LECTURES ON DEFORMATIONS OF GALOIS REPRESENTATIONS

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Lecture 3: Representability

(3.1) Quotients by finite (formal) group actions are often representable, and indeed there are general results which guarantee this in certain situations.

In this section we assume that G satisfies the condition Φ_p. We begin with a result from Lecture 1, whose proof had been postponed.

Theorem (3.1.1). *Suppose* $\mathrm{End}_{\mathbb{F}[G]} V_{\mathbb{F}} = \mathbb{F}$. *Then* $D_{V_{\mathbb{F}}}$ *is representable.*

Proof. We already saw that $D^{\square}_{V_{\mathbb{F}}}$ is representable by $\mathrm{Spf}\, R^{\square}_{V_{\mathbb{F}}} =: X_{V_{\mathbb{F}}}$ where $R^{\square}_{V_{\mathbb{F}}}$ is a complete local $W(\mathbb{F})$-algebra.

Let $\widehat{\mathrm{PGL}}_d$ denote the formal completion of the $W(\mathbb{F})$-group scheme PGL_d along its identity section. Then $\widehat{\mathrm{PGL}}_d$ acts on $X_{V_{\mathbb{F}}}$ and we have

$$X_{V_{\mathbb{F}}} \times \widehat{\mathrm{PGL}}_d \rightrightarrows X_{V_{\mathbb{F}}}; \quad (x,g) \mapsto (x, gx).$$

The action of $\widehat{\mathrm{PGL}}_d$ on $X_{V_{\mathbb{F}}}$ is *free* which means that the induced map

$$X_{V_{\mathbb{F}}} \times \widehat{\mathrm{PGL}}_d \to X_{V_{\mathbb{F}}} \times X_{V_{\mathbb{F}}}.$$

is a closed immersion.

We would like to take the quotient of $X_{V_{\mathbb{F}}}$ by this action. To do this we need a little preparation.

(3.1.2) Let $\widehat{\mathfrak{AR}}_W$ denote the category of complete local Noetherian $W(\mathbb{F})$-algebras, so the opposite category $(\widehat{\mathfrak{AR}}_W)^\circ$ is equivalent to the category of formal spectra of such $W(\mathbb{F})$-algebras.

An *equivalence relation* $R \rightrightarrows X$ in $(\widehat{\mathfrak{AR}}_W)^\circ$ is a pair of morphisms such that

(1) $R \to X \times X$ is a closed embedding.

(2) For all T in $(\widehat{\mathfrak{AR}}_W)^\circ$ $R(T) \subset (X \times X)(T)$ is an equivalence relation.

For example let G be a group object in $(\widehat{\mathfrak{AR}}_W)^\circ$, and $G \times X \to X$ a free action. Then the map

$$G \times X \rightrightarrows X; \quad (g,x) \mapsto (x, gx)$$

is an equivalence relation.

A flat morphism $X \to Y$ in $(\widehat{\mathfrak{AR}}_W)^\circ$ is said to be a quotient of X by R, if the embedding $R \to X \times X$ induces an isomorphism $R \xrightarrow{\sim} X \times_Y X$.

Theorem (3.1.3). *(SGA 3, VIIb, Thm. 1.4): Let* $R \overset{p_0}{\underset{p_1}{\rightrightarrows}} X$ *be an equivalence relation in* $(\widehat{\mathfrak{AR}}_W)^\circ$ *such that the first projection* $R \to X$ *is flat. Then the quotient of* X *by* R *exists. If* $X = \operatorname{Spf} B$ *and* $R = \operatorname{Spf} C$*, then* $X/R = \operatorname{Spf} A$*, where*

$$A = \{b \in B : p_0^*(b) = p_1^*(b)\}.$$

We can now complete the proof of Theorem (3.1.1) by applying (3.1.3) to the equivalence relation $X_{V_{\mathbb{F}}} \times \widehat{\mathrm{PGL}}_d \rightrightarrows X_{V_{\mathbb{F}}}$. □

(3.2) Now fix a pseudo-representation $\bar{\tau} : G \to \bar{\tau}$. We now want to construct some representable subgroupoids of $\mathrm{Rep}_{\bar{\tau}} \to D_{\bar{\tau}}$.

Suppose that for $i = 1, \dots, s$ $\bar{\rho}_i : G \to \mathrm{GL}_{d_i}(\mathbb{F})$ are pairwise distinct absolutely irreducible representation of G, such that $\bar{\tau} = \sum_{i=1}^s \bar{\rho}_i$.

Let $\mathrm{Rep}'_{\bar{\tau}} \subset \mathrm{Rep}_{\bar{\tau}}$ be the full subgroupoid of $\mathrm{Rep}_{\bar{\tau}}$ such that $\mathrm{Rep}'_{\bar{\tau}}(A, B)$ consists of the objects V_B in $\mathrm{Rep}_{\bar{\tau}}(A, B)$ such that

$$V_B \otimes_B B/\mathfrak{m}_A B \sim \begin{pmatrix} \bar{\rho}_1 & c_1 & \cdots & \cdots \\ & \bar{\rho}_2 & c_2 & \cdots \\ & & \cdots & \cdots \\ & & & \bar{\rho}_s \end{pmatrix}. \tag{3.2.1}$$

where for $i = 1, \dots, s-1$ c_i is a *non-trivial* extension of $\bar{\rho}_{i+1}$ by $\bar{\rho}_i$.

Note that the isomorphism in (3.2.1) is uniquely determined as out conditions imply that the representation on the right has no non-trivial automorphisms.

Theorem (3.2.2). *The groupoid* $\mathrm{Rep}'_{\bar{\tau}} \to D_{\bar{\tau}}$ *is representable by a proper formal scheme over* $\operatorname{Spf} R_{\bar{\tau}}$.

Proof. Let

$$\mathrm{Rep}^{\square,\prime}_{\bar{\tau}} = \mathrm{Rep}'_{\bar{\tau}} \times_{\mathrm{Rep}_{\bar{\tau}}} \mathrm{Rep}^{\square}_{\bar{\tau}} \subset \mathrm{Rep}^{\square}_{\bar{\tau}}.$$

This is a locally closed subspace. The group PGL_d acts freely on $\mathrm{Rep}^{\square,\prime}_{\bar{\tau}}$ as $V_B \in \mathrm{Rep}'_{\bar{\tau}}(A, B)$ has no non-trivial automorphisms. To take the quotient we need the following

Theorem (3.2.3). *Let* $S = \operatorname{Spec} A$*, with* A *a local Artin ring, and let* X/S *be a finite type* S*-scheme equipped with a free action by a reductive group* G/S.[1] *Suppose that every* $x \in X$ *is contained in an affine,* G*-stable open subset of* X*. Then the quotient* X/G *exists.*

x When S is a field this is explained in Mumford's book [GIT, Ch 1, §4, Prop 1.9]. The general case will be explained by Brian Conrad in another lecture.

We want to apply the theorem to the quotient $\mathrm{Rep}^{\square,\prime}_{\bar{\tau}}/\mathrm{PGL}_d$.

For simplicity we will consider only the case when $d = 2$ and $\bar{\rho}_1, \bar{\rho}_2$ are characters, which we will denote by χ_1 and χ_2 respectively. We have to check the condition that every point of $\mathrm{Rep}^{\square,\prime}_{\bar{\tau}}$ has an affine PGL_d-stable neighbourhood.

Let $U \subset \mathrm{Ext}^1(\chi_1, \chi_2)\backslash\{0\}$ be affine open and define $\mathrm{Rep}^{\square,U}_{\bar{\tau}} \subset \mathrm{Rep}^{\square,\prime}_{\bar{\tau}}$ the subgroupoid consisting of those (V_B, β) in $\mathrm{Rep}^{\square,\prime}_{\bar{\tau}}$ such that $V_B \xrightarrow{\sim} \begin{pmatrix} \chi_1 & c \\ 0 & \chi_2 \end{pmatrix}$ with $c \in U$. Then $\mathrm{Rep}^{\square,U}_{\bar{\tau}}$ is stable by PGL_2, and its fibre over the closed point of $\operatorname{Spf} R_{\bar{\tau}}$

[1]A reductive group over a base S is a smooth group scheme with reductive fibres.

is isomorphic to U. In particular $\mathrm{Rep}_{\bar{\tau}}^{\square,U}$ is affine, and we may apply (3.2.3) with $A = R_{\bar{\tau}}/\mathfrak{m}^n_{R_{\bar{\tau}}}$ for $n = 1, \dots$.

We obtain an (a priori non-separated) scheme $\mathcal{E}_{\bar{\tau},n}$ over $R_{\bar{\tau}}/\mathfrak{m}^n_{R_{\bar{\tau}}}$, and hence a formal scheme $\mathcal{E}_{\bar{\tau}} = \lim_n \mathcal{E}_{\bar{\tau},n}$. Since $\mathcal{E}_{\bar{\tau},1} \xrightarrow{\sim} \mathbb{P}(\mathrm{Ext}^1(\chi_2, \chi_1))$ is proper, we see also that $\mathcal{E}_{\bar{\tau}}$ is proper. $\square$

Remark (3.2.4) I expect that the representing formal scheme is projective and therefore arises from a *scheme* of finite type over $Spec R_{\bar{\tau}}$.

Corollary (3.2.5). *Let $x \in \mathrm{Rep}^{\square}_{\bar{\tau}}(\mathbb{F})$ and $V_{\mathbb{F}}$ the corresponding representation of G. Then the complete local ring at of $\mathrm{Rep}^{\square}_{\bar{\tau}}$ at x pro-represents $D^{\square}_{V_{\mathbb{F}}}$.*

On the situation of (3.2), if $x \in \mathrm{Rep}^{\square}_{\bar{\tau}}(\mathbb{F})$ then the complete local ring at of $\mathrm{Rep}^{\square}_{\bar{\tau}}$ at x is a quotient of $R^{\square}_{V_{\mathbb{F}}}$.

Proof. This follows immediately from the definitions. $\square$

Corollary (3.2.6). *Let $E/W(\mathbb{F})[1/p]$ be a finite extension and $x : Rep'_{\bar{\tau}} \to E$ a point such that the corresponding E-valued pseudo-representation τ_x is absolutely irreducible. Then the map*

$$\mathrm{Rep}'_{\bar{\tau}} \to \mathrm{Spf}\, R_{\bar{\tau}}$$

is a closed embedding over a formal neighbourhood of τ_x.[2]

Proof. First we remark that x is the only point of $\mathrm{Rep}'_{\bar{\tau}}$ lying over τ_x. To see this suppose x' is another such point and let denote by V_x and $V_{x'}$ the corresponding G-representations. Then, by the properness of $\mathrm{Rep}'_{\bar{\tau}}$ x and x' arise from $\mathcal{O}_E$ valued points, which in turn correspond to G-stable lattice $L_x \subset V_x$ and $L'_x \subset V_{x'}$. Since V_x and $V_{x'}$ are absolutely irreducible with the same trace, they are isomorphic. We may choose this isomorphism so that it induces a non-zero map $L_x \to L_{x'}$, whose reduction modulo the radical (π_E) of $\mathcal{O}_E$ is non-zero. Now as $L_x/\pi_E L_x$ and $L_{x'}/\pi_E L_{x'}$ are both non-zero described in (3.2.1), the only non-zero maps between them are isomorphisms. Hence we find that $L_x \xrightarrow{\sim} L_{x'}$.

Next let $\widehat{R}_{\tau_x}$ be the complete local ring at τ_x. By (3.2.4) $\widehat{R}_{\tau_x}$ is the universal deformation ring of τ_x. Similarly the complete local ring at x, $\widehat{R}_x$ is a quotient of the universal deformation ring of V_x, by Lecture 1, Exercise 4 and (3.2.6). Hence the map $\widehat{R}_{\tau_x} \to \widehat{R}_x$ is a surjection by the Theorem of Nyssen-Rouquier (2.3.1). $\square$

Exercises:

Exercise 1: Verify that the quotient produced in the proof of (3.1.1) does represent $R_{V_{\mathbb{F}}}$.

Exercise 2: Complete the proof of (3.2.2) for arbitrary d.

References

[GIT] D. Mumford, *Geometric invariant theory*, Ergeb. der Math. 34, Springer, 1994.

[SGA 3] M. Demazure, A. Grothendieck, *Schemas en Groupes I, II, III*, Lecture notes in math. 151-153, Springer, 1970.

[2]Note that when we refer to E-valued points of formal schemes over $W(\mathbb{F})$, and complete local rings at such points, what we really mean is the complete local ring at the corresponding point of the p-adic analytic space