LECTURES ON DEFORMATIONS OF GALOIS REPRESENTATIONS

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Lecture 4: Presenting global deformation rings over local ones

(4.1) Let F be a number field and S a finite set of primes of F containing the primes dividing p. Fix an algebraic closure $\bar{F}$ of F and denote by $F_S \subset \bar{F}$ the maximal extension of F unramified outside S. Write $G_{F,S} = \mathrm{Gal}(F_S/F)$.

Let $\Sigma \subset S$ and fix an algebraic closure $\bar{F}_v$ of F_v for each $v \in \Sigma$, as well as an embedding $\bar{F} \hookrightarrow \bar{F}_v$. We write $G_{F_v} = \mathrm{Gal}(\bar{F}_v/F_v)$.

Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$ and uniformizer $\pi_{\mathcal{O}}$. We fix a finite dimensional $\mathbb{F}$-vector space $V_{\mathbb{F}}$ equipped with a continuous action of $G_{F,S}$, and a continuous character $\psi : G_{F,S} \to \mathcal{O}^\times$ such that $\det_{\mathbb{F}} V_{\mathbb{F}} \sim \psi$.

For simplicity we assume in the following that $p \nmid \dim V_{\mathbb{F}}$, although this is not really necessary (see [Ki 2]).

(4.1.1) For each $v \in \Sigma$ fix a basis β_v of $V_{\mathbb{F}}$. For A in $\mathfrak{AR}_W(\mathbb{F})$ denote by $D_v^{\square,\psi}(A)$ the category of framed deformations of $(V_{\mathbb{F}}|_{G_{F_v}}, \beta)$ to A, with determinant ψ. This is pro-representable by a complete local $\mathcal{O}$-algebra $R_v^{\square,\psi}$.

Similarly we denote by $D_{F,S}^{\square,\psi}(A)$ the category whose objects consist of a deformation of $V_{\mathbb{F}}$ to V_A together with a lifting of *each* basis β_v to an A-basis of V_A.

We also have the analogous functors D_v^{ψ} and $D_{F,S}^{\psi}$ for unframed deformations and the universal $\mathcal{O}$-algebras R_v^{ψ} and $R_{F,S}^{\psi}$ when these functors are representable.

We set

$$R_\Sigma^{\square,\psi} = \widehat{\otimes}_{\mathcal{O},v\in\Sigma} R_v^{\square,\psi} \text{ and } R_\Sigma^{\psi} = \widehat{\otimes}_{\mathcal{O},v\in\Sigma} R_v^{\psi}$$

when the latter ring exists.

Finally we denote by $\mathfrak{m}_\Sigma^\square$ and $\mathfrak{m}_{F,S}^\square$ the radicals of $R_\Sigma^{\square,\psi}$ and $R_{F,S}^{\square,\psi}$ respectively, and similarly for $\mathfrak{m}_\Sigma$ and $\mathfrak{m}F, S$.

Proposition (4.1.2). *For $i \geq 1$ let h_Σ^i and c_Σ^i denote respectively the dimension of the kernel and cokernel of*

$$\theta^i : H^i(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) \to \prod_{v\in\Sigma} H^i(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}})$$

where $\mathrm{ad}^0 V_{\mathbb{F}} \subset \mathrm{ad} V_{\mathbb{F}}$ denotes the space of endomorphisms with trace 0. If R_Σ^ψ exists then $R_{F,S}^\psi$ is a quotient of $R_\Sigma^\psi[\![x_1, \dots, x_{h_\Sigma^1}]\!]$ by $c_\Sigma^1 + h_\Sigma^2$ relations.

In general let

$$\eta : \mathfrak{m}_\Sigma^\square/(\mathfrak{m}_\Sigma^{\square,2}, \pi_{\mathcal{O}}) \to \mathfrak{m}_{F,S}^\square/(\mathfrak{m}_{F,S}^{\square,2}, \pi_{\mathcal{O}}).$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Then $R^{\psi,\square}_{F,S}$ *is a quotient of a power series ring over* $R^{\square}_{\Sigma}$ *in* $\dim\operatorname{coker}\eta$ *variables by* $h^2_\Sigma + \dim\ker\eta$ *relations.*

Proof. We prove only the first statement. Choose a surjection

$$\tilde{R} := R^{\psi}_{\Sigma}[\![x_1, \dots, x_{h^1_\Sigma}]\!] \to R^{\psi}_{F,S} \tag{4.1.3}$$

which induces a surjection on reduced tangent spaces and denote by J its kernel. We denote by $\tilde{\mathfrak{m}}$ the radical of $\tilde{R}$.

The universal representation $\rho_{R^{\psi}_{F,S}} : G_{F,S} \to \mathrm{GL}_d(R^{\psi}_{F,S})$ has a set theoretic lifting $\tilde{\rho} : G_{F,S} \to G_{F,S} \to \mathrm{GL}_d(\tilde{R}/\tilde{\mathfrak{m}}J)$ such that $\det\tilde{\rho}(\gamma) = \psi(\gamma)$ for all $\gamma \in G_{F,S}$. Such a lifting exists as the fibres of $\det : \mathrm{GL}_d \to \mathrm{GL}_1$are torsors over SL_d, and in particular smooth.

Define

$$c : G^2_{F,S} \to J/\tilde{m}J \otimes_{\mathbb{F}} \mathrm{ad}^0 V_{\mathbb{F}}; \quad c(g_1, g_2) = \tilde{\rho}(g_1 g_2)\tilde{\rho}(g_2)^{-1}\tilde{\rho}(g_1)^{-1}$$

where we regard

$$J/\tilde{m}J \otimes_{\mathbb{F}} \mathrm{ad}^0 V_{\mathbb{F}} \xrightarrow{\sim} \ker(\mathrm{GL}_d(\tilde{R}/\tilde{m}J) \to \mathrm{GL}_d(\tilde{R}/J))$$

Then $[c] \in H^2(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) \otimes J/\tilde{\mathfrak{m}}J$ depends only on $\rho_{R^{\psi}_{F,S}}$ and not on $\tilde{\rho}$, and $[c] \sim 0$ if and only if $\tilde{\rho}$ can be chosen to be a homomorphism (see Exercise 2 below).

As $\rho_{R^{\psi}_{F,S}}|_{G_{F_v}}$ is induced by the universal representation over R^{ψ}_{v}, $\rho_{R^{\psi}_{F,S}}|_{G_{F_v}}$ lifts to $\tilde{R}/\tilde{\mathfrak{m}}J$ and hence $[c|_{G_{F_v}}] \sim 0$. Hence $[c] \in H^2_\Sigma(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) \otimes J/\tilde{\mathfrak{m}}J$ where

$$H^2_\Sigma(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) := \ker(H^2(G_{F,S}, \mathrm{ad}^- V_{\mathbb{F}}) \to \prod_{v \in \Sigma} H^2(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}})),$$

and we obtain a map

$$(J/\tilde{\mathfrak{m}}J)^* \to H^2_\Sigma(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}); \quad u \mapsto \langle [c], u \rangle. \tag{4.1.4}$$

Let

$$\begin{array}{ccc} I = \ker(\mathfrak{m}_\Sigma/(\mathfrak{m}^2_\Sigma, \pi_{\mathcal{O}}) & \longrightarrow & \mathfrak{m}_{F,S}/(\mathfrak{m}^2_{F,S}, \pi_{\mathcal{O}})) \\ \big\downarrow \sim & & \big\downarrow \sim \\ \oplus D_{V_{\mathbb{F}}|_{G_{F_v}}}(F[\epsilon])^* & \longrightarrow & D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon]) \end{array}$$

Note that $I \xrightarrow{\sim} \ker(\tilde{\mathfrak{m}}/(\tilde{\mathfrak{m}}^2, \pi_{\mathcal{O}}) \to \mathfrak{m}_{F,S}/(\mathfrak{m}^2_{F,S}, \pi_{\mathcal{O}}))$, so reducing mod $\tilde{\mathfrak{m}}$ we get a surjection $J/\tilde{\mathfrak{m}}J \to I$ and an injection $I^* \hookrightarrow (J/\tilde{\mathfrak{m}}J)^*$.

We claim that I^* contains the kernel of (4.1.4). If $0 \neq u \in (J/\tilde{\mathfrak{m}}J)^*$ let $\tilde{R}_u$ be the pushout of $\tilde{R}/\tilde{\mathfrak{m}}J$ by u. Then $R^{\psi}_{F,S} = \tilde{R}_u/I_u$ where $I_u \subset \tilde{R}_u$ is an ideal with $I_u \cdot \tilde{\mathfrak{m}}$ and which is 1-dimensional as an $\mathbb{F}$-vector space. If $\langle [c], u \rangle = 0$ then $\rho^{\psi}_{F,S}$ lifts to a representation into $\mathrm{GL}_d(\tilde{R}_u)$ with determinant ψ. Hence $\tilde{R}_u \to R^{\psi}_{F,S}$ has a section and $\tilde{R}_u = R^{\psi}_{F,S} \oplus I_u$. This implies that $\tilde{R}_u \to R^{\psi}_{F,S}$ does not induce a bijection on reduced tangent spaces. In particular, the composite

$$\ker(J/\tilde{\mathfrak{m}}J \to I) \to J/\tilde{\mathfrak{m}}J \to I_u$$

is not surjective, and is therefore 0. This means that u factors through I, which proves our claim.

It follows that

$$\dim_{\mathbb{F}}(J/\tilde{\mathfrak{m}}J)^* \leqslant \dim_{\mathbb{F}} I + h^2_\Sigma = c^1_\Sigma + h^2_\Sigma.$$

□

Theorem (4.2). *Suppose $\{v|p\} \subset \Sigma$, $\{v|\infty\} \subset S$, and $S\backslash\Sigma$ contains a finite prime. Then*

$$R^{\psi}_{F,S} \xrightarrow{\sim} R^{\psi}_{\Sigma}[\![x_1,\dots,x_r]\!]/(f_1,\dots,f_{r+s})$$

for some $r \geq 0$, and $s = \sum_{v|\infty, v\notin\Sigma} \dim_{\mathbb{F}}(\mathrm{ad}^0 V_{\mathbb{F}})^{G_{F_v}}$, provided the rings $R^{\psi}_{F,S}$ and R^{ψ}_{Σ} exist.

Moreover,

$$R^{\psi}_{F,S} \xrightarrow{\sim} R^{\psi}_{\Sigma}[\![x_1,\dots,x_{r+\Sigma-1}]\!]/(f_1,\dots,f_{r+s})$$

(4.2.1) To prove the theorem, we need the Poitou-Tate sequence. Late X be a finite abelian p-group equipped with an action of $G_{F,S}$. We denote by $X^{\vee}$ the Pontryagin dual of X, and by $X^* = X^{\vee}(1)$ its Tate dual. Then there is an exact sequence

(PT(X))

$$0 \to H^0(G_{F,S}, X) \to \prod_{v|\infty} \widehat{H}^0(G_{F_v}, X) \times \prod_{v\in S_f} H^0(G_v, X) \to H^0(G_{F,S}, X^*)^{\vee}$$
$$\to H^1(G_{F,S}.X) \to \prod_{v\in S} H^1(G_{F_v}, X) \to H^1(G_{F,S}, X^*)^{\vee}$$
$$\to H^2(G_{F,S}, X) \to \prod_{v\in S} H^2(G_{F_v}, X) \to H^0(G_{F,S}, X^*)^{\vee} \to 0$$

Here $\widehat{H}^0(G_{F_v}, X)$ denotes $H^0(G_{F_v}, X)$ modulo the subgroup of norms in X.

Local Tate duality provides an isomorphism

$$H^i(G_{F_v}, X)^{\vee} \xrightarrow{\sim} H^{2-i}(G_{F_v}, X^*)$$

for v a finite prime and $i = 0, 1, 2$. Using this, one can identify the Pontryagin dual of the sequence $PT(X)$ with $PT(X^*)$.

Proof of (4.2). We will prove only the first statement since the proof of the second statement requires only some extra book keeping.

We apply the above sequence with $X = \mathrm{ad}^0 V_{\mathbb{F}}$. First note that, using the remark on the duality of $PT(X)$ and $PT(X^*)$ one sees that the map

$$\prod_{S\backslash\Sigma} H^2(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}}) \to H^0(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}(1))^{\vee}$$

induced by the final map of $PT(X)$ is surjective, as $S\backslash\Sigma$ contains a finite prime. Hence the map

$$H^2(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) \to \prod_{v\in\Sigma} H^2(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}})$$

is surjective and

$$h^2_{\Sigma} = h^2(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) - \sum_{v\in\Sigma} h^2(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}}).$$

Here we use the convention that $h^i = \dim H^i$.

By (4.1.2) we have $R^{\psi}_{F,S}[\![x_1,\dots,x_r]\!]/(f_1,\dots f_{r+s})$. with

$$(4.2.2)\quad -s = h^1_\Sigma - c^1_\Sigma - h^2_\Sigma$$
$$= h^1(G_{F,S}.\mathrm{ad}^0 V_{\mathbb{F}}) - \sum_{v\in\Sigma} h^1(G_{F_v},\mathrm{ad}^0 V_{\mathbb{F}}) - h^2(G_{F,S}.\mathrm{ad}^0 V_{\mathbb{F}}) - \sum_{v\in\Sigma} h^2(G_{F_v},\mathrm{ad}^0 V_{\mathbb{F}})$$
$$= -\chi(G_{F,S},\mathrm{ad}^0 V_{\mathbb{F}}) + \sum_{v\in\Sigma}\chi(G_{F_v},\mathrm{ad}^0 V_{\mathbb{F}}).$$

Here we have used that fact that the existence of $R^{\psi}_{G_{F,S}}$ and $\mathcal{R}^{\psi}_{\Sigma}$ implies that $\mathrm{ad}^0 V_{\mathbb{F}}$ has no $G_{F,S}$ or G_{F_v} invariants for $v\in\Sigma$.

Now we use Tate's global Euler characteristic formula which says that

$$(4.2.3)\quad \chi(G_{F,S},\mathrm{ad}^0 V_{\mathbb{F}}) = \sum_{v|\infty}(h^0(G_{F_v},\mathrm{ad}^0 V_{\mathbb{F}}) - [F_v:R]\mathrm{dim}_{\mathbb{F}}\mathrm{ad}^0 V_{\mathbb{F}})$$
$$= (\sum_{v|\infty} h^0(G_{F_v},\mathrm{ad}^0 V_{\mathbb{F}})) - [F;\mathbb{Q}]\mathrm{dim}_{\mathbb{F}}\mathrm{ad}^0 V_{\mathbb{F}}.$$

The local Euler characteristic $\chi(G_{F_v},\mathrm{ad}^0 V_{\mathbb{F}})$ is 0 if $v\nmid p$ is a finite prime. Hence the contributions of the local terms in (4.2.2) is

$$(4.2.4)\quad \sum_{v|\infty, v\in\Sigma} h^0(G_{F_v},\mathrm{ad}^0 V_{\mathbb{F}}) - \sum_{v|p}[F_v:\mathbb{Q}_p]\mathrm{dim}_{\mathbb{F}}\mathrm{ad}^0 V_{\mathbb{F}}$$
$$= \sum_{v|\infty, v\in\Sigma} h^0(G_{F_v},\mathrm{ad}^0 V_{\mathbb{F}}) - [F:\mathbb{Q}]\mathrm{dim}_{\mathbb{F}}\mathrm{ad}^0 V_{\mathbb{F}}.$$

Subtracting (4.2.4) from (4.2.3) one finds

$$s = \sum_{v|\infty, v\notin S} h^0(G_{F_v},\mathrm{ad}^0 V_{\mathbb{F}}).$$

□

Exercises:

Exercise 1: Prove the second statement in Proposition (4.1.2).

Exercise 2: Check the statements about the cocycle c in Proposition: That $[c]$ does not depend on $\tilde{\rho}$ and is trivial if and only if $\tilde{\rho}$ can be chosen to be a homomorphism.

Exercise 3: Prove the second statement in Theorem (4.2)

Exercise 4: (This is more difficult.) Formulate and prove Theorem (4.2) without assuming $p\nmid \mathrm{dim}V_{\mathbb{F}}$.

References

[Ki 2] M. Kisin, *Modularity of* 2*-dimensional Galois representations*, Current developments in mathematics 2005, 2008.