

ON ATIYAH-SINGER AND ATIYAH-BOTT FOR FINITE ABSTRACT SIMPLICIAL COMPLEXES

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ABSTRACT. A linear or multi-linear valuation on a finite abstract simplicial complex can be expressed as an analytic index $\dim(\ker(D)) - \dim(\ker(D^*))$ of a differential complex $D : E \rightarrow F$. In the discrete, a complex D can be called elliptic if a McKean-Singer spectral symmetry applies as this implies $\text{str}(e^{-tD^2})$ is t -independent. In that case, the analytic index of D is $\chi(G, D) = \sum_k (-1)^k b_k(D)$, where b_k is the k 'th Betti number, which by Hodge is the nullity of the $(k + 1)$ 'th block of the Hodge operator $L = D^2$. It can also be written as a topological index $\sum_{v \in V} K(v)$, where V is the set of zero-dimensional simplices in G and where K is an Euler type curvature defined by G and D . This can be interpreted as a Atiyah-Singer type correspondence between analytic and topological index. Examples are the de Rham differential complex for the Euler characteristic $\chi(G)$ or the connection differential complex for Wu characteristic $\omega_k(G)$. Given an endomorphism T of an elliptic complex, the Lefschetz number $\chi(T, G, D)$ is defined as the super trace of T acting on cohomology defined by D and G . It is equal to the sum $\sum_{v \in V} i(v)$, where V is the set of zero-dimensional simplices which are contained in fixed simplices of T , and i is a Brouwer type index. This Atiyah-Bott result generalizes the Brouwer-Lefschetz fixed point theorem for an endomorphism of the simplicial complex G . In both the static and dynamic setting, the proof is done by heat deforming the Koopman operator $U(T)$ to get the cohomological picture $\text{str}(e^{-tD^2}U(T))$ in the limit $t \rightarrow \infty$ and then use Hodge, and then by applying a discrete gradient flow to the simplex data defining the valuation to push $\text{str}(U(T))$ to the zero dimensional set V , getting curvature $K(v)$ or the Brouwer type index $i(v)$.

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1. SIMPLICIAL COMPLEXES

1.1. A **finite abstract simplicial complex** is a finite set G of non-empty sets which is closed under the operation of taking finite non-empty subsets. A set $x \in G$ with $k + 1$ elements is called a **k -simplex** or k dimensional face; its **dimension** is k . If $H \subset G$, we say H is a **sub-complex**. The set of sub-complexes of G is a **Boolean lattice** because both the union and intersection of a complex is a complex and the empty complex is a complex. We often just write **simplicial complex** for a finite abstract simplicial complex.

1.2. An example of a simplicial complex is the **Whitney complex** of a finite simple graph (V, E) . In that case, $G = \{A \subset V \mid \forall a, b \in A, (a, b) \in E\}$. The **Barycentric refinement** G_1 of a complex G is the subset G_1 of the power set 2^G consisting of sets of sets in G , in which any pair a, b either satisfies $a \subset b$ or $b \subset a$. It is the Whitney complex of the graph $(V, E) = (G, \{(a, b) \mid a \subset b \text{ or } b \subset a\})$. In topology, one can therefore mostly focus on Whitney complexes of graphs which are more intuitive than sets of sets. The Barycentric argument shows that almost nothing is lost by looking at Whitney complexes of graphs.

1.3. Not all simplicial complexes are Whitney complexes. We can for example truncate a given complex at dimension d , removing all sets of cardinality larger than $d + 1$ to get the **d -dimensional skeleton complex** of G . For a Whitney complex of a graph, such a skeleton is no more a Whitney complex in general. Take $G = K_3$ for example which is $G = \{(1, 2, 3), (1, 2), (2, 3), (1, 3), (1), (2), (3)\}$. The 1-dimensional skeleton is the subcomplex $H = \{(1, 2), (2, 3), (1, 3), (1), (2), (3)\}$ which is no more the Whitney complex of a graph. The subcomplex H is a discrete circle with Euler characteristic 0 while the complex G itself is a two dimensional disc with Euler characteristic 1. Complexes can appear in other ways also. The **graphical complex** of a graph consists of all non-empty forests in G , subgraphs for which every connected component is a tree. As any non-empty subset of a forest is a forest, this is a simplicial complex. More general **graph complexes**, where the sets are subsets of the edge set are considered in [2], which is also a good introduction for abstract simplicial complexes.

2. VALUATIONS

2.1. Assume we are given a simplicial complex G . An integer-valued function X from the Boolean lattice of sub-complexes to the integers is called a **valuation** if $X(A \cup B) + X(A \cap B) = X(A) + X(B)$. Examples of valuations are $v_k(H)$ counting the number of k -simplices in

H. We don't really insist in general to have X integer valued. One could assume X to take values in an Abelian group. According to the **discrete Hadwiger theorem** [3], the linear space of valuations of a complex is $(d + 1)$ -dimensional, where d is the maximal dimension of the complex. A basis is $\{v_k(G)\}_{k=0}^d$. If $f(A) = (v_0(A), \dots, v_d(A))$ is the **f -vector** of A , then $X(A) = \langle X, f(A) \rangle = \sum_{i=0}^d X_i v_i(A)$.

2.2. An example of a valuation is the **Euler characteristic** $\chi(G) = \sum_{x \in G} \omega(x)$ with $\omega(x) = (-1)^{\dim(x)}$ for a simplex x . It is invariant under Barycentric refinements and comes from the only eigenvector of eigenvalue 1 of the transpose S^T of the **Barycentric refinement operator** $S_{ij} = i! \text{Stirling}_2(j, i)$ mapping the f -vector of G to the f -vector of its Barycentric refinement G_1 . Other examples of valuations are the number of 0-dimensional points in G or the **volume**, the number of **facets**, sets in G of largest cardinality $d + 1$ if G has dimension d . Also the other eigenvectors of the Barycentric refinement operator can be of topological interest. For geometric graphs, discrete manifolds, in particular, half of the eigenvectors lead to valuations which are zero. They are related to **Dehn-Sommerville** invariants. On the other hand, a **Betti number** $A \rightarrow b_i(A)$ is no valuation in general.

2.3. A **multi-linear valuation** X is function of $G^k = G \times G \times \dots \times G$ which is a valuation in each of the k coordinates [12]. Examples of **bilinear valuations** are $v_{kl}(G)$, counting the number of ordered pairs of simplices (a, b) of dimension k and l which intersect. By a generalization of the discrete Hadwiger theorem, if G has dimension d , then the valuations $v_{k,l}(G)$ with $0 \leq k \leq l \leq d + 1$ form a basis for the linear space of bilinear valuations. If $v_{kl}(G)$ is the symmetric **f -matrix** encoding the intersection data, then every bilinear valuation can be written $X(A, B) = \sum_{i,j} X_{ij} v_{ij}(G)$, where X_{ij} is a symmetric $(d + 1) \times (d + 1)$ matrix. Similar statements hold for any k -linear valuation.

2.4. An example of a bilinear valuation is the **Wu intersection number** [12] $\omega(A, B) = \sum_{x \sim y} \omega(x) \omega(y)$, where the sum is over all ordered pairs of elements $x \in A, y \in B$ which intersect. The **Wu characteristic** is then $\omega(G) = \omega(G, G)$. Wu characteristic has many important properties: it is invariant under Barycentric refinements, if x is a simplex, then $\omega(x) = (-1)^{\dim(x)}$, and if G is a discrete manifold with $(d - 1)$ dimensional boundary, then $\omega(G) = \chi(G) - \chi(\delta G)$. Also higher order **Wu characteristics** ω_k exist. The first one, ω_1 , is the Euler characteristic χ , the second ω_2 is the bilinear Wu characteristic. Each of these characteristic has its own differential complex and its own cohomology.

This is useful, as now not only the valuations, but also the Betti numbers are combinatorial invariants. Unlike simplicial cohomology which is invariant under homotopy, the finer connection cohomologies related to ω_k are not.

3. CALCULUS

3.1. Given a finite abstract simplicial complex G , equipped with an orientation of the simplices, one can look at the linear space $\Lambda_k(G)$ of anti-symmetric functions on the set of k -simplices. The initial choice of the orientation of G is a basis selection. It is a gauge choice as custom in linear algebra which does not affect interesting quantities. The orientations on each simplex $x \in G$ do not have to be compatible so that every abstract simplicial complex G can be oriented. A complex can be called **compatibly orientable**, if there is a choice of orientation which is compatible: if $x \subset y$ are simplices in G , then the orientation of x is inherited from the orientation of y . As in the continuum, examples like the Möbius strip or Klein bottle show that not all complexes possess a **compatible orientation**. But every simplicial complex can be oriented similarly as any vector space can be equipped with a basis.

3.2. The space $\Lambda_k(G)$ is also called the space of **discrete k -forms** on G . It has dimension $v_k(G)$. We think of each $\Lambda_k(G)$ as a fiber bundle over G . We can extend a form f from sets in G to sub-complexes of G to get **signed valuations** $f(A) = \sum_{x \in A} f(x)$ which still satisfy $f(A \cap B) + f(A \cup B) = f(A) + f(B)$. Evaluating a signed valuation is **integration** $f(A) = \int_A f$.

3.3. The integration of signed valuations corresponds to **geometric measure theory**, while the integration of valuations corresponds to **geometric probability theory=integral geometry**. The former is orientation sensitive like line or flux integrals in school calculus. The later does not depend on the orientation and relates to integrations in calculus, like arc length or surface area. In the discrete, it can be important to be aware of the difference and distinguish integration of valuations and integration of signed valuations.

3.4. The **exterior derivatives** $d_k : \Lambda_k(G) \rightarrow \Lambda_{k+1}(G)$ are linear transformations which extend to a linear transformation d on the graded vector space $\Lambda = \bigoplus \Lambda_k$, the vector space of **discrete differential forms**. A differential form is just a function from G to \mathbb{R} satisfying $f(T(x)) = \text{sign}(T)f(x)$ for any permutation T of the simplex x . The boundary operation δ maps a sub complex A to its **boundary**

chain δA , where the orientation of δA is now compatible with the orientation of A . The image δA of a complex is not a complex any more in general. For $A = xu + yu + zu + x + y + z + u$ for example, then $\delta A = (u - x) + (u - y) + (u - z) = 3u - x - y - z$. A signed valuation f can be extended linearly to the group of chains on G however.

3.5. The exterior derivative d defines what we will call a discrete **elliptic complex** $D = (d + d^*) : E \rightarrow F$: the non-zero eigenvalues of the Hodge operator D^2 restricted to the space E of even forms, are in bijective correspondence with the eigenvalues of D^2 restricted to the space F of odd forms. When talking about an elliptic complex, we have three things: a linear map $D = d + d^*$ incorporating all exterior derivatives d_k , where $d^2 = 0$, a domain E and a co-domain F . The exterior derivatives d_k can be rather general however.

3.6. If f is a signed valuation, **Stokes theorem** is $f(\delta A) = df(A)$. For example, if A is a two-dimensional connected sub-complex of G equipped with a compatible orientation and having the property that every unit sphere $S(x)$ in A is either a circular graph C_n with $n \geq 4$ or a linear graph L_n with $n \geq 2$, then A is called a **surface**. The boundary δA is then a **curve**, a one-dimensional complex consisting of finitely many circular directed graphs. Stokes theorem is then the classical Stokes theorem from school calculus. If A is a sub-complex for which every unit sphere is either a 0-sphere (a 2-point graph without edges), or a 1-point graph P_1 , then every connected component has either 0 or 2 boundary points and $\int_A df = \int_{\delta A} f$ is the **fundamental theorem of line integrals**. If A is a sub-complex for which every unit sphere is a 2-sphere or a 2-disc, then the boundary δA is a discrete closed 2-manifold, a complex for which every unit sphere is a circular graph. In that case, Stokes theorem corresponds to the **classical divergence theorem**.

3.7. Every differential complex $D = d + d^*$ defines a **flavor of calculus**. In each case, Stokes theorem is the defining relation for the **boundary operation** δ . The boundary δx of a simplex x is now no more given by a collection of simplices or chain. It must be probed with functions $f(\delta x) = df(x)$. The boundary δx can now be more extended unlike in classical or in **connection calculus**, where in the quadratic case, the boundary δ acts on pairs of intersecting simplices as $\delta(x, y) = (\delta x, y) - (x, \delta y)$.

3.8. A example where the boundary δ is no more in sync with the geometric boundary is if $D(t) = d_t + d_t^* + b_t$ is an isospectral Lax deformation $D' = [B, D]$ with $B = d - d^*$ of the Dirac operator $d + d^*$.

[10, 9] This is a deformation $d_t + d_t^*$ of a complex. The boundary $\delta_t A$ of a geometric object A is now not a linear combination of simplices any more. Stokes theorem $f(\delta A) = df(A)$ is still a definition. But now, a traditional line integral of df along a closed loop is no more zero in general as a closed loop can not be written as a linear combination of boundaries of simplices any more. In the deformed calculus, fields have appeared. We see that if we let a geometric object evolve freely in its isospectral set, then the geometry dynamically produces sources for each space of differential forms. This works in the same way for Riemannian manifolds, where the deformed exterior derivative $d(t)$ is a pseudo differential operator describing an expanding space.

3.9. One can speculate that the isospectral deformation of the differential complex produces fields which are relevant in physics. Start with the classical point of view of Gauss writing a force field f as a **Poisson equation** $\text{div}(f) = -4\pi G\sigma$, where σ is the mass density. If the field f is a 2-form satisfying $df = 0$, then $0 = f(\delta A)$ for the **deformed boundary** of a region G . For a **classical boundary** $\delta_0 A$ of a ball A however, the flux $f(\delta_0 A)$ is not necessarily zero. In the deformed differential complex, it appears as if some field f has been generated from geometry alone. It would be good to explore how the strength of this field depends on the geometry and especially on the curvature.

4. DIFFERENTIAL COMPLEXES

4.1. A **discrete differential complex** is defined as a sequence of linear maps $d : E_k \rightarrow E_{k+1}$ with $d_{k+1}d_k = 0$ with $E = \bigcup_k E_{2k}$ and $F = \bigcup_k E_{2k+1}$. To be more concrete, the finite dimensional vector spaces E_k are required to be subspaces of tensor products of the de-Rham complex Λ_k or connection complex. We ask this so that the individual fibers are local. A complex defines a **Dirac operator** $D = \sum_i d_i + d_i^*$ with domain E and co-domain F . In the case of the de Rham complex Λ_k , we can take E the set of even forms and F the set of odd forms.

4.2. Given a differential complex, the analytic index of the Dirac operator $D : E \rightarrow F$. is defined as $\dim(\ker(D)) - \dim(\ker(D^*)) = \dim(E) - \dim(F)$. For example, for the connection complex of order n , for which $\Lambda_k(G)$ has as dimension the number of n -tuples of simultaneously intersecting simplices adding to dimension k , then the analytic index of D is the n 'th Wu characteristic. If $E = F = \Lambda(G)$, then the analytic index is zero as the kernel of D and D^* agree.

4.3. Examples.

- 1) Let E be the linear space of even differential forms and F the linear space of odd differential forms. The analytic index of $D = (d + d^*)$ is the Euler characteristic $\chi(G)$.
- 2) Let $E = \Lambda_k(G)$ and $F = \{0\}$. Then, the analytic index of $D = d_k$ is v_k .
- 3) Let $E = \Lambda = \bigoplus_k \Lambda_{2k}$ and $F = \{0\}$ and $D = \sum_i a_{2i} d_{2i}$ with $a_i \neq 0$, $d_i = 0$ for all i , then the analytic index of D is $\sum_i v_{2i}$. This is clearly not an elliptic complex.
- 4) The analytic index $d_0 + d_1$ mapping even forms $E = \bigoplus \Lambda_{2k}$ to odd forms $F = \bigoplus \Lambda_{2k+1}$ is $b_0 - b_1$, if G is equipped with the 1-skeleton simplicial complex. For $G = K_3$ with this complex $\chi(G) = 0$ as the two dimensional simplex does not count, the complex is a discrete circle.

4.4. If A is a sub simplicial complex of G , we have a **sub differential complex** $D|A : E|A \rightarrow F|A$. Any subcomplex A of G with the same base $V = \bigcup_{x \in G} x$ has its own Euler characteristic $\chi(A) = \sum_{x \in A} \omega(x)$ which can be written as $\sum_i (-1)^k b_i(A)$, where b_i are the Betti numbers of the sub differential complex.

Lemma 1. *For any discrete differential operator $D : E \rightarrow F$, the map which assigns to a sub-complex A the analytic index of $D|A$ is a valuation. Every linear or multi-linear valuation can be represented as an analytic index of some differential complex.*

Proof. The kernels $A \rightarrow \ker(d_i|A)$ of d_i are valuations. So are the kernels of d_i^* and the sums. To get a linear valuation $v_k(A)$ counting the number of k -dimensional simplices in A , take $E_k = \Lambda_k(G)$ and all other $E_j = 0$. Then let all $d_j = 0$. Now, $\dim \ker(d_k|A) - \dim \ker(d_k^*|A) = \dim(E_k) = v_k(A)$.

To get v_{ij} for example, let all E_k be zero except E_{i+j} which is the vector space of all functions on (x, y) with $\dim(x) = i, \dim(y) = j$. Let all $F_k = \{0\}$. Let all $d_k = 0$. Now, $\ker(D) - \ker(D^*) = v_{ij}$. □

4.5. The reason to focus on Fredholm indices rather than the nullity of the operator itself is that they have a chance of staying bounded in continuum limits and also because $i(AB) = i(A) + i(B)$. In finite dimensions, the Fredholm indices is just $i(A) = \dim(E) - \dim(F)$, independent of A . This follows from the **rank-nullity theorem** and the fact that the row and column ranks of a finite matrix A are the same.

4.6. In the discrete, Atiyah-Singer or Atiyah-Bott like results still have some interest as we can equate both with cohomological data as well as topological data with the valuation, at least if the complex is elliptic. Classically, ellipticity is defined by the **symbols** of the differential operators. Instead of trying to translate a continuum definition to the discrete, we have chosen to define ellipticity in the simplest way to have the proofs work.

4.7. A **differential complex** (D, E, F) defined by maps $\Lambda_k \rightarrow^{d_k} \Lambda_{k+1}$ is called an **elliptic complex** if $L = (d + d^*)^2$ has the property that the spectrum of non-zero eigenvalues of L on E is the same than the spectrum of non-zero eigenvalues of L on the odd forms F . We wrote "spectrum" rather than "set" to stress that also the multiplicities of the eigenvalues have to be the same. The simplest proof of McKean-Singer [15] relies on this symmetry [1] and can be adapted to the discrete [7].

5. THEOREMS

5.1. The discrete version stated here only requires knowledge of finite sets and finite matrices. It is a first attempt to emulate those classical theorems, risking of course to appear preposterous.

5.2. Given an elliptic differential complex $D : E \rightarrow F$ over a simplicial complex G . The **analytic index** of D is $\dim(\ker(D)) - \dim(\ker(D^*))$. The **cohomological index** of D is $\sum_i (-1)^k b_k(G, D, E, F)$, where b_k are the Betti numbers defined by the cohomology of D . The curvature of a pairwise intersecting simplex tuple $x = (x_1, \dots, x_k)$ is $\prod_j \omega(x_j)$. The curvature of a vertex $v \in V$ is defined as $K(v) = (\sum_{x \in G^k(v)} i(x)) / (\sum_{x \in G^k(v)} 1)$, where both sum is over the set $G^k(v)$ of pairwise intersecting k tuples $(x_1, \dots, x_k), v \in \bigcup x_j$. The **topological index** is then defined as $\sum_{v \in V(G)} K(v)$. In the Gauss-Bonnet case, $K(v) = 1 - \frac{V_0}{2} + \frac{V_1}{3} - \frac{V_2}{4} + \dots$, where $V_k(v)$ is the number of k -simplices in the **unit sphere** $S(x)$ (often called link in the simplicial complex literature). This formula [4] appeared already in [14]. Almost a hundred year old is the planar case where the curvature is $1 - d(v)/6$ with vertex degree $d(v)$ appeared in the context of graph colorings.

Theorem 1 (Atiyah-Singer like). *The analytic index of D is equal to the cohomological index and equal to the topological index.*

Proof. The super trace $\text{str}(e^{-tD^2})$ is independent of t . For $t = 0$ it is the super trace of 1 which is the analytic index of D . Now apply the heat kernel deformation to make the Euler-Poincaré equivalence to cohomological data. For $t \rightarrow \infty$, the non-zero eigenspace of D^2 dies

out and only the kernels survive. Since by Hodge, the dimensions of the kernels of L restricted to k -forms is $b_k(G)$, the super trace in the limit $t \rightarrow \infty$ is the cohomological index.

The topological index is obtained by pushing the defining combinatorial data located on simplices to the zero dimensional part of space. This can be done in various ways. The above definition of $K(v)$ does this by distributing each value equally to its vertices [4]. An other extreme case is Poicaré-Hopf [5] which lets the curvature flow along the gradient of a function f having the effect that curvature remains as integer index. \square

5.3. Atiyah-Bott is a Lefschetz type result which relates a cohomologically defined Lefschetz number with a sum of indices of fixed points of the endomorphism of an elliptic complex. The proof in [8] for Whitney complexes works for general simplicial complexes. Simpler is a heat deformation approach.

5.4. Let T be an automorphism of the elliptic complex. The **Lefschetz number** of T is the super trace of the linear map induced on cohomology. Let $\text{Fix}(T)$ be the set of elements in G which are fixed under T and let $V(T)$ denote the set of vertices which contain an element in $\text{Fix}(T)$. Define for $x \in \text{Fix}(T)$ the index $i(x)$ as $(-1)^{\dim(x)} \det(T|x) \text{Tr}(D|x)$. The index of $v \in V$ is then defined as $i(v) = \sum_{x \in G, v \in x} i(x) / (\dim(x) + 1)$.

Theorem 2 (Atiyah-Bott like). *The Lefschetz number of T is equal to the index sum over all fixed points.*

Proof. Also here, there are two deformations: the first Euler-Poincaré deformation equates a sum of fixed point indices with the Lefschetz number, the second, the Gauss-Bonnet or Poincaré-Hopf deformation expresses this Lefschetz number as an integral of curvature over space, where in the Poincaré-Hopf case, the curvature is a divisor. Averaging the curvature over all locally injective functions gives the Euler curvature. \square

5.5. In the elliptic case, the cohomological data do not change when making a topological deformation like a Barycentric refinement. The curvature data however change. This can be exploited for fixed point results. The Atiyah-Singer or Atiyah-Bott theorems allow for more flexibility as one can chose also the elliptic complex. The de Rham complex or the more general connection complexes are the guiding examples.

6. REMARKS

6.1. Both Atiyah-Singer and Atiyah-Bott are milestones in geometry which require a decent amount of technical background in functional analysis, differential geometry and topology [17, 19]. Heat approaches have been established in the continuum [16], first by V.K. Patodi. The above results have much less structure as they are defined for general abstract finite simplicial complexes and don't assume that the geometries have any manifold type. The analytic index $\dimker(D) - \dimker(D^T)$ for a finite dimensional linear operator $D : E \rightarrow F$ is always just $\dim(E) - \dim(F)$ and so independent of D . So, if we look at a discrete analogue of an elliptic complex, then the analytic index is already the combinatorial quantity under consideration.

6.2. Whether there is in the manifold case a refinement- averaging procedure which produces the classical results, is not clear at the moment. Especially the topological index has some flexibility still. The curvature should depend naturally on the elliptic complex. If $D = RD_0R^*$, where D_0 is the connection Dirac operator, then the curvature can be obtained as the expectation of the Poincaré-Hopf indices $E[i_f]$ where the measure on the functions is the push forward of the uniform measure by R . So at least in the case of a Hamiltonian deformation of the complex there is a natural way to deform the curvature by deforming the measure on the space of functions and getting curvature through a deformed expectation.

6.3. The mathematics underlying the geometry of finite sets is about a century older than Atiyah-Singer. Combinatorial versions hardly replace the continuum results but this note could one day lead to a **pedagogical entry point** into the topic. I personally still struggle to understand the Atiyah-Singer and the Atiyah-Bott index theorems. The theorems ¹ have not yet entered undergraduate courses. It will probably need an other half of a century to achieve that. And this is important: to cite Atiyah from [18]: *"The passing of mathematics on to subsequent generations is essential for the future, and this is only possible if every generation of mathematicians understands what they are doing and distills it out in such a form that it is easily understood by the next generation."*

¹We mean of course a treatment with full proofs.

6.4. The connection of discrete with the continuum needs still to be explored. Maybe the discrete case can in a limiting case become the continuum case. It is also possible that the discrete case remains a **caricature**². We think however that taking Barycentric limits combined with a suitable smoothing process can lead to classical differential operators which are Fredholm and have a finite index. But there are other battles which need to be fought first in the discrete.

6.5. An other link between the discrete and continuum is integral geometry: if $\gamma : [a, b] \rightarrow M, t \rightarrow r(t)$ is a smooth curve in a smooth connected manifold M , let $L(\gamma) = \int_a^b |r'(t)| dt$ denote its length. Given a probability space (Ω, P) of smooth functions ω on M one can look at the random variable $N_\gamma(\omega)$ counting the number of intersections of surfaces $\{\omega = 0\}$ with γ . Counting the number N_γ of transitions from $\omega \leq 0$ to $\omega > 0$ defines a **Crofton pseudo metric** $d(x, y) = \inf_{\gamma(x,y), N_\gamma \in L^1(\Omega, P)} \mathbb{E}[N_\gamma]$, where the infimum is taken over all curves connecting x with y with the understanding that $d(x, y) = \infty$, if there should be no γ for which N_γ is in L^1 . The **Kolmogorov quotient** (M_P, d_P) consists of all equivalence classes of the equivalence relation given by $d(x, y) = 0$. For discrete measures P , one gets like this discrete metric spaces and so finite simplicial complexes. Nash embedding M into an ambient Euclidean space and taking a rotationally invariant measure P leads to the Riemannian metric on M because it is the Euclidean metric in the ambient space. As curvature in the discrete can be expressed integral geometrically [6, 11], also Gauss-Bonnet type results should go over. Bridging the functional analysis of the Dirac and Laplace operators and the topological index both remain complicated tasks and as the continuum is technical, no real short cut might exist.

6.6. Elliptic differential complexes can be added and multiplied and so extended to a ring over a fixed simplicial complex G . As we can also look at the **strong ring** generated by simplicial complexes, there is an other possibility: extend the strong ring of simplicial complexes to a strong ring of differential complexes [13]. Now it appears that not only the category of differential complexes over simplicial complexes but also the sub-category of elliptic differential complexes is a cartesian closed category.

²There is an SNL sketch from 1992 with Tom Hanks of an “prize is right” show, where a **cordless telephone** is sold to contestants. The deal was a traditional phone, from which the cord has been cut.

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