k,L[aut]}];
  LG=Sum[A[[k]],{k,L[A]}]/L[aut];
  product[x_]:=Product[If[x[[i]]==0,1,x[[i]]],{i,L[x]}];
  Z[p_]:=Module[{zz=1,fold={},cycl,nn,TT,f,g,u1,u2},
      cycl=PermutationCycles[p][[1]];nn=product[Map[L,cycl]];
     Do[ pp=iter[p,j];
        TT[x_]:=Table[pp[[x[[j]]]],{j,L[x]}];
        f=F[pp];  f=Map[Sort,f];  ff={};
        Do[If[Not[MemberQ[fold,f[[k]]]],
            ff=Append[ff,f[[k]]]],{k,L[f]}];
        f=ff;  fold=Union[fold,f];
        g[q_,y_]:=Sig[Table[Po[y,q[[k]]][[1,1]],{k,L[q]}]];
        u1=Table[g[f[[k]],TT[f[[k]]]],{k,L[f]}];
        u2=Table[(-1)^(L[f[[k]]]+1),{k,L[f]}];  u=u1*u2;
        zz=zz*Product[(1-u1[[k]](z^j))^(-u2[[k]]/j),{k,L[u]}],
        {j,nn}];  zz];
  B=InputForm[Table[Z[aut[[k]]],{k,L[aut]}]];
  ZZ={B,InputForm[Simplify[Product[B[[k]],{k,L[B]}]]]};
  {A,LG,ZZ}]
Lefschetz[CycleGraph[4]]
```

Example: Get $L(T)$ and $\zeta_T(z)$ for the 8 automorphisms T of C_4:

T	L(T)	$\zeta_T(z)$
(1,2,3,4)	0	$(1-z^2)/(1-z)^2$
(1,5,4,2)	2	$(1+z)^2/(1-z^2)$
(2,1,4,3)	2	$(1+z)^2/(1-z^2)$
(2,3,4,1)	0	1
(3,2,1,4)	2	$(1-z^2)/(1-z)^2$
(3,4,1,2)	0	1
(4,1,2,3)	0	1
(4,3,2,1)	2	$(1+z)^2/(1-z^2)$

We get $L(G) = 1$ and $\zeta(G) = (1+z)^4/(-1+z)^4$. For the automorphism $T = (4,3,2,1)$ for example, we have the following prime periodic points:

j	dimension	multiplicity	zeta factor
p=1	dim=1	2	$(1+z)^2$
p=2	dim=0	2	$1/(1-z^2)^2$
p=2	dim=1	2	$(1+z)^2$

And the product is $\zeta_T(z) = (1+z)^2/(1-z^2)$.

References

[1] O. Knill. A Brouwer fixed point theorem for graph endomorphisms. *Fixed Point Theory and Applications*, 85, 2013.

E-mail address: knill@math.harvard.edu

Department of Mathematics, Harvard University, Cambridge, MA, 02138