LECTURES ON DEFORMATIONS OF GALOIS REPRESENTATIONS

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Lecture 2: Pseudo-representations

(2.1) we saw in the previous lecture that if $\bar{\rho} : G \to \mathrm{GL}_n(\mathbb{F})$ is absolutely irreducible then its deformations are determined by their traces.

The idea of pseudo-representations, introduced by Wiles [Wi] for odd two dimensional representations and by Taylor [Ta] for an arbitrary group, is to try to characterize those functions on G which are traces and to study deformation theory via deformations of the trace functions.

Definition (2.1.2). *Let R be any (topological) ring. An E-valued (continuous) pseudo-representation of dimension $d \in \mathbb{N}^+$ is a continuous function $T : G \to R$ such that*

(1) $T(1) = d$.
(2) $T(g_1 g_2) = T(g_1)T(g_2)$ *for* $g_1, g_2 \in G$.
(3) *If S_{d+1} denotes the symmetric group on $d+1$ letters and $\epsilon : S_{d+1} \to \{\pm 1\}$ denotes its sign character then*

$$\sum_{\sigma \in S_{d+1}} \varepsilon(\sigma) T_\sigma(g_1, \dots, g_{d+1}) = 0$$

for $g_1, \dots, g_{d+1} \in G$. Here if $\sigma \in S_{d+1}$ has cycle decomposition

$$\sigma = (i_1^{(1)}, \dots, i_{i_{r_1}}^{(1)}) \dots (i_1^{(s)}, \dots, i_{r_s}^{(1)})$$

then $T_\sigma : G^{d+1} \to R$ is the function

$$(g_1, \dots, g_{d+1}) \mapsto T(g_{i_1^{(1)}} \dots g_{r_1}^{(1)}) \dots T(g_{i_1}^{(s)} \dots g_{i_{r_s}}^{(s)}).$$

It will often be convenient to form the R-linear extension of a pseudo-representation $T : R[G] \to R$. The relations in the definition are then satisfied for $g_1, \dots, g_{d+1} \in R[G]$.

Theorem (2.1.3). *(Taylor)*

(1) *If $\rho : G \to \mathrm{GL}_d(R)$ is a representation then $\mathrm{tr}\rho$ is a pseudo-representation of dimension d.*
(2) *If R is an algebraically closed field of characteristic 0 and T is a pseudo-representation of dimension d, then there exists a unique semi-simple representation $\rho : G \to \mathrm{GL}_d(R)$ with $\mathrm{tr}\rho = T$.*

(3) *If G is (topologically) finitely generated then for every integer $d \geq 1$, there is a finite subset $S \subset G$, such that a pseudo-representation $T : G \to R$ of dimension d valued in an $\mathbb{Z}[1/d]$-algebra R is determined by its restriction to S.*

Proof. Taylor proves this using results of Procesi on invariant theory [Pr]. A self contained proof was given by Rouquier [Ro]. We sketch some of the arguments for (1) here.

For $g_1, \dots, g_{d+1} \in M_d(R)$ let

$$\Theta(g_1, \dots, g_{d+1}) = \sum_{\sigma \in S_{d+1}} \varepsilon(\sigma) T_\sigma(g_1, \dots, g_{d+1}).$$

We want to show $\Theta \equiv 0$. Writing R as a quotient of a domain of characteristic 0, one sees it is enough to consider such rings, and then that it suffices to consider the case of R a field of characteristic 0. Thus we assume this from now on.

Let $V = R^d$ and $W = V \otimes V^* = \operatorname{End} V$. Note that Θ is invariant by the action of S_{d+1}. Hence if we extend Θ to a multi-linear map $\Theta : W^{\otimes d+1} \to R$, it is determined by its value on $\mathrm{Sym}^{d+1} W \subset W^{\otimes d+1}$. Since $\mathrm{Sym}^{d+1} W$ is spanned by the image of the diagonal map

$$\Delta : W \to W^{\otimes d+1}; \quad w \mapsto w \otimes w \otimes \cdots \otimes w$$

it suffices to show that $\Theta(\Delta(w)) = 0$ for all $w \in W$.

As the semi-simple elements in $\mathrm{Aut} V$ are Zariski dense in W, it is enough to show $\Theta(\Delta(w)) = 0$ for w semi-simple.

Choose a basis for V in which w is diagonal. Then a simple computation shows that $w \sum_{\sigma \in S_{d+1}} \varepsilon(\sigma)\sigma$ acting on $V^{\otimes d+1}$ has trace $\Theta(\Delta(w))$. Here S_{d+1} acts on $V^{\otimes d+1}$ by permutation of the factors in the tensor product, while w acts as $\Delta(w)$. Since

$$(\sum_{\sigma \in S_{d+1}} \varepsilon(\sigma)\sigma) V^{\otimes d+1} \subset \wedge^{d+1} V = 0,$$

the proposition follows. □

(2.2) Deformations: Let $\tau : G \to \mathbb{F}$ be a pseudo-representation. For A in $\mathfrak{AR}_W(\mathbb{F})$ define

$$D_\tau(A) = \{\text{pseudo-representations } \tau_A : G \to A \text{ lifting } \tau\}.$$

We suppose from now on that $p \nmid d!$.

Proposition (2.2.1). *Suppose G satisfies Φ_p. Then D_τ is pro-representable by a complete local Noetherian $W(\mathbb{F})$-algebra R_τ.*

This would be immediate of G were topologically finitely generated. In general we need some preparation.

Definition (2.2.2). *For any pseudo-representation $T : G \to R$ define*

$$\ker T = \{h \in G : T(gh) = T(g) \text{for all } g \in G\}.$$

If we view T as an R-linear map $\tilde{T} : R[G] \to R$ then we set

$$\ker \tilde{T} = \{h \in R[G] : T(gh) = 0 \text{for all } g \in G.\}$$

Lemma (2.2.3). *Let A be in $\mathfrak{AR}_{W(\mathbb{F})}$ and $\tau_A \in D_\tau(A)$. If $G' = \ker \tau$, and G'/H is the maximal pro-p quotient of G', then $\ker \tau_A$ contains H.*

Proof. If $\ker \tau_A$ does not contain H, then the finite group $G'/\ker(\tau_A|_{G'})$ contains a non-trivial element of order coprime to p $\gamma \in G'/\ker(\tau_A|_{G'})$.

Suppose $g \in A[G']$ satisfies $\tau_A(g^i)^2 = 0$ for $i \geq 1$. Then taking $g_1 = g_2 = \cdots = g$ in (2.1.2)(3), we find $tau_A(g^{d+1}) = 0$. Taking $g_1 = h$, $g_2 = g_3 = \cdots = g^{d+1}$ in (2.1.2)(3) we get $\tau_A(hg^{(d+1)^2}) = 0$ for all $h \in A[G']$. Hence $g^{(d+1)^2} \in \ker \tilde{\tau}_A$.

By induction on the length of A, we may assume that $\tilde{\tau}_A((\gamma-1)^i)\mathfrak{m}_A = 0$ for $i \geq 1$. Hence $(\gamma-1)^{(d+1)^2} \in \ker \tilde{\tau}_A$. But since γ has prime to p order, $(\gamma-1)A[\langle\gamma\rangle]$ is an idempotent ideal, so $\gamma - 1 \in \ker \tilde{\tau}_A$ and $\gamma \in \ker \tau_A$. □

Proof. By Lemma (2.2.3), we map consider τ_A as a pseudo-representation of G/H. This is a topologically finitely generated group so the proposition follows from (2.1.3)(3). □

(2.3) Relationship between pseudo-representations and representations:

For absolutely irreducible representations we have the following

Theorem (2.3.1). *(Nyssen-Rouquier) Suppose that G satisfies Φ_p and that $\bar{\rho} : G \to \mathrm{GL}(V_\mathbb{F})$ is absolutely irreducible. IF $\bar{\tau} = \mathrm{tr}(\bar{\rho})$, then there is an isomorphism of functors on $\mathfrak{AR}_{W(\mathbb{F})}$, $D_{V_\mathbb{F}} \xrightarrow{\sim} D_{\bar{\tau}}$.*

(2.3.2) For any $\bar{\rho} : G \to \mathrm{GL}(V_\mathbb{F})$ and $\bar{\tau} - \mathrm{tr}(\bar{\rho})$, one has, of course the morphism of functors $D^{\square}_{V_\mathbb{F}} \to D_{\bar{\tau}}$, however there is something more interesting:

Suppose for example that $\chi_1, \chi_2 : G \to \mathbb{F}^times$ are characters and $c_1, c_2 \in \mathrm{Ext}^1(\chi_2, \chi_1)$. Then $\begin{pmatrix} \chi_1 & c+1+Tc_2 \\ 0 & \chi_2 \end{pmatrix}$ is a representation $G \to \mathrm{GL}_n(\mathbb{F}[T])$. More naturally, one obtains a family of representations of G over $\mathbb{P}(\mathrm{Ext}^1(\chi_1, \chi_2))$, the projectivization of $\mathrm{Ext}^1(\chi_2, \chi_1)$, all have the pseudo-character $\chi_1 + \chi_2$.

(2.3.3) To fully express the relationship between representations and pseudo-representations it will be convenient to work with groupoids see the appendix of [Ki] for a summary of what is needed. Briefly, if $\mathcal{C}$ is a category, a groupoid over $\mathcal{C}$ is a morphism of categories $\pi : \mathcal{F} \to \mathcal{C}$. The morphism π is required to satisfy certain axioms, the most important of which is that a morphism in $\mathcal{F}$ covering an identity map in $\mathcal{C}$ is an isomorphism.

We will often specify groupoids by specifying the categories $\pi^{-1}(c)$ for $c \in \mathcal{C}$. For example, let $\mathcal{C} = \mathfrak{AR}_{W(\mathbb{F})}$, and $D_{V_\mathbb{F}}(A)$ the *category* of deformations of $V_\mathbb{F}$ to a finite free A-module V_A. Then the $D_{V_\mathbb{F}}(A)$ form a groupoid over $\mathfrak{AR}_{W(\mathbb{F})}$; for $\psi : A \to A'$ a map in $\mathfrak{AR}_{W(\mathbb{F})}$ a morphism in $D_{V_\mathbb{F}}$ covering ψ is an isomorphism $V_A \otimes_A A' \xrightarrow{\sim} V_{A'}$ where $V_A, V_{A'}$ are deformations of $V_\mathbb{F}$ to A and A' respectively.

Previously we also denoted by $D_{V_\mathbb{F}}$ the functor of isomorphism classes of the groupoid, which assigns to A in $\mathfrak{AR}_{W(\mathbb{F})}$ the set of isomorphism classes of $D_{V_\mathbb{F}}$. When $V_\mathbb{F}$ has non-trivial automorphisms, then so do the objects in the categories $D_{V_\mathbb{F}}(A)$. In this situation and groupoid $D_{V_\mathbb{F}}$ captures the geometry of the deformation theory of $V_\mathbb{F}$ more accurately than its functor of isomorphism classes.

(2.3.4) For a pseudo-representation $\bar{\tau} : G \to \mathbb{F}$.

Let $\mathfrak{Aug}_{W(\mathbb{F})}$ denote the category of pairs (A, B) where A is in $\mathfrak{AR}_{W(\mathbb{F})}$ and B is an A-algebra. We consider such a B with the topology induced by the radical $\mathfrak{m}_A$ of A.

Define a groupoid $\mathrm{Rep}_{\bar\tau}$ on $\mathfrak{Aug}_{W(\mathbb{F})}$ by setting $\mathrm{Rep}_{\bar\tau}(B)$ equal to the category of finite free B-modules V_B equipped with a continuous action of G such that $\mathrm{tr}(\sigma|V_B \otimes_A A/\mathfrak{m}_A) = \bar\tau(\sigma)$ for $\sigma \in G$.

Similarly we define $\mathrm{Rep}^{\square}_{\bar\tau}(B)$ to be the category are pairs (V_B, β) where V_B is in $\mathrm{Rep}_{\bar\tau}(B)$ and β is a basis for V_B.

Finally we extend $D_{\bar\tau}$ to a groupoid on $\mathfrak{Aug}_{W(\mathbb{F})}$ by setting

$$D_{\bar\tau}(A, B) = \lim_{A' \subset B} D_{\bar\tau}(A')$$

where the limit runs over subalgebras $A' \subset B$ which are local, Artinian, A-algebras. This limit is also equal to the set of continuous pseudo-representations $\tau_B : G \to B$ such that $\tau_B \otimes_B B/\mathfrak{m}_A B = \bar\tau$.

Lemma (2.3.5). *If G satisfies Φ_p, then $\mathrm{Rep}^{\square}_{\bar\tau}$ is representable by a formal scheme over $\mathrm{Spf}\, R_{\bar\tau}$ which is formally of finite type.*

Proof. This is an exercise. See the exercises below for the definition of representability of a groupoid. □

(2.4) Exercises:

Exercise 1: Check carefully all the steps in the proof of Proposition (2.1.3)(1).

Exercise 2: If $\pi : \mathcal{F} \to \mathcal{C}$is a groupoid and $\xi \in \mathrm{Ob}(\mathcal{F})$ define a groupoid $\tilde\xi \to \mathcal{C}$ as follows: An object of $\tilde\xi$ is a morphism $\xi \to \eta$ in $\mathcal{F}$, and $\tilde\xi \to \mathcal{C}$ is given by $(\xi \to \eta) \mapsto \pi(\eta)$.

We say that ξ represents $\mathcal{F}$ if

$$\tilde\xi \to \mathcal{F}; \quad (\xi \to \eta) \mapsto \eta$$

is an equivalence of categories. If such a ξ exists we say that $\mathcal{F}$ is representable.

Show that if $\mathcal{F}$ is representable then $\mathrm{Aut}(\eta) = \mathrm{id}$ for all η in $\mathrm{Ob}(\mathcal{F})$.

Exercise 3: If $\epsilon' : \mathcal{F}' \to \mathcal{F}$ and $\epsilon'' : \mathcal{F}'' \to \mathcal{F}$ are morphisms of categories, let $\mathcal{F}' \times_{\mathcal{F}} \mathcal{F}''$ be the category whose objects are triples (η', η'', θ) where $\eta' \in \mathrm{Ob}(\mathcal{F}')$, $\eta'' \in \mathrm{Ob}(\mathcal{F}'')$ and θ is an isomorphism $\theta : \epsilon'(\eta') \xrightarrow{\sim} \epsilon''(\eta'')$.

For examples if $\mathcal{F}' \to \mathcal{F}$ is a morphism of groupoids over $\mathcal{C}$ and $\xi \in \mathcal{F}$ we can form $\mathcal{F}'_\xi = \mathcal{F}' \times_{\mathcal{F}} \tilde\xi$.

Let S be a scheme. Then we may consider an S-scheme as a groupoid over S-schemes, using the construction in Exercise 2. If $X \to Y$ and $X' \to Y'$ be morphisms of S-schemes, show that there is an isomorphism $\tilde X \times_{\tilde Y} \tilde X' \xrightarrow{\sim} (X \times_Y X')^{\sim}.xs$

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