CLASSICAL MATHEMATICAL STRUCTURES WITHIN TOPOLOGICAL GRAPH THEORY

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Abstract. Finite simple graphs are a playground for classical areas of mathematics. We illustrate this by looking at some results.

1. Introduction

These are slightly enhanced preparation notes for a talk given at the joint AMS meeting of January 16, 2014 in Baltimore. It is a pleasure to thank the organizers, **Jonathan Gross** and **Tom Tucker** for the invitation to participate at the special section in topological graph theory.

A first goal of these notes is to collect some results which hold unconditionally for any **finite simple graph** without adding more structure. Interesting are also results which hold for specific classes of graphs like **geometric graphs**, graphs of specific dimension d for which the unit spheres are $(d-1)$-dimensional geometric graphs. Such graphs behave in many respects like manifolds. We use that in full generality, any finite simple graph has a natural higher simplex structure formed by the presence of complete subgraphs present in the graph. Finite simple graphs are intuitive and harbor translations of theorems in mathematics which in the continuum need machinery from tensor analysis, functional analysis, complex analysis or differential topology. Some concepts in mathematics can be exposed free of technicalities and could be taught very early on. Here is an example of a result which mirrors a Lefschetz fixed point theorem in the continuum and which explains why seeing graphs as higher dimensional objects is useful. The result which follows from [24] is a discretization of a fixed point result in the continuum:

Given an automorphism T of a triangularization of an n-dimensional sphere. Assume T is orientation reversing if n is odd and orientation

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preserving if n is even. Then T has at least two fixed simplices. The reason is that the Lefschetz number of T is 2 and that the sum of the indices of fixed point simplices adds up to 2. For an orientation preserving automorphism of a triangularization of a 2-sphere for example, fixed points are either triangles, vertices or edges. An other consequence of [24] is that any automorphism of a tree has either a fixed vertex or edge, a result which follows also from [43].

2. Results

The following theorems hold unconditionally for any finite simple graph $G = (V, E)$. They are well known in the continuum, but are relatively new within graph theory. Here T is an automorphism of the graph with Lefschetz number $L(T) = \sum_{k=0}(-1)^k \mathrm{tr}(T_k)$, where T_k is the linear map induced on the k'th cohomology group $H^k(G)$. Note that H^k is not only considered for $k = 0, 1$ as in some graph theory literature. A triangularization of a n-dimensional sphere for example has $\dim(H^0(G)) = \dim(H^n(G)) = 1$ and all other $H^k(G) = \{0\}$ leading to the Betti vector $\vec{b} = (1, 0, \dots, 0, 1)$. Only for triangle-free graphs, where $v_0 = |V|, v_1 = |E|$ determine the Euler characteristic $\chi(G) = v_0 - v_1$ we have by Euler-Poincaré $\chi(G) = b_0 - b_1$, the difference between the number of components and genus. For us, triangle-free graphs are special geometric objects with dimension ≤ 1 and dimension 1 if there are no isolated vertices. In other words, only for triangle free graphs, we consider graphs as "curves". The function $K(x) = \sum_{k=0}^{\infty}(-1)^k V_{k-1}(x)/(k+1)$ is the Euler curvature, where $V_k(x)$ is the number of K_{k+1} subgraphs of $S(x)$ and $V_{-1}(x) = 1$. The integer $i_T(x) = \mathrm{sign}(T|x)(-1)^{k(x)}$ is the **degree** of T for the simplex x and $i_f(x) = 1 - \chi(S(x) \cap \{y \mid f(y) < f(x)\ \})$ is the **index** of a function f at a vertex x. If d is the exterior derivative matrix, the matrix $L = dd^* + d^*d$ is the form-Laplacian which when restricted to k-forms is denoted L_k. For a complete graph $G = K_{n+1}$ for example, L is a $2^n \times 2^n$ matrix which decomposes into blocks L_k of $B(n, k) \times B(n, k)$ matrices, where $B(n, k) = n!/(k!(n-k)!)$. An injective function f on V has a critical point x, if $S^-(x) = \{y \in S(x) \mid f(y) < f(x)\}$ is not contractible. Let $\mathrm{crit}(G)$ the minimal number of critical points, an injective function f on the vertices can have. Let tcap denote the minimal number of in G contractible graphs which cover G. A graph is **contractible** if a sequence of homotopy steps, consisting either of pyramid extensions or removals, brings it down to the one point graph K_1. Let $\mathrm{cup}(G)$ be the **cup length** of G as used in the discrete in [15]. We compute the cup length in an example given in the section with remarks. The minimal number m of k-forms f_j with $k \geq 1$ in this algebra with the property that $f_1 \wedge f_2 \cdots \wedge f_k$ is not zero in $H^m(G)$ is called the **cup length**. It is an algebraic invariant of the graph.

Theorem 1 (Gauss-Bonnet). $\sum_{x \in V} K(x) = \chi(G)$.

Theorem 2 (Poincaré-Hopf). $\sum_{x \in V} i_f(x) = \chi(G)$.

Theorem 3 (Index expectation). $\mathrm{E}[i_f(x)] = K(x)$.

Theorem 4 (Lefschetz). $L(T) = \sum_{T(x)=x} i_T(x)$.

Theorem 5 (Brower). *G contractible, then T has a fixed simplex.*

Theorem 6 (McKean-Singer). $\mathrm{str}(\mathrm{e}^{-\mathrm{tL}}) = \chi(\mathrm{G})$.

Theorem 7 (Hodge-DeRham). $\dim(\ker(L_k)) = \dim H^k(G) = b_k$

Theorem 8 (Ljusternik-Schnirelmann). $\mathrm{cup}(G) \leq \mathrm{tcap}(G) \leq \mathrm{crit}(G)$.

For Theorem (1) see [18, 17], for Theorem (2) see [19], for Theorem (3) see [21], for Theorem (4) see [24]. In [34] is a Kakutani version. For Theorem (5), see [22] where also Theorem (7) appears. As shown in the appendix of that paper, the result is very close to the classical result [40]. Theorem (8) in [15] is especially striking because it relates an **algebraic quantity** cup with a **topological quantity** tcap and an **analytic quantity** crit. The discrete result [15] is identical to the continuum result, in the continuum however often other counting conditions are used. We can make both tcap and crit homotopy invariant by minimizing over all graphs homotopic to G. Then the two inequalities relate three homotopy invariants: and algebraic, a topological and an analytical one.

The following theorems were known for graphs already, sometimes in other incarnations. We assume the graph to be connected. For the statement of Kirchoff's theorem we avoid pseudo determinants by using a **Google damping matrix** $P_{ij} = 1/n$ which when added just shifts the eigenvalue 0 to 1 so that $\mathrm{Det}(\mathrm{L}) = \det(\mathrm{P} + \mathrm{L})$, where Det is the pseudo determinant [25], the product of nonzero eigenvalues. Lets write shortly r-forest for **rooted spanning forests**. For the Riemann-Hurwitz statement, we assume that G/A is a graph and A a subgroup of order n of the Automorphism group of the graph.

Theorem 9 (Ivashchenko). *Cohomology is a homotopy invariant.*

Theorem 10 (Kirchoff). $\det(P + L)/n$ *is the # of maximal trees.*

Theorem 11 (Chebotarev-Shamis). $\det(1+L)$ *is the # of r-forests.*

Theorem 12 (Stokes-Gauss). $\sum_{x\in\mathcal{G}} df(x) = \sum_{x\in\delta\mathcal{G}} f(x)$.

Theorem 13 (Euler-Poincaré). $\sum_{k=0}^{\infty}(-1)^k \mathrm{dim}(\mathrm{H^k(G)}) = \sum_{\mathrm{k=0}}^{\infty}(-1)^{\mathrm{k}}\mathrm{v_k}$.

Theorem 14 (Riemann-Roch). $r(D) - r(K-D) = \chi(G) + \mathrm{deg(D)}$.

Theorem 15 (Riemann-Hurwitz). $\chi(G) = n\chi(G/A) - \sum_{x\in\mathcal{G}}(e_x - 1)$.

For a short exposition on Stokes, see [23]. Ivashcheko's result [13] is of great importance because it allows us to take a large complex network, homotopy deform it to something smaller, then compute the cohomology using linear algebra. Better even than homotopy shrinking procedures is a Čech approach: find a suitable topology on the graph [34] and compute the cohomology of the nerve graph. Kirkchoff's theorem gives the order of the Jacobian group of a graph and was primarily the reason we got more interested in it [25]. For Chebotarev-Shamis [45], there is an elegant proof in [26] using classical multilinear algebra. For Riemann-Roch, see [1], for Riemann-Hurwitz [38] we can assure that G/A is a graph if we take **simple group actions** which prevent that the quotient graph gains higher dimensional simplices. Riemann-Hurwitz holds for pretty arbitrary graphs. The ramification points can be higher-dimensional simplices so that the result holds for pretty general group actions on graphs. Stokes holds more generally for **chains** as Poincaré knew already. It is part of graph theory if δG is a graph. Assume f is a k-form. Summing over the set of simplices means that we sum over all k-dimensional simplices in $\mathcal{G}$ or its boundary $\delta\mathcal{G}$, which is the set of simplices in δG. Stokes can be abbreviated as $\langle G, df\rangle = \langle \delta G, f\rangle$, indicating that the **exterior derivative** d is dual to the **boundary operation** δ. [2] have formulated a Riemann-Roch theorem for 1-dimensional multi-graphs. It is formulated here for triangle-free graphs to indicate that we neglect the higher-dimensional structure. The **principal divisor** K is -2 times the curvature indicated that Riemann-Roch is related to Gauss-Bonnet, but the result is definitely deeper. Divisors are integer-valued functions on vertices. As in the continuum, Riemann-Roch turns out to be a sophisticated Euler-Poincaré formula relating analytically and combinatorically defined quantities. A higher-dimensional version will need a discrete analogue of sheaf cohomology.

Riemann-Hurwitz holds in full generality for chains and just reflects the Burnside lemma for each dimension [38]. For the theorem to work within graph theory we have to insist that the "orbifold" G/A is a graph. This is not always the case as already a Z_n action on C_n shows, for which the quotient is no more a finite simple graph. Riemann-Hurwitz has appeared in graph theory before, but only in the case when graphs are looked at as discrete analogues of algebraic curves or Riemann surfaces see [41] (a reference, I owe Roman Nedela). Assume a finite group A of order n acts on a finite graph G by automorphisms. The theorem sees G as a ramified cover of the chain $H = G/A$ and ramification indices $e_x - 1$ with $e_x = 1 + \sum_{a \neq 1, a(x)=x} (-1)^{k(x)}$, where $k(x)$ is the dimension of the simplex x.

Lets call a function f on the vertices a **Morse function** if adding a new point along the filtration defined by f changes none or exactly one entry b_m in the **Betti vector** $\vec{b} = (b_0, b_1, \dots)$. If the entry b_m is increased, this corresponds to add a m-dimensional "handle" when adding the vertex. Now, the index $i_f(x)$ of each critical point is by definition either 1 or -1. Adding a zero-dimensional handle for example increases b_0 and also the number of connected components, adding a one-dimensional handle increases b_1 and has the effect of "closing a loop". One can write $i_f(x) = (-1)^{m(x)}$ where $m(x)$ is the **Morse index**, which is the integer $m \geq 0$ at which the Betti number b_m has changed. In the continuum, $m(x)$ is the dimension of the stable manifold at the critical point. For the minimum, $m(x) = 0$ and for a maximum $m(x) = n$, where n is the dimension. Denote by c_m the number of critical points of Morse index m. Then

Theorem 16 (Weak Morse)**.** $\chi(G) = \sum_k (-1)^k c_k$ *and* $b_m \leq c_m$.

Theorem 17 (Strong Morse)**.** $\sum_{k=0}^m (-1)^k b_{m-k} \leq \sum_{k=0}^m (-1)^k c_{m-k}$.

The proof is by induction by adding more and more points to the graph. The definition of Morse function has been made in such a way that the inequalities remain true under the induction step of adding an other vertex. The induction foundation holds because the results hold for a one point graph K_1. Discrete Morse theory has been pioneered in a different way [7, 8] and is more developed.

Discrete PDE dynamics reduces to almost trivial linear algebra in the graph theoretical setup if the PDE is linear and involves the Laplacian on the geometry as many problems do. We mention this because in the continuum, there is a relatively large technical overhead with integral operators, distributions and functional analysis just to make sense of objects like Greens functions. One reason is that in the continuum, the involved operators are unbounded, making the use of functional analysis unavoidable for example just to be able to establish spectral properties like elliptic regularity or even establish the existence of solutions of the PDEs in suitable function spaces. The matrix L_h is the form-Laplacian restricted to the complement of the kernel. It operates on general k-forms, and these matrices are easy to write down for any finite simple graph. We could write $d_0 = \text{grad}, d_0^* = \text{div}, d_0^* d_0 = L_0 = \Delta, d_1 = \text{curl}$. Electromagnetic waves, heat, gravity all make sense on a general finite simple graph. The **Newton gravitational potential** V for example satisfies $LV = \rho$, where ρ is the **mass density**, a function on the vertices. We can get from the mass density the **gravitational field** $F = dV$, a one-form. Note that this works on any finite simple graph: the Laplacian defines a natural Newton potential. For electromagnetism, we get from the **charge-current one-form** j the **electromagnetic potential** A and from that the **electromagnetic field** $F = dA$. When studiing the **wave equation** it becomes apparent how useful it is to see graphs as discrete Riemannian manifolds. While the Dirac operator $D = d + d^*$ is a cumbersome object in the continuum, it is natural in the discrete as it leads immediately to a basis for the cohomology groups $H^k(G)$ - which are vector spaces - by computing the kernel of each block matrix L_k of $L = D^2 = dd^* + d^*d$. The matrix D is the crux of the story because it encodes the full exterior derivative d in the upper triangular part, its adjoint d^* in the lower triangular part. Evolving partial differential equations on a graph reduces to matrix exponentiation in linear algebra. Gravity lives on zero-forms, electromagnetism on one-forms, the weak force on two-forms and the strong force on three-forms. We mention this because in the continuum, there are chapters of books dedicated to the problem just to find the electromagnetic field to a current and charge distribution. To find the gravitational field F of a mass distribution ρ on a Riemannian manifold, we have to compute the **Green's kernel** which already uses the language of distributions in the continuum. The Poisson equation is a system of linear equations. The heat equation is $u' = -Lu$, the wave equation is $u'' = -Lu$ the Maxwell equations are $dF = 0, d^*F = j$ for a one form j. A Coulomb gauge $d^*A = 0$ reduces Maxwell to a Poisson equation for 1-forms, as in the continuum.

Theorem 18 (Fourier). *$e^{-Lt}u(0)$ solves the heat equation $u' = -Lu$.*

Theorem 19 (d'Alembert). *$\cos(Dt)u_0 + \sin(Dt)D^{-1}u_0'$ solves wave.*

Theorem 20 (Poisson). *$L_h^{-1}g$ solves the Poisson equation $Lu = g$*

Theorem 21 (Maxwell). *$A = L_h^{-1}j, F = dA$ solves Maxwell.*

Theorem 22 (Gauss). *$d^*F = \rho$ defines gravity $F = dV$ by $V = L_h^{-1}\rho$.*

Theorem 23 (Hopf-Rynov). *For $x, y \in V$, exists v with $\exp_x(v) = y$.*

Theorem 24 (Toda-Lax). *$D' = [B, D]$ with $B(0) = d - d^*$ is integrable.*

The heat equation is important because we could use it to find harmonic k-forms if it were not just given already as an eigenvalue problem. The wave equation $(d^2/dt^2 + L) = 0$ can be factored $(d/dt + iD)(d/dt - iD)\psi = 0$ leading to **Schrödinger equations** $\psi' = \pm iD\psi$ for the Dirac operator D,and complex quantum wave $\psi(t) = u(t) + iD^{-1}u'(t)$ encoding position $u(t) = \mathrm{Re}(\psi(t))$ and velocity $u'(t) = D\mathrm{Im}(\psi(t))$ of the classical wave. We write $\exp_x(v) = \psi(t)$ if $\psi(0) = x + D^{-1}v$. It is convenient for example to use the wave flow as a discrete analogue of the **geodesic flow** which in physical contexts has always been given by light evolution: we measure distances with light. What happens in the discrete is that Hopf-Rynov only can be realized by looking at a quantum dynamical frame work. It is absolutely futile to try to find a notion of tangent space and exponential map on a discrete level by only using paths on the graph. The reason is that for any pair of points x, y we want to have an element v in the tangent space so that $exp_x(v) = y$. If the graph is large then the tangent space needs to be large. One can try to look at equivalence classes of paths through a vertex but things are just not working naturally. Linear algebra is not only easier, it is also how nature has implemented the geodesic flow: as motions of particles satisfying quantum mechanical rules. Hopf-Rynov in the graph case is now very easy: for any two vertices x, y there is a unique initial velocity of a wave localized initially at x so that at a later time T it is at y. This is just linear algebra. As in the continuum there is an uncertainty principle: while we can establish to have a particle passing through two points ("knowing the velocity") prevents us to know the position exactly at other times. But we can have a classical motion on the graph as the vertex on which the probability density of the particle position is highest. In the graph case, the quantum unitary evolution happens in a compact unitary group so that there is always Poincaré recurrence evenso as in classical mechanics, the return times are huge already for relatively small graphs. We can use the wave equation naturally to measure distances, which does this much better than the very naive geodesic distance. Naturally, different types of particles - as waves of differential forms - travel with different velocities too. The point of view that a graph alone without further input allows to study relatively complex physics has been mentioned in [29, 31]. We don't even need to establish initial conditions since the isospectral deformation of $D(t)$ does that already. Which isospectral deformation do we chose? It turns out not to matter. The physics is similar. A finite simple graph leads to physics without further assumptions. The only input is the graph. The details of the dynamical system looks difficult: find relations between the speeds with which different discrete differential forms $f \in \Omega_k$ move. Since the Dirac operator links different forms, this can not be studied on each k-form sector Ω_k separately. Only the Laplacians L_k leave those sectors invariant - sectors which informally could be thought of as gravitational, electric, weak and strong. The symmetry translations given by isospectral deformations $D(t)$ allows to study the column vectors of $D(t)$ which asymptotically solve the wave equation and especially explore the geometry on $\Omega = \oplus\Omega_k$ by computing distances between various simplices. This leads to speed relations between various particles. Even for smaller graphs, the dynamics is complex without the need for any further input. The dynamics can be simulated on the computer. Evenso the picture is naive, it looks like a wonderful playground for experimentation. In trying to figure out, which graphs are natural we have looked in [29] at the Euler characteristic is a natural functional. The isospectral deformation of the Dirac operator $D = d + d^*$ can be done in such a way that it becomes in the limit a wave evolution. It is interesting that this modification of the flow, where we take $D(t) = d + d^* + b$. The deformation of the Dirac operator studied in [31] can be modified so that it becomes complex. We have then more symmetry with a selfadjoint operator $D = d+d^*+b$ and an antisymmetric operator $B = d - d^* + ib$. The dynamics (and geometry given by the differential forms d) now becomes complex even if we start with a real structure. With or without the modification, it provides an isospectral deformation of D. The Laplacian L stays the same so that the geometric evolution can not be seen on a classical level, except when looking at the d'Alembert solution of the wave equation, where D enteres in the initial condition. The symmetry group of isospectral Dirac operators on a graph provides a natural mechanism for explaining why geometry expands with an inflationary start. See also [29]. This is not due to some special choice of the deformation but is true for any deformation starting with a Dirac operator which has no diagonal part. The deformed operator will have some diagonal part which leads to "dark matter" which is not as geometric as the side diagonal part. This simple geometric evolution system [30, 31] allows to deform both Riemannian manifolds for which the deformed d are pseudo differential operators. It is an exciting system and can not be more natural because one is forced to consider it when taking quantum symmetries of a graph seriously. Besides establishing basic properties like universal expansion with inflationary start and camouflaged supersymmetry, establishing the limiting properties of the system has not yet been done.

A graph has **positive curvature** if all sectional curvatures are positive. In the following, a **positive curvature graph** is a **geometric positive curvature graph**, meaning that there is d such that for all vertices x the sphere $S(x)$ is a $d-1$ dimensional geometric graph which is a sphere in the sense that the minimal number of critical points, an injective function on the graph $S(x)$ can take is 2. The set of all d-dimensional positive curvature graphs is called $\mathcal{P}_d$. A **sectional curvature** is the curvature of an embedded wheel graph. We denote by $K(x)$ the Euler curvature, with $\mathrm{Aut}(G)$ the group of automorphisms and with $\mathrm{Aut}_+(G)$ the group of orientation-preserving automorphisms of a geometric graph G and with $\mathcal{F}$ the set of fixed simplices of T.

Theorem 25 (Flatness)**.** *Odd dimensional graphs have* $K(x) \equiv 0$.

Theorem 26 (Bonnet)**.** *Positive curvature graphs have diameter* ≤ 3.

Theorem 27 (Synge)**.** $2m$*-dim positive curvature* $\Rightarrow$ *simply connected.*

Theorem 28 (Bishop-Goldberg)**.** $G \in \mathcal{P}_4 \Rightarrow b_0 = b_3 = 1, b_1 = b_2 = 0$.

Theorem 29 (Weinstein)**.** *In* $G \in \mathcal{P}_4$, $T \in \mathrm{Aut}_+(G)$ *has* $|\mathcal{F}| \geq 2$.

For Theorem (25), see [20].

A connection with different fields of mathematics comes through zeta functions $\zeta(s) = \sum_{\lambda>0} \lambda^{-s}$, where λ runs over all positive eigenvalues of the Dirac operator D of a graph G. These entire functions leads to connections to basic complex analysis. Why using the Dirac operator? One reason is that for circular graphs $G = C_n$, the zeta function has relations with the corresponding zeta function of the Dirac operator $i\partial_x$ of the circle $M = T^1$, for which the zeta function is the **standard Riemann zeta function**. Also, we have $\zeta(2s)$ as the zeta function of the Laplace operator. It is better to start with $\zeta(s)$ and go to $\zeta(2s)$ than looking at the Laplace zeta function $\tilde{\zeta}(s)$ and then have to chose branches when the square root $\tilde{\zeta}(s/2)$. Again, many technicalities are gone since we deal with entire functions so that there is no surprise that for discrete circles C_n. While there is no relation with the Riemann hypothesis - the analogue question for the circle S^1 - it is still interesting because the proof [35] has relations with single variable calculus.

Theorem 30 (Baby Riemann)**.** *Roots of* $\zeta_{C_n}(s) \to \mathrm{Re}(s) = 1/2$.

Finally, lets mention an intriguing class of **orbital networks** which we discovered first together with Montasser Ghachem and which has much affinities to realistic classical networks and leads to questions of number theoretical nature. We consider k polynomials maps T_k on the ring $V = Z_m$ or multiplicative group Z_m^* which we take as vertices of a graph. Two vertices x, y are connected if there is k such that $T_k(x) = y$ or $T_k(y) = x$. Here are "miniature example results" [28, 32, 33]:

Theorem 31. *$(Z_n^*, 2x)$ is connected iff $n = 2^m$ or 2 is primitive root.*

Theorem 32. *(Z_n^*, x^2) is connected iff $n = 2$ or n is Fermat prime.*

Theorem 33. *$(Z_n, \{2x, 3x+1\})$ has 4 triangles for prime $n > 17$.*

Theorem 34. *$(Z_n^*, \{x^2, x^3\})$ is connected iff n is Pierpont prime.*

Theorem 35. *(Z_n, T) has no K_4 graphs and $\chi \geq 0$.*

Theorem 36. *(Z_p, x^2+a, x^2+b) with $a \neq b$ has $\chi < 0$ for large primes.*

Theorem 37. *$(Z_p, x^2 + a)$ has zero, one or two triangles.*

More questions are open than answered: we still did not find a Colatz graph $(Z_n, 2x, 3x + 1)$ which is not connected. We have found only one example of a quadratic orital network with 3 different generators which is not connected: the only case found so far $(Z_{311}, x^2 + 57, x^2 + 58, x^2 + 213)$ and checked until $p = 599$ but since billions of graphs have to be tested for connectivity, this computation has slowed down considerably, now using days just for dealing with one prime. We also see that for primes $p > 23$ and $p \leq 1223$ all quadratic orbital networks with two different generators are not planar. Also here, we quickly reach computational difficulties for larger p. These are serious in the sense that the computer algebra system refuses to decide whether the graph is planar or not.

3. Remarks

Having tried to keep the previous section short and the statements concise, the following remarks are less polished. The notes were used for preparation similarly as [27] for the more linear algebra related issues.

Dimension. Classically, the **Hausdorff-Uhryson inductive dimension** of a topological space X is defined as $\mathrm{ind}(\emptyset) = -1$ and $\mathrm{ind}(X)$ as the smallest n such that for every $x \in X$ and every open set U of x, there exists an open $V \subset U$ such that the boundary of V has dimension $n-1$. For a finite metric space and a graph in particular, the inductive dimension is 0 because every singleton set $\{x\}$ is open. It can become interesting for graphs when modified: dimension [18] for graphs is defined as $\dim(\emptyset) = -1$, $\dim(G) = 1 + \frac{1}{|V|} \sum_{v \in V} \dim(S(v))$, where $S(v)$ is the unit sphere. Dimension associates a rational number to every vertex x, which is equal to the dimension of the sphere graph $S(x)$ at x. The dimension of the graph itself is the average over the dimensions for all vertices x. We can compute the expected value $d_n(p)$ of the dimension on Erdoes-Rényi probability spaces $G(n,p)$ recursively: $d_{n+1}(p) = 1 + \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} d_k(p)$, where $d_0 = -1$. Each d_n is a polynomial in p of degree $\binom{n}{2}$. [16] Triangularizations of d-dimensional manifolds for example have dimension d.

Homotopy. Classically, two topological spaces X, Y are homotopic, if there exist continuous maps $f : X \to Y$, $g : Y \to X$ and $F : X \times [0,1] \to X, G : Y \times [0,1] \to Y$ such that $F(x,0) = x, F(x,1) = g(f(x))$ and $G(y,0) = y, G(y,1) = f(g(y))$. For graphs, there is a notion which looks different at first: it has been defined by Ivashchenko [13] and was refined in [4]. We came to this in work with Frank Josellis via Morse theory as we worked on [19]: it became clear that the set $S^-(x) = \{ y \mid f(y) < f(x) \}$ better has to be contractible if nothing interesting geometrically happens. Indeed, we called in [19] the quantity $1 - \chi(S^-(x))$ the index. Given a function f on a graph, we can build a Morse filtration $\{f \leq c\}$ and keep track of the moment, when something interesting happens to the topology. Values for which such a thing happens are **critical values**. If we keep track of the Euler characteristic, using the fact that unit balls have Euler characteristic 1 and $\chi(A \cap B) = \chi(A) + \chi(B) - \chi(A \cap B)$ by counting, we immediately get $\sum_x i_f(x) = \chi(G)$ which is the Poincaré-Hopf theorem. We can look at extension steps $X \to Y$, in which a new vertex z is attached to a **contractible** part of the graph. This is a homotopy step and is defined recursively because **contractible** means homotopic to K_1. The reverse step is to take a vertex z with contractible unit sphere and remove both z and all connections to z. A graph G for which we can apply a finite sequence of such steps is called a **homotopy**. Homotopy preserves cohomology and Euler characteristic. But as in the classical case, homotopy does not preserve dimension: all complete graphs K_n are homotopic.

Continuity. While natural notions of homotopy and cohomology exist for graphs, we need a notion of homeomorphism which allows to deform graphs in a rubber geometry way. We want dimension, homotopy and cohomology and connectivity to be preserved as in the continuum.

Any notion of topology on a graph which breaks one of these invariants is deficient. Topology is also needed for discrete sheave constructions. Classically, a notion of homeomorphism allows to do subdivisions of edges of a graph to get a homeomorphic graph. This notion is crucial for the classical Kuratowski theorem which tells that a graph is not planar if and only it contains a homeomorphic copy of either K_5 or $K_{3,3}$. This classical notion however does not preserve dimension, nor cohomology, nor Euler characteristic in general: the triangle is homotopic to a point, has Euler characteristic 1 and trivial cohomology. After a subdivision of the vertices we end up with the graph C_6 which has Euler characteristic 0, is not homotopic to a point and has the betti vector $b_1 = 1$. In [34], we have introduced a notion of homeomorphism which fulfills everything we wish for. The definition: a **topological graph** $(G = (V, E), \mathcal{B}, \mathcal{O})$ is a standard topology $\mathcal{O}$ on V generated by a subbasis $\mathcal{B}$ of contractible sets which have the property that if two basis elements A, B intersect under the **dimension condition** $\dim(\mathrm{A} \cap \mathrm{B}) \geq \min(\dim(\mathrm{A}), \dim(\mathrm{B}))$ then the intersection is contractible. The **nerve graph** $(\mathcal{B}, \mathcal{E})$ is given by the set $\mathcal{E}$ consisting of all pairs A, B for which the dimension assumption holds. This nerve graph is asked to be homotopic to the graph. A **homeomorphism** of two-topological graphs is a classical homeomorphism between the topologies so that the subbasis elements correspond and such that dimension is left constant on $\mathcal{B}$. Two graphs G, H are homeomorphic if there are topologies $(\mathcal{B}, \mathcal{O})$ and $(\mathcal{C}, \mathcal{P})$ on each for which they are homeomorphic. Homeomorphic graphs are homotopic, have the same cohomology and Euler characteristic. Lets call a topological graph $(G, \mathcal{B}, \mathcal{O})$ **connected** if $\mathcal{B}$ can not be split into two disjoint sets $\mathcal{B}_1 \cup \mathcal{B}_2$ such that for every $B_i \in \mathcal{B}_i$ the sets $B_1 \cap B_2$ are empty. For any topology this notion of connectivity is equivalent to path connectivity. (Also in classical point set topology, a space X is connected if and only if there is a subbasis of connected sets for which the nerve graph is connected.) Every graph has a topology for which the nerve graph is the graph itself. With the just given definition of topology, any two cyclic graphs C_n, C_m are homeomorphic if $n, m \geq 4$. Also, the octahedron and icosahedron are homeomorphic. Contractible graphs can carry weak topologies so that all trees are homeomorphic with respect to this indiscrete topology. Graphs are isomorphic as graphs if we take the discrete topology on it, the topology generated by star graphs centered at wedges. Under the fine topology, two graphs are homeomorphic if and only if they are isomorphic. As in the continuum, natural topologies are neither discrete nor indiscrete. For triangularizations of manifolds, we can take $\mathcal{B}$ to consist of discs, geometric graphs of the same dimension than the graph for which the boundary is a Reeb sphere.

Trees and forests. The matrix tree theorem of Kirkhoff is important because the number of spanning tree is the dimension of the Jacobian group of a graph. Higher dimensional versions of this theorem will be relevant in higher dimensions. The **Chebotarev-Shamis forest theorem** follows from a general Cauchy-Binet identity, if F, G are two $n \times m$ matrices, then $\det(1 + xF^T G) = \sum_P x^{|P|} \det(F_P)\det(G_P)$, where the sum is over all minors [25]. This implies $\det(1 + F^T G) = \sum_P \det(F_P)\det(G_P)$ and especially a cool **Pythagorean identity** $\det(1+A^T A) = \sum_P \det^2(A_P)$ which is true for any $n \times m$ matrix. See [45]. We are not aware that even that special Pythagorean identity has appeared already. Note that the sum is over all square sub patterns P of the matrix.

Curvature. For a general finite simple graph, the **curvature** is defined as $K(x) = \sum_{k=0}^{\infty}(-1)^k V_{k-1}(x)/(k+1)$, where $V_k(x)$ be the number of K_{k+1} subgraphs of $S(x)$ and $V_{-1}(x) = 1$. It leads to the **Gauss-Bonnet-Chern** theorem $\sum_{x \in V} K(x) = \chi(G)$ [17]. These results are true for any finite simple graph [23]. Gauss-Bonnet-Chern in the continuum tells that for a compact even dimensional Riemannian manifold M, there is a local function K called Euler curvature which when integrated produces the Euler characteristic. Other curvatures are useful. In Riemannian geometry, one has sectional curvature, which when known along all possible planes through a point allows to get the Euler curvature, Ricci curvature or scalar curvature. In the discrete, the simplest version of sectional curvature is the curvature of an embedded wheel graph. if the center of the later has degree d, then the curvature is $1-d/6$. This means that wheel graphs with more than 6 spikes have negative curvature and wheel graphs with less than 6 spikes have positive curvature. It is easy to see that any graph for which all sectional curvatures are positive must be a finite graph. More generally, if one assumes that positive curvature graphs have positive density on any two dimensional geometric subgraph, then one can give bounds of the diameter of the graph similarly as in the continuum. Unlike as in differential geometry, curvature appears to be a pretty rigid notion. This is not the case: as indicated below, we have proven that curvature is the average over all indices $i_f(x)$ when we average over a probability space of functions. This is an integral geometric point of view which allows deformation. If we deform the functions f, for example using a discrete partial differential equation, then the probability measure changes and we obtain different curvatures which still satisfy Gauss-Bonnet [30, 31]. This holds also in the continuum: any probability measure on Morse functions on a Riemannian manifold defines a curvature by taking the index expectation on that probability space. For manifolds embedded in a larger Euclidean space (no restriction of generality by Nash), one can take a compact space of linear functions in the ambient space and integrate over the natural measure on the sphere to get Euler curvature. What is still missing is an **intrinsic measure** on the space of Morse functions of a Riemannian manifold such that the average index density is the curvature.

Morse filtration. A function f on the vertex set V of a finite simple graph defines a filtration of the graph into subgraphs $\emptyset = G_0 \subset G_1 \cdots \subset G_n = G$, where each $G_k = \{v \mid f(v) \leq c_k\}$ is obtained from G_{k-1} by adding a new vertex for which the value of f is the next bigger number in the range of f. The sphere $S(x_k)$ in G_k is $S^-(x_k)$. Define the index of x to be $i_f(x) = 1 - \chi(S^-(x))$. We see that $\chi(G_k) - \chi(G_{k-1}) = i_f(x_k)$ so that $\sum_{x \in V} i_f(x) = \chi(G)$. This is the Poincaré-Hopf formula for the gradient field of f. As a side remark, me mention that also the **Poincaré-Hopf theorem** can be considered for more general spaces. Assume we have a compact metric space (X, d) for which Euler characteristic is defined and for which small spheres $S_r(x)$ are homeomorphic for small enough positive r. Given a function f, we call x a **critical point** if for arbitrary small positive r, the set $S_r^-(x) = S_r(x) \cap \{f(y) < f(x)\}$ is not contractible or empty and the index $i_f(x)$ at a critical point $1 - \chi(S_r^-(x))$ exists for small enough r and stays the same. In that case, the Poincaré-Hopf formula $\sum_x i_f(x) = \chi(X)$ holds.

Integral geometry. There is a great deal of integral geometry possible on finite simple graphs. If we take a probability measure on the space of functions like taking f with the uniform distribution $[-1, 1]$ independent on each node, then the expectation of $i_f(x)$ is curvature. This also works for discrete measures like the uniform distribution on all permutations of $\{1, \dots, n\}$. By changing the probability measure, we can get different curvatures, which still satisfy the Gauss-Bonnet theorem. This principle is general and holds for Riemannian manifolds, even in singular cases like **polytopes** where the curvature is concentrated on finitely many points. Lets look at a triangle ABC in the plane and the probability space of linear maps $\{f(x, y) = x\cos(t) + b\sin(t) \mid 0 \leq t < 2\pi\}$ on the plane which induces functions on the triangle. Define the index

$i_f(x) = \lim_{r \to 0} \chi(S_r(x) \cap \{y \mid f(y) < f(x)\})$. The probability to have a positive index away from the corner is zero. This implies that the curvature has its support on the vertices. The probability to have index 1 at A is the curvature $K(A) = 1 - \alpha/\pi$. The sum of all curvatures is 2. That the sum of the angles $\alpha + \beta + \gamma$ in a triangle is equal to π can be seen in an integral geometric way as an index averaging result. The same is true for polyhedra [3] and so by a limiting procedure for general Riemannian manifolds. We have not yet found a natural **intrinsic** probability space of Morse functions on a manifold for which the index expectation is the curvature. Taking an even dimensional Riemannian manifold M with normalized Riemannian volume m we can take the probability space (M, m). For any point x one can look at the **heat signature function** $f(y) = K(x, y)$ of the **heat kernel** $K(x, y)$. The functions f are parametrized by x and are Morse functions. Intuitivly, $f(y)$ is the temperature at a point y of the manifold if the heat source is localized at x. The index expectation of this probability space of Morse function certainly defines a curvature on the manifold for which Gauss-Bonnet-Chern holds. The question is whether it is the standard Euler curvature. It certainly is so in symmetric situations like constant curvature manifolds.

Divisors. Lets look at an example for Riemann-Roch. If $G = C_5$ is a cyclic graph then $\chi(G) = 1 - g = 0$ and the **principal divisor** is constant 0. The Riemann Roch theorem now tells $r(D) - r(-D) = \deg(D)$. Given the divisor $D = (a, b, c, d, e)$ we have $\deg(D) = a + b + c + d + e$. The linear system $|D|$ consists of all divisors E for which $D - E$ is equivalent to an effective divisor. $r(D)$ is -1 if $|D| = \emptyset$ and $r(D) \geq s$ if and only if $|D - E| \neq \emptyset$ or all effective divisors E of degree s. If $\deg(D) = 0$, then clearly $r(D) = r(-D) = 0$. We can assume $\deg(D) > 0$ without loss of generality. Otherwise just change D to $-D$ which flips the sign on both sides. The set of zero divisors is 4 dimensional. The set of principal divisors is four dimensional too. The Jacobian group $\mathrm{Jac}(G) = \mathrm{Div}_0(G)/\mathrm{Prin}(G)$ has order 5 and agrees with the number of spanning trees in C_5 is 5. It has representatives like $(1, -1, 0, 0, 0)$. Given a vertex x the Abel-Jacobi map is $S(v) = [(v) - (x)]$ [1].

Geometric graphs. The emergence of the definition of **polyhedra** and **polytopes** is fascinating. The struggle is illustrated brilliantly in [39] or [46]. Many authors nowadays define polyhedra embedded in some geometric space like [11]. We have in [18] tried to give a purely graph theoretical definition: a **polyhedron** is a graph, which can be completed or truncated to become a two dimensional geometric graph.

The only reason to allow truncations of vertices is so that we can include graphs like the tetrahedron into the class of polyhedra. With this notion, all platonic solids are polyhedra: the octahedron and icosahedron are already two-dimensional geometric graphs, the dodecahedron and cube can be stellated to become a two dimensional geometric graph. The notion in higher dimensions is similar. A polytop is a graph which when completed becomes a d-dimensional geometric graph. Now, in higher dimensions, one can still debate whether one would like to assume that the unit spheres $S(x)$ are all Reeb sphers, or whether one would like to weaken this and allow unit sphere to be a discrete torus for example. In some sense this is like allowing exotic differential structures on manifolds.

Positive curvature. A classical theorem of Bonnet tells that a Riemannian manifold of positive sectional curvature is compact and satisfies an upper diameter bound $\pi/\sqrt{k}$. For two-dimensional graphs we can give a list of all the graphs with strictly positive curvature and all have diameter 2 or 3. We would like to have Schoenberg-Myers which needs **Ricci curvature**. The later notion can be defined in the discrete as a function $R(e)$ on edges e given by the average of the wheel graph curvature containing the edge. The **scalar curvature** at a vertex v would then just be average of all $R(e)$ with $v \in e$. We have not proven yet but expect that only finitely many graphs exist which have strictly positive Ricci curvature and fixed dimension. Lets go back to positive curvature which implies that every dimensional subgraph has positive curvature. Positive sectional curvature is a strong assumption in the discrete which prevents to build a discrete positive curvature projective plane. Since every two dimensional positive curvature graph is a sphere, results are stronger. Why is there no projective plane of positive curvature? Because the curvature constraint in the discrete produces too strong pitching. Identifying opposite points of icosahedron (the graph with minimal possible positive curvature $1/6$ at every vertex) would become higher dimensional. The statements in the theorems all follow from the following "geomag lemma": *a closed* 2*-geodesic in a positive curvature graph can be extended within G to a two-dimensional orientable positive curvature graph.* Proof: extend the graph and possibly reuse the same vertices again to build a two dimensional graph which is locally embedded. The wheel graphs in this 2-geodesic 2-dimensional subgraph has 4 or 5 spikes due to the positive curvature assumption. By projecting onto G we see that this is a finite cover of a two-dimensional embedded graph of diameter ≤ 3. The later is one of a finite set of graphs which all are orientable and have Euler characteristic 2. By Riemann-Hurwitz, the cover must be $1:1$ since there are no two dimensional connected graphs with Euler characteristic larger than 2. Bonnet immediately follows from the Geomag lemma since a geodesic is contained in a two dimensional geodesic surface which has diameter ≤ 3. Synge follows also follows from the geomag lemma because all the two dimensional positive curvature graphs are simply connected and any homotopically nontrivial closed curve can be extended to a two dimensional surface. Bishop-Goldberg follows now from the Poincaré duality. Weinstein immediately follows from Bishop-Goldberg and the Lefschetz theorem because the Lefschetz number is 2. In the continuum, Synge has odd dimensional projective spaces as counter examples for positive curvature manifolds which are not simply connected. The discrete Bishop-Goldberg result implies that Euler characteristic is 2 and so positive. In $d \geq 8$ dimensions this is unsolved in the continuum but the sectional curvature assumption in the discrete is of course much stronger. In the continuum, one has first tried to relate positive Euler curvature with positive sectional curvature which fails in dimensions 6 and higher [50, 9]. **Ricci curvature** $R(e)$ at an edge e is as the average over all curvatures of wheel graphs which contain e. By extending embedded geodesics to two-dimensional immersed surfaces within G, one might get Schoenberg-Myers type results linking positive Ricci curvature everywhere with diameter bounds. It could even lead to **sphere theorems** like Rauch-Berger-Klingenberg: Positive curvature graphs are triangularizations of spheres. These toy results in the discrete could become more interesting (and get closer to the continuum) if the curvature statements are weakened. A graph has **positive curvature of type** (M, δ) if the total curvature for any geodesic two-dimensional subgraph of diameter $\geq M$ is $\geq \delta$. Unlike for Riemannian manifolds, where geodesic two-dimensional surfaces in general do not exist, the notion of **geodesic surface** is interesting in the discrete: a subgraph H of G is called **L-geodesic** if for any two vertices x, y with distance $\leq L$ in H, the geodesic distance of x and y within H is the same than the geodesic distance within G. Any curve in a graph is a 1-geodesic and a 2-geodesic if we shorten each corner in a triangle by the third side. In other words, a curve is a 2-geodesic, if we can not do localized homotopy transformations shortening the curve. A wheel subgraph is an example of a two-dimensional 2-geodesic subgraph. In the continuum, geodesic two dimensional surfaces in $d \geq 3$ dimensional manifolds M do not exist in general: we can form the two-dimensional surfaces $\exp_D(x)$, where D is a two-dimensional disc in the tangent space T_xM but for y in this surface the surface $\exp_D(y)$ intersects $\exp_D(x)$ only in a one dimensional set in general. The geodesic surfaces do not match up.

Structures. Graph theory allows to illustrate how different fields of mathematics like algebra, calculus, analysis, topology, differential geometry, number theory or algebraic geometry interplay. There are properties of graphs which are of **metric nature** and this reflects through the automorphism group of the graph consisting of isometries which are graph isomorphisms. **Spectra** of the Laplacian L or Dirac operator D are examples which are metric properties. The spectrum changes under topological deformations or even homotopies. Then there is the group of homeomorphisms [34] of a graph with respect to some topology. This group is in general much richer and more flexible; as homeomorphisms do not have to be isometries any more. They still preserve everything we want to be preserved like homotopy, homology and dimension. Topological symmetries can be weakened when looking at homotopies which provides a much weaker equivalence relation between graphs. Homotopy still preserves homology and connectivity but does no more preserve dimension. Anything contractible is homotopically equivalent to the one point graph K_1. An even weaker notion is formed by the equivalence classes of graphs with the same homology. This algebraic equivalence relation and still weaker than homotopy as we know from the continuum: there are nonhomotopic graphs which have the same cohomology groups: examples are discrete Poincaré spheres. There is still room to experiment with **differentiability structures** on a graph. One possibility for an analogue of a C^r-diffeomorphism between two graphs is a homeomorphism which has the property that the nerve graph has discs of radius r for which the boundary is a geometric sphere. One can consider [18] as an attempt to look at curvature in a smoother setup and as the Hopf Umlaufsatz proven there shows, things are already subtle in **discrete planimetry**, where one looks at regions in the discrete plane. Second order curvatures obtained by measuring the size of spheres of radius 1 and 2 often fail to satisfy Gauss-Bonnet, already in two dimensions. Since it often also works, it is an interesting question to find "smoothness conditions" under which the discrete curvature $K(p) = 2|S_1(p)| - |S_2(p)|$ [18] works for geometric graphs. The just mentioned higher order curvature could be used as sectional curvature when the spheres are restricted to two dimensional subgraphs. The curvature is motivated from Puiseux type formulas $K = \lim_{r \to 0} \frac{2|S_r| - |S_{2r}|}{2\pi r^3}$ for two dimensional Riemannian surfaces.

Network theory. Graph theory has strong roots in computer science, where networks or geometric meshes in computer graphics are considered which are in general very complex. Social networks [14], biological networks [47], the web, or triangularizations of surfaces used to display computer generated images are examples. Many mathematicians look at graphs as analogues of Riemann surfaces, as one-dimensional objects therefore. Similarly like the GAGA principle seeing correspondences between algebraic and analytic geometry, there is a principle which parallels the geometry of graphs and the geometry of manifolds or varieties. It is also a place, where some Riemann themes come together: **Riemannian manifolds**, **Riemann-Roch**, **Riemann-Hurwitz**; studying the **Riemann Zeta function** on cyclic graphs leads naturally to **Riemann sums**.

Dynamics. We have seen dynamical systems on graphs given by discrete partial differential equations. It is the dynamics when studying the evolution of functions or discrete differential forms on graphs. This is heat or wave dynamics or quantum dynamics. Dynamics also appears is as **geometric evolution equations**, which describe families of geometries on a graph. While the graph itself can be deformed by discrete time steps which preserve continuity, one can also deform the exterior derivative. If this is done in a way so that the spectrum of the Laplacian stays the same, we study symmetries of the quantum mechanical system. Heat, wave, Laplace, Maxwell or Poisson equations are based on the Laplacian L. Classically, Gauss defined the gravitational field F as the solution to the Poisson equation $LF = \sigma$, where σ is the mass density. The Newton equations describe particles $u(t)$ satisfying the equation $u'' = L^{-1}\sigma$. Better is a Vlasov description where particles and mass are on the same footing and where we look at deformations $q(t)$ which satisfy the differential equation $q''(x) = -L^{-1}q$, a differential equation which makes sense also in the discrete.

Algebra. There are other places where graphs can be seen from the dynamical system point of view. It is always possible to see a graph as an orbit of a monoid action on a finite set. Sometimes this is elegant. For example, look at the graph on Z_p generated by the maps $T(x) = 3x + 1$ and $S(x) = 2x$ or the system generated by a quadratic map $T(x) = x^2 + c$. This orbital network construction produces deterministic realistic networks. These graphs relate to **automata**, edge colored directed graphs with possible self loops and multiple loops and encode a monoid acting on the finite set. [48] who has developed a theory of finite transformation monoids calls a graph generated by T

an **orbital digraph** which prompted us to call the graph **orbital networks**. That the subject has some number theoretical flavour has been indicated in [28, 32, 33]. It shows that elementary number theory matters when trying to understand connectivity properties of the graphs. We would not be surprised to see many other connections.

Cup product. Exactly in the same way as in the continuum, one can define on the space Ω of antisymmetric functions a product $f \cup g$ which maps $\Omega_p \times \Omega_q$ to Ω_{p+q} and which has the property that the product of two cocycles is a cocycle extending so to a product $H^k \times H^l \to H^{k+l}$. The definition is the same: assume $f \in \Omega^p$ and $g \in \Omega^q$, then define $f \cup g(x_0, ..., x_{p+q}) = f(x_0, ...x_p)g(x_p, ...x_{p+q})$. Now check that $d(f \cup g) = df \cup g + (-1)^p f \cup dg$. For example, if $p = q = 1$, then $h(x_0, x_1, x_2) = (f \cup g)(x_0, x_1, x_2) = f(x_0, x_1)g(x_1, x_2)$ and $dh(x_0, x_1, x_2) = f(x_1, x_2)g(x_2, x_3) - f(x_0, x_2)g(x_2, x_3) + f(x_0, x_1)g(x_1, x_3) - f(x_0, x_1)g(x_1, x_2)$ which agrees with $df \cup g - f \cup dg = (f(x_1, x_2) - f(x_0, x_2) + f(x_0, x_1))g(x_2, x_3) - f(x_0, x_1)(g(x_1, x_2) - g(x_1, x_3) + g(x_2, x_3))$. We see that if $df = 0$ and $dg = 0$, then $d(f \cup g) = 0$. Lets take the example, where $G = (V, E)$ is the octahedron, where $V = \{a, b, c, d, p, n\}$ and E consists of 12 edges. Define first a 1-form f which is equal to 1 on the equator $a \to b \to c \to d \to a$ and zero everywhere else. Assume that the remaining edges are oriented so that all point to the north or south pole n rsp p. The second 1-form g is now chosen to be 1 on each of the remaining 8 edges and zero on the equator. We can fix an orientation now on the triangles (x, y, z) so that $f \cup g(x, y, z) = 1$ for every triangle. This is a nonvanishing 2-form is a discrete area form and showing that $\text{cup}(G) = 2$. Of course, one has also $\text{tcap}(G) = 2$ since we can find two in G contractible sets which cover G. The function h given by $h(p) = 0, h(a) = 1, h(b) = 2, h(c) = 3, h(d) = 4, h(n) = 5$ has exactly two critical points $\{n, p\}$ because p is the minimum with $i_h(x) = \chi(S^-(p)) = 1 - \chi(\emptyset) = 1$ and n is the maximum with $i_h(n) = 1 - \chi(S^-(n)) = 1 - \chi(\{a, b, c, d\}) = 1 - \chi(C_4) = 1$. This is an example, where both Ljusternik-Schnirelman inequalities are equalities: $\text{cup}(G) = \text{tcap}(G) = \text{crit}(G)$.

Calculus of variations. One can study various variational problems for graphs. The most important example is certainly the Euler characteristic $\chi(G)$. It actually can be seen as a discrete Hilbert action [29]. Network scientists look at the characteristic length $\mu(G)$ or the mean clustering $\nu(G)$. One can also look at the dimension $\iota(G)$, edge density $\epsilon(G)$, scale measure $\sigma(G)$, or spectral complexity $\xi(G)$ of a graph.

There is a lot of geometry involved. The cluster coefficient $C(x)$ for example is closely related to characteristic length. If μ denotes the mean distance and ν is the mean clustering coefficient and η is the average of scalar curvature $S(x)$, a formula $\mu \sim 1 + \log(\epsilon)/\log(\eta)$ of Watts and Strogatz [42] relates μ with the edge density ϵ and average scalar curvature η telling that large curvature implies small average length, a fact familiar for spheres. We often see statistical relations $\mu \sim \log(1/\nu)$ holds for random or deterministic constructed networks, indicating that small clustering is often associated to large characteristic lengths. Clustering ν, edge density ϵ and curvature average η therefore can relate with average length μ on a statistical level.

Pedagogy. Graphs are intuitive objects. We know them from networks like street maps or subway networks. While calculus needs some training, especially in higher dimensions, the fundamental theorem of calculus in any dimension can be formulated and proven in a couple of minutes within graph theory. Without any technicalities, it illustrates the main idea of Stokes theorem. While differential forms need some time to be mastered by students learning analysis, functions on simplices of graphs are very concrete in the sense that we always can get concrete results. A couple of lines in a computer algebra system provides code, which universally works in principle for any finite simple graph. Procedures spitting out a basis for the cohomology groups for example is convenient since we just have to compute the eigenvalues and eigenvectors of a concrete matrix L and take the eigenvectors belonging to the eigenvalue 0. The subject therefore has **pedagogical merit**, also in physics, where wave and heat evolution on geometric spaces can be visualized. If the spaces are discretizations of manifolds, the discretization provides a numerical scheme. Of course, the classical numerical methods to solve PDEs can do this often more effectively but then, one has in each case to deveop code for specific situations. Evolving waves in two and three dimensions effectively for example needs completely different techniques. The graph case is simple because the a dozen lines can deal with an arbitrary graph, as long as the machine can handle the matrices involved. For the computation of cohomology for example, we can first homotopically simplify the graph as good as possible, then find a good Čech cover and then deal with a much smaller nerve graph.

Quantum calculus. Graph theory leads to relations between discrete mathematics, analysis and discrete differential geometry, analysis, topology and algebra. In a calculus setting, it leads to calculus without limits which is also called **quantum calculus** because of commutation relations $[Q_e, P_e] = i$ which position $Q_e = q(x)\sigma^*$ and momentum operators $P_e f(x) = i[\sigma, f] = [f(x+e) - f(x)]\sigma$ attached to an edge $e = (x, y)$, where $q(y) = 1$ and $q(x) = 0$. For every path in the graph γ we can define Q_γ and have a **linear approximation formula** $f(z) = f(x) + D_e f(x)$ for neighboring points x, z and a **Tylor formula** x, z if we chose a path from x to z and have $f(z) = f(x) + D_e f(x) + D_{e_1} D_{e_2} f(x)/2! + ... = \exp(-iD)f$. This is analogue to the continuum where the Taylor equation means $\exp(-itP)f(x) = f(x+t)$ if $P = i\partial_x$ is the momentum operator. Discrete Taylor formulas have already been known to Newton and his contemporary Gregory and are now mostly known in the context of numerical analysis. One can see the analysis of graphs as a "quantization" since position and momentum operators stop to commute if space is discretized.

Illustrations. Graph theory allows to illustrate how different fields of mathematics like algebra, calculus, analysis, topology, differential geometry, number theory or algebraic geometry can interplay. Graph theory is also an applied topic in mathematics with strong roots in computer science, where networks or geometric meshes in computer graphics are considered which are in general very complex. Social networks, biological networks, the web, or triangularizations of surfaces used to display computer generated images are examples. Many mathematicians look at graphs as analogues of Riemann surfaces, as one dimensional objects therefore. Similarly like the GAGA principle seeing correspondences between algebraic and analytic geometry, there is principle which sees graphs and one dimensional possibly complex curves and varieties as on the same footing. But there is more to it: we can treat in many respects graphs like Riemannian manifolds or varieties. It is also a place, where Riemann themes come together: **Riemannian manifolds**, **Riemann-Roch**, **Riemann-Hurwitz**; studying the **Riemann Zeta function** on cyclic graphs leads naturally to **Riemann sums**. Which is one reason why I like the topic because Riemann is one of my favorate mathematicians.

The point of view. Most mathematicians in geometric graph theory consider graphs as objects of **one-dimensional nature**: edges are one-dimensional arcs which connect the zero-dimensional vertices. Indeed, many results from algebraic curves parallel in graph theory. An example is the Riemann-Roch theorem [1] or the Nowakowski-Rival fixed point theorem of 1979 which is a special case of a discrete Brouwer fixed point theorem [24]. The Riemann-Roch theorem is of more **algebro-geometric** nature, fixed point theorems are by nature interesting in **dynamical systems theory** or **game theory**. In the last couple of years, the point of view started to shift and geometric questions which are traditionally asked for higher dimensional geometric objects started to pop up in graph theory. An example is **discrete Morse theory** and **discrete differential geometry**. Morse theory deals studies critical points of functions on a graph and relates them to geometry. Morse inequalities relate cohomology with the number of critical points of certain type, relating so algebra with analysis. An other example is **category theory** in topology, which LS-category is a fascinating and central topic in topology because it links analysis, topology and algebra. An algebraic notion of how "rich" a space is the **cup length**. It is defined in the cohomology ring of a graph. The topological notion is called LS-category which roughly tells with how many contractible sets one can patch up a space. The third notion is analytic and gives the minimal number of critical points which a function can have on a graph.

Higher dimensional structures. The fact that notions of curvature, homotopy, degree or index [24], or critical points can be defined for graphs so that classical results hold also in the discrete is a strong indication that there is more higher dimensional structure on a graph than anticipated. Until recently, this was only studied for smaller dimensions. For triangularizations of two-dimensional surfaces for example, there is a simple notion of curvature which involves the degree of the graph. This has been known for a while [6, 12, 10] but is probably much older in special cases. The triangular lattice with hexagonal symmetry where every node has uniform degree 6 for example is the prototype of a "flat" geometry. The Gromov type curvature for planar graphs might have proposed in [49]. This curvature was generalized to arbitrary finite simple graphs [17]. As indicated in [18], first order notions of curvature can be refined. One can slso average first order notions over smaller neighborhoods to get more refined notions. New definitions of curvature based on probabilistic Markov chain concepts were introduced in [44] and pursued further in work like [51]. The Olivier curvature has proven to be fruitful and is under heavy investigation. It looks promising to get results close to classical differential geometry like that. The proposal under consideration uses other curvatures and plans to investigate the relations between different curvatures. We have seen that virtually any topic in differential geometry can be studied in the discrete. Gauss-Bonnet, Riemann-Hurwitz, fixed point theorems, Poincaré-Hopf, differential equations, isospectral graphs for the Dirac operator, integrable systems obtained by doing isospectral deformations, zeta functions for graphs or a McKean-Singer result using the heat kernel in graph theory.

Networking with structures. Since finite simple graphs are networks, there are connections with network geometry, which is itself close to fields like combinatorics, statistical mechanics or complexity theories. What is particularly exciting is that the topic allows with modest techniques to illustrate how different fields of mathematics can overlap and play together. Gauss-Bonnet illustrates how the metric property of curvature is not topological but related to the topological property Euler characteristic. The probabilistic relation with Poincaré-Hopf shows a integral geometric statistical angle. Integral calculus has penetrated classical differential geometry a long time ago and there are generations of mathematicions like Blaschke-Chern-Banchoff which illustrate this. Ljusternik-Schnirelmann shows that algebraic, topological and analytic notions work together. The cohomologically defined cup product, the topologically defined cup length using covers and the analytically defined minimal number of critical points. In geometry, analysis deals with spaces of functions on a geometric space and is a theory called functional analysis. Calculus and differetial topology deals with especially critical points. In the discrete as well as in more general situations like metric spaces, a notion homotopy leads immediately leads to a notion of critical points as points where the part of small spheres for which the function value is smaller are not contractible.

The Dirac operator. An other important connection is through the Dirac operator D which is more fundamental than the Laplacian $L = D^2$ because its symmetry group is much larger than the one of the Laplacian. It allows to be deformed in an isospectral way without that the Laplacian is affected. This means that these internal symmetries do not affect classical quantum mechanics but do affect the wave equation in a particular way. Space expands. We can see that in the D'Alembert solution of the wave equation where D appears in the solution. In the expanded space, we have to prepare larger initial velocities to get to the same solution. One can see the expansion also in the Connes reformulation of Riemannian geometry. Also in the Riemannian manifold case we have isospectral deformations of the Dirac operator $d + d^*$ and so a deformation of the exterior derivative which allows geometric evolutions to happen. Cohomology of course does not change. This is natural since these deformations are fundamental symmetries of the underlying geometry. Both in the continuum as well as in the discrete it produces an expansion of the metric with inflationary start.

What's next? As mentioned in [22], one goal is to have a translation of Atiah-Singer which can be taught without much technical overhead. Already Gauss-Bonnet can be seen as a very special case of an index theorem, where the analytic index of the Dirac matrix $D : \Omega_{even} \to \Omega_{odd}$ is the cohomological Euler characteristic and the average curvature can be seen as a topological index. While Gauss-Bonnet-Chern is an almost trivial case in the discrete, the next step is Hirzebruch-Riemann-Roch. It has been stated as an open problem in [1] and it is not expected to be so simple also in the discrete because more structure is needed like higher dimensional versions of divisors and an adaptation of sheaf cohomology. Since the discrete case of classical theorems like Gauss-Bonnet or Poincaré-Hopf is so short that full proofs could be presented in the first part of this 20 minutes Baltimore talk and where concrete and short computer algebra implementations [37, 36] containing detailed code doing this this for arbitrary graphs, I would not be surprised to finally see and understand Riemann-Roch in a discrete setup. This means that we can program it for an arbitrary graph and divisor and get two quantities from it, which the theorem shows to be equal. While I had to learn as a student the Patodi proof of Gauss-Bonnet-Chern in [5], I don't understand Riemann-Roch yet in higher dimensions. It is an other level of difficulty and there are so many different aspects of the theorem. Thinking about the discrete could be a way to learn it. Having mentioned positive curvature graphs, there is a whole more to be explored, if the positive curvature assumption is relaxed to average positive curvature assumptions for subgraphs. One combinatorial question in the positive curvature case is to make a list of all positive curvature graphs in dimensions d and to get sphere theorems. The most exciting problem by far is to understand the isospectral deformation of the Dirac operator. It is important because the deformation also works for Riemannian manifolds. The deformed exterior derivative defines a deformed geometry on the graph or manifold. The question is: what geometry are obtained asymptotically if the expansion is rescaled?

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