LECTURES ON DEFORMATIONS OF GALOIS REPRESENTATIONS

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Lecture 5: Flat deformations

(5.1) Flat deformations: Let $K/\mathbb{Q}_p$ be a finite extension with residue field k. Let $W = W(k)$ and $K_0 = \mathrm{Fr}W$. We consider a finite dimensional $\mathbb{F}$-vector space equipped with a continuous action of G_K. Fix an algebraic closure $\bar{K}$ of K, and let $G_K = \mathrm{Gal}(\bar{K}/K)$.

Recall that a representation of G_K on a finite abelian p-group is called *flat* if it arises from a finite flat group scheme over $\mathcal{O}_K$.

The following result is due to Ramakrishna [Ra]:

Proposition (5.1.1). *Let $A \in \mathfrak{AR}_W$ and V_A in $D_{V_\mathbb{F}}(A)$. There exists a quotient A^{fl} of A such that for any morphism $A \to A'$ in $\mathfrak{AR}_W$, $V_{A'} =: V_A \otimes_A A'$ is flat if and only if $A \to A'$ factors through A^{fl}.*

Proof. First note that if V is a finite abelian p-group equipped with a continuous action of G_K and V is flat, then any G_K-stable subgroup $V' \subset V$ is flat. Indeed, suppose $V = \mathcal{G}(\mathcal{O}_{\bar{K}})$ think of V, V' as finite étale group schemes over K. If $\mathcal{G}'$ is the closure of V' in $\mathcal{G}$, then $V' = \mathcal{G}'(\mathcal{O}_{\bar{K}})$.

This remark shows that if $\theta : A \to A'$ is a morphism in $\mathfrak{AR}_W$ then $V_{A'}$ is flat if and only if $V_{\theta(A)}$ is flat. Similarly, if $I, J \subset A$ are ideals and $V_{A/I}$ and $V_{A/J}$ is flat then $V_{A/I\cap J} \subset V_{A/I} \oplus V_{A/J}$ is flat. $\square$

Corollary (5.1.2). *Let $D^{\mathrm{fl}}_{V_\mathbb{F}} \subset D_{V_\mathbb{F}}$ denote the sub-functor corresponding to flat deformations. Then $D^{\mathrm{fl}}_{V_\mathbb{F}} \subset D_{V_\mathbb{F}}$ is relatively representable.*

Proof. In the language of groupoids this just means that if ξ in $D_{V_\mathbb{F}}$, then $(D^{\mathrm{fl}}_{V_\mathbb{F}})_\xi$ is representable, and this follows from (5.1.1). $\square$

(5.2) Weakly admissible modules and Smoothness of the generic fibre:

Proposition (5.2.1). *Suppose that $D_{V_\mathbb{F}}$ is pro-represented by $R_{V_\mathbb{F}}$ and let $R^{\mathrm{fl}}_{V_\mathbb{F}}$ be the quotient of $R_{V_\mathbb{F}}$ which pro-represents $D^{\mathrm{fl}}_{V_\mathbb{F}}$. Let $E/W(\mathbb{F})[1/p]$ be a finite extension and $x : R^{\mathrm{fl}}_{V_\mathbb{F}}[1/p] \to E$ be a point such that* $\ker x$ *has residue field E. Write $\widehat{R}^{\mathrm{fl}}_x$ (resp. $\widehat{R}_x$) for the completion of $R^{\mathrm{fl}}_{V_\mathbb{F}}[1/p]$ (resp. $R_{V_\mathbb{F}}[1/p]$) at x.*

For any Artinian quotient $\epsilon : \widehat{R}_x \to B$ denote the specialization of the universal deformation by $R_{V_\mathbb{F}} \to B$. Then ϵ factors through $R^{\mathrm{fl}}_{V_\mathbb{F}}$ if and only if V_B factors through $\widehat{R}^{\mathrm{fl}}_x$ if and only if V_B arises from a p-divisible group. Moreover this condition holds if and only if V_B is crystalline.

Proof. Let B be in $\mathfrak{AR}_E$ and denote by Int_B the set of finite $\mathcal{O}_E$-subalgebras $A \subset B$. Then $R_{V_{\mathbb{F}}} \to B$ factors through some A in Int_B. Denote by V_A the induced G_K representation.

Then V_B arises from a p-divisible group if and only if V_A does. By a result of Raynaud [Ray, 2.3.1], V_A arises from a p-divisible group if and only if V_A/p^nV_A is flat for $n \geq 1$. This is equivalent to asking that $V_A \otimes_A A/\mathfrak{m}_A^n$ be flat for $n \geq 1$, or that $V_A \otimes_A A/\mathfrak{m}_A^n$ is in $D^{\mathrm{fl}}(A/\mathfrak{m}_A^n)$.

For the final statement we use Breuil's result that a crystalline representation with all Hodge-Tate weights equal to $0, 1$ arises from a p-divisible group [Br, Thm. 5.3.2], [Ki 4, 2.2.6]. □

(5.2.2) The deformation theoretic description of $\widehat{R}_x^{\mathrm{fl}}$ allows us to show that this ring is always formally smooth over E, and hence that $R^{\mathrm{fl}}_{V_{\mathbb{F}}}[1/p]$ is formally smooth over $W(\mathbb{F})[1/p]$. To prove this we need a little preparation (see [Ki 3]).

For any weakly admissible filtered φ-module D over K, denote by $C^\bullet(D)$ the complex $C^\bullet(D)$ the complex

$$D \stackrel{(1-\varphi,\mathrm{id})}{\longrightarrow} D \oplus D_K/\mathrm{Fil}^0 D_K$$

concentrated in degrees $0, 1$.

Lemma (5.2.3). *There is a canonical isomorphism*

$$\mathrm{Ext}^1_{w.adm}(\mathbf{1}, D) \xrightarrow{\sim} H^1(C^\bullet(D)).$$

where $\mathbf{1} = K_0$ *denotes the unit object in the category of weakly admissible module.*

Proof. Let

$$0 \to D \to \tilde{D} \to \mathbf{1} \to 0$$

be an extension. Let $\tilde{d} \in \mathrm{Fil}^0\tilde{D}$ be a lift of $1 \in \mathbf{1}$. Note that $D_K/\mathrm{Fil}^0 D_K \xrightarrow{\sim} \tilde{D}_K/\mathrm{Fil}^0\tilde{D}_K$, so we may regard $\tilde{d} \in D_K/\mathrm{Fil}^0 D_K$. Moreover $(1-\varphi)(\tilde{d}) \in D$. We associate the class

$$((1-\varphi)\tilde{d}, \tilde{d}) \in H^1(C^\bullet(D))$$

to the given extension.

If $(d_0, d_1) \in D \oplus D_K/\mathrm{Fil}^0 D_K$ we construct an extension of $\mathbf{1}$ by D by setting $\tilde{D} = D \oplus 1$ on underlying K_0-vector spaces and defining φ on $\tilde{D}$ by $\varphi(1) = 1 + d_0$ and the filtration by

$$\mathrm{Fil}^i\tilde{D}_K = \mathrm{Fil}^i D_K + K \cdot d_1 \quad i \leqslant 0$$

and $\mathrm{Fil}^i\tilde{D}_K = \mathrm{Fil}^i D_K$ if $i > 0$.

One checks that these two constructions induce the required isomorphism and its inverse. □

(5.2.4) Let $E/\mathbb{Q}_p$ be a finite extension, and D_E a weakly admissible filtered φ-module over K, equipped with an action of E. For B in $\mathfrak{AR}_E$ we denote by $D_{D_E}(B)$ the category of weakly admissible filtered filtered φ-modules D_B, equipped with an action of B, and an isomorphism $D_B \otimes_B E \xrightarrow{\sim} D$, such that D and $\mathrm{gr}^\bullet D_K$ are free B-modules.

For V_E a crystalline representation on a finite dimensional E-vector space, and B in $\mathfrak{AR}_E$ we denote by $D_{\mathrm{cris}}(V_B)$ the category of crystalline deformations of V_E to B.

Lemma (5.2.5). *Let V_E be as above and $D_E = D_{\mathrm{cris}}(V_E)$. Then D_{cris} induces an equivalences of groupoids over $\mathfrak{AR}_E$ $D_{V_E} \xrightarrow{\sim} D_{D_E}$. Moreover each of these groupoids is formally smooth.*

Proof. The proof of the first statement is formally similar to (5.3.5) below, and left as an exercise to the reader. The formal smoothness can be proved by a deformation theoretic argument using (5.2.3) and the fact that $H^2(C^\bullet(\mathrm{ad}D_E)) = 0$. □

Proposition (5.2.6). *In the notation of (5.2.1), let $D_E = D_{\mathrm{cris}}(V_E)$. The E-algebra $\widehat{R}_x^{\mathrm{fl}}$ is formally smooth of dimension*

$$\dim_E H^1(\mathcal{C}^\bullet(\mathrm{ad}D_x)) = 1 + \dim_E D_K/\mathrm{Fil}^0 D_K.$$

Proof. By (5.2.1) $\widehat{R}_x^{\mathrm{fl}}$ pro-represents D_{V_E}, which is equivalent to D_{D_E} and hence formally smooth by (5.2.5). The dimension of $\widehat{R}_x^{\mathrm{fl}}$ is equal to

$$\begin{aligned}\dim_E \mathrm{Ext}^1_{\mathrm{cris}}(V_E, V_E) = \dim_E \mathrm{Ext}^1_{w.adm}(D_E, D_E) = \\ \dim_E \mathrm{Ext}^1_{w.adm}(\mathbf{1}, \mathrm{ad}D_x) = \dim_E H^1(C^\bullet(\mathrm{ad}D)) = 1 + \dim_E D_K/\mathrm{Fil}^0 D_K.\end{aligned}$$

Here the first term means crystalline self extensions of V_E, as a representation of $E[G_K]$, the second last equality follows from (5.2.3), and the final one from the fact that the Euler characteristic of a finite complex is equal to that of its cohomology. □

(5.3) The Fontaine-Laffaille functor and Smoothness when $e = 1$: Recall the Fontaine-Laffaille category $\mathrm{MF}^1_{\mathrm{tor}}$ whose objects consist of a finite, torsion W-module M, together with a submodule $M^1 \subset M$, and Frobenius semi-linear maps

$$\varphi : M \to M \text{ and } \varphi^1 : M^1 \to M$$

such that

(1) $\varphi|_{M^1} = p\varphi^1$.
(2) $\varphi(M) + \varphi^1(M^1) = M$.

$\mathrm{MF}^1_{\mathrm{tor}}$ is an abelian subcategory of the category of filtered W-modules of finite length [FL, 1.9.1.10]. In particular, any morphism on $\mathrm{MF}^1_{\mathrm{tor}}$ is strict for filtrations.

Note also that if $p \cdot M = 0$, then $\varphi(M^1) = 0$, and so comparing the lengths of the two sides of (2) above shows that φ^1 is injective and

$$\varphi(M) \oplus \varphi^1(M^1) \xrightarrow{\sim} M.$$

Theorem (5.3.1). *(Fontaine-Laffaille, Raynaud) Suppose that $K = K_0$ and $p > 2$. Then there exist equivalences of abelian categories*

$$\mathrm{MF}^1_{\mathrm{tor}} \xrightarrow[\mathrm{FL}]{\sim} \{\text{f.flat group schemes}/W\} \xrightarrow{\sim} \{\text{flat repns. of } G_K\}.$$

Proof. The first equivalence is obtained by composing the anti-equivalence [FL, 9,11] with Cartier duality. The second follows from Raynaud's result [Ray, 3.3.6] that when $e(K/K_0) < p - 1$ the functor $\mathcal{G} \mapsto \mathcal{G}(\mathcal{O}_{\bar{K}})$ is fully faithful and the category of finite flat group schemes over $\mathcal{O}_K$ is abelian. □

(5.3.2) We will need a little more information about the functor FL. For a finite flat group scheme $\mathcal{G}$ over W we denote by $t_{\mathcal{G}}$ the tangent space of $\mathcal{G} \otimes_W k$, and by $\mathcal{G}^*$ the Cartier dual of $\mathcal{G}$.

The contravariant version of the functor is constructed via the theory of Honda systems, which is an extension of Dieudonné theory (which classifies finite flat group schemes over k) [FL, 9.7].

In particular, if $\mathcal{G}$ killed by p, and $M = \mathrm{FL}(\mathcal{G})$, there is an exact sequence

$$0 \to (t_{\mathcal{G}^*}{}^\vee \to \sigma^{-1*}M \to t_{\mathcal{G}} \to 0$$

where σ denote the Frobenius on W. Moreover the linear map $1\otimes\varphi_1 : M^1 \to \sigma^{-1*}M$ identifies M^1 with with $t^\vee_{\mathcal{G}^*}$. In particular

$$\dim_k M^1 = \dim_k \varphi^1(M_1) = \dim_k t_{\mathcal{G}^*}.$$

Now suppose that $\mathcal{G}$ is a *p-divisible group*, $T_p\mathcal{G}$ its Tate module, and $\mathcal{G}^*$ its Cartier dual. Write $V_p\mathcal{G} = T_p\mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then

$$D(\mathcal{G}) := D_{\mathrm{cris}}(T_p\mathcal{G})(1) \xrightarrow{\sim} \mathrm{Hom}_{G_K}(T_p\mathcal{G}^*, B_{\mathrm{cris}})$$

is a weakly admissible module whose associated graded is zero except in degrees $0, 1$. The lattice $T_p\mathcal{G} \subset V_p\mathcal{G}$ corresponds to a *strongly divisible lattice* $M \subset D_{\mathrm{cris}}(T_p\mathcal{G})$ with M/pM canonically isomorphic to $\mathrm{FL}(\mathcal{G}[p])$ as an object of $\mathrm{MF}^1_{\mathrm{tor}}$. In particular

$$\dim_{K_0} \mathrm{Fil}^1 D(\mathcal{G}) = \mathrm{rk}_{\mathcal{O}_{K_0}} \mathrm{Fil}^1 M = \dim_k M^1 = \dim_k t_{\mathcal{G}^*[p]}. \tag{5.3.3}$$

Theorem (5.3.4). *Suppose $K = K_0$ and $p > 2$. Then $D^{\mathrm{fl}}_{V_{\mathbb{F}}}$ is formally smooth. If $\mathcal{G}_{\mathbb{F}}$ denote the unique finite flat model of $V_{\mathbb{F}}$, $\mathcal{G}^*_{\mathbb{F}}$ denotes its Cartier dual, and $t_{\mathcal{G}_{\mathbb{F}}}$ denotes the tangent space of $\mathcal{G}_{\mathbb{F}}$, then*

$$\dim_{\mathbb{F}} D^{\mathrm{fl}}_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon]) = 1 + \dim_{\mathbb{F}} t_{\mathcal{G}_{\mathbb{F}}} \dim_{\mathbb{F}} t_{\mathcal{G}^*_{\mathbb{F}}}.$$

Proof. Let $M_{\mathbb{F}}$ in $\mathrm{MF}^1_{\mathrm{tor}}$ denote the object corresponding to $V_{\mathbb{F}}$. Then $M_{\mathbb{F}}$ is naturally an $\mathbb{F}$-vector space by the full faithfulness of (5.2.1). Let $D_{M_{\mathbb{F}}}$ denote the groupoid over $\mathfrak{AR}_{W(\mathbb{F})}$ such that $D_{M_{\mathbb{F}}}(A)$ is the category of objects M_A in $\mathrm{MF}^1_{\mathrm{tor}}$ equipped with an action of A, such that such that M_A is a finite free A-module and M^1 is an A-module direct summand, and an isomorphism $M_A \otimes_A \mathbb{F} \xrightarrow{\sim} M_{\mathbb{F}}$ in $\mathrm{MF}^1_{\mathrm{tor}}$.

Lemma (5.3.5). *The Fontaine-Laffaille functor of (5.2.1) induces an equivalence of categories*

$$\mathrm{FL} : D_{M_{\mathbb{F}}} \xrightarrow{\sim} D^{\mathrm{fl}}_{V_{\mathbb{F}}}.$$

Proof. If M_A is in $D_{M_{\mathbb{F}}}(A)$ let V_A be its image under FL. As FL is exact V_A is finite free over A. Indeed, for any finite A-module N we have

$$\mathrm{FL}(M_A) \otimes_A N \xrightarrow{\sim} \mathrm{FL}(M_A \otimes_A N). \tag{5.3.4}$$

This is obvious if N is free over A, and the general case follows by choosing a presentation of N by free modules. As the right hand side is an exact functor in N, so is the left hand side, which shows that V_A is a free A-module. Applying (5.3.4) with $N = \mathbb{F}$, one also sees that V_A is naturally a deformation of $V_{\mathbb{F}}$.

Conversely if V_A is in $D^{\mathrm{fl}}_{V_{\mathbb{F}}}$, and $M_A \in \mathrm{MF}^1_{\mathrm{tor}}$ satisfies $\mathrm{FL}(M_A) \xrightarrow{\sim} V_A$, then M_A is an A-module by the full faithfulness of FL, and since $\mathrm{MF}^1_{\mathrm{tor}}$ is abelian, the same argument as above shows that M_A is free over A, and that $M^1_A \subset M_A$ is an A-module direct summand □

(5.3.6) We return to the proof of (5.3.4). By the lemma to prove the formal smoothness of $D^{\mathrm{fl}}_{V_{\mathbb{F}}}$ it suffices to prove the formal smoothness of $D_{M_{\mathbb{F}}}$. Let A be in $\mathfrak{AR}_{W(\mathbb{F})}$, $I \subset A$ an ideal and $M_{A/I}$ in $D_{M_{\mathbb{F}}}(A/I)$. We have to show that $M_{A/I}$ lifts to an object of $D_{M_{\mathbb{F}}}(A)$.

First choose a lifting of the A/I-module $M_{A/I}$ to an A-module M_A, and a submodule $M^1_A \subset M_A$ which is a direct summand and lifts $M^1_{A/I}$. Next let $L_{A/I} = \varphi^1(M^1_{A/I})$, and choose a lift of $L_{A/I}$ to a direct summand $L_A \subset M_A$ and a lift of the composite

$$\varphi^*(M_A) \to \varphi^*(M_{A/I}) \stackrel{1\otimes\varphi^1}{\to} M_{A/I}$$

to L_A. Finally one checks that the map $1\otimes p\varphi^1 : \varphi^*(M^1_A) \to M_A$ admits an extension to a map $\varphi^*(M_A) \to M_A$ which induces the given map $\varphi^*(M_{A/I}) \to M_{A/I}$.

(5.3.7) It remains to check that the dimension of $D^{\mathrm{fl}}_{V_{\mathbb{F}}}$ has the claimed dimension. We could do this directly by computing the dimension of $D_{M_{\mathbb{F}}}(\mathbb{F}[\epsilon])$, however it is simpler to use our computation of the dimension of the generic fibre of $R^{\mathrm{fl}}_{V_{\mathbb{F}}}$.

Let $x : R^{\mathrm{fl}}_{V_{\mathbb{F}}}[1/p] \to E$ be a surjective map, where E is a finite extension of $\mathbb{Q}_p$. Write V_x for the corresponding crystalline representation. Since we already know that $R^{\mathrm{fl}}_{V_{\mathbb{F}}}$ is smooth, we need to compute the dimension of the tangent space of $\mathcal{R}^{\mathrm{fl}}_{V_{\mathbb{F}}}[1/p]$ at x. Let $D_x = D_{\mathrm{cris}}(V_x)$.

Using (5.2.6) and (5.3.3)one sees that this dimension is[1]

$$1 + \dim_E(D_x/\mathrm{Fil}^1 D_x)\dim_E \mathrm{Fil}^1 D_x = 1 + \dim_{\mathbb{F}} t_{\mathcal{G}} \dim_{\mathbb{F}} t_{\mathcal{G}^*}.$$

□

Exercises:

Exercise 1: Formulate and prove Proposition (5.2.1) for framed deformations.

Exercise 2: Check that the two constructions used to define the isomorphism

$$\mathrm{Ext}^1_{w.adm}(\mathbf{1}, D) \xrightarrow{\sim} H^1(C^\bullet(D))$$

in (5.2.3) are well defined and inverse.

Exercise 3: Give an explicit description of the isomorphism

$$\mathrm{Ext}^1_{w.adm}(\mathbf{1}, \mathrm{ad}D) \xrightarrow{\sim} \mathrm{Ext}^1_{w.adm}(D, D)$$

used in (5.2.6).

Exercise 4: Show that the functor D_{D_E} in (5.2.5) is formally smooth.

Exercise 5: Show that the category $\mathrm{MF}^1_{\mathrm{tor}}$ is equivalent to the category of *finite Honda systems* defined in Conrad's lectures. This is slightly tricky [FL, 9.4].

[1]The computation becomes slightly easier if $E = W(\mathbb{F})[1/p]$, which we may assume as $R^{\mathrm{fl}}_{V_{\mathbb{F}}}$ is formally smooth.

References

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