

AN INDEX FORMULA FOR SIMPLE GRAPHS

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ABSTRACT. We prove that any odd dimensional geometric graph $G = (V, E)$ has zero curvature everywhere. To do so, we prove that for every injective function f on the vertex set V of a simple graph the index formula $\frac{1}{2}[1 - \chi(S(x))/2 - \chi(B_f(x))] = (i_f(x) + i_{-f}(x))/2 = j_f(x)$ holds, where $i_f(x)$ is a discrete analogue of the index of the gradient vector field ∇f and where $B_f(x)$ is a graph defined by G and f . The Poincaré-Hopf formula $\sum_x j_f(x) = \chi(G)$ allows so to express the Euler characteristic $\chi(G)$ of G in terms of smaller dimensional graphs defined by the unit sphere $S(x)$ and the "hypersurface graphs" $B_f(x)$. For odd dimensional geometric graphs, $B_f(x)$ is a geometric graph of dimension $\dim(G)-2$ and $j_f(x) = -\chi(B_f(x))/2 = 0$ implying $\chi(G) = 0$ and zero curvature $K(x) = 0$ for all x . For even dimensional geometric graphs, the formula becomes $j_f(x) = 1 - \chi(B_f(x))/2$ and allows with Poincaré-Hopf to write the Euler characteristic of G as a sum of the Euler characteristic of smaller dimensional graphs. The same integral geometric index formula also is valid for compact Riemannian manifolds M if f is a Morse function, $S(x)$ is a sufficiently small geodesic sphere around x and $B_f(x) = S(x) \cap \{y \mid f(y) = f(x)\}$.

1. INTRODUCTION

While for compact Riemannian manifolds M , the curvature $K(x)$ satisfying the Gauss-Bonnet-Chern relation $\int_M K(x) d\nu(x) = \chi(M)$ is defined only in even dimensions, the discrete curvature

$$K(x) = \sum_{k=0}^{\infty} (-1)^k \frac{V_{k-1}(x)}{k+1},$$

satisfying Gauss-Bonnet is defined for arbitrary simple graphs $G = (V, E)$. In the definition of curvature, $V_k(x)$ is the number of K_{k+1} subgraphs in the sphere $S(x)$ at a vertex x and $V_{-1}(x) = 1$. With this curvature, the Gauss-Bonnet theorem

$$\sum_{x \in V} K(x) = \chi(G)$$

holds [62], where $\chi(G) = \sum_{k=0}^{\infty} (-1)^k v_k$ is the **Euler characteristic** of the graph, and where v_k is the number of K_{k+1} subgraphs of G . Since for all geometric odd dimensional graphs we have seen, the curvature has shown to be constant zero, we wondered whether it is constant zero in general. For 5-dimensional geometric graphs for example, the curvature reduces to $K(x) = -E(x)/6 + F(x)/4 - C(x)/6$, where $E(x)$ is the number of edges, $F(x)$ the number of faces and $C(x)$ the number

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of three dimensional chambers in the unit sphere $S(x)$ of a vertex x . We show here that $K(x)$ is constant 0 for any odd dimensional graph for which the Euler characteristic of spheres is 2 at every point and for which every $S(x)$ inherits the property of being geometric.

The key tool to analyze the curvature $K(x)$ is to represent it as an expectation of an index [65]. This integral geometric approach shows that the curvature $K(x)$ of a finite simple graph $G = (V, E)$ is equal to the expectation of indices $i_f(x) = 1 - \chi(S_f^-(x))$ when averaging over a probability space of injective scalar functions f on V and where $S_f^-(x) = \{y \in V \mid f(y) < f(x)\}$. This links Poincaré-Hopf

$$\sum_{v \in V} i_f(v) = \chi(G)$$

with Gauss-Bonnet $\sum_{v \in V} K(v) = \chi(G)$ and gives a probabilistic interpretation of curvature as expectation $K(x) = 1 - \mathbb{E}[\chi(S_f^-(x))]$, where $\mathbb{E}[\chi(S_f^-(x))]$ is the average Euler characteristic of the subgraph $S_f^-(x)$ of $S(x)$, when integrating over all injective functions. Proof summaries of [62, 64, 65] are included in the Appendix A.

While Gauss-Bonnet, Poincaré-Hopf and index expectation hold in full generality for any finite simple graph, we can say more if graphs are geometric in the sense that unit spheres share properties from unit spheres in the continuum. Similarly as manifolds appear in different contexts like topological or differentiable manifolds, one can distinguish different types of geometric graphs by imposing properties on unit spheres $S(x)$ or larger spheres. In the present paper we see what happens if G is geometric in the sense that for all vertices x , the unit spheres $S(x)$ are $(d-1)$ -dimensional geometric graphs with Euler characteristic $1 - (-1)^d$.

The topological features of unit spheres in a graph define a discretized local Euclidean structure on a graph. For triangularizations of manifolds, the unit spheres are $(d-1)$ -dimensional graphs which share properties of the unit spheres in the continuum. Without additional assumptions, the unit spheres can be arbitrarily complicated because pyramid constructions allow any given graph to appear as a unit sphere of an other graph. A weak geometric restriction is to assume that all the unit spheres $S(x)$ are $d-1$ dimensional, have Euler characteristic $1 - (-1)^d$, and share the property of being geometric.

When trying to prove vanishing curvature for odd dimensional graphs, one is lead to an alternating sum $W_1(x) - W_2(x) + W_3(x) - \dots + (-1)^d W_{d-1}(x)$, where $W_k(x)$ is the number of k -dimensional complete graphs K_{k+1} in $S(x)$ which contain both vertices in $S^-(x) = \{y \mid f(y) < f(x)\}$ and $S^+(x) = \{y \mid f(y) > f(x)\}$. We show that W_k can be interpreted as the number $(k-1)$ -dimensional faces in a $d-2$ dimensional polytop $A_f(x)$, a graph which when completed is a $d-2$ dimensional geometric graph $B_f(x)$ and therefore has Euler characteristic 0 by induction if d is odd. When starting with a three dimensional graph for example, then $B_f(x)$ is a one dimensional geometric graph, which is a finite union of cyclic graphs.

This analysis leads to an index formula which holds for a general simple graph G :

$$(1) \quad j_f(x) = \frac{1}{2}[1 - \chi(S(x))/2 - \chi(B_f(x))],$$

where $B_f(x)$ is a graph defined by G, f and x . For geometric graphs G , the graph $B_f(x)$ is geometric too of dimension $d - 2$ and the formula simplifies to

$$(2) \quad j_f(x) = \frac{1}{2}[1 + (-1)^d - \chi(B_f(x))].$$

For odd dimensions d , it simplifies further to $j_f(x) = -\chi(B_f(x))/2 = 0$, while for even dimensions d , it becomes $j_f(x) = 1 - \chi(B_f(x))/2$.

The formula (2) holds classically for compact Riemannian manifolds M if $j_f(x) = [i_f(x) + i_{-f}(x)]/2$ for a Morse function f on M , where $i_f(x)$ is the classical index of the gradient vector field ∇f at x and where $S(x) = S_r(x)$ is a sufficiently small geodesic sphere of radius r . Here r can depend on f . The continuum version is simpler than in the discrete so that it can be given and proved in this introduction:

Let (M, g) be a compact d -dimensional Riemannian manifold and let $f \in C^2(M)$ be a Morse function. If $x \in M$ is a critical point of f with critical value $c = f(x)$, then for small enough $r > 0$, the formula (2) holds with $B_f(x) = S_r(x) \cap f^{-1}(c)$. Also if x is a regular point then the formula holds for small enough positive r and gives $j_f(x) = 0$.

Proof: if x is a critical point of f , then $i_f(x) = (-1)^{m(x)}$, where $m(x)$ is the Morse index, the number of negative eigenvalues of the Hessian of f at x . If d is odd, then $B_f(x)$ is a $d - 2$ dimensional manifold which must have Euler characteristic 0 and at a critical point x we have $j_f(x) = [(-1)^{m(x)} + (-1)^{d-m(x)}]/2 = 0$.

If d is even, then the Morse lemma tells that in suitable coordinates near x that the function is $f(x) = -x_1^2 - \dots - x_m^2 + x_{m+1}^2 + \dots + x_d^2$ and the level surfaces $B_f(x) = \{f = 0\}$ are locally homeomorphic to quadrics intersected with small spheres $S_r(x)$. The equations $2x_1^2 + \dots + 2x_m^2 = r^2, x_{m+1}^2 + \dots + x_d^2 = r^2$ show that this is $S_{m-1} \times S_{d-1-m}$ topologically. Therefore $\chi(B_f(x)) = 4$ or $\chi(B_f(x)) = 0$ depending on whether m is odd or even. This implies that $1 - \chi(B_f(x))/2 = -1$ if m is odd and that it is equal to 1 if m is even. If x is a regular point for f , then $B_f(x)$ is a $d - 2$ dimensional sphere which has Euler characteristic 2 and $1 - \chi(B_f)/2 = 0$. An alternative proof uses a property of Euler characteristic and the Poincaré-Hopf index formula for $i_f(x)$: since

$$S(x) = \{f(y) \leq f(x)\} \cup \{f(y) = f(x)\} \cup \{f(y) \geq f(x)\},$$

we have by the inclusion-exclusion principle

$$\begin{aligned} 0 = 1 - (-1)^d &= \chi(S(x)) = \chi(S_f^-(x)) - \chi(B_f(x)) + \chi(S_f^+(x)) \\ &= 1 - i_f(x) + 1 - i_{-f}(x) - \chi(B_f(x)) = 2 - 2j_f(x) - \chi(B_f(x)), \end{aligned}$$

where $S_f^-(x) = \{y \in S(x) \mid f(y) \geq f(x)\}$. (In the discrete graph case, the \leq in $S_f^-(x)$ is equivalent to $<$ because f is assumed to be injective). Since $r(x) > 0$ can be chosen to be continuous in x , there exists by compactness of M for every f an $r_f > 0$ so that the index formula holds for all $x \in M$ if $0 < r < r_f$.

Examples:

- 1) For $d = 2$ dimensional surfaces M , the space $B_f(x)$ is a discrete set on the circle $S_r(x)$, where f is zero. If we denote the cardinality with $s_f(x)$, the formula tells $j_f(x) = 1 - s_f(x)/2$. For a Morse function on the surface M , there are three possibilities: either f is a maximum or minimum, where $s(x) = 0$ and $j_f(x) = 1$ or f has a saddle point in which case $s(x) = 4$ and $j_f(x) = -1$.
- 2) For a 3-dimensional manifold, the formula tells $j_f(x) = -\chi(B_f(x))/2$ and $B_f(x)$ is either empty or consists of two circles, the intersection of a 2 dimensional cone with a sphere. In either case, the Euler characteristic is zero.
- 3) Proceeding inductively, since $B_f(x)$ is $(d - 2)$ -dimensional, we get inductively with Poincaré-Hopf that all odd dimensional manifolds have zero Euler characteristic.
- 4) For a four dimensional manifold, where $j_f(x) = 1 - \chi(B_f(x))/2$, we either have that $B_f(x)$ is empty which implies $j_f(x) = 1$ or that it is the union of a two 2 dimensional spheres which implies $j_f(x) = -1$. Or that it is a single 2 dimensional sphere so that $j_f(x) = 0$.

In order to prove formula (2) for graphs it is necessary to define a discrete version of a **hyper surface** $G_f = \{f = 0\}$ in a graph. For any simple graph $G = (V, E)$ and any nonzero function f on the vertex set there is such a graph G_f or G_f is empty. We see examples for some random graphs in Figure 6. The idea is motivated but not equivalent to an idea used in statistical mechanics, where "contours" are used to separate regions of spins in a ferromagnetic graph. We use the edges of the graph which connect vertices for which f takes different sign. In graph theory jargon, the graph G_f is a subgraph of the line graph of G . The mixed edges become the vertices of G_f and are connected if they are in a common mixed triangle of G .

What will be important for us is to see that if G is geometric, then G_f can be completed to become a geometric graph G'_f . For a d -dimensional geometric graph G , the new graph G'_f is then $d - 1$ dimensional. We will apply this construction to hyper surfaces $S(x)_f$ in the unit spheres $S(x)$ of a d dimensional graph G and call $B_f(x)$ the completion of $S(x)_f = A_f(x)$. Since $S(x)$ is $(d - 1)$ -dimensional, the graphs $B_f(x)$ are $(d - 2)$ -dimensional. We see some pictures of $G_f(x)$ when G is three-dimensional in Figures 1, and where G is four dimensional in Figure 2.

To conclude this introduction with the remark that it is quite remarkable how close simple graphs and Riemannian manifolds are. While in the previous papers on Gauss-Bonnet and Poincaré-Hopf as well as in integral geometric representation of curvature, the discrete story is much simpler, we see here that in the more geometric part, we have to work harder in the discrete case. The continuous analogue has been proven above. But we should note that in the continuum, the above analysis relies partly on Morse theory.

2. GEOMETRIC GRAPHS

In this section, we define geometric graphs inductively and formulate the main result. The induction starts with one-dimensional graphs which are called geometric, if every unit sphere $S(x)$ in G has Euler characteristic 2. A one-dimensional geometric graph is therefore a finite union of cyclic graphs. For a two dimensional

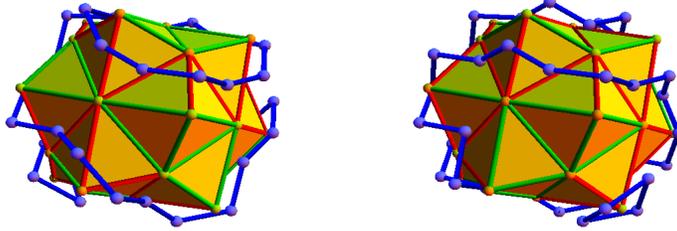


FIGURE 1. Two examples of the one-dimensional polytop $B_f(x)$ for two random functions defined by the unit sphere of a three dimensional graph G . We see the two dimensional unit sphere $S(x)$ of a vertex x , where triangles have been filled in for clarity and above it the one dimensional graph $B_f(x)$. Every vertex of $B_f(x)$ belongs to mixed edges of $S(x)$ and two vertices in $B_f(x)$ are connected, if they are part of a mixed triangle in $S(x)$. For odd dimensional graphs the Euler characteristic of $B_f(x)$ is proportional to the symmetric index $j_f(x)$ in the original graph. It is zero, implying that the three-dimensional graph G has zero curvature.

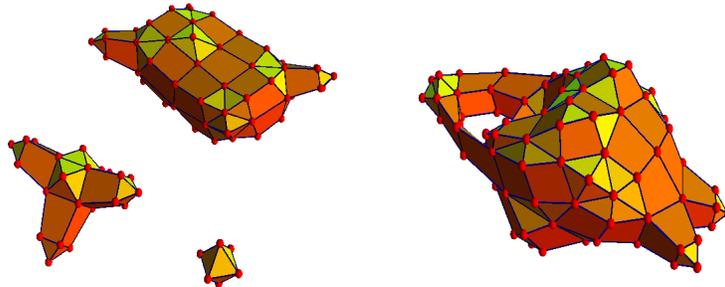


FIGURE 2. Two examples of the polytop $B_f(x)$ for random functions defined on a four dimensional graph G . The left example has three components and $\chi(B_f(x)) = 6$. The right example has one component and a hole so that $\chi(B_f(x)) = 0$. The three dimensional unit sphere $S(x)$ of a vertex x is not displayed in this figure. We only see examples of the "surface" $B_f(x)$. We have $j_f(x) = 1 - \chi(B_f(x))/2$ and the curvature $K(x)$ is the expectation $E[j_f(x)]$ when integrating over the finite dimensional parameter space of functions f , for which all $f(x_i)$ are independent random variables.

geometric graph, each sphere $S(x)$ must be a one dimensional cyclic graph. In general:

Definition. A graph is called a **geometric graph** of dimension d , if the dimension $\dim(x)$ is constant d for every vertex v and if each unit sphere $S(x)$ is a $(d - 1)$ -dimensional geometric graph satisfying $\chi(S(x)) = 1 - (-1)^d$. A one-dimensional graph is geometric if every unit sphere has Euler characteristic 2.

Remarks.

1) We could assume connectivity of each sphere $S(x)$ for $d \geq 2$ in order to avoid Hensel type situations like gluing two icosahedra at a single vertex. The unit sphere at this vertex would then consist of two disjoint cyclic graphs. But this does no matter for us in this paper, so that we do not require connectivity of $S(x)$.

2) Geometric graphs resemble topological manifolds because their unit spheres have the same Euler characteristic as small spheres in a d -dimensional manifold. The octahedron and icosahedron are geometric graphs of dimension $d = 2$. While not pursued in this paper, it can be useful to strengthen the assumption a bit and assume that the unit spheres share more properties from the continuum. Euler characteristic, dimension and connectedness are not sufficient. The following definition is motivated by the classical Reeb's theorem: a graph G is called **sphere-like** if it is the empty graph or if the minimal number $m(G)$ of critical points among all Morse functions is 2 and inductively every unit sphere $S(x)$ in G is sphere-like. Inductively, a geometric graph is called **manifold like** if every unit sphere $S(x)$ is sphere like and manifold like. In dimensions $d \geq 4$, there are geometric graphs which are not manifold-like because their unit spheres can have the correct Euler characteristic different from a sphere. For us, it is enough to assume that the graph is geometric in the sense that we fix the dimension and the Euler characteristic as well as assume that $S(x)$ is geometric for every vertex $x \in V$.

3) A cyclic graph C_n is geometric for $n > 3$. The complete graphs K_n are not geometric because the unit spheres have Euler characteristic 1. From the 5 platonic solids, the octahedron and icosahedron are geometric. The 600 cell is an example of a three dimensional sphere type graph and so geometric because every unit sphere is an icosahedron which is sphere like. We can place G into R^4 with an injective function $r : V \rightarrow R^4$. For most unit vectors v the function $f(x) = v \cdot r(x)$ is injective with two critical points and f restricted to the unit spheres has the same properties.

Theorem 1 (Zero index for odd dimensional geometric graphs). *For any odd-dimensional geometric graph $G = (V, E)$ and any injective function f on V , the symmetric index function $j_f(x)$ is zero for all $x \in V$.*

The proof of Theorem 1 is given later after polytopes are introduced. The theorem implies the main goal of this article:

Corollary 2. *Any odd dimensional geometric graph G has constant zero curvature $K(x) = 0$ and zero Euler characteristic.*

Proof. From the fact that curvature is equal to the index expectation [65] and since $j_f(x)$ is zero, the curvature is zero. From Gauss-Bonnet or Poincaré-Hopf, we see that the Euler characteristic is zero. \square

Remarks.

- 1) The result implies that for odd dimensional geometric graphs with boundary, the curvature is confined to the boundary. For a one dimensional interval graph I_n for example with $v = n + 1$ vertices and $e = n$ edges with boundary v_0, v_n the curvature is $1/2$ at the boundary points and adds up to $\chi(I_n) = v - e = 1$.
- 2) The previous remark also shows that for odd dimensional orbigraphs, geometric graphs modulo an equivalence relation on vertices given by a finite group of graph automorphisms, the curvature is not necessarily zero in odd dimensions. Geometric d dimensional graphs with $d - 1$ dimensional boundary are special orbigraphs because the interior can be mirrored onto the "other side" leading to an orbigraph defined by a Z_2 graph automorphism action. Because orbigraphs are just special simple graphs, curvature and Euler characteristic are defined as before and Gauss-Bonnet of holds as it is true for any simple graph.
- 3) The zero Euler characteristic result is known for triangularizations of odd dimensional manifolds by Poincaré duality or Morse theory. There are higher dimensional geometric graphs in the sense defined here which are not triangularizations of odd dimensional manifolds. For such exotic geometric graphs, the unit sphere has the correct dimension and Euler characteristic but is not a triangularization of a sphere. For such geometric graphs, zero Euler characteristic does not directly follow from the continuum.

3. POLYTOPES

Graphs like the cube or dodecahedron need to be completed to become 2-dimensional. To do so, one can triangulate some polygonal faces. A triangulation of a face means that a subgraph which is the boundary of a ball is replaced with the ball. In two dimensions in particular, this means to replace geometric cyclic subgraphs using pyramid extensions. This changes the Euler characteristic. We call the polytop Euler characteristic the Euler characteristic of the as such completed graph.

The K_n graphs have dimension $n - 1$ but are not geometric graphs. They can not be completed in the above way. It can be desirable therefore to extend the operations on polyhedra a bit. A tetrahedron G for example is a three dimensional graph because each unit sphere is a triangle which is two dimensional. It is not a geometric graph because the Euler characteristic of the unit sphere is always 1. If we truncate the vertices and replace each vertex with a triangle, we obtain a two dimensional graph which we can completed to become a two dimensional geometric graph. Such considerations are necessary when giving a comprehensive definition of "polytop" which is graph theoretical and agrees with established notions of "polytop". In this paper, graphs K_n are not considered polytopes.

Definition. Given an arbitrary simple graph G , we can define a new graph by adding one vertex z and adding vertices connecting z with each vertex v to G . The new graph is called a **pyramid extension** of G .

Remarks.

- 1) The order of the pyramid extension exceeds the order of G by 1 and the size of the extension exceeds the old size by the old order.
- 2) The pyramid construction shows that any graph can appear as a unit sphere of a larger graph.

Lemma 3. *The pyramid construction defines from a d -dimensional geometric graph G a $d + 1$ dimensional geometric graph. The extended graph always has Euler characteristic 1.*

Proof. The construction changes the cardinalities as follows v_0 changes to $v_0 + 1$ and v_1 changes to $v_1 + v_0$, v_2 becomes $v_2 + v_1$ etc until v_d becomes $v_d + v_{d-1}$ and $v_{d+1} = 0$ becomes v_d .

Therefore,

$$\chi(G') = \chi(G) + 1 - (v_0 - v_1 + \cdots + (-1)^d v_d) = 1 .$$

□

Examples. The pyramid construction of a zero dimensional discrete graph P_n is a star graph, a tree. The pyramid construction of a cycle graph C_n is a wheel graph W_n . The pyramid construction of the complete graph K_n is the complete graph K_{n+1} .

Definition. A graph G is called a **d -dimensional polytop**, if we can find a finite sequence of completion steps such that the end product is a d -dimensional geometric graph G' called the **completion** of G . We require that the end product G' does not depend on the completion process. A single **completion step** takes a $0 < k < d$ dimensional geometric subgraph of G and performs a pyramid construction so that it becomes a $k + 1$ dimensional geometric graph.

Remarks.

- 1) The subgraphs considered in the completion step need to have dimension smaller than d because otherwise, we could take any already d -dimensional geometric graph and add an other completion step and get a geometric graph of dimension $d + 1$.
- 2) The completion of a geometric graph is the graph itself. This can be seen by induction on dimension, because if we complete a subgraph, then we complete each unit sphere in that subgraph. In other words, a graph in which we can make a completion step is not geometric yet.
- 3) It follows that the completion of a completed graph G' is equal to G' .

Definition. The **polytop Euler characteristic** of a polytop G is defined as the Euler characteristic of the completed graph G' .

Examples.

- 1) The polytop Euler characteristic of the cube is 2 because the completed graph has $v = 14$ vertices, $e = 36$ edges and $f = 24$ faces and $v_0 - v_1 + v_2 = v - e + f = 2$. The graph Euler characteristic was $8 - 12 = -4$ due to the lack of triangles. The 6 squares are $k = 1$ -dimensional subgraphs. The completion has added 6 faces of Euler characteristic 1 and we can count $f_0 - f_1 + f_2 = 8 - 12 + 6 = 2$. This is also an example to Proposition 5.
- 2) A wheel graph W_k with $k \geq 4$ is a two-dimensional polytop because we can make a pyramid construction on the boundary which leads to a polyhedron. For $G = W_4$, the completed graph G' is the octahedron. Because the subgraph C_4 which was completed is odd dimensional, the completion has increased the Euler characteristic to 2.
- 3) A two-dimensional graph G is called a geometric graph with boundary, if a vertex is either an interior point, where the unit sphere $S(x)$ is a cyclic graph or a

boundary point, for which the unit sphere $S(x)$ is an interval graph. The boundary of G is a one dimensional graph, a union of finitely many cyclic graphs. We can replace each of these with a corresponding wheel graph to get a geometric graph without boundary or what we call a geometric graph for short.

4) The complete graph K_n and especially the triangle K_3 is not a polytop in the above sense. To enlarge the class of polytopes (but not done in this paper) if we would have to allow first the truncation of corners.

5) A hypercube K_2^4 initially has $v = 16$ vertices and $e = 32$ edges but no triangles nor tetrahedral parts, so that the graph Euler characteristic is $v - e = -16$. To complete it, replace each of the 8 cubes with its completed two dimensional versions, then add 8 central points each reaching out to the 14 vertices of the completed two dimensional cubes, to get a 3 dimensional geometric graph. The polytop Euler characteristic is $v - e + f - c = 0$ with $v = 48, e = 240, f = 384, c = 192$. We could have computed the polytop Euler characteristic more quickly by counting faces. There are $v = 16$ vertices, $e = 32$ edges, and 24 square faces and 8 cubic faces. The polytop Euler characteristic is again $16 - 32 + 24 - 8 = 0$.

4. PRODUCT GRAPHS

In this section we look at graph products of complete graphs K_n and show in Proposition 4 that if both have positive dimension, then the product graph is a polytop.

Definition. The **graph product** $G \times H$ of two graphs G, H is a graph which has as the vertex set the Cartesian product of the vertex sets of G and H and where two new vertices (v_1, w_1) and (v_2, w_2) are connected if either

- i) $v_1 = v_2$ and w_1, w_2 are connected in H or
- ii) $w_1 = w_2$ and v_1, v_2 are connected in G .

Examples.

- 1) Figure 3 shows for example the prism $K_2 \times K_3$.
- 2) The product graph of two cyclic graphs $C_k \times C_l$ with $k, l \geq 4$ is called a grid graph. It is a graph G of constant curvature -1 with Euler characteristic $\chi(G) = -kl$. It can be completed by stellating the square faces to become a two dimensional graph with Euler characteristic 0.

Definition. A graph is called a **d -dimensional face** if it appears as a unit ball $B_1(x)$ of a geometric d -dimensional graph G .

Remark.

Since $B_1(x)$ is the pyramid construction of $S(x)$, a d dimensional face always has Euler characteristic 1.

Examples.

- 1) Every complete graph K_{d+1} is a d -dimensional face.
- 2) Given a $d - 1$ dimensional geometric graph G , then its pyramid extension G' is a d -dimensional face.
- 3) Every wheel graph W_n is a two dimensional face. Any two dimensional face is either a triangle K_3 or a wheel graph W_n with $n \geq 4$.

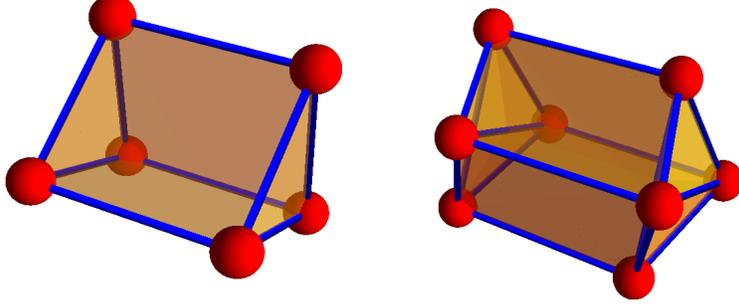


FIGURE 3. The left figure shows the graph $K_2 \times K_3$ is a two-dimensional polytope. The 3 faces with 4 vertices can be completed. The right figure shows the graph $K_2 \times K_4$ which is a three dimensional polytope. To complete it, first complete the quadrilaterals, then complete the three dimensional spaces.

The next lemma tells that the product graph of two complete graphs is a polytope and that then an other pyramid construction of the completion renders it a face.

Proposition 4. *For $l, m > 0$, the completion K' of the graph $K_{l+1} \times K_{m+1}$ is a $d = l + m - 1$ -dimensional polytope of graph Euler characteristic $1 - (-1)^d$. A further pyramid construction produces a $l+m$ -dimensional face, a graph \overline{K}' of Euler characteristic 1. If $l = 0$ or $m = 0$, then $K_{l+1} \times K_{m+1}$ is already a $l+m$ -dimensional face of Euler characteristic 1.*

Proof. For $l, m = 1$, we get $K_2 \times K_2$, where the product is a one dimensional square C_4 which is already geometric. It is a polytope because the completion is unique. A pyramid construction produces the 2 dimensional wheel graph W_4 .

To show the claim for general l, m we proceed by induction. Assume it is proven for (l, m) it is enough to cover the case $(l + 1, m)$ since the other case $(l, m + 1)$ is similar. For any subgraph $H = K_{l+1}$ of K_{l+2} , we can by induction uniquely complete $H \times K_{m+1}$ to become a $l + m$ dimensional face. All these faces are part a $l + m$ dimensional geometric graph. A further pyramid construction produces a $l + m + 1$ dimensional face of Euler characteristic 1. \square

The examples $K_2 \times K_3$ and $K_2 \times K_4$ are shown in Figure (3). The graph $K_2 \times K_3$ is a prism, a two dimensional geometric graph. A pyramid construction renders it into a three dimensional face.

We will now count faces in the $(d-2)$ -dimensional geometric graph $B_f(x)$ which are obtained from a d -dimensional graph $G = (V, E)$ and a function $f : V \rightarrow \mathbf{R}$. The graph $B_f(x)$ is made of faces obtained from products $K_k \times K_l$, where $k + l = d - 2$. For example, if G is four dimensional, then each face in $B_f(x)$ consists of triangles $K_3 \times K_1$ and squares $K_2 \times K_2$. The dimension of $B_f(x)$ is 2 and the Euler characteristic of each face is 1.

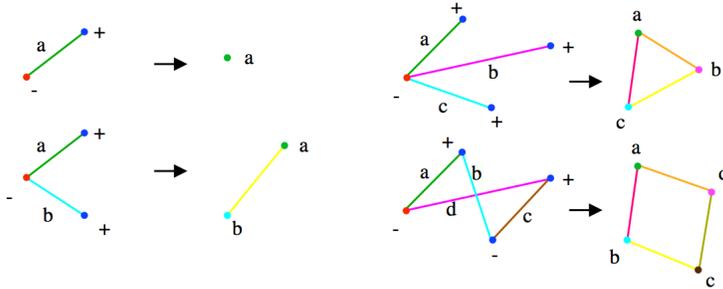


FIGURE 4. The left upper figure illustrates that every edge in G defines a point in G_f . The left lower figure illustrates that a mixed triangle in G defines an edge in G_f . The upper right figure shows the situation when only one of the vertices in the tetrahedron K_4 have a different sign. This leads to a triangle in G_f . The lower right figure shows that if two have positive and two have negative sign, then we get a square $K_2 \times K_2$ in G_f . We see in figure (2) examples of polyhedra which were obtained like that. All faces are triangles $K_3 \times K_1 = K_3$ or squares $K_2 \times K_2$.

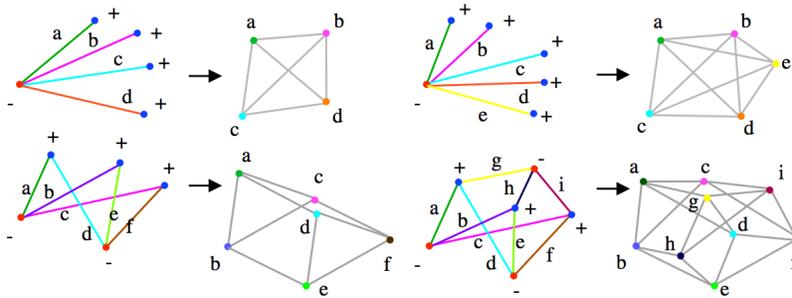


FIGURE 5. The W_4 subgraphs K_5 in $S(x)$ define either tetrahedra $K_4 \times K_1$ or prisms $K_3 \times K_2$. The W_5 subgraphs K_6 in $S(x)$ define either $K_5 \times K_1$ graphs or prisms $K_4 \times K_2$ (not shown) or a product $K_3 \times K_3$ which is a graph of order 9.

If G is a polytop with completion G' , the polytop Euler characteristic of G was defined as $\chi(G')$. The later can also be computed by counting faces:

Proposition 5. *Assume G is a polytop with completion G' and assume that G' is even dimensional. If f_k is the number of k dimensional faces in G' , then*

$$\chi(G') = \sum_{i=0}^{\infty} (-1)^k f_k .$$

Proof. The Euler characteristic $\chi(G')$ is a sum $\sum_k (-1)^k v_k$, where v_k counts the complete subgraphs K_{k+1} of G' . Every k -dimensional face has Euler characteristic 1. Counting the K_{k+1} subgraphs of the completed graph G' gives the same result as counting the number f_k of k -dimensional faces with $(-1)^k f_k$. When writing the

Euler characteristic 1 of each face as an alternating sum of cardinalities of complete subgraphs, we double count each boundary of a face. But each of these boundaries are geometric $2d - 1$ dimensional graphs with Euler characteristic 0 if G was even dimensional. \square

Example: the dodecahedron G has the graph Euler characteristic $\sum_{k=0}^{\infty} (-1)^k v_k = 20 - 30 = -10$ because there are no triangles. The graph G is a "sphere with 12 holes". The polytop Euler characteristic is $32 - 90 + 60 = 2$ because the two dimensional completed graph G' has $20 + 12 = 32$ vertices, $30 + 60 = 90$ edges and $5 \cdot 12 = 60$ triangles so that the polytop Euler characteristic is $\chi(G') = 2$. This is the Euler characteristic which Descartes has counted: $f_0 = 20, f_1 = 30, f_2 = 12$ and $\chi(G') = 20 - 30 + 12 = 2$. In this example, each face is a wheel graph W_5 with 6 vertices, 10 edges and 5 triangle. Each double counted boundary is the cyclic graph C_5 .

This concludes the **graph theoretical definitions** of dimension, geometric graph, completion, polytop, face, Euler characteristic and polytop Euler characteristic.

5. THE GRAPH G_f

In this section we define for a general graph $G = (V, E)$ and any function $f : V \rightarrow R$ a new graph G_f . We will see that if G is geometric of dimension d then G_f has a completion which is geometric of dimension $d - 1$. The graph G_f is a discrete analogue of the "hyper surface" $\{f = 0\} = f^{-1}(0)$ in a Riemannian manifold M . The vertices of G_f consist of edges of G , where f changes sign. The edges of G_f are triangles of G in which f takes different values. We will see that while the new graph G_f is not geometric, it can be completed to become geometric.

Given a graph $G = (V, E)$ and a function $f : V \rightarrow R$ which is nowhere zero, we can partition the vertex set V into two sets $V_f^+ = \{x \mid f(x) > 0\}$ and $V_f^- = \{x \mid f(x) < 0\}$ of cardinality s and t .

Definition. A subgraph H of G is called **mixed**, if it contains both vertices from V_f^+ and V_f^- .

Two mixed edges in a triangle forces the triangle to be mixed and the third edge to be not mixed. Every mixed triangle of G therefore has exactly 2 edges in V_f^+ or two edges in V_f^- .

Definition. Given an simple graph $G = (V, E)$ and a nonzero function f on V , a new **hypersurface graph** G_f is defined as follows:

- The vertices of G_f are the mixed edges of G .
- The edges of G_f are the mixed triangles in G .

Each mixed triangle connects exactly two mixed edges.

Examples:

1) If G is the octahedron and the function is positive on two antipodal points and negative everywhere else, then G_f is the union of two cyclic graphs C_4 . If f is positive everywhere or negative everywhere, then G_f is empty. If f is positive only on one vertex, then G_f is a cyclic graph C_4 . We can also realize C_6 . Because G is a Hamiltonian graph, we can find a coloring so that $G_f = C_8$.

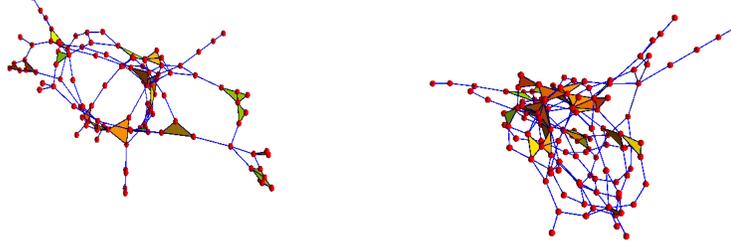


FIGURE 6. The figure shows the graph G_f for two random graphs G in the Erdős-Renyi probability space $G(40, 0.3)$. The function f was also chosen randomly. The dimension of G_f tends to be smaller than $\dim(G) - 1$. For geometric graphs of dimension d , the dimension of the completion of G_f is exactly $d - 1$.

- 2) If $G = K_3$ and $s = 1$ or $s = 2$ then $G_f = K_2 \times K_1$ which is K_2 .
- 3) If $G = K_4$ and $s = 1$ or $s = 3$ then $G_f = K_3$. If $s = t = 2$, then $G_f = K_2 \times K_2$.
- 4) If $G = K_5$ then we have a tetrahedron with $4 = \binom{4}{1}$ vertices if $s = 1$ or $t = 1$. If $s = 2$ or $s = 3$ we have a prism with 6 vertices.
- 5) For $G = K_6$ and $s = 2$ or $s = 4$ we have 8 vertices in B_f . It is a prismatic graph connecting two tetrahedra. If $s = 3$, we have 9 vertices and 18 edges. This is $K_3 \times K_3$.

The next lemma tells that any hypersurface graph G_f of a complete graph K_k is a product of two complete graphs. It is therefore either a complete graph or a polytop as defined in the last section.

Lemma 6. *For a complete graph $G = K_k = (V, E)$ and any nonzero function $f : V \rightarrow \mathbb{R}$, the graph G_f is isomorphic to the graph product $K_s \times K_t$, where $s = |V_f^+|$ and $t = |V_f^-|$. The graph G_f is a polytop in this case.*

Proof. Use induction with respect to in k . The induction starts at K_2 , where for $s = t = 1$, the graph G_f is K_1 . Assume the claim is settled for k and all $s + t = k$. Take now $G = K_{k+1}$ and s, t satisfying $s + t = k + 1$. To use the induction assumption, we assume v is the additional vertex added to K_k to get K_{k+1} . We can assume $f(v) > 0$ because the other case is similar. Let v_1, \dots, v_{s+1} be the vertices in V_+ and let w_1, \dots, w_t be in V_- . We want to show that the new graph G_f is $K_{s+1} \times K_t$. The graph $G_{f,k}$ defined by K_k is a subgraph of $G_{f,k+1}$ the graph defined by K_{k+1} . We additionally have got t new vertices (v, w_j) in G_f . The order of G_f and $K_{s+1} \times K_t$ are the same. The new graph gets new edges (v, v_i, w_j) which connect all the $s \cdot t$ old vertices (v_i, w_j) with the new t vertices (v, w_j) . There are no connections between the new vertices in the same way as $K_s \times K_t$ is extended to $K_{s+1} \times K_t$. \square

We have now seen that for any d -dimensional geometric graph and any nonzero function f , the graph G_f is a polytop which can be completed to become a $(d-1)$ dimensional geometric graph. For $d = 1$, this can be seen easily because a one dimensional geometric graph has no triangles and the graph G_f has no edges and the dimension of G_f therefore is uniformly 0. For a two-dimensional geometric graph, the graph G_f is a union of closed cycles. Some notation:

Definition. For unit spheres $G = S(x)$ and injective f we use the name $A_f(x)$ for the graph $S(x)_g$ with $g(y) = f(y) - f(x)$. The injectivity of f implies that g is nonzero. We denote by $B_f(x)$ the completion of $A_f(x)$.

6. THE INDEX FORMULA

The main result in this paper is:

Theorem 7 (Index formula). *If $G = (V, E)$ is a simple graph and f is an injective function on V then*

$$j_f(x) = [1 - \chi(S(x))/2 - \chi(B_f(x))]/2 .$$

The proof of Theorem 7 is given below. First a lemma:

Lemma 8 (Counting W_k). *The number W_k of mixed k -dimensional simplices K_{k+1} in G satisfies the formula*

$$\chi(G'_f) = W_1 - W_2 + W_3 - \dots - W_{2d} ,$$

where G'_f is the completion of G_f .

Proof. We know that each face is of the form $K_s \times K_t$ with $s + t = k$ which can be completed. The faces of an odd dimensional geometric polytop G_f fit together and form a completed polytop G'_f . Therefore, the sum $W_1 - W_2 + W_3 - \dots + (-1)^{2d-1} W_{2d}$ the polytop Euler characteristic of G_f which is $\chi(G'_f)$. See Proposition 5. \square

Corollary 9. *If $G = (V, E)$ is a d dimensional geometric graph, then*

$$\chi(G) = \sum_x (1 + (-1)^d)/2 - \chi(B_f(x))/2 .$$

Epecially,

$$\chi(G) = - \sum_{x \in V} \chi(B_f(x))/2$$

for odd dimensional graphs G and

$$\chi(G) = \sum_{x \in V} 1 - \chi(B_f(x))/2$$

for even-dimensional graphs G

Here is the proof of Theorem 7:

Proof. Adding

$$i_f^-(x) = (1 - \chi(S^-(x)))$$

and

$$i_f^+(x) = (1 - \chi(S^+(x)))$$

gives

$$2j_f(x) = [2 - \chi(S^-(x)) - \chi(S^+(x))] = 2 - \chi(S(x)) - \sum_{k=1}^{\infty} (-1)^k W_k(x) .$$

By Lemma 8 applied to the sphere $S(x)$, we see that $\sum_{k=1}^{\infty} (-1)^k W_k(x)$ is the polytop Euler characteristic of $S(x)_f = A_f(x)$ which is equal to $\chi(B_f(x))$. It follows that

$$j_f(x) = 1 - \chi(S(x))/2 - \chi(B_f(x))/2 .$$

□

The proof of Corollary 9 follows immediately:

Proof. Use Theorem 7 and the assumption $\chi(S(x)) = 1 - (-1)^d$ if G is d dimensional. □

Now the proof of the Theorem 1:

Proof. We use induction with respect to d . Assume we know that the symmetric index $j_f(x)$ is zero everywhere for all $(d-2)$ -dimensional graphs. Then the Euler characteristic of any $d-2$ dimensional geometric graph is zero by Poincaré-Hopf

$$\sum_{x \in V} j_f(x) = \chi(G) .$$

Because the dimension d is odd, we have $\chi(S(x)) = 2$ so that $1 - \chi(S(x))/2 = 0$. The sum simplifies therefore to

$$2j_f(x) = - \sum_{k=1}^{\infty} (-1)^k W_k(x)$$

which is of course a finite sum. By Lemma 8, this is the polytop Euler characteristic of a $(d-2)$ -dimensional polytop and so zero by induction. □

Examples:

1) For the 3D cross polytop (the pyramid construction of an octahedron), we have $v_0 = 8, v_1 = 24, v_2 = 32, v_3 = 16$ with $\chi = v_0 - v_1 + v_2 - v_3 = 0$. and $V_0 = 6, V_1 = 12, V_2 = 8$ at every vertex x . The curvature by definition is

$$K(x) = V_{-1}/1 - V_0/2 + V_1/3 - V_2/4 = 1 - 6/2 + 12/3 - 8/4 = 1 - 3 + 4 - 2 = 0 .$$

2) For a 5D cross polytop G , with $v_0 = 12$ vertices, $v_1 = 60$ edges, $v_2 = 160$ triangles and $v_3 = 240$ tetrahedra and $v_4 = 192$ spaces and $v_5 = 64$ halls. The Euler characteristic is $12 - 60 + 160 - 240 + 192 - 64 = 0$. We can compute W_k data for a typical function f and check that $\sum_{v \in V} \sum_{k=1}^5 (-1)^{k+1} W_k(v) = 0$.

APPENDIX A

Here are the proofs of the three preprints [62, 64, 65]. The index expectation result appears here slightly generalized in that the probability measure on functions is allowed to have a general continuous distribution. Let v_k the number of complete K_{k+1} subgraphs in G . Denote by $V_k(x)$ the number of K_{k+1} subgraphs in the sphere $S(x)$ with the convention $V_{-1}(x) = 1$.

Lemma 10 (Transfer equations). $\sum_{x \in V} V_{k-1}(x) = (k+1)v_k$.

Proof. This generalizes Euler's hand shaking lemma $\sum_{x \in V} V_0(x) = 2v_1$: Draw and count handshakes from every vertex to every center of any k -simplex in two different ways. A first count sums up all connections leading to a given vertex, summing then over all vertices leading to $\sum_{x \in V} V_{k-1}(x)$. A second count is obtained from the fact that every simplex has $k+1$ hands reaching out and then sum over the simplices gives $(k+1)v_k$ handshakes. \square

Define curvature at a vertex x as $K(x) = \sum_{k=0}^{\infty} (-1)^k \frac{V_{k-1}(x)}{k+1}$.

Theorem 11 (Gauss-Bonnet). $\sum_{x \in V} K(x) = \chi(G)$.

Proof. From the definition $\sum_{x \in V} K(x) = \sum_{x \in V} \sum_{k=0}^{\infty} (-1)^k \frac{V_{k-1}(x)}{k+1}$ we get by (10)

$$\sum_{x \in V} K(x) = \sum_{k=0}^{\infty} \sum_{x \in V} (-1)^k \frac{V_{k-1}(x)}{k+1} = \sum_{k=0}^{\infty} (-1)^k v_k = \chi(G).$$

\square

Given f , let $W_k(x)$ denote the number of all mixed k simplices in the sphere $S(x)$, simplices for which $f(y)$ takes both values smaller and larger than $f(x)$.

Lemma 12 (Intermediate equations). $\sum_{x \in V} W_k(x) = kv_{k+1}$

Proof. For each of the v_{k+1} simplices K_{k+2} in G , there are k vertices x which have neighbors in K_{k+2} with both larger and smaller values. For each of these k vertices x , we can look at the unit sphere $S(x)$ of v . The simplex K_{k+2} defines a k -dimensional simplex K_{k+1} in that unit sphere. Each of them adds to the sum $\sum_{x \in V} W_k(x)$ which consequently is equal to kv_{k+1} . \square

Define $i_f(x) = 1 - \chi(S_f^-(x))$, where $S_f^-(x) = \{y \in S(x) \mid f(y) < f(x)\}$.

Lemma 13 (Index stability). *The index sum $\sum_{x \in V} i_f(x)$ is independent of f .*

Proof. Deform f at a vertex x so that only for $y \in S(x)$, the value $f(y) - f(x)$ changes from positive to negative. Now $S^-(x)$ has gained a point y and $S^-(y)$ has lost a point. To show that $\chi(S^-(x)) + \chi(S^-(y))$ stays constant, show that $V_k^+(x) + V_k^-(y)$ stays constant, Since $i(x) = 1 - \sum_k (-1)^k V_k^-(x)$, the lemma is proven if $V_k^-(x) + V_k^-(y)$ stays constant. Let $U_k(x)$ denote the number of K_{k+1} subgraphs of $S(x)$ which contain y . Similarly, let $U_k(y)$ denote the number of K_{k+1} subgraphs of $S(y)$ which do not contain x but are subgraphs of $S^-(y)$ with x . The sum of K_{k+1} graphs of $S^-(x)$ changes by $U_k(y) - U_k(x)$. Summing this over all vertex pairs x, y gives zero. \square

Theorem 14 (Poincaré-Hopf). $\sum_{x \in V} i_f(x) = \chi(G)$.

Proof. The number $V_k^-(x)$ of k -simplices K_{k+1} in $S^-(x)$ and the number $V_k^+(x)$ of k -simplices in $S^+(x)$ are complemented within $S(x)$ by the number $W_k(x)$ of mixed k -simplices. By definition, $V_k(x) = W_k(x) + V_k^+(x) + V_k^-(x)$. By Lemma 13, the index $i_f(x)$ is the same for all injective functions $f : V \rightarrow \mathbf{R}$. Let $\chi'(G) = \sum_{x \in V} i_f(x)$. By the symmetry $f \leftrightarrow -f$ switching $S^+ \leftrightarrow S^-$, we can prove $2v_0 -$

$\sum_{x \in V} \chi(S^+(x)) + \chi(S^-(x)) = 2\chi'(G)$ instead. Lemma 10 and Lemma 12 give

$$\begin{aligned} \chi'(G) &= v_0 + \sum_{k=0}^{\infty} (-1)^k \sum_{x \in V} \frac{V_k^-(x) + V_k^+(x)}{2} = v_0 + \sum_{k=0}^{\infty} (-1)^k \sum_{x \in V} \frac{V_k(x) - W_k(x)}{2} \\ &= v_0 + \sum_{k=0}^{\infty} (-1)^k \frac{(k+2)v_{k+1} - kv_{k+1}}{2} = v_0 + \sum_{k=1}^{\infty} (-1)^k v_k = \chi(G). \end{aligned}$$

□

Define a probability space Ω of all functions from V to \mathbf{R} , where the probability measure is the product measure and where $f(x)$ has a continuous distribution. The later assures that injective functions have probability 1. Denote by $E[X]$ the expectation of a random variable X on (Ω, P) .

Lemma 15 (Clique stability). *For all $x \in V$ and all $k \geq 0$,*

$$E[V_{k-1}^-(x)] = \frac{V_{k-1}(x)}{k+1}.$$

Proof. Since $V_{k-1}(x)$ is the number of K_k subgraphs of $S(x)$ and every node is knocked off with probability p , we have $V_{k-1}^-(x) = p^k V_{k-1}(x)$. The statement in the lemma is proven if we integrate this from 0 to 1 with respect to p . □

Theorem 16 (Index expectation = curvature). *For every vertex x , the expectation of $i_f(x)$ is $K(x)$:*

$$E[i_f(x)] = K(x).$$

Proof.

$$\begin{aligned} E[1 - \chi(S^-(x))] &= 1 - \sum_{k=0}^{\infty} (-1)^k E[V_k^-(x)] = 1 + \sum_{k=1}^{\infty} (-1)^k E[V_{k-1}^-(x)] \\ &= 1 + \sum_{k=1}^{\infty} (-1)^k \frac{V_{k-1}(x)}{(k+1)} = \sum_{k=0}^{\infty} (-1)^k \frac{V_{k-1}(x)}{(k+1)} = K(x). \end{aligned}$$

□

The curvature function $K(x)$ can depend on the probability measure P of functions but does not if the function values at different vertices are independent.

APPENDIX B

Here are some references.

Poincaré proved the index theorem in the eighth chapter of [86]. Hopf extended it to arbitrary dimensions in [54] and mentions that Hadamard had stated the result without proof in 1910. [71] mentions some contribution of Brouwer. The result has been generalized by Morse to manifolds with boundary, where the vector field is directed inwards at the boundary [74] and generalized further in [90, 60]. More about the history of Poincaré-Hopf is contained in [97, 71, 13, 44, 53].

The index of a vector field traces back to a winding number defined by Cauchy in 1831 for holomorphic functions. Cauchy in 1937 generalized the notion to differentiable self maps of the plane. Kronecker generalized the index to C^1 maps of R^n

in 1869 [30]. The index of a vector field at an isolated singularity p is traditionally defined as the degree of the self map $x \rightarrow F(p + \epsilon x)/|F(p + \epsilon x)|$ on the sphere. The degree of a continuous map $f : N \rightarrow N$ was first defined topologically in [14] and can be traced back to Kronecker [44]. It can for smooth maps be computed as $\sum_{y \in f^{-1}(x)} \text{sign}(\det(df(y)))$ at a regular value x .

The Euler curvature for Riemannian manifolds was first defined by Hopf in his 1925 thesis for hypersurfaces [54, 18] where it is the product of the product of principal curvatures. The general definition is due to Allendoerfer [4] and independently by [37] for Riemannian manifolds embedded in an Euclidean space. Since we know today that this includes all Riemannian manifolds, the Allendoerfer-Fenchel is the general definition. It is also called Lipschitz-Killing curvature [100] or Euler curvature. It is now part of a larger frame work, the Euler class being one of the characteristic classes.

The first higher dimensional generalization of Gauss-Bonnet was obtained by Hopf (and earlier by Dyck in a special case in 1888 [85, 44]) for compact hypersurfaces in R^{2n+1} who showed that the degree of the normal map of a hypersurface is $\chi(M)/2$ [97], a formula which can be seen topological since it does not involve curvature. Gauss-Bonnet in higher dimensions was obtained first independently by Allendoerfer [4] and Fenchel [37] for surfaces in Euclidean space using tube methods and extended jointly by Allendoerfer and Weil [3] to closed Riemannian manifolds which from modern perspective is the full theorem already thanks to the Nash's embedding theorem (i.e. [49]). Chern gave the first intrinsic proof in [17]. The proof is given in [45, 97, 85]. See [25, 93] for a textbook versions of Patodi's proof using Witten's deformed Laplacian. Gauss-Bonnet has a long history. Special cases (see [43]) are the fact known by ancient Greeks that the sum of the angles in a triangle are π , Descartes lost theorem about the sum of the polyhedral curvatures in a convex polyhedron or Legendre's theorem stating that the sum of the angles in a spherical triangle in a unit sphere is π plus the area of the triangle. Gauss generalized this Harriot-Girard theorem [73] to geodesic triangles on a surface proving that the integral of Gauss curvature over the interior is the angle defect. For expositions of polyhedral Gauss-Bonnet results see [87, 6]). By triangularization, this Gauss-Formula leads to Gauss-Bonnet too [97].

Discrete differential geometry is useful in many parts of mathematics like computational geometry [28, 8], the analysis of polytopes [94, 22, 47, 57], integrable systems [11], networks [20, 78, 103]. computer graphics or computational-numerical methods [26, 82, 105].

For Morse theory see [70, 71, 72] or any differential geometry textbook like [48, 97, 53, 51, 31, 101, 58, 9, 67]. A discrete Morse theory for cell complexes which is closer to classical differential geometry is developed in [39, 40]. There, Morse theory is built for simplicial complexes. It is close to classical Morse theory and many classical results like the Reeb's sphere theorem is proven in this framework. The theory has concrete applications in graph theory.

Discrete differential geometry is used also in gravitational physics [27, 91, 24, 42, 75] and statistical mechanics [16] and other parts [27]. The elementary approach to discrete differential geometry followed here is graph theoretical [12, 7, 19] and therefore different from the just mentioned approaches which include more structure, usually structure from Euclidean embeddings like angles or lengths. These concepts are always related because any simplicial complex defines a graph and conversely, graph are embedded in Euclidean space to model geometric objects there. Setting things up purely graph theoretically can reveal how much is combinatorial and independent of Euclidean embeddings.

Discrete curvature is by Higushi traced back to a combinatorial curvature defined in [46]. [Added May 4, 2012: the formula $K(p) = 1 - V_1(p)/6$ and for graphs on the sphere appears in [88, 89], where it is also pointed out that $\sum_p K(p) = 2$ is Gauss-Bonnet formula.] It is there up to normalization defined by $K(p) = 1 - \sum_{j \in S(p)} (1/2 - 1/d_j)$, where d_j are the cardinalities of the neighboring face degrees for vertices j in the sphere $S(p)$. For two dimensional graphs, where all faces are triangles, this simplifies to $d_j = 3$ so that $K = 1 - |S|/6$, where $|S|$ is the cardinality of the sphere $S(p)$ of radius 1 appears in [63]. Discrete curvature was used in [52] who uses a definition of an unpublished talk by Ishida from 1990. Higushi's curvature is used in large-scale data networks [77] Gauss-Bonnet for planar graphs appears in the form $\sum_n (6 - n)f_n = 12$ in [33] and in [12] in the form $\sum_g C(g) = -12$ where $C(g) = d(g) - 6$ is called the **charge** at x , where $d(g)$ is the degree. Estimates for the size of graphs with positive combinatorial curvature [99]. The general Euler curvature was first defined in [62]. It appears to be the first curvature defined for general simple graphs which satisfies Gauss-Bonnet. Other curvatures which refer to Euclidean embeddings are sometimes called discrete curvature and used in computer graphics both for surfaces as well as in image analysis for numerical general relativity like Regge calculus.

Other unrelated curvatures for graphs have been defined: in [68], the curvatures of the surface $2z = \langle x, Ax \rangle$ in $R^{\text{ord}(G)+1}$ are considered, where A is the adjacency matrix of G . The local clustering coefficient defined by Watts and Strogatz [102] is $c(x) = V_1(x)2/(V_0(x)(V_0(x) - 1))$. Ricci curvatures for Markov chains on metric spaces has been defined by [81] and studied for graphs in [106, 59] for pairs of vertices x, y . A distance between probability distributions m_1, m_2 on the vertex set V can be defined as $W(m_1, m_2) = \sup_f \sum_x f(x)[m_1(x) - m_2(x)]$, where the supremum is taken over all 1-Lipshitz functions f . Now define $k_\alpha(x, y) = 1 - W(m_x, m_y)/d(x, y)$, where $m_x(v) = \alpha$ for $v = x$ and $m_x(v) = (1 - \alpha)/V_0(x)$ for $v \in S(x)$. The Ollivier Ricci curvature is the limit $k(x, y) = \lim_{\alpha \rightarrow 1} k_\alpha(x, y)$.

The inductive dimension for graphs [63] is formally related to the inductive Brouwer-Menger-Urysohn dimension for topological spaces which goes back to the later work of Poincaré [55, 34, 84, 76] and was given by Menger in 1923 and Urysohn in 1925. It also is related to Brouwer's Dimensionsgrad [15] given in 1913. Any natural topology on a graph renders a graph zero-dimensional with the Menger-Urysohn as well as the Brouwer definition. The Menger-Uryson dimension for example is zero if there is a basis for open sets for which every basis element is both closed and open. The inductive dimension for graphs was already defined in [63] is a rational

number. It is natural also in probabilistic contexts as its expectation can be computed easily in random graphs [61]. There are other unrelated notions of dimension in graph theory like the EHT embedding dimension [83] which is based on the smallest dimensional Euclidean space in which the graph can be embedded, with variants like Euclidean dimension [95] or faithful dimension [35]. The metric dimension of a graph [50] is the minimal number of points $x_i \in V$ allowing to characterize every vertex by its distance coordinates $a_i = d(x, x_i)$. The metric dimension of K_n is $n-1$ of a path it is 1, for a cyclic graph or tree which is not a path it is larger than 1.

The history of the notion of polyhedron and polytop is complicated and "a surprisingly delicate task" [28]. "Agreeing on a suitable definition is surprisingly difficult" [92]. Coxeter [22] defines it as a convex body with polygonal faces. Grunbaum [47] also works with convex polytopes, the convex hull of finitely many points in R^n . The dimension is the dimension of its affine span. The perils of a general definition have been pointed out since Poincaré (see [92, 23, 66]). Topologists started with new definitions [2, 38, 21, 96], and define first a simplicial complex and then polyhedra as topological spaces which admit a triangularization by a simplicial complex. In this paper, we use a graph theoretical definition slightly modified from [63]: a polytop is a finite simple graph which can be completed to become a d -dimensional geometric graph. Since we would like to include K_n as polytopes, one can add a voluntary truncation process of vertices first. The graph theoretical definition it reaches all uniform polytopes or duals as well as graphs defined by convex hulls of finitely many points in R^n or graphs defined from simplicial complexes. The assumption on the unit sphere of the geometric graph will decide what polyhedra are included. If we insist for a two dimensional polyhedron every unit sphere to be connected then the Skilling's figure G is not included. If we insist the unit spheres just to be one dimensional graphs with zero Euler characteristic, then G is a polytop.

Euler characteristic for polytopes was first used by Descartes in a letter of 1630 to Leibnitz [98], but he did not take the extra step to prove the polyhedon formula $v - e + f = 2$. Euler sketched the first proof of the polyhedra formula in a November 1750 letter to Goldbach [36] where he gives a triangularization argument. The limitations of Euler's proof are now well understood [66, 41]. In modern language, Euler's formula tells that the combinatorial Euler characteristic $v - e + f$ coincides with the cohomological Euler characteristic $b_0 - b_1 + b_2 = 2$ for two dimensional connected, simply connected geometric graphs, where "geometric" means that $S(x)$ is a connected cyclic graph at every point and that G can be oriented. The connectedness assures $b_0 = 1$, the orientability implies $b_0 = b_2$ and the simply connectedness forces $b_1 = 0$. While we now know that all this can be defined purely graph theoretically without assuming any Euclidean embeddings, Euler - evenso himself a pioneer in graph theory - did not consider polyhedra as graphs. This was a step only taken by Cauchy [69]. The story about the Euler characteristic are told in [1, 92, 32, 66]. The average Euler characteristic in random graphs was computed in [61].

Random methods in geometry is part of integral geometry as pioneered by Crofton and Blaschke [10, 79]. Integral geometry has been used in differential

geometry extensively by Chern in the form of kinematic formulae [79]. This is not surprising since Chern is a student of Blaschke. It was also used by Milnor in proving total curvature estimates for knots made widely known by Spivak's textbook [97]. Banchoff used integral geometric methods in [5] and got analogue results for Polyhedra and surfaces similar to what was obtained in [65] for graphs. Whether integral geometric methods have been used in graph theory before [65] is unknown to this author.

The history of topology [29] and graph theory [80, 56]. For an introduction to Gauss-Bonnet with historical pointers to early discrete approaches, see [104]. More historical remarks about Gauss-Bonnet are in [18].

REFERENCES

- [1] A. Aczel. *Descartes's secret notebook, a true tale of Mathematics, Mysticism and the Quest to Understand the Universe*. Broadway Books, 2005.
- [2] P. Alexandrov. *Combinatorial topology*. Dover books on Mathematics. Dover Publications, Inc, 1960.
- [3] C. Allendoerfer and A. Weil. The gauss-bonnet theorem for riemannian polyhedra. *Transactions of the American Mathematical Society*, 53:101–129, 1943.
- [4] C.B. Allendoerfer. The Euler number of a Riemann manifold. *Amer. J. Math.*, 62:243, 1940.
- [5] T. Banchoff. Critical points and curvature for embedded polyhedra. *J. Differential Geometry*, 1:245–256, 1967.
- [6] T. Banchoff. Critical points and curvature for embedded polyhedral surfaces. *Amer. Math. Monthly*, 77:475–485, 1970.
- [7] B.Bollobás. *Modern Graph Theory*. Graduate Texts in Mathematics. Springer, New York, 1998.
- [8] M.de Berg, M.van Kreveld, M.Overmars, and O.Schwarzkopf. *Computational Geometry*. Springer Verlag, second edition, 1998.
- [9] M. Berger and B. Gostiaux. *Differential geometry: manifolds, curves, and surfaces*, volume 115 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1988.
- [10] W. Blaschke. *Vorlesungen über Integralgeometrie*. Chelsea Publishing Company, New York, 1949.
- [11] A. Bobenko and Y. Suris. *Discrete Differential Geometry, Integrable Structure*, volume 98 of *Graduate Studies in Mathematics*. AMS, 2008.
- [12] J. Bondy and U. Murty. *Graph theory*, volume 244 of *Graduate Texts in Mathematics*. Springer, New York, 2008.
- [13] B.O'Neill. *Elementary Differential Geometry*. Elsevier, revised second edition, 2006.
- [14] L. Brouwer. Über Abbildung von Mannigfaltigkeiten. *Math. Ann.*, 71(1):97–115, 1911.
- [15] L. Brouwer. Über den natürlichen Dimensionsbegriff. *Journal für die reine und angewandte Mathematik*, 142:146–152, 1913.
- [16] J-M. Drouffe C. Itzikson. *Statistical Field Theory*. Cambridge University Press, 1989.
- [17] S.-S. Chern. A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds. *Annals of Mathematics*, 45, 1944.
- [18] S-S. Chern. Historical remarks on Gauss-Bonnet. In *Analysis, et cetera*, pages 209–217. Academic Press, Boston, MA, 1990.
- [19] F. Chung. *Spectral graph theory*, volume 92 of *CBMS Regional Conf. Series*. AMS, 1997.
- [20] R. Cohen and S. Havlin. *Complex Networks, Structure, Robustness and Function*. Cambridge University Press, 2010.
- [21] J.B. Conway. *Mathematical Connections: A Capstone Course*. American Mathematical Society, 2010.
- [22] H.S.M. Coxeter. *Regular Polytopes*. Dover Publications, New York, 1973.
- [23] P.R. Cromwell. *Polyhedra*. Cambridge University Press, 1997.
- [24] K.S. Thorne C.W. Misner and J.A. Wheeler. *Gravitation*. Freeman, San Francisco, 1973.

- [25] H.L. Cycon, R.G.Froese, W.Kirsch, and B.Simon. *Schrödinger Operators—with Application to Quantum Mechanics and Global Geometry*. Springer-Verlag, 1987.
- [26] M. Desbrun, E. Kanso, and Y. Tong. Discrete differential forms for computational modeling. In A. Bobenko, P. Schroeder, J. Sullivan, and G. Ziegler, editors, *Discrete Differential Geometry*, Oberwohlfach Seminars, 2008.
- [27] M. Desbrun and K. Polthier. Discrete differential geometry: An applied introduction. In Ari Stern Peter Schröder, Eitan Grinspun, editor, *SIGGRAPH 2006*, 2006.
- [28] S. Devadoss and J. O'Rourke. *Discrete and Computational Geometry*. Princeton University Press, 2011.
- [29] J. Dieudonne. *A History of Algebraic and Differential Topology, 1900-1960*. Birkhäuser, 1989.
- [30] G. Dinca and J. Mawhin. *Brouwer Degree and Applications*. http://www.ann.jussieu.fr/~smets/FULM/Brouwer_Degree_and_applications.pdf, retrieved April, 2012, 2009.
- [31] M.P. do Carmo. *Differential forms and applications*. Universitext. Springer-Verlag, Berlin, 1994. Translated from the 1971 Portuguese original.
- [32] B. Eckmann. The Euler characteristic - a few highlights in its long history. In *Mathematical Survey Lectures: 1943-2004*, 1999.
- [33] T. Gowers (Editor). *The Princeton Companion to Mathematics*. Princeton University Press, 2008.
- [34] R. Engelking. *Dimension Theory*. North-Holland Publishing Company, 1978.
- [35] P. Erodes and M. Simonovits. On the chromatic number of geometric graphs. *Ars Comb.*, pages 229–246, 1980.
- [36] L. Euler. Letter to Goldbach of 14. November, 1750
. <http://eulerarchive.maa.org/correspondence/letters/OO0863.pdf>, 1750.
- [37] W. Fenchel. On total curvatures for riemannian manifolds (i). *J. London Math. Soc.*, 15:15, 1940.
- [38] A. Fomenko. *Visual Geometry and Topology*. Springer-Verlag, Berlin, 1994. From the Russian by Marianna V. Tsaplina.
- [39] R. Forman. A discrete Morse theory for cell complexes. In *Geometry, topology, & physics*, Conf. Proc. Lecture Notes Geom. Topology, IV, pages 112–125. Int. Press, Cambridge, MA, 1995.
- [40] R. Forman. Morse theory for cell complexes. *Adv. Math.*, page 90, 1998.
- [41] C. Franceses and D. Richeson. The flaw in Euler's proof of his polyhedral formula. *Amer. Math. Monthly*, 114(4):286–296, 2007.
- [42] J. Fröhlich. Regge calculus and discretized gravitational functional integrals. In *Advanced Series in Mathematical Physics*, volume 15. World Scientific, 1981.
- [43] D. Fuchs and S. Tabachnikov. *Mathematical Omnibus: Thirty Lectures on Classic Mathematics*. AMS, 2007.
- [44] D. Gottlieb. All the way with Gauss-Bonnet and the sociology of mathematics. *Amer. Math. Monthly*, 103(6):457–469, 1996.
- [45] A. Gray. *Tubes*. Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, 1990.
- [46] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, 1987.
- [47] B. Grünbaum. *Convex Polytopes*. Springer, 2003.
- [48] V. Guillemin and A. Pollack. *Differential topology*. Prentice-Hall, Inc., New Jersey, 1974.
- [49] M. Günther. Zum Einbettungssatz von J. Nash. *Math. Nachr.*, 144:165–187, 1989.
- [50] F. Harary and R.A. Melter. On the metric dimension of a graph. *Ars Combinatoria*, 2:191–195, 1976.
- [51] M. Henle. *A combinatorial Introduction to Topology*. Dover Publications, 1994.
- [52] Y. Higuchi. Combinatorial curvature for planar graphs. *J. Graph Theory*, 38:220–229, 2001.
- [53] M.W. Hirsch. *Differential topology*. Graduate texts in mathematics. Springer-Verlag, Berlin, 1976.
- [54] H. Hopf. Über die Curvatura integra geschlossener Hyperflaechen. *Mathematische Annalen*, 95:340–367, 1926.
- [55] W. Hurewicz and H. Wallman. *Dimension Theory*. Princeton mathematical series, 1941.
- [56] J. James. History of topology. In *History of Topology*, 1999.

- [57] C. Goodman-Strauss J.H. Conway, H.Burgiel. *The Symmetries of Things*. A.K. Peterse, Ltd., 2008.
- [58] J. Jost. *Riemannian Geometry and Geometric Analysis*. Springer Verlag, 2005.
- [59] J. Jost and S. Liu. Ollivier's Ricci curvature, local clustering and curvature dimension inequalities on graphs. arXiv:1103.4037v2, April 1, 2011, 2011.
- [60] B. Jubin. A generalized Poincaré-Hopf index theorem. ArXiv:0903.0697v2, 2009.
- [61] O. Knill. The dimension and Euler characteristic of random graphs. <http://arxiv.org/abs/1112.5749>, 2011.
- [62] O. Knill. A graph theoretical Gauss-Bonnet-Chern theorem. <http://arxiv.org/abs/1111.5395>, 2011.
- [63] O. Knill. A discrete Gauss-Bonnet type theorem. *Elemente der Mathematik*, 67:1–17, 2012.
- [64] O. Knill. A graph theoretical Poincaré-Hopf theorem. <http://arxiv.org/abs/1201.1162>, 2012.
- [65] O. Knill. On index expectation and curvature for networks. <http://arxiv.org/abs/1202.4514>, 2012.
- [66] I. Lakatos. *Proofs and Refutations*. Cambridge University Press, 1976.
- [67] J.M. Lee. *Riemannian manifolds*, volume 176 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. An introduction to curvature.
- [68] L.Ja.Beresina. The normal curvatures of a graph. *J. Geom.*, 18(1):54–56, 1982.
- [69] J. Malkevich. Euler's polyhedral formula I,II. *AMS feature articles*, 2004-2005. <http://www.ams.org/samplings/feature-column/fcarc-eulers-formulaii>.
- [70] J. Milnor. *Morse theory*, volume 51 of *Annals of Mathematics Studies*. Princeton University press, Princeton, New Jersey, 1963.
- [71] J. Milnor. *Topology from the differential viewpoint*. University of Virginia Press, Charlottesville, Va, 1965.
- [72] M.Morse. The critical points of a function of n variables. *Trans. Amer. Math. Soc.*, 33(1):72–91, 1931.
- [73] F. Morgan. *Riemannian Geometry, A Beginner's Guide*. Jones and Bartlett Publishers, 1993.
- [74] M. Morse. Singular points of vector fields under general boundary conditions. *American Journal of Mathematics*, 51, 1929.
- [75] P. Mullen, A. McKenzie, D. Pavlov, L. Durant, Y. Tong, E. Kanso, J. E. Marsden, and M. Desbrun. Discrete Lie advection of differential forms. *Found. Comput. Math.*, 11(2):131–149, 2011.
- [76] J.R. Munkres. *Topology, a first course*. Princeton Hall, New Jersey, 1975.
- [77] O. Narayan and I. Saniee. Large-scale curvature of networks. *Physical Review E*, 84, 2011.
- [78] M. E. J. Newman. *Networks*. Oxford University Press, Oxford, 2010. An introduction.
- [79] L. Nicolaescu. *Lectures on the Geometry of Manifolds*. World Scientific, second edition, 2009.
- [80] R.J. Wilson N.L. Biggs, E.K. Lloyd. *Graph Theory, 1736-1936*. Clarendon Press, Oxford, second edition, 1998.
- [81] Y. Ollivier. Ricci curvature of Markov chains on metric spaces. *J. Funct. Anal.*, 256:810–864, 2009.
- [82] R. Rieben P. Castilo, J. Koning and D. White. A discrete differential forms framework for computational electromagnetism. *CMES*, 1:1–15, 2002.
- [83] F. Harary P. Erdos and W. Tutte. On the dimension of a graph. *Mathematika*, 12:118–122, 1965.
- [84] A.R. Pears. *Dimension Theory of General Spaces*. Cambridge University Press, London, New York, 1975.
- [85] P. Petersen. *Riemannian Geometry*. Springer Verlag, second edition, 2006.
- [86] H. Poincaré. Sur les courbes définies par les equation differentielle III. *Journal de Mathématique pures et appliquées*, pages 167–244, 1885.
- [87] G. Polya. An elementary analogue to the Gauss-Bonnet theorem. *American Mathematical Monthly*, pages 601–603, 1954.
- [88] E. Presnov and V. Isaeva. Positional information as symmetry of morphogenetic fields. *Forma*, 5:59–61, 1990.
- [89] E. Presnov and V. Isaeva. Local and global aspects of biological morphogenesis. *Speculations in Science and Technology*, 14:68, 1991.

- [90] C. Pugh. A generalized Poincaré index formula. *Topology*, 7:217–226, 1968.
- [91] T. Regge. General relativity without coordinates. *Nuovo Cimento (10)*, 19:558–571, 1961.
- [92] D.S. Richeson. *Euler's Gem*. Princeton University Press, Princeton, NJ, 2008. The polyhedron formula and the birth of topology.
- [93] S. Rosenberg. *The Laplacian on a Riemannian Manifold*, volume 31 of *London Mathematical Society, Student Texts*. Cambridge University Press, 1997.
- [94] L. Schlafli. *Theorie der Vielfachen Kontinuität*. Cornell University Library Digital Collections, 1901.
- [95] A. Soifer. *The Mathematical Coloring Book*. Springer Verlag, 2009.
- [96] E.H. Spanier. *Algebraic Topology*. Springer Verlag, 1966.
- [97] M. Spivak. *A comprehensive Introduction to Differential Geometry I-V*. Publish or Perish, Inc, Berkeley, third edition, 1999.
- [98] J. Stillwell. *Mathematics and its history*. Springer, 2010.
- [99] T. Réti, E. Bitay, Z. Kosztolányi. On the polyhedral graphs with positive combinatorial curvature. *Acta Polytechnica Hungarica*, 2:19–37, 2005.
- [100] V.V. Trofimov and A.T. Fomenko. Riemannian geometry. *Journal of Mathematical Sciences*, 109:345–1501, 2002.
- [101] F. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate texts in mathematics*. Springer, New York, 1983.
- [102] D. J. Watts and S. H. Strogatz. Collective dynamics of 'small-world' networks. *Nature*, 393:440–442, 1998.
- [103] E. Weinan, J. Lu, and Y. Yao. The landscape of complex networks. arXiv:1204.6376v1, 2012.
- [104] P. Wilson. *Curved Spaces: From Classical Geometries to Elementary Differential Geometry*. Cambridge University Press, 2008.
- [105] X. Yin, M. Jin, F. Luo, and X. Gu. Discrete curvature flow for hyperbolic 3-manifolds with complete geodesic boundaries. Department of Mathematics, Rutgers University preprint, 2008.
- [106] Y.Lin, L.Lu, and S-T.Yau. Ricci curvature of graphs. To appear in *Tohoku Math. J.*, 2010.