# Gromov-Witten Theory

Fall 2013, taught by Yaim Cooper.

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1 Overview

1.1 Classical Enumerative Geometry

Schubert calculus: enumerative geometry in projective space. For example, how many lines in $\mathbb{P}^3$ meet 4 fixed lines in general position?

Hurwitz numbers: how many degree $d$ covers $f : C \to \mathbb{P}^1$ (C of genus) $g$ are there ramified over $2d + 2g - 2$ fixed points?

Also counted genus 0, 1, and 2 in $\mathbb{P}^3$.

For example, there are 2 lines in $\mathbb{P}^3$ touching 4 general lines, 27 lines on a cubic, 2875 lines on a quintic 3-fold.

One of the first results of Gromov-Witten theory: Kontsevich’s count of degree $d$, geometric genus 0 curves in $\mathbb{P}^2$ passing through $3d - 1$ general points. Later genus 0 curves on a quintic 3-fold were counted.

1.2 Moduli Spaces of Stable Maps

Recall that $\overline{M}_{g,n}(X, \beta)$ parameterizes $f : C \to X$ along with $p_1, \ldots, p_n \in C$ such that $C$ is nodal with arithmetic genus $f$, $f_*[C] = \beta$, and $\text{Aut}(C)$ is finite (modulo reparameterization).

Examples include:
\[ \overline{\mathcal{M}}_{0,4}(pt) = \overline{\mathcal{M}}_{0,4}. \] We can send three of the points to 0, 1, and \( \infty \). The locus where the four points are distinct is \( \mathbb{P}^1 \) minus three points. In the limit as two points come together, we obtain two rational branches, each with two marked points on it.

There is one for each partition of the four points into two pairs (three of them), so \( \overline{\mathcal{M}}_{0,4} = \mathbb{P}^1 \).

\[ \overline{\mathcal{M}}_0(\mathbb{P}^2, 2) \] is the space of complete conics. These correspond exactly to conics except in the case of a double line, where we obtain a curve with a degree 2 map to the line.

\[ \overline{\mathcal{M}}_1(\mathbb{P}^2, 3) \] has 3 components.

Gromov-Witten invariants:

For \( \alpha_1, \ldots, \alpha_k \) cohomology classes in \( X \), \( N_{g,\beta}^{GW}(\alpha_1, \ldots, \alpha_k) \) “is the number of genus \( g \) curves in \( X \) with class \( \beta \) passing through generic representations of \( \alpha_1, \ldots, \alpha_k \)”. The formula is given by

\[
\int_{[\overline{\mathcal{M}}_g(X,\beta)]^{vir}} \pi_*(f^*(\alpha_1^*)) \cap \cdots \cap \pi_*(f^*(\alpha_k^*))
\]  

This is not quite an exact count since we need to divide by the number of automorphisms. In general, \( N_{g,\beta}^{GW} \in \mathbb{Q} \), not necessarily \( \mathbb{Z} \). On the other hand, GW theory is deformation invariant.

### 1.3 Geometry of Moduli Spaces of Stable Maps

When \( \overline{\mathcal{M}}_{g,n}(X, \beta) \) has multiple components, people know little. But much more is known when \( \overline{\mathcal{M}}_{g,n}(X, \beta) \) has a single component, for example \( g = 0 \) and \( X = \mathbb{P}^r \).

We would like to understand the Picard group and top intersections of divisor classes. Pand (’90s) computed generators of Pic and computed all top intersections of divisors. For example, he could compute that there are 80160 twisted cubics meeting 12 given lines.

### 1.4 Birational Geometry of \( \overline{\mathcal{M}}_{g,n}(X, \beta) \)

The first step to understanding birational geometry is to determine the ample and effective cones in the Picard group. The birational geometry of \( \overline{\mathcal{M}}_{g,n} \) has been studied a lot. Also work has been done on \( \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) \) (Coskun-Harris-Starr found ample and effective cones, and Chen-Coskun-Chrissman worked out the birational geometry in some cases) and \( \overline{\mathcal{M}}_1(\mathbb{P}^r, d) \) (Viscardi).
1.5 Gromov-Witten Theory of a Point

Psi classes $\psi_i$ tell you about the cotangent bundle at $p_i$. We consider the descendants

$$\int_{\overline{M}_{g,n}(X,\beta)} \psi_1^{\alpha_1} \cap \cdots \cap \psi_n^{\alpha_n} \cap (\pi_* f^* \alpha_1) \cap \cdots$$  \hspace{1cm} (1.3)

In the case of a point, only the $\psi$ classes will appear.

**Conjecture 1.1** (Witten, proved by Kontsevich). Define $F = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g$. Then for all $n \geq -1$, $L_n(\exp F) = 0$.

1.6 Main Computational Tools

Localization: The idea is that given $M$ with a $\mathbb{C}^\times$ action, to compute an integral over $M$, consider various curves (hypersurfaces?) $F_1, F_2, \ldots$, and then we obtain a sum of (more complicated) integrals over the $F_i$.

For Gromov-Witten theory of $\mathbb{P}^r$, $\overline{M}_{g,n}(\mathbb{P}^r, d)$ is related to $\prod_i \overline{M}_{g_i,n_i}$. We need Hodge integrals on $\overline{M}_{g,n}$.

1.7 Gromov-Witten Theory of a Point Plus Hodge Structure

Over $\overline{M}_{g,n}$, consider $L_i$ the cotangent bundle at $p_i$, and $E \to \overline{M}_{g,n}$ fibered by $H^0(C, \omega_C)$. Let $\psi_i = c_1(L_i)$ and $\lambda_j = c_j(E)$.

**Theorem 1.2** (Faber-Pandharipande). On $\overline{M}_{g,1}$, define

$$F(t, k) = 1 + \sum_{g \geq 1} \sum_{i=0}^{g} t^{2g} k^i \int_{\overline{M}_{g,1}} \psi_1^{2g-2+i} \lambda_{g-i}.$$  \hspace{1cm} (1.4)

Then $F(t, k) = \left( \frac{t/2}{\sin(t/2)} \right)^{k+1}$.

1.8 Mirror Symmetry

For $X$ a Calabi-Yau manifold (for example, a quintic 3-fold), we can associate a mirror Calabi-Yau $\tilde{X}$. Then

$$e^{HT} + \frac{H^2}{5} \sum_{d=1}^{\infty} n_d d^3 \sum_{k=1}^{\infty} \frac{e^{(H+kd)T}}{(H + kd)^2}$$  \hspace{1cm} (1.5)

where $n_d$ is the number of degree $d$ rational curves in $X$, should equal $\sum_{i=0}^{3} J_i H^i$, where $J_i$ come from solutions of Picard-Fuchs equations on $\tilde{X}$. 

4
1.9 The Main Tool of Degeneration

The idea is given $X$, split into $Y$ and $Z$ with common boundary $D$, then relate the absolute Gromov-Witten theory of $X$ to the relative Gromov-Witten theory of $Y$ relative to $D$. The relative theory was worked out by Li-Tian and Parker-Ionel.

For example, the Gromov-Witten theory of curves was worked out by Pandharipande-Okounkov. There are two degenerations which pinch a tube to a point: this may reduce the geometric genus of the curve (if the tube lies in a single handle for example). The answer is that the Gromov-Witten theory of curves is governed by Virasaro constraints.

1.10 Other Counting Theories

Donaldson-Thomas theory:

Idea: if $X$ is a smooth 3-fold, define a moduli space of sheaves $I_n(X, \beta)$. Then it is possible to construct a virtual class. The advantage of this theory is that in many cases, the Donaldson-Thomas invariants will in fact be integers. However, forcing $X$ to be 3-dimensional is quite a limitation, and even then, there can be pathological behavior with “free points breaking off”. For example a twisted cubic can degenerate into a nodal plane cubic, which in turn can degenerate into a smooth cubic along with a point not on the cubic.

Stable pairs:

A stable pair $(F, s)$ is a coherent sheaf $F$ with 1-dimensional support in $X$ and a section $s \in H^0(X, F)$. This theory does not have free points.

Stable quotients:

This can be viewed as a hybrid between Gromov-Witten theory and stable pairs. We write $\overline{\mathfrak{M}}_{g, n}(G(k, r), d)$ to denote the space parameterizing $C, p_1, \ldots, p_n$ along with

$$0 \to S \to \mathcal{O}_C^{\oplus r} \to Q \to 0$$

with $S$ of rank $k$ and degree $d$. This has the advantage of being a more efficient compactification than stable maps (we have a map $\overline{\mathfrak{M}} \to \overline{\mathfrak{M}}$). However, it’s defined only for a limited set of targets (such as $G(k, r)$).

The web of conjectured relations is

$$\begin{align*}
\text{Stable Quotients} & \quad \text{Stable Pairs} \\
Z^{S\mathfrak{M}}(Q) = J(Q), Z^{GW}(q) = J(Q) & \quad Z^D_{\bar{\beta}}(q) = Z^{red}_{\bar{\beta}}(q) \\
\text{Gromov-Witten} & \quad \text{Donaldson-Thomas}
\end{align*}$$

$Z^{GW}(u) = Z^{red}_{\bar{\beta}}(e^u)$

(1.7)

Here $q = f(Q)$ for $f$ involving the mirror transform (in the case of a quintic 3-fold).
2 Moduli Spaces of Stable Maps

To do Fill in Tuesday’s gap. (1)

Recall that $\overline{\mathcal{M}}_{g,n}(X, \beta)$ parameterizes arithmetic genus $g$ curves with appropriate stable maps. For example, consider the domain curves appearing in $\overline{\mathcal{M}}_1(\mathbb{P}^2, 3)$. We could have:

- A smooth genus 1 curve with degree 3.
- A rational nodal curve with degree 3.
- One of the possibilities below. (Unless otherwise specified, the degree is 1 and the genus is 0.)
- These are not the only possibilities.

\begin{align}
&\begin{array}{ccc}
3 & 3 & 0 \\
g = 1 & g = 1 & 3
\end{array} \\
&\begin{array}{ccc}
2 & 0 & 0 \\
g = 1 & g = 1 & 0
\end{array}
\end{align}

This space can be thought of as a resolution of $\mathbb{P}^9$ where the highly singular curves in $\mathbb{P}^9$ have large preimages (as with the case of complete conics).

For example, there are two domain curves whose image is a nodal cubic (both the nodal cubic, and its normalization along with an elliptic tail which gets contracted to a point (We could use a curve of arithmetic genus 1 instead of just an elliptic curve). There are three whose image is a line and a conic. Two were described above, or we could have a genus 0 curve intersect two other genus 0’s (of degrees (2,1)) and a genus 1. And so on.

Consider just the closure of the locus whose domain curve is a genus 1 curve with a degree 1 map. This contains nodal cubics, intersections of lines and conics, “triangles” of lines, etc.

There turn out to be three components. The one above is 9-dimensional, the closure of the nodal cubics with embedded points is 10-dimensional, and the closure of the intersections of lines and conics with embedded points is 9-dimensional.

The first two components have a nontrivial intersection given by a curve whose image is a cuspidal cubic. The domain curve can be found by stable reduction. This locus is actually 8-dimensional (7
dimensions of cuspidal cubics, and 1 dimension of elliptic tails which get contracted), which is as large as possible.

The second and third components also intersect in dimension 8, as do the first and third. (The first and third consist of image curves with tacnodes.) The triple intersection is 7-dimensional, and consists of cases where the image curve is three concurrent lines, and the lines and an elliptic tail are all connected by a genus 0 curve which is contracted.

For \( f : \Sigma \to X \), we have an exact sequence of sheaves

\[
0 \to T_\Sigma \to f^* T_X \to N_{\Sigma \setminus X} \to 0 \quad (2.2)
\]

for which the long exact sequence in cohomology is

\[
0 \to H^0(\Sigma, T_\Sigma) \to H^0(\Sigma, f^* T_X) \to H^0(\Sigma, N_{\Sigma \setminus X}) \to H^1(\Sigma, T_\Sigma) \to H^1(\Sigma, f^* T_X) \to H^1(\Sigma, N_{\Sigma \setminus X}) \to 0. \quad (2.3)
\]

For the \( T_\Sigma \) parts, \( H^0(\Sigma, T_\Sigma) \) is the infinitesimal automorphisms of \( \Sigma \), and \( H^1(\Sigma, T_\Sigma) \) is the space of deformations of \( \Sigma \). For the \( f^* T_X \) parts, \( H^0(\Sigma, f^* T_X) \) is the space of deformations of \( f \). For given \( s \in H^0(\Sigma, f^* T_X) \), we can consider \( f_{s,\epsilon} : \Sigma \to X \) by \( f_{s,\epsilon}(p) = f(p) + \epsilon s(p) \). Meanwhile, \( H^1(\Sigma, f^* T_X) \) measures the obstructions to finding deformations of \( f \).

Finally, consider the parts with the normal bundle. \( H^0(\Sigma, N_{\Sigma \setminus X}) \) consists of the space of simultaneous deformations of \( \Sigma \) and \( f \). \( H^1 \) is the obstruction space.

Replacing \( T_\Sigma \) by \( T_\Sigma(-p_1, \ldots, -p_n) \) gives pointed deformations and obstructions.

The tangent space to \( \overline{M}_{g,n}(X, \beta) \) at \( (\Sigma, p_1, \ldots, p_n, f) \) is given by \( \text{Def}(\Sigma, p_1, \ldots, p_n, f) \). Suppose that \( \text{Ob}(\Sigma, p_1, \ldots, p_n, f) = 0 \). Then we can compute its dimension as

\[
\dim \text{Def}(\Sigma, p_1, \ldots, p_n, f) = \dim \text{Aut}(\Sigma, p_1, \ldots, p_n, f) - \dim \text{Aut}(\Sigma, p_1, \ldots, p_n) + \dim \text{Def}(f) + \dim \text{Def}(\Sigma, p_1, \ldots, p_n) - \dim \text{Ob}(f). \quad (2.4)
\]

The group of infinitesimal automorphisms is 0 by stability. Next,

\[
\dim \text{Def}(\Sigma, p_1, \ldots, p_n) - \dim \text{Aut}(\Sigma, p_1, \ldots, p_n) = \dim \overline{M}_{g,n} = 3g - 3 + n. \quad (2.5)
\]

Finally, \( h^0(\Sigma, f^* T_X) - h^1(\Sigma, f^* T_X) \), by Riemann-Roch, is \( \beta \cdot c_1(T_X) + (\dim X)(1 - g) \). Hence if \( \text{Ob}(\Sigma, p_1, \ldots, p_n, f) = 0 \), then

\[
\dim \overline{M}_{g,n}(X, \beta) = 3g - 3 + n + (\dim X)(1 - g) + \beta \cdot c_1(T_X). \quad (2.10)
\]

We would like to know when \( \text{Ob}(\Sigma, f) = 0 \). This will certainly hold if \( \text{Ob}(f) = 0 \). We say that \( X \) is convex if \( h^1(\Sigma, f^* T_X) = 0 \) for every genus 0 stable map \( f : \Sigma \to X \). If \( X \) is convex, then \( \overline{M}_{0,n}(X, \beta) \) is a smooth Deligne-Mumford stack (there are at worst finite quotient singularities). Examples of convex spaces include projective space, Grassmannians, and flag varieties.
For example, we now know based on general principles that $\mathcal{M}_{0,0}(\mathbb{P}^2, 2)$ is smooth of dimension

$$\beta \cdot c_1(T_X) + 2(1) + (-3) = 2L \cdot 3H + 2 - 3 = 5. \quad (2.11)$$

In general, though, $\text{Ob}(\Sigma, f)$ may not be zero. The expected dimension of $\mathcal{M}_{g,n}(X, \beta)$ is $\dim(\text{Def}(\Sigma, f)) - \dim(\text{Ob}(\Sigma, f))$.

In the case of $\mathcal{M}_1(\mathbb{P}^2, 3)$, the expected dimension equals

$$3g - 3 + n + (\dim X)(1 - g) + \beta \cdot c_1(T_X) = 3L \cdot 3H = 9. \quad (2.12)$$

Suppose $X$ is a smooth degree 7 hypersurface in $\mathbb{P}^7$ that contains a $\mathbb{P}^3$. The expected dimension of $\mathcal{M}_{0,0}(X, 2)$ then equals

$$3g - 3 + n + (\dim X)(1 - g) + \beta \cdot c_1(T_X) = -3 + 6(1) + 2L \cdot (8H - 7H)|_X = 5. \quad (2.13)$$

There turns out to be a component of dimension 5, but we could also map a genus 0 curve into the $\mathbb{P}^3$. Hence the moduli space contains $\mathcal{M}_{0,0}(\mathbb{P}^3, 2)$. $\mathbb{P}^3$ is convex, so this dimension equals its expected dimension, equal to 8.

3 Examples of $\mathbb{C}^\times$ Fixed Loci

Consider the cases of $\mathcal{M}_0(\mathbb{P}^2, 2)$ and $\mathcal{M}_1(\mathbb{P}^2, 3)$. Recall that the first space is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at three points, but the second is highly singular.

Recall that any $(\alpha_0, \ldots, \alpha_n) \in \mathbb{Z}^{n+1}$ gives a $\mathbb{C}^\times$-action on $\mathbb{P}^n$ by

$$\lambda[x_0 : \cdots : x_n] = [\lambda^{\alpha_0}x_0 : \cdots : \lambda^{\alpha_n}x_0]. \quad (3.1)$$

If all $\alpha_i$ are distinct, the fixed locus will consist of $n + 1$ points.

A given $\mathbb{C}^\times$-action $\mu$ on $\mathbb{P}^n$ induces one on $\mathcal{M}_{g,n}(\mathbb{P}^2, d)$ by sending $f$ to $\mu \circ f$.

The fixed loci of $\mathcal{M}_{0,0}(\mathbb{P}^2)$ should have the following properties:

1. The image should be a union of invariant lines.

2. Any special points (which could be nodes, marked points, or collapsed components) should be mapped to fixed points of $\mathbb{P}^2$.

For example, we could have a degree 2 map to the line between $[1 : 0 : 0]$ and $[0 : 1 : 0]$ ramified at those two points. There are two other similar maps (taking the other two pairs of three points), giving three fixed loci. We could also have a union of two of these lines, or even a nodal curve with both branches mapping onto the same line. In any case, the fixed locus is a finite set of points.

Now consider $\mathcal{M}_1(\mathbb{P}^2, 3)$. Any possible $\mathbb{C}^\times$-fixed locus must collapse curves of genus greater than 0 to a point (since the curves which aren’t contracted can have only two branch points).

Consider the case of a degree 3 map of a genus 0 curve ramified at two of the special points, with an arbitrary elliptic tail contracted to one of these points. This gives a $\mathbb{P}^1$ inside the fixed locus.
Recall that the above are the generic cases for one of the components. The other components have some fixed loci (for example the triangle of three lines is contained in the “nice” component). For the last component, we can have two maps to lines, one of degree 2 and one of degree 1, with the genus 0 curves connected by a genus 1 curve contracted to a point. This gives a surface $\overline{M}_{1,2}$ inside the fixed locus.

Finally, consider three genus 0 curves, all connected by a genus 1 curve. This can map into $\mathbb{P}^2$ by mapping the genus 0 curves to two of the special lines. Here the fixed locus is given by $\overline{M}_{1,3}$.

## 4 Gromov-Witten Theory of a Point

Recall that the universal curve $\pi_0 : C \rightarrow \overline{M}_{g,n}$ is given by $\overline{M}_{g,n+1}$ parameterized by the $(n+1)$st point, which we call $p_0$. However, when $p_0$ coincides with one of the other $p_i$, then the curve corresponding to the associated point of $\overline{M}_{g,n+1}$ has a “bubble” containing $p_0$ and $p_i$.

The relative dualizing sheaf $\omega_{\pi}$ is a line bundle on $C$ whose fiber over $p \in C$ is the cotangent line at $p$ to the fiber of $\pi$. In general, given $\pi : C \rightarrow B$, we have

$$H^0(\omega_{\pi}(\pi^{-1}(U))) = H^0(\text{coker}(\pi^*\omega_B(U) \rightarrow \omega_C(U))).$$

(4.1)

$\pi_0$ has sections $\sigma_i$ by setting $p_0$ to $p_i$. Set $L_i = \sigma_i^*\omega_{\pi}$. Then $L_i$ is a line bundle over $\overline{M}_{g,n}$, whose fiber over $(C, p_1, \ldots, p_n)$ is the cotangent line to $C$ at $p_i$. Define the $\psi$-class $\psi_i = c_1(L_i)$.

If $\gamma \in H^*(X)$, we have evaluation maps $ev^*_i(\gamma)$ on $\overline{M}_{g,n}(X, \beta)$, and we obtain $ev^*_i(\gamma) \in H^*(\overline{M}_{g,n}(X, \beta)).$ If $\gamma$ is Poincare dual to $\Gamma \in H_*(X)$, then $ev^*_i$ is the locus of maps where $p_i$ maps to $\Gamma$.

We have $\psi$-classes for moduli of stable maps similar to before. Consider $\pi_0 : \overline{M}_{g,n+1}(X, \beta) \rightarrow \overline{M}_{g,n}(X, \beta)$ and again define the sections $\sigma_i$, the line bundles $L_i = \sigma_i^*\omega_{\pi_0}$, and $\psi_i = c_1(L_i)$. This is related to tangency conditions explicitly worked out in the case of curves.

On $\overline{M}_{0,3}$, each $\psi$-class is 0, and $\int_{\overline{M}_{0,3}} \psi_0^0 \psi_1^0 \psi_2^0 = 1$.

Now consider $L_i$ on $\overline{M}_{0,4}$. This can be determined by induction. In general, given $L_i$ on $\overline{M}_{g,n}$, we want to compare $\pi^*L_i$ to the $L_i$ on $\overline{M}_{g,n+1}$.

After applying $\sigma_i$, each point in $\overline{M}_{g,n}$ gets mapped to a point corresponding to a curve with a “bubble” containing $p_0$ and $p_i$. The set of these curves form a divisor $D_{i,0}$.

Now we want to compare $L_i^{(n+1)}$ to $\pi_{0,1}^*L_i^{(n)}$. Observe that:

1. On $\overline{M}_{g,n+1} \setminus D_{i,0}$, $L_i^{(n+1)} = \pi_0^*L_i^{(n)}$, since the cotangent lines are canonically isomorphic. (No contraction will occur outside a bubble.)

   Hence, for some $r$, we must have

   $$L_i^{(n+1)} = \pi_0^*L_i^{(n)} \otimes \mathcal{O}(rD_{i,0}).$$

   (4.2)

2. Applying $\sigma_i^*$, we obtain

   $$\mathcal{O} = L_i^{(n)} \otimes \mathcal{O}(D_{i,0})^{\otimes r}|_{D_{i,0}}.$$  

   (4.3)

But $\mathcal{O}(D_{i,0})$ is given by $N_{D_{i,0}/\overline{M}_{g,n+1}}$. This is $(L_i^{(n)})^\vee$, so we must have $r = 1$. 

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Proposition 4.1. On $\overline{M}_{0,n}$, $\psi_1 = (p_1|p_2,p_3)$, the union of all boundary divisors with $p_1$ on one component and $p_2,p_3$ on the other. Specifically,

$$\psi_1 = \pi^* (p_1|p_2,p_3) + D_{1,n+1}$$

where $D_{I\setminus J}$ is the divisor where the points corresponding to $I$ is contained in the first component and those corresponding to $J$ is contained in the other.

Proof. Induct on $n$. Assume that $\psi_1 = (p_1|p_2,p_3)$ on $\overline{M}_{0,n}$. On $\overline{M}_{0,n+1}$,

$$\psi_1 = \pi^* (p_1|p_2,p_3) + D_{1,n+1}$$

Theorem 4.2 (String equation).

$$\int_{\overline{M}_{g,n+1}} \psi_1^{\beta_1} \cdots \psi_n^{\beta_n} \psi_0 = \sum_{i=1}^n \int_{\overline{M}_{g,n}} \psi_1^{\beta_1} \cdots \psi_i^{\beta_i-1} \cdots \psi_n^{\beta_n}. \quad (4.8)$$

To prove this:

Fact. (I) $D_{i,0} D_{j,0} = 0$ if $i \neq j$.

(II) $\psi_i D_{i,0} = 0$. Hence $D_{i,0}^2 = -(\pi^* \psi_i) D_{i,0}$.

(III) (Push-pull formula) Given $f : X \to Y$, $\beta \in H^*(X)$, and $\alpha \in H^*(Y)$,

$$f_*(f^* \alpha \cdot \beta) = \alpha f_* \beta. \quad (4.9)$$

We want to compute $\pi_* \psi_1^{\beta_1} \cdot \psi_n^{\beta_n}$, where $\pi = \pi_0$. As a warm-up, consider just $\psi_1$:

$$\pi_* \psi_1 = \pi_*(\pi^* \psi_1 + D_{1,0})$$

$$= \pi_*(\pi^* \psi_1) + \pi_* D_{1,0}$$

$$= \psi_1 \pi_1 + \pi_* D_{1,0}$$

$$= 1. \quad (4.13)$$

Similarly, we can work out higher powers inductively:
\[ \pi_\ast \psi_1^2 = \pi_\ast((\pi^* \psi_1 + D_{1,0})\psi_1) \] \hfill (4.14)
\[ = \pi_\ast(\pi^* \psi_1 \cdot 1 + D_{1,0} \cdot 1) \] \hfill (4.15)
\[ = \psi_1. (\pi_\ast \psi_1) \] \hfill (4.16)
\[ = \psi_1.1. \] \hfill (4.17)

In general,

Claim. \( \pi_\ast \psi_1^\beta = \psi_1^{\beta-1} \).

This is done by essentially the same calculation as above:

\[ \pi_\ast \psi_1^\beta = \pi_\ast((\pi^* \psi_1 + D_{1,0})\psi_1^{\beta-1}) \] \hfill (4.18)
\[ = \pi_\ast((\pi^* \psi_1).\psi_1^{\beta-1}) \] \hfill (4.19)
\[ = \psi_1. \pi_\ast \psi_1^{\beta-1} \] \hfill (4.20)
\[ = \psi_1^{\beta-1}. \] \hfill (4.21)

Now we add in a second \( \psi \)-class, one power at a time.

\[ \pi_\ast(\psi_1^\beta \psi_2) = \pi_\ast(\psi_1^\beta (\pi^* \psi_2 + D_{2,0})) \] \hfill (4.22)
\[ = \pi_\ast(\pi^* \psi_1 \cdot 1 + D_{2,0} \cdot 1) \] \hfill (4.23)
\[ = \psi_2 \pi_\ast \psi_1^\beta + \pi_\ast((D_{2,0}(\pi^* \psi_1 + D_{1,0})\beta)) \] \hfill (4.24)
\[ = \psi_1^{\beta-1} \psi_2 + \pi_\ast((D_{2,0}(\pi^* \psi_1))\beta) \] \hfill (4.25)
\[ = \psi_1^{\beta-1} \psi_2 + \psi_1^\beta. \] \hfill (4.26)

Now if \( \beta_1, \beta_2 > 1 \),

\[ \pi_\ast(\psi_1^\beta_1 \psi_2^\beta_2) = \pi_\ast((\pi^* \psi_1 + D_{1,0})\psi_1^{\beta_1-1} \psi_2^{\beta_2}) \] \hfill (4.27)
\[ = \pi_\ast((\pi^* \psi_1).\psi_1^{\beta_1-1} \psi_2^{\beta_2}) \] \hfill (4.28)
\[ = \psi_1 (\pi_\ast(\psi_1^{\beta_1-1} \psi_2^{\beta_2})) \] \hfill (4.29)
\[ = \psi_1 (\psi_1^{\beta_1-2} \psi_2^{\beta_2} + \psi_1^{\beta_1-1} \psi_2^{\beta_2-1}) \] \hfill (4.30)
\[ = \psi_1^{\beta_1-1} \psi_2^{\beta_2} + \psi_1^{\beta_1} \psi_2^{\beta_2-1}. \] \hfill (4.31)

Similar computations will show that

\[ \pi_\ast \psi_1^{\beta_1} \cdots \psi_n^{\beta_n} \cdot 0 = \sum_{i=1}^n \psi_1^{\beta_1} \cdots \psi_i^{\beta_i-1} \cdots \psi_n^{\beta_n}. \] \hfill (4.32)
Now we can prove the string equation. For $P'$ and $P$ the maps from $\overline{M}_{g,n}$ and $\overline{M}_{g,n+1}$ to a point, we have

$$\int_{\overline{M}_{g,n+1}} \psi_1^{\beta_1} \cdots \psi_n^{\beta_n} \psi_0^0 = P_* \psi_1^{\beta_1} \cdots \psi_n^{\beta_n}$$

$$= (P' \circ \pi_0)_* \psi_1^{\beta_1} \cdots \psi_n^{\beta_n} \psi_0^0$$

$$= P'_* \pi_* \psi_1^{\beta_1} \cdots \psi_n^{\beta_n}.$$  

(4.33)

We conclude that

$$\int_{\overline{M}_{g,n+1}} \psi_1^{\beta_1} \cdots \psi_n^{\beta_n} \psi_0^0 = \int_{\overline{M}_{g,n}} \pi_* \psi_1^{\beta_1} \cdots \psi_n^{\beta_n} \psi_0^0.$$  

(4.34)

(4.35)

$$\int_{\overline{M}_{g,n+1}} \psi_1^{\beta_1} \cdots \psi_n^{\beta_n} \psi_0^0 = \int_{\overline{M}_{g,n}} \pi_* \psi_1^{\beta_1} \cdots \psi_n^{\beta_n} \psi_0^0.$$  

(4.36)

Proposition 4.3. If $\sum \beta_i = n - 3$, then

$$\int_{\overline{M}_{0,n}} \psi_1^{\beta_1} \cdots \psi_n^{\beta_n} = \frac{(n - 3)!}{\beta_1! \cdots \beta_n!}$$

(4.37)

This follows from the multinomial analogue of Pascal’s identity:

$$\frac{p!}{q_1! \cdots q_n!} = \sum_{j=1}^{n} \frac{(p - 1)!}{(q_j - 1)! \prod_{i \neq j} q_i!}$$

(4.38)

and also the observation that in genus 0, $\sum \beta_i = n - 3$ forces at least one $\beta_i$ to be 0, so the string equation can be applied on at least one point (an appropriate $i$ replacing the role of the index 0). Now use induction. The base case $\int_{\overline{M}_{0,3}} \psi_1^1 \psi_2^0 \psi_3^0 = 1$ follows from $\overline{M}_{0,3} = pt$.

We might want to ask what happens if the genus is, say, one instead. Then the nontrivial case is $\sum \beta_i = n$. If one of the $\beta_i$ is 0, we can apply the string equation. Otherwise, at least one (in fact, all of them!) equals 1. It turns out that we have the Dilaton equation:

$$\int_{\overline{M}_{g,n}} \psi_1^{\beta_1} \cdots \psi_n^{\beta_n} = \int_{\overline{M}_{g,n-1}} (2g - 2 + n - 1) \psi_2^{\beta_2} \cdots \psi_n^{\beta_n}.$$  

(4.39)

We will first prove this in genus 0. Now let $\pi = \pi_1$. Then

$$\pi_* \psi_1 = \pi_* \left( \sum_{1 \subseteq I \subseteq 2, 3 \subseteq J} D_{\{I\} \{J\}} \right).$$  

(4.40)

Now if $|I| \geq 3$, then $\pi_* D_{\{I\} \{J\}} = 0$; pushing forward will give a divisor on $\overline{M}_{g,n}$. On the other hand, if $|I| = 2$, the push-forward has image exactly $\overline{M}_{g,n}$, so $\pi_* D_{\{I\} \{J\}} = 1$. Hence $\pi_* \psi_1$ equals the number of such $I, J$ with $|I| = 2$, which is $n - 2$.

Now consider $\pi_* \psi_1$ in higher genus. In the case $n = 1$, $\psi_1$ is the zero locus of a generic section of $L_1$. But $L_1|_C = \omega_C$, having degree $2g - 2$.

In general, we will show $\pi_* \psi_1 = 2g - 2 + n$. 

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Fact. $\pi_i \pi_j^* \alpha = \pi_j^* \pi_i \alpha$.

Now use induction. Assume $\pi \psi_1 = 2g - 2 + n - 1$. Then

\[
\begin{align*}
\pi_1 \psi_1 &= \pi_1 (\pi_{n+1}^* \psi_1 + D_{1,n+1}) \\
&= \pi_1 (\pi_{n+1}^* \psi_1) + \pi_1 (D_{1,n+1}) \\
&= \pi_{n+1}^* (\pi_1 \psi_1) + 1 \\
&= 2g - 2 + n.
\end{align*}
\]

Proof of Dilaton equation.

\[
\begin{align*}
\pi_1 (\psi_1 \psi_2^{\beta_2} \cdots \psi_n^{\beta_n}) &= \pi_1 (\psi_1 (\pi^* \psi_2 + D_{12})^{\beta_2} \cdots (\pi^* \psi_n + D_{1n})^{\beta_n}) \\
&= \psi_1 (\psi_1 \pi^* \psi_2^{\beta_2} \cdots \pi^* \psi_n^{\beta_n}) \\
&= \psi_2^{\beta_2} \cdots \psi_n^{\beta_n} \pi_1 \psi_1 \\
&= (2g - 2 + n) \psi_2^{\beta_2} \cdots \psi_n^{\beta_n}.
\end{align*}
\]

Now integrate.

We can reduce the $\psi$-integrals for genus 1 to the case $\int_{\mathcal{M}_{1,1}} \psi_1$. On $\mathcal{M}_{1,1}$,

- $\psi_1 = \lambda_1 = \frac{1}{12} D$.

- More generally, on $\mathcal{M}_{1,n}$, $\psi_i = \frac{1}{12} D + \beta_i$ where $\beta_i$ is the sum of all boundary components with $i \in I$.

In higher genus, we have $\sum \beta_i = 3g - 3 + n$. The string or Dilaton equation can be applied unless every $\beta_i$ is at least 2. This is only possible for $n \leq 3g - 3$, so we may reduce to a finite number of base cases.

Witten introduced the notation

\[
\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g = \int_{\mathcal{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}.
\]

We may also write $\tau_i^{a_i}$ for $\tau_1 \cdots \tau_i$. We may sometimes drop $g$ because $g$ is uniquely determined as

\[
g = \frac{1}{3} \left( 3 - n + \sum a_i k_i \right).
\]

Now introduce the generating function
\[ F(t, \lambda) = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t) \quad (4.51) \]

\[ F_g(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{i=0}^{n} t_i \tau_i \right)^n \]

\[ = \sum_{\{n_i\}} \left( \prod_{i} \frac{t_i^{n_i}}{n_i!} \right) \langle \tau_0^{n_0} \tau_1^{n_1} \cdots \rangle_g \quad (4.52) \]

Also introduce the notation

\[ \ll \tau_{k_1} \cdots \tau_{k_n} \gg = \frac{\partial}{\partial t_{k_1}} \cdots \frac{\partial}{\partial t_{k_n}} F. \quad (4.54) \]

Now look at \( \ll \tau_{k_1} \cdots \tau_{k_n} \gg \big|_{t_i=0} \). The term \( t_{\beta_1} \cdots t_{\beta_n} \langle \tau_{\beta_1} \cdots \tau_{\beta_n} \rangle \) will produce a nonzero contribution under this differentiation if and only if the \( \beta_i \) are a permutation of the \( k_i \). We end up with

\[ \ll \tau_{k_1} \cdots \tau_{k_n} \gg \big|_{t_i=0} = \lambda^{2g-2} \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g. \quad (4.55) \]

We can express the string and Dilaton equations using this notation:

\[ \left\langle \tau_0 \prod_{i=1}^{n} \tau_{k_i} \right\rangle_g = \sum_{i=1}^{n} \left\langle \tau_{k_1} \cdots \tau_{k_{i-1}} \cdots \tau_{k_n} \right\rangle_g \quad (4.56) \]

\[ \left\langle \tau_1 \prod_{i=1}^{n} \tau_{k_i} \right\rangle_g = (2g - 2 + n) \left\langle \prod_{i=1}^{n} \tau_{k_i} \right\rangle_g \quad (4.57) \]

\( F \) turns out to determined by systems of differential equations.

Formulation one: the KdV equation: for all \( n \geq 1 \), \( F(t, \lambda) \) satisfies

\[ (2n+1)\lambda^{-2} \ll \tau_n \tau_0^2 \gg = \ll \tau_{n-1} \tau_0 \gg \ll \tau_0^3 \gg + 2 \ll \tau_{n-1} \tau_0^2 \gg \ll \tau_0^2 \gg + \frac{1}{4} \ll \tau_{n-1} \tau_0^4 \gg. \quad (4.58) \]

As an example, consider \( n = 3 \) and all \( t_i = 0 \). Then

\[ 7\lambda^{-2} \langle \tau_3 \tau_0^2 \rangle_1 = \lambda^{-2} \langle \tau_2 \tau_0 \rangle_1 \langle \tau_0^3 \rangle_0 + 2\lambda^{-2} \langle \tau_2 \tau_0 \rangle_0 \langle \tau_0^2 \rangle_1^2 + \frac{1}{4} \lambda^{-2} \langle \tau_2 \tau_0 \rangle_0^4 \quad (4.59) \]

where a required fractional genus means the bracket must equal 0. Simplifying and then applying the string and Dilaton equations:

**To do** Work this out. (2)

We end up with \( \langle \tau_1 \rangle_1 = \frac{1}{24} \). (This could be derived directly.)
Second formulation: there is one differential operator $L_i$ for each $i \geq -1$. However, $L_i$ acts on $\exp(F)$, not $F$.

The first two operators are

\begin{align*}
L_{-1} &= -\frac{\partial}{\partial t_0} + \frac{1}{2} \lambda^{-2} t_0^2 + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial}{\partial t_i} \\
L_0 &= -\frac{3}{2} \frac{\partial}{\partial t_1} + \sum_{i=0}^{\infty} \frac{2i + 1}{2} t_i \frac{\partial}{\partial t_i} + \frac{1}{16}.
\end{align*}

This indexing arises from the Virasoro algebra $V$: if $L_n = -z^{n+1} \frac{\partial}{\partial z}$, then

$$[L_n, L_m] = (n - m) L_{n+m}.$$  

The claim is that the $L_i$ satisfy the above relations, and

$$L_i(\exp(F)) = 0.$$  

The general formula for the $L_n$ for $n \geq 1$ is

\begin{align*}
L_n &= -\frac{(2n + 3)!!}{2^{n+1}} \frac{\partial}{\partial t_{n+1}} && (4.64) \\
&\quad + \sum_{i=0}^{\infty} \frac{(2i + 2n + 1)!!}{(2i - 1)!!2^{n+1} t_i} \frac{\partial}{\partial t_{i+n}} \\
&\quad + \frac{\lambda^2}{2} \sum_{i=0}^{n-1} \left( \frac{(2i + 1)!![2n - (2i + 1)!!]}{2^{n+1}} \right) \frac{\partial^2}{\partial t_i \partial t_{n-1-i}} \\
&\quad (4.65) \\
&\quad (4.66)
\end{align*}

As an example, the differential equation $L_3(\exp(F)) = 0$ means

\begin{align*}
0 &= -\frac{945}{16} \frac{\partial}{\partial t_4} F \\
&\quad + \sum_{i=0}^{\infty} t_i \frac{\partial}{\partial t_{i+3}} F \quad (4.67) \\
&\quad + \lambda^2 \frac{15}{16} \left( \frac{\partial^2 F}{\partial t_0 \partial t_2} + \frac{\partial F}{\partial t_0} \frac{\partial F}{\partial t_2} \right) \\
&\quad + \frac{9}{32} \lambda^2 \left( \frac{\partial^2 F}{\partial t_1 \partial t_1} + \frac{\partial F}{\partial t_1} \frac{\partial F}{\partial t_1} \right) \quad (4.68) \\
&\quad (4.69) \\
&\quad (4.70)
\end{align*}

Taking constant terms, $\langle \tau_4 \rangle_2 = \frac{1}{1152}.$

Observe that if $D$ is a derivation, then $D(\exp(F)) = \exp(F) D(F).$ Meanwhile, the $L_n$ differ from a derivation by only finitely many terms (the second partials, or the term with no differentiation in the case $n = 0, -1$).
In the case \( n = -1 \), write \( L_{-1}^{\text{der}} \) for the part which is a derivation. Then

\[
e^{-F}L_{-1}(e^F) = L_{-1}^{\text{der}}(F) + \frac{\lambda^2}{2} t_0^2. \tag{4.71}
\]

So \( L_{-1}(e^F) = 0 \) is equivalent to

\[
[L_{-1}^{\text{der}} F]_{\lambda = 0} t_0^{a_0} t_1^{a_1} \cdots = \begin{cases} \frac{1}{2} & a = -2, a_0 = 2, a_i = 0 \text{ for } i \geq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.72}
\]

Observe that for

\[
F(t) = \sum_{a_0, a_1, \ldots} \frac{1}{\prod_j a_j!} t_0^{a_0} t_1^{a_1} \cdots \langle \tau_0^{a_0} \tau_1^{a_1} \cdots \rangle
\]

we have

\[
\frac{d}{dt_i} F(t) = \sum_{a_0, a_1, \ldots} \frac{1}{\prod_j a_j!} t_0^{a_0} t_1^{a_1} \cdots \langle \tau_0^{a_0} \cdots \tau_i^{a_i+1} \cdots \rangle. \tag{4.74}
\]

We see that differentiating with respect to \( t_i \) “multiplies” the coefficients by \( \tau_i \). We also have

\[
t_i F(t) = \sum_{a_0, a_1, \ldots} \frac{1}{\prod_j a_j!} t_0^{a_0} t_1^{a_1} \cdots \langle a_i \tau_0^{a_0} \cdots \tau_i^{a_i-1} \cdots \rangle
\]

So multiplication by \( t_i \) “differentiates” the coefficients with respect to \( \tau_i \).

Consider now the equation \( L_{-1}^{\text{der}} F = 0 \) (which is almost \( L_{-1} e^F = 0 \)). Then we require

\[
\frac{\partial}{\partial t_0} F = \sum_{i=0}^{\infty} t_{i+1} \frac{\partial}{\partial t_i} F. \tag{4.76}
\]

This means that for each \( a_i \),

\[
\langle \tau_0^{a_0+1} \tau_1^{a_1} \rangle = \sum_{i=0}^{\infty} a_{i+1} \langle \tau_0^{a_0} \cdots \tau_i^{a_i+1} \tau_{i+1}^{a_{i+1}-1} \cdots \rangle \tag{4.77}
\]

which is the string equation.

Now consider \( [L_{-1}^{\text{der}} F]_{\lambda = -2 t_0^2} = -\frac{1}{2} \). This equates to \( \langle \tau_0^3 \rangle = 1 \).

For \( L_0 \), write

\[
e^{-F}L_0 e^F = L_0^{\text{der}} F + \frac{1}{16}. \tag{4.78}
\]

Setting the nonconstant terms equal to 0 gives the Dilaton equation. The constant term gives \( \langle \tau_1 \rangle = 1 \).

For \( k \geq 1 \), we can examine \( L_k \). We have \( e^{-F} D_1 D_2 (e^F) = D_1 D_2 (F) + D_1 (F) D_2 (F) \). Write

\[
\tilde{L}_k F = e^{-F} L_k e^F. \tag{4.78}
\]

Then
\[
\tilde{L}_k F = L^\text{der}_k F + \frac{\lambda^2}{2} \sum_{i=0}^{n-1} \left( \cdots \right) \left( \frac{\partial^2 F}{\partial t_i \partial t_{k-1-i}} F + \frac{\partial F}{\partial t_i} \frac{\partial F}{\partial t_{k-1-i}} \right).
\] (4.79)

We get a linear relation between:

- \( \langle \tau_0^a \cdots \tau_{k+1}^a \cdots \rangle \)
- \( \langle \tau_0^a \cdots \tau_i^a \cdots \tau_{k+1}^a \cdots \rangle \)
- \( \langle \tau_0^a \cdots \tau_i^a \cdots \tau_{k-1-i}^a \cdots \rangle \)
- Products of \( \langle \tau_j^b \rangle \) and \( \langle \tau_j^c \rangle \) where \( b_j + c_j = a_j \) except for \( b_i + c_i = a_i + 1 \) and \( b_{k-1-i} + c_{k-1-i} = a_{k-1-i} + 1 \).

5 \( \psi \) Classes on \( \overline{M}_{g,n}^\bullet \)

The genus of a curve of components having genera \( g_i \) is the arithmetic genus, equal to \( 1 + \sum (g_i - 1) \). Let \( \overline{M}_{g,n}^\bullet \) denote the moduli space of disconnected (meaning not necessarily connected) \( n \)-pointed genus \( g \) curves. This is a disjoint union of products of moduli spaces of pointed curves. The \( \psi \)-classes on products are generated by pullbacks of \( \psi \)-classes from each of the factors.

We use the notation

\[
\langle \tau_{k_1} \cdots \tau_{k_n} \rangle^g = \int_{\overline{M}_{g,n}} \psi^{k_1}_1 \cdots \psi^{k_n}_n .
\] (5.1)

Our generating function will be

\[
G(t, \lambda) = \sum_{g,n} \lambda^{2g-2} \frac{1}{n!} \sum_{k_1, \ldots, k_n} t_{k_1} \cdots t_{k_n} \langle \tau_{k_1} \cdots \tau_{k_n} \rangle^g .
\] (5.2)

Claim. \( G(t, \lambda) = e^{F(t, \lambda)} \).

For example, to check the \( t_0 t_1 t_2 \) coefficient,

\[
\int_{\overline{M}_{1,3}} \psi_1^0 \psi_2^1 \psi_3^2 = \int_{\overline{M}_{1,3}} \psi_1^0 \psi_2^1 \psi_3^2 + \sum \text{choice of pt} \int_{\overline{M}_{1,2} \times \overline{M}_{1,1}} \psi_1^0 \psi_2^1 \psi_3^2
\] (5.3)

\[
+ \int_{\overline{M}_{2,3}} \psi_1^0 \psi_2^1 \psi_3^2
\] (5.4)

\[
+ \int_{\overline{M}_{1,1} \times \overline{M}_{1,1} \times \overline{M}_{1,1}} \psi_1^0 \psi_2^1 \psi_3^2
\] (5.5)

\[
+ \int_{\overline{M}_{1,1}} \psi_1^0 \psi_2^1 \psi_3^2
\] (5.6)

The only contributions from the nonzero terms are
\[ \int_{\mathfrak{M}_{1,3}} \psi_1^0 \psi_1^1 \psi_2^3 + \left( \int_{\mathfrak{M}_{1,1,2}} \psi_1^0 \psi_2^3 \right) \left( \int_{\mathfrak{M}_{1,2}} \psi_2^1 \right) \] (5.7)

so that

\[ \langle \tau_0 \tau_1 \tau_2 \rangle_1 = \langle \tau_0 \tau_1 \tau_2 \rangle_1 + \langle \tau_0 \tau_2 \rangle_1 \langle \tau_1 \rangle_1. \] (5.8)

Likewise, when expanding \( e^F \), we get the same partition into \( \tau \)'s (and many of them happen to be zero).

Now examine the \( L_k \) on disconnected parts. The \( \frac{\partial}{\partial t_{k+1}} \) part multiplies by \( \tau_{k+1} \), the \( t_i \frac{\partial}{\partial t_{i+k}} \) replaces a \( \tau_i \) with \( \tau_{i+k} \), and \( \frac{\partial}{\partial t_i \partial t_{k-i-1}} \) multiplies by \( \tau_i \) and \( \tau_{k-i-1} \).

### 6 The KdV Formulation

For a polynomial \( U(t_0, t_1, t_2, \ldots) \), recursively define the polynomials \( R_i \) in \( U, \dot{U}, \ddot{U}, \ldots \) by \( R_1 = U \) and

\[ \frac{\partial}{\partial t_0} R_{i+1} = \frac{1}{2i+1} \left( R_i \frac{\partial}{\partial t_0} U + 2 \frac{\partial}{\partial t_0} R_i U + \frac{1}{4} \frac{\partial^3}{\partial t_0^3} U \right) \] (6.1)

The KdV equations are then

\[ \frac{\partial}{\partial t_i} U = \frac{\partial}{\partial t_0} R_{i+1}(U, \dot{U}, \ddot{U}, \ldots) \] (6.2)

Witten's conjecture claims this holds for \( U = \frac{\partial^2}{\partial t_0^2} F \) and \( R_{i+1} = \frac{\partial}{\partial t_i} \frac{\partial}{\partial t_{0}} F \).

These equations, along with the string equation, can be used to determine all of the descendant invariants. For example, the string equation relates \( \langle \tau_0^2 \tau_i \rangle \) with \( \langle \tau_i \rangle \), so knowledge of \( \langle \tau_{i+2} \tau_0^2 \rangle \) (which can be expressed in terms of already known terms and some terms which end up being zero) gives \( \langle \tau_i \rangle \). In fact

\[ \langle (2i+1) \tau_i \rangle = \frac{1}{4} \langle \tau_{i-3} \rangle \] (6.3)

for \( i = 3g - 2 \). So \( \langle \tau_{3g-2} \rangle = \frac{1}{24g!} \).

To get a term like \( \langle \tau_0^2 \tau_i \tau_j \rangle \), we can differentiate \( \langle \tau_0^2 \tau_i \rangle \) with respect to \( t_j \) and then take the constant term.

It turns out that the Virasoro conditions are equivalent to KdV, the string equation, and the base cases \( \langle \tau_0^3 \rangle = 1 \) and \( \langle \tau_1 \rangle = \frac{1}{24} \).

The Lie algebra constraints on the \( L_k \) can be used to show that

\[ L_k = \frac{(-1)^{k-2}}{(k-2)!} \text{ad} (L_1)^{k-2} L_2 \] (6.4)
for $k \geq 2$. Also $L_1 = -\frac{1}{3} [L_{-1}, L_2]$. So knowing that $e^F$ is annihilated by $L_0, L_{-1}, L_2$ implies annihilation by all other $L_k$.

Write the recursion for $R_i$ as $\frac{\partial}{\partial t_0} R_{i+1} = \frac{2}{2i+1} (K) \cdot R_i$, where

$$
K = \frac{1}{2} \frac{\partial}{\partial t_0} U + U \frac{\partial}{\partial t_0} + \frac{1}{8} \frac{\partial^3}{\partial t_0^3}.
$$

To show that KdV implies Virasoro, let $Z_k = L^\text{conn}_k F$.

**Theorem 6.1.** $\frac{\partial}{\partial t_0} K \frac{\partial}{\partial t_0} Z_{k-1} = \frac{\partial^3}{\partial t_0^3} Z_k$.

**Theorem 6.2.** If $\frac{\partial^3}{\partial t_0^3} Z_k = 0$ for every $k \leq M$, then $Z_k = 0$ for every $k \leq M$.

$Z_{-1} = 0$ is the string equation. Then the rest can be shown to be zero by induction.

Now Virasoro can also be used to uniquely determine the descendants, so implies KdV.

## 7 Gromov-Witten Theory in Higher Dimensions

Consider the diagram

$$
\begin{array}{c}
\overline{\mathcal{M}}_{g,n+1}(X,\beta) \xrightarrow{\text{ev}_0} X \\
\sigma_i \uparrow \\
\overline{\mathcal{M}}_{g,n}(X,\beta)
\end{array}
$$

(7.1)

Let $\psi_i = c_1(\sigma_i^* \omega_{\pi_0})$, for $\omega_{\pi_0}$ the dualizing sheaf on the universal curve. Define

$$
\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle^X_{g,\beta} = \int_{\overline{\mathcal{M}}_{g,n}(X,\beta)^\text{vir}} \psi_1^{k_1} \circ \text{ev}_1^*(\gamma_1) \cdots \psi_n^{k_n} \circ \text{ev}_n^*(\gamma_n).
$$

(7.2)

Let $T_0$ denote the fundamental class of $X$. The string equation is then

$$
\langle \tau_0(T_0) \prod_{i=1}^n \tau_{k_i}(\gamma_i) \rangle^X_{g,\beta} = \sum_{j=1}^n \langle \tau_{k_{j-1}}(\gamma_j) \prod_{i \neq j} \tau_{k_i}(\gamma_i) \rangle^X_{g,\beta}.
$$

(7.3)

A special base case is given by

$$
\langle \tau_0(T_0) \tau_0(\gamma_i) \tau_0(\gamma_j) \rangle^X_{0,0} = \int_X \gamma_i \circ \gamma_j.
$$

(7.4)

The Dilaton equation becomes

$$
\langle \tau_0(T_0) \prod_{i=1}^n \tau_{k_i}(\gamma_i) \rangle^X_{g,\beta} = (2g-2+n) \langle \prod_{i=1}^n \tau_{k_i}(\gamma_i) \rangle^X_{g,\beta}.
$$

(7.5)
A base case is given by

\[ \langle \tau_1(T_0) \rangle_{1,0}^X = \frac{1}{24} \chi(X). \]  

(7.6)

We also have the divisor relation:

\[
\left\langle \tau_0(D) \prod_{i=0}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,\beta}^X = d \left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,\beta}^X 
+ \sum_{j=1}^{n} \left\langle \tau_{k_{i-1}}(\gamma_i - D) \prod_{i \neq j} \tau_{k_i}(\gamma_i) \right\rangle_{g,\beta}^X
\]  

(7.7)

(7.8)

where \( d = \beta \cdot D \).

We claim that these relations allow us to reduce to a finite number of base cases. We have

\[
\dim \overline{\mathcal{M}}_{g,n}(X,\beta) = (1 - g) \dim X + 3g - 3 + n + \int_\beta c_1(X).
\]  

(7.9)

In the case of a Calabi-Yau threefold, this dimension is \( n \). Hence in the case of a nontrivial integral

\[
\int \psi^{a_1}_{1} \mathrm{ev}_1^* (\gamma_1) \psi^{a_2}_{2} \mathrm{ev}_2^* (\gamma_2) \cdots
\]  

(7.10)

one of the \( a_i \) or the codim \( \gamma_i \) is 0 or 1. codim \( \gamma_i \) = 0 is not a condition, and in the other situations, we can use either the string equation, dilaton equation, or divisor relation. So we can reduce all of the Gromov-Witten theory of a Calabi-Yau 3-fold in this way.

Observe that pullback and integration are linear, so Gromov-Witten invariants are linear in the \( \gamma_i \). From now on, we will fix a basis \( T_0, T_1, \ldots, T_r \) of cohomology classes of \( X \), with \( T_0 \) as before.

Define \( \tau_{k,a} = \psi^k_{i} \mathrm{ev}_1^* (T_a) \) for \( k \in \mathbb{N} \) and \( a \in \{0, \ldots, r\} \). Then define \( \tau^s = \prod_{k,a} \tau_{k,a}^s \). Then \( n = \sum_{k,a} s_{k,a} \) is the number of marked points. The codimension of \( \tau^s \) is \( \sum_{k,a} s_{k,a} (k + \text{codim } T_a) \).

Take a formal variable \( q = (q_1, \ldots, q_r) \), and for each \( \beta \), associate a vector \( d = (d_1, \ldots, d_r) \) where \( d_i = \int_\beta T_i \). Then define

\[
\langle \tau^s \rangle = \sum_{\beta} q^d \langle \tau^s \rangle_{g,\beta}^X
\]  

(7.11)

and the generating function

\[
\Phi(t) = \sum_{s} \frac{t^s}{s!} \langle \tau^s \rangle
= \sum_{\beta} q^d \sum_{s} \frac{t^s}{s!} \langle \tau^s \rangle_{g,\beta}^X.
\]  

(7.12)

(7.13)
7.1 \( \mathbb{P}^1 \)

In this case, \( \Phi(t) \) satisfies the 2-Toda hierarchy. \( \Phi(t) \) also satisfies Virasoro constraints. These were proven by Pand-Okounkov (directly!).

The Virasoro constraints have the form

\[
L_k = \sum_{m=0}^{\infty} \sum_{i=0}^{k+1} (\text{mult by } c_1(X)^i) \frac{\partial}{\partial t_m} \frac{\partial}{\partial t_{b,m+k-1}} \quad (7.14)
\]

\[
+ (\ast)(C) \frac{\partial}{\partial t_{a,m}} \frac{\partial}{\partial t_{b,k-m-i-1}} \quad (7.15)
\]

\[
+ \frac{\lambda^2}{2} (C^{k+1})_{ij} t_i t_j \quad (7.16)
\]

\[
+ \text{correction term}. \quad (7.17)
\]

7.2 \( \mathbb{P}^2 \)

First consider the genus 0 case. This gives an answer to the enumeration of degree \( d \), genus \( g \) curves through \( 3d - 1 + g \) points. Define \( N_d \) to be this count in genus 0. For now, set \( n = 3d \) and consider the Kontsevich space \( \mathcal{M}_{0,n}(\mathbb{P}^2, d) \).

Fact. \( \ast \cdot \)

\[ N_d = \int_{\mathcal{M}_{0,n}(\mathbb{P}^2, d)} \ev_1^*(pt) \sim \cdots \sim \ev_n^*(pt). \quad (7.18) \]

\( \bullet \) \( D(ij|k\ell) \cong D(ik|j\ell) \). For \( \mathcal{M}_{0,4} \cong \mathbb{P}^1 \) and the divisors are points, which are linearly equivalent. Now we can map \( \mathcal{M}_{0,n}(\mathbb{P}^2, d) \) down to \( \mathcal{M}_{0,4} \). Call this map \( \pi \).

In fact, \( D(ij|k\ell) \) is reducible. It splits as \( \sum_{i,j \in A} D(A, B|d_1, d_2) \) where \( D(A, B|d_1, d_2) \) is the divisor of the maps from a curve with two rational branches, one having marked points of \( A \), the other having marked points of \( B \), mapping to \( \mathbb{P}^2 \) by degree \( d_1 \) in the \( A \) branch and \( d_2 \) in the \( B \) branch. This shows that \( D(ij|k\ell) = \pi^*(D(12|34)) \).

Now define a curve \( Y \) by

\[ Y = \ev_1^*(z_1) \cap \cdots \cap \ev_{n-4}^*(z_{n-4}) \cap \ev_q^*(\ell_q) \cap \ev_r^*(\ell_r) \cap \ev_s^*(z_s) \cap \ev_t^*(z_t) \quad (7.19) \]

where the \( z \)'s are general points and the \( \ell \)'s are general lines. We will use the fact that \( Y \cap D(qr|st) = Y \cap D(qs|rt) \).

First consider the intersection \( Y \cap D(qr|st) \). We break up \( D(qr|st) \) into its irreducible components. First consider the case \( d_1 = 0, d_2 = d, A = \{q, r\}, \) and \( B = \{1, \ldots, n - 4, s, t\} \). Then the branch with \( q \) and \( r \) is contracted. The contraction point (necessarily the intersection of \( \ell_q \) and \( \ell_r \)) and the other \( n - 2 \) points give \( n - 1 \) points on the degree \( d \) image of the second branch. Hence the count from this intersection is \( N_d \), for \( n - 1 = 3d - 1 \).

Now look at \( d_1 \geq 1 \) and \( d_2 = d - d_1 \geq 0 \). The claim is that the intersection count is
\[
\left( \frac{3d - 4}{3d_1 - 1} \right) N_{d_1} N_{d_2} d_1^3 d_2.
\]  

(7.20)

There are \(3d - 4\) points besides \(q, r\), of which we must make a choice of \(3d_1 - 1\) of them to lie on the \(A\) branch. There are \(d_1\) choices for the image of \(q\) (one of the intersection points of \(\ell_q\) and the first image curve), \(d_2\) choices for where \(r\) is sent, and \(d_1 d_2\) choices for the image of the node (the intersection of the two branches of the domain curve, which must be sent to a point of the intersection of the two image curves). Finally, choosing where to send the other points gives the \(N\)'s.

Now consider \(Y \cap D(qs|rt)\). If \(1 \leq d_1 \leq d_1 - 1\), there is a nonempty intersection only if \(|A| = 3d_1\), and there are \(\binom{3d - 4}{3d_1 - 2}\) partitions of the other points with \(|A| = 3d_1\). \(N_{d_1}\) and \(N_{d_2}\) arise from choosing the images of the other points. Now the images of \(q, r\), and the nodal point have \(d_1, d_2\), and \(d_1 d_2\) choices. Hence the count for each \(d_1\) and \(d_2 = d - 1\) is

\[
\left( \frac{3d - 4}{3d_1 - 2} \right) d_1^2 d_2^2 N_{d_1} N_{d_2}.
\]  

(7.21)

Now we can equate to obtain a recursive formula:

\[
N_d + \sum_{d_1, d_2 > 0} \left( \frac{3d - 4}{3d_1 - 1} \right) N_{d_1} N_{d_2} d_1^3 d_2 = \sum_{d_1, d_2 > 0} \left( \frac{3d - 4}{3d_1 - 2} \right) d_1^2 d_2^2 N_{d_1} N_{d_2}.
\]  

(7.22)

And of course \(N_1 = 1\).

8 The Theory for Curves

First consider \(\mathbb{P}^1\). When determining a Gromov-Witten invariant, each \(\gamma_i\) is sent to either the fundamental class \([1]\) or the class of a point, which we will denote by \(\omega\). Our generating function will have the form

\[
F(t^0, t^1, \ldots) = (\cdots) t_i^0 \tau_i \ev_i^*(1) + t_j^1 \tau_j \ev_j^*(\omega).
\]  

(8.1)

We write down the Virasoro operators for \(\mathbb{P}^1\):
\[ L_{-1} = -\frac{\partial}{\partial t_0^0} + \sum_{\ell \geq 0} \left( t_{\ell+1}^0 \frac{\partial}{\partial t_{\ell}^0} + t_{\ell+1}^1 \frac{\partial}{\partial t_{\ell}^1} \right) + t_{0}^1 t_{0}^1 \]  

(8.2)

\[ L_{k} = -(k+1)! \frac{\partial}{\partial t_{k+1}^0} - \chi(\mathbb{P}^1)(1)_{k+1} \sum_{r=1}^{k+1} \frac{1}{r} \frac{\partial}{\partial t_{r}^1} \]  

(8.3)

\[ + \sum_{\ell \geq 0} \left( (\ell)_{k+1} t_{0, k+1}^0 \frac{\partial}{\partial t_{k+1}^0} + (\ell + 1)_{k+1} t_{1, k+1}^1 \frac{\partial}{\partial t_{k+1}^1} \right) \]  

(8.4)

\[ + \chi(\mathbb{P}^1) \sum_{\ell \geq 0} (\ell)_{k+1} \sum_{r=\ell}^{k+1} \frac{1}{r} \frac{\partial}{\partial t_{k+1}^1} \]  

(8.5)

\[ + \frac{\chi(\mathbb{P}^1)}{2} \sum_{\ell \geq 0} (\ell+1)!(k-\ell-1)! \frac{\partial}{\partial t_{1}^0} \frac{\partial}{\partial t_{1}^1} \]  

(8.6)

Here \((a)_b = \frac{(a+b-1)!}{(a-1)!}\).

\(\tau_{k+1}(1)\) acts as the following: \(1 \rightarrow \tau_k(\omega), \tau_k(1) \rightarrow \tau_{k+1}(\omega), \) and \(1 \rightarrow \tau_k(\omega)\tau_{k-1}(\omega)\) (extensive), and \(\tau_k(1) \rightarrow \tau_{k+1}(1)\) and \(\tau_k(\omega) \rightarrow \tau_{k+1}(\omega)\) (intensive).

Now, to compute a Gromov-Witten invariant, Virasoro can be used to take out all \(\tau_i(1)\). Then we are left with a stationary sector (only \(\tau_i(\omega)\) appear). Then GW/Hurwitz correspondence claims this can be computed by Hurwitz theory (and then representation theory of \(S_n\)).

Hurwitz theory counts degree \(d\) curves of \(\mathbb{P}^1\) with specified ramification. We consider degree \(d\) maps \(\pi : C \rightarrow \mathbb{P}^1\) where \(C\) is a nonsingular curve of genus \(g\) and \(\pi\) is dominant on each component. A partition \(\eta\) is a tuple of integers \((\eta_1, \eta_2, \ldots)\) with \(\eta_i \geq \eta_{i+1}\), and we consider partitions \(\eta\) with \(|\eta| = d\). A Hurwitz cover of \(\mathbb{P}^1\) of genus \(g\), degree \(d\), and monodromy \(\eta\) at \(q_i\) is a morphism where:

(i) \(C\) is nonsingular of genus \(g\).

(ii) \(\pi\) has profile \(\eta\) over \(q_i\).

(iii) \(\pi\) is unramified over \(\mathbb{P}^1 \setminus \{q_1, \ldots, q_n\}\).

The associated Hurwitz number \(H_d^{\mathbb{P}^1}(\eta^1, \ldots, \eta^n)^\bullet\) is a weighted count of distinct (possibly disconnected) covers \(\pi\), weighted by dividing by the number of automorphisms of \(\pi\).

Due to the Riemann Existence Theorem, Hurwitz covers with profile \(\eta\) over \(q_i\) in \(\mathbb{P}^1\), up to automorphisms, are in bijection with \(n\)-tuples of permutations \((s_1, \ldots, s_n)\) in \(S_d\) where each \(s_i\) has cycle type \(\eta^i\) and \(s_1 \cdots s_n = 1\), up to conjugation in \(S_d\). Hence the Hurwitz number can be computed as

\[ H_d^{\mathbb{P}^1}(\eta^1, \ldots, \eta^n) = \sum_{|\lambda| = d} \left( \frac{\dim \lambda}{d!} \right)^2 \prod_{i=1}^{n} f_{\eta^i}(\lambda) \]  

(8.7)

where the \(f_{\eta^i}\) are certain representatives of the center of \(\mathbb{Q}[S_d]\).

Now we relate this theory to that of stationary Gromov-Witten invariants. Consider the open locus \(\mathcal{M}_{g,n}^\bullet(\mathbb{P}^1, d)\) where the domain curve is smooth. The algebraic cycle class
\[ \prod_i (k_i! c_1(L_i)^{k_i} \ev_i^*(\omega)) \cap [\mathcal{M}_{g,n}^*(\mathbb{P}^1, d)] \]  

(8.8)

is represented by the locus of covers enumerated by \( H_d^\mathbb{P}((k_1 + 1) \cdots (k_n + 1)) \).

Let \( V \subseteq \mathcal{M}_{g,n}^*(\mathbb{P}^1, d) \) be defined as \( V = \prod_i \ev_i^*(\omega) \). Then we will show that \( \prod_i k_i! c_1(L_i)^{k_i} \cap [V] \) represents Hurwitz covers. Consider a marked point \( p_1 \); we construct a section \( s \in H^0(V, L_1) \). We have

\[ \pi^*: T_{q_1}(\mathbb{P}^1) \cong m_{q_1}/m_{q_1}^2 \to m_{p_1}/m_{p_1}^3 = T_{p_1}(C). \]  

(8.9)

\( T_{q_1}(\mathbb{P}^1) \), being trivial, is then mapped to \( L_1 \). This gives a section, and its zero locus \( Z(s) \) is the locus of maps ramified to order at least 1 at \( p_1 \). Inside \( Z(s) \), we have a map

\[ \pi^*: m_{q_1}/m_{q_1}^2 \to m_{p_1}/m_{p_1}^3 \cong L_1^2. \]  

(8.10)

Hence we get a section \( s' \) of \( L_1^2 \mid Z(s) \), and \( Z(s') \) is the locus of maps ramified to order at least 2 at \( p_1 \). \( Z(s') \) represents the class \( c_1(L_1^2) = 2c_1(L_1) \) in \( Z(s) \), and so \( 2c_1(L_1) \cdot c_1(L_1) \) in \( V \). We can iterate to obtain higher orders of ramification.

We find that the Hurwitz number describes the number of points in the cycle class

\[ \prod_i k_i! c_1(L_i)^{k_i} \ev_i^*(\omega) \cap [\mathcal{M}_{g,n}^*(\mathbb{P}^1, d)]. \]  

(8.11)

We want to understand the intersection with \( [\mathcal{M}_{g,n}^*(\mathbb{P}^1, d)]^{\vir} \); this will give us the stationary sector. The recipe for going between Gromov-Witten numbers and Hurwitz numbers involves completed cycles. Examples include

\[ (1) = (1) - \frac{1}{24}( ) \]  

(8.12)

\[ (2) = (2) \]  

(8.13)

\[ (3) = (3) + (1, 1) + \frac{1}{12}(1) + \frac{7}{2880}( ) \]  

(8.14)

\[ (4) = (4) + \cdots \]  

(8.15)

We have \( \tau_k(\omega) = \frac{1}{k^2}(k + 1) \). The general formula is

\[ k_1! \cdots k_n!(\tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega))_{\mathcal{M}_{g,n}^*(\mathbb{P}^1, d)} = \left( H_d^\mathbb{P}((k_1 + 1), (k_2 + 1), \ldots) \right). \]  

(8.16)

We can extend Hurwitz numbers to the situation of partitions having different total degrees:

- \( H_0^\mathbb{P}((\emptyset, \ldots, \emptyset) = 1. \)

- If \( \eta_i > d \) for some \( i \), then \( H_0^\mathbb{P}(\eta \cdots) = 0. \)
• Otherwise,

\[
H^\mathbb{P}_1_d(\eta^1 \cdots \eta^n) = \prod_{i=1}^n \left( \frac{m_1(\tilde{\eta}^i)}{m_1(\eta^i)} \right) H^\mathbb{P}_1_d(\tilde{\eta}^1 \cdots \tilde{\eta}^n) \tag{8.17}
\]

where the \(\tilde{\eta}^i\) are formed from \(\eta^i\) by adding 1’s to obtain the correct degree, and \(m_1\) is the number of 1’s.

To see the formula for (1), we consider \(\tau_0(\omega)\). \(\langle \tau_0(\omega) \rangle_{\mathfrak{M}_{1,1}(\mathbb{P}^1)}\). The moduli space has (expected) dimension 1 and is isomorphic to \(\mathbb{P}^1\), while \(\tau_0(\omega)\) has codimension 1. Being the class of a point, we get \(\int_{\mathfrak{M}_{0,1}(\mathbb{P}^1)} ev^*_1(\omega) = 1\).

However, we also need to consider maps from a disjoint union of a genus 0 curve and a pointed genus 1 curve to \(\mathbb{P}^1\) with a marked point, where the genus 1 curve is contracted to the marked point. This map is not dominant on every component, so only contributes to the Gromov-Witten number. The corresponding moduli space \(\mathfrak{M}\) is then

\[
\mathfrak{M} = \mathfrak{M}_{1,1}(\mathbb{P}^1, 0) \times \mathfrak{M}_{0,0}(\mathbb{P}^1, 1) \tag{8.18}
\]

Although \(\mathfrak{M}\) has dimension 2, its expected dimension is 1.

Look at \(\mathfrak{M}_{1,1}(\mathbb{P}^1, 0) \cong \mathfrak{M}_{1,1} \times \mathbb{P}^1\). We then have

\[
[\mathfrak{M}_{1,1}(\mathbb{P}^1, 0)]^\text{vir} = c_1(\mathcal{E}^\vee \boxtimes T_{\mathbb{P}^1}) \tag{8.19}
\]

where \(\mathcal{E}\) is the Hodge bundle. Now we have

\[
\int_{[M]^\text{vir}} ev^*_1(pt) = \int_M ev^*_1(pt) c_1(\mathcal{E}^\vee \boxtimes T_{\mathbb{P}^1}) \tag{8.20}
\]

\[
= \int_{\mathfrak{M}_{1,1}} c_1(\mathcal{E}^\vee) \tag{8.21}
\]

\[
= -\frac{1}{24} \tag{8.22}
\]

Now look at (2). We will compute \(\langle \tau_1(pt) \rangle_{\mathfrak{M}_{0,2}(\mathbb{P}^1, 2)}\). In addition to having a genus 0 curve mapping of degree 2, we could move one (or both) of the marked points to separate genus 1 components. We could also (in addition) take a genus 2 curve which will contract, along with two genus 0 components with maps of degree 1.

Considering just the genus 0 curve of degree 2 gives a Hurwitz number of 1. On the other spaces, we claim that \(\int \psi_1 ev^*_1(pt) \psi_2^*(pt) = 0\)! Indeed, one of the marked points lives on at least one component with expected dimension not equal to 2, even though we integrate a cycle of codimension 2. The exception is the case where both of the marked points lie on a genus 2 curve, where both the points must map to the same point on the image \(\mathbb{P}^1\).
Given a moduli space $\mathcal{M}$, we would like to consider all of its possible compactifications. Any two such compactifications must be birational to each other.

Fix a ($\mathbb{Q}$-factorial, normal, projective) variety $X$. Two divisors $D_1, D_2$ on $X$ are numerically equivalent if $C.D_1 = C.D_2$ for every curve $C \subseteq X$. Let $N^1(X)$ be the vector space of numerical equivalence classes of divisors on $X$. We have a similar concept of numerical equivalence of curves; define $N_1(X)$ in this way.

Inside $N_1(X)$, consider its effective cone and (inside of it) its ample cone, the closure of the ample cone being the nef cone. The closure of the effective cone is called the pseudo-effective cone.

Given any effective divisor $D$, the section ring $\bigoplus_{m \geq 0} H^0(mD)$ is a graded ring. When it is finitely generated, taking Proj gives a variety. If $D$ is ample (so some multiple of $D$ is very ample), this variety is isomorphic to $X$. In general, the variety will consist of contracting certain loci in $X$.

Under the intersection pairing, the nef cone of divisors is dual to the effective cone of curves. The pseudo-effective cone of divisors is dual to the mobile cone of curves. Here $\gamma \in N_1(X)$ is mobile if there exists a projective birational map $f : X' \to X$ and ample classes $a_1, \ldots, a_{n-1}$ in $N^1(X')$ such that $\gamma = f_*(a_1 \cdots a_{n-1})$.

The stable base locus of a divisor $D$ is

$$B(D) = \bigcap_{n \geq 1} Bs(|mD|),$$

where $Bs(|mD|)$ is the locus where all sections of $H^0(mD)$ vanish.

The minimal model program for moduli space goes as follows:

1. Determine the ample and effective cones on $\mathcal{M}$.
2. Decompose the effective cone using the stable base locus.
3. For each effective $D$ in a chamber, describe

$$\overline{\mathcal{M}}(D) = \text{Proj} \left( \bigoplus H^0(\mathcal{M}, mD) \right).$$

4. Describe these models, give a modular description, and describe maps.

This comes from Izzet Caskin’s notes.
3. For each divisor class $D$ in a chamber, describe

$$\overline{\mathcal{M}}(D) = \text{Proj} \left( \bigoplus_{m \geq 0} H^0(\overline{\mathcal{M}}, mD) \right)$$

(10.1)

4. Describe a sequence of elementary birational maps (flips and divisorial contractions) with end result $\overline{\mathcal{M}} \to \overline{\mathcal{M}}(D)$.

5. Interpret $\overline{\mathcal{M}}(D)$ as moduli spaces.

These steps can be extremely hard in general.

Consider the example of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2)$, the moduli space of stable maps $\alpha : C \to \mathbb{P}^2$ of degree 2, for $C$ a semistable curve of genus 0.

0. The divisor theory can be determined intrinsically or extrinsically (ad hoc). Consider the universal family

$$C \xrightarrow{\alpha} \mathbb{P}^2 \times \overline{\mathcal{M}} \xrightarrow{f} \mathbb{P}^2 \xrightarrow{g} \overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2)$$

(10.2)

Using this family, we can construct various types of divisors:

- $\kappa = f_*(c_1(\omega_f)^2)$.
- $\lambda = c_1(f_*(\omega_f))$.
- $H = f_*(c_1(\alpha^*(\mathcal{L}))^2)$ for some line bundle $\mathcal{L}$ on $\mathbb{P}^2$.
- $f_*(c_1(\alpha^*\mathcal{L}) \omega_f)$ (possibly another class).
- $\Delta$, the singular domain.

We’d expect these divisors to generate Pic.

In our case, $\kappa$ and $\lambda$ are both trivial. On the other hand, we can construct divisors by an ad hoc argument:

$$H = \{ \alpha : C \to \mathbb{P}^2 | \alpha(C) \text{ hits a fixed point } p \in \mathbb{P}^2 \}$$

(10.3)

$$T = \{ \alpha : C \to \mathbb{P}^2 | \alpha \text{ “tangent” to a fixed line } \ell \subseteq \mathbb{P}^2 | \alpha^*(\ell) \text{ is nonreduced} \}$$

(10.4)

$$\Delta = \text{ singular domain}$$

(10.5)

$$D_{\text{deg}} = \{ \alpha : C \to \mathbb{P}^2 | \text{im}(\alpha) \text{ is contained in a linear subspace, } \alpha \text{ 2:1 onto a line} \}$$

(10.6)

We have $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2) \setminus D_{\text{deg}} \cong \mathbb{P}^5 \setminus \text{(Veronese } \mathbb{P}^2)$, which has $\text{Pic}_Q$ generated by $H$. Hence we get a surjection $Q(\langle D_{\text{deg}}, H \rangle) \to \text{Pic} \overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2)$.

We want to show that $D_{\text{deg}}$ and $H$ are independent, so that this map will be an isomorphism. We can show this by considering test families. Suppose we had $aD_{\text{deg}} + bH = 0$. Intersect
with a test curve. For example consider $P$ a general pencil of conics. Then we have $P.H = 1$, $P.\Delta = 3$ (there are three ways of expressing four points as the intersection of a pair of lines), $P.D_{\deg} = 0$, $P.T = 2$ (the number of branch points of the 2:1 map $\ell \to \mathbb{P}^1$).

So $b = 0$. On the other hand, we cannot have $aD_{\deg} = 0$ for some $a \neq 0$, since $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2)$ is smooth projective.

1. We first determine the effective cone. We know that $H$ and $\Delta$ are effective. We claim that $\Delta$ is an edge (this usually happens). Consider the test family $S$ (“scissors”) where the image curve is a union of a fixed line $M$ and a varying line $N_t$ moving around a fixed point on $M$. We claim $S.H = 1$ and $S.\Delta = -1$. Consider the scroll $\mathbb{F}_1$, and glue $\mathbb{P}^1 \times \mathbb{P}^1$, where the $-1$ curve in $\mathbb{F}_1$ is glued to a horizontal section of $\mathbb{P}^1 \times \mathbb{P}^1$. This maps to $\mathbb{P}^2$, $\mathbb{F}_1$ being $\mathbb{P}^2$ blown up at a point, and $\mathbb{P}^1 \times \mathbb{P}^1$ having image a family of lines.

The glued surface is singular, so $S \subseteq \Delta$. $S.\Delta = \deg N_{\text{section/its surface}} \otimes N_{\text{other section/its surface}} = -1$.

By varying $S$, we can cover any general point of $\Delta$. So suppose $\Delta - \epsilon H$ were effective. Then its intersection with $S$ would have to be nonnegative, since $S$ is not contained in any representative for $\Delta - \epsilon H$ ($H$ not being a multiple of $\Delta$). This is a contradiction.

We claim the other edge of the cone is $D_{\deg}$. This divisor is clearly effective.

**Proposition 10.1.** $D_{\deg}$ is linearly equivalent to a multiple of $3H - \Delta$.

For we know $D_{\deg} \sim aH - b\Delta$, and then intersect with the test family $P$.

Now we will show $D_{\deg}$ is actually an edge. The families $P$ suffice; this shows $D_{\deg} - \epsilon H$ cannot be effective. For the $P$ sweep out a large portion of the moduli space, which includes $D_{\deg}$. But $(D_{\deg} - \epsilon H). P < 0$.

We have determined the effective cone. Now we will find the ample cone. The divisor $H$ is nef (basepoint free). The same is true of the divisor $T$ (and, by test curve arguments, $T \sim \frac{1}{2}(H + \Delta)$). Then test curves can be used to show that $T$ and $H$ are the edges of the nef cone. (Use the families $P$, $S$, and the families $C$ of double covers of a fixed line.)

2. For stable base loci: to the right of $H$ (in the direction of $-\Delta$), the stable base locus is $D_{\deg}$.

To the left of $T$ (in the direction of $\Delta$), the stable base locus is $\Delta$.

3. To find models for $H$ and $T$, we reinterpret $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2)$ of the space of “complete conics” in $\mathbb{P}^5 \times (\mathbb{P}^5)^*$ (the closure of the space $\{(C, C^*) | C \text{ smooth conic}\}$). The dual map $\mathbb{P}^5 \dashrightarrow (\mathbb{P}^5)^*$ fails to be defined on the Veronese surface parameterizing double lines. The space of complete conics is then isomorphic to $\mathbb{P}^5$ blown up at the Veronese surface.

The two projections of the space of complete conics to $\mathbb{P}^5$ and $(\mathbb{P}^5)^*$ correspond to the divisors $H$ and $T$. So the birational models corresponding to $H$ and $T$ are $\mathbb{P}^5$ and $(\mathbb{P}^5)^*$.

4. The projections are contractions.

5. The given models can be expressed as Hilbert schemes.

A much more difficult example is $\overline{\mathcal{M}}_g$.

First, $\text{Pic}_g \overline{\mathcal{M}}_g$ is known to be generated by $\lambda$ and the boundary divisors $\delta_i$, but there is no known algebraic proof.
The ample and effective cones are very complicated, and we restrict our attention to a 2-dimensional slice. We know \( K = 13 \lambda - 2 \delta \), for \( \delta \) the sum of the \( \delta_i \), so we consider only the cone inside the space generated by \( \lambda \) and \( \delta \). 11\( \lambda \) – \( \delta \) is one of the edges of the nef cone (the other one being \( \lambda \)). For the effective cone, however, less is known. \( \delta \) is one edge. For the other side, let \( \frac{\delta}{5} \) be the slope of \( a \lambda - b \delta \). We know there are effective divisors of various slopes. The Brill-Noether divisor has slope \( 6 + \frac{12}{g+1} \).

We next consider the ample and effective cones of \( \overline{M}_{0,0}(\mathbb{P}^r,d) \), and the Mori program for \( \overline{M}_{0,0}(\mathbb{P}^3,3) \). The Picard rank of \( \overline{M}_{0,0}(\mathbb{P}^3,3) \) is 2. Consider the divisors \( \Delta \) consisting of reducible domain curves and \( D_{\text{deg}} \) consisting of degenerate image curves. Coskun-Harris-Starr showed that the effective cone is generated by \( \Delta \) and \( D_{\text{deg}} \).

Also consider the divisor \( H \) where \( \text{im} f \) hits a fixed codimension 2 locus (here a line), and the divisor \( T \) consisting of those maps where \( f^{-1}(H) \) is not \( d \) distinct points for a fixed hyperplane \( H \). The ample cone is generated by \( H \) and \( T \).

We next consider the stable base loci. Between \( T \) and \( \Delta \), the stable base locus is \( \Delta \). The region between \( H \) and \( D_{\text{deg}} \) is split by a divisor \( F \); between \( F \) and \( D_{\text{deg}} \), the stable base locus is \( D_{\text{deg}} \), but between \( H \) and \( F \), it is smaller.

As an example of a stable base locus calculation, consider the chamber between \( T \) and \( \Delta \). Define a test curve \( B \) being the locus where, the domain has two branches, and the image is union of a line and a fixed conic, with fixed intersection point, and the line coming from a pencil. Then \( B.\Delta = -1 \) and \( B.H = 1 \), so \( B.T = 0 \). Hence for any \( D \) in our given chamber, \( B.D < 0 \). The union of all such \( B \) is then contained in the stable base locus of \( D \). The union is open in \( \Delta \), so the stable base locus contains \( \Delta \). On the other hand, \( \Delta \) must also be contained in the stable base locus. So we have equality.

To describe \( F \), fix \( p \in H \) in \( \mathbb{P}^3 \). Then we have

\[
F = \{(f,C) \mid \exists p_1, p_2 \in f(C) \cap H \text{ such that } p, p_1, p_2 \text{ collinear}\}. \tag{10.7}
\]

Taking \( H \) and \( \Delta \) as generators of the Picard group, we then have

\[
T = \frac{2}{3}(H + \Delta) \tag{10.8}
\]
\[
D_{\text{deg}} = \frac{2}{3}(H - \frac{1}{2}\Delta) \tag{10.9}
\]
\[
K = -\frac{8}{3}(H + \frac{1}{4}\Delta) \tag{10.10}
\]
\[
F = \frac{5}{3}(H - \frac{1}{5}\Delta) \tag{10.11}
\]

In particular, \(-K\) is ample, so \( \overline{M}_{0,0}(\mathbb{P}^3,3) \) is Fano. BCHM then implies it is a Mori dream space, meaning Mori’s program can be carried out for any divisor on \( \overline{M}_{0,0}(\mathbb{P}^3,3) \). Namely, Proj of the section rings are varieties for all effective divisors, there are finitely many chambers, and all maps are divisorial contractions and flips.

We now introduce some varieties:

- The Hilbert scheme \( \mathcal{H}^{3m+1}(\mathbb{P}^3) \) of degree 3 curves having arithmetic genus 0 in \( \mathbb{P}^3 \). The extraneous component consists of a smooth plane cubic with an additional given point. This
The moduli spaces associated to the various chambers are:

- The closure $C$ of twisted cubics in the Chow variety. Here recall that

$$\text{Chow}_{r,d}(X) = \{ r\text{-cycles of degree } d \text{ in } X \}$$

(10.12)

where an $r$-cycle is $\sum_i a_i[V_i]$ where $[V_i]$ is a reduced subvariety of dimension $r$ in $X$.

There is a map $g_0 : \mathcal{H} \dashrightarrow C$ which forgets the scheme structure, and a map $f_0 : \overline{M}_{0,0}(\mathbb{P}^3, 3) \dashrightarrow C$ which forgets the map.

We will instead consider $C'$, since $C$ is not a normal variety. This is a consequence of Zariski's Main Theorem: given $\phi : X \dashrightarrow Y$ with $Y$ normal, $\phi$ must have connected fibers. But for $[C] \in C$ given by a line and a conic contained in a plane, $g_0^{-1}([C])$ has two points, since the curve in the Hilbert scheme has an embedded point at one of the intersection points. Also $f_0^{-1}([C])$ consists of curves where the domain is a nodal $\mathbb{P}^1$, and the node can go to either intersection point. Again there are two preimages.

Let $N \subseteq \mathcal{H}$ be the locus of curves having a nonreduced 1-dimensional component, and $M \subseteq \overline{M}_{0,0}(\mathbb{P}^3, 3)$ given by

$$M = \{ [C, f] | \exists C_1 \text{ component of } \text{im} \ f \text{ such that the degree of } f \text{ over } C_1 \text{ is at least 2} \}.$$  

(10.13)

$N$ is irreducible of dimension 9, but $M$ has two components: $M_{1,2}$ of dimension 9 and $M_3$ of dimension 8.

- We now define the variety $\mathcal{H}(2)$. A twisted cubic $C$ is cut out by three quadrics on $\mathbb{P}^3$. Thus to $C$ we may associate a net of quadrics in $\mathbb{P}^3$. For $[C] \in \mathcal{H}$, $H^0(I_C(2))$ is 3 dimensional. This gives a morphism $\mathcal{H} \to \mathcal{H}(2)$.

Define the spaces $\overline{M}_{0,0}(\mathbb{P}^r, d, k)$ for $k \leq d$ (originally introduced by Mustata-Mustata-Parker) as follows: the space parameterizes $f : C \to \mathbb{P}^r$ along with $i : S \to O_C^{r+1}$ such that $S^*$ is a line bundle of degree $d$, the map to $\mathbb{P}^r$ induced from $O_C^{r+1} \to S^*$ agrees with $f$, and we require stability conditions: $S^* \otimes \omega^{d-k+\epsilon}$ is ample for every $\epsilon \in (0, \delta)$ for some positive $\delta$, and if $Q = \text{coker} \ i$ has torsion, then $\dim Q_p \leq d - k$ for every $p \in C$.

For $k = d$, this agrees with $\overline{M}_{0,0}(\mathbb{P}^r, d)$. In fact, in $\overline{M}_{g,n}(\mathbb{P}^r, d)$, stability given by finiteness of automorphisms is equivalent to $\omega \otimes f^*(O(3))$ being ample. At the other extreme, if $k \leq \frac{d}{2}$, then the first stability condition cannot be satisfied.

In our case, $\overline{M}_{0,0}(\mathbb{P}^3, 3, 2)$ gives a different moduli space.

The moduli spaces associated to the various chambers are:

- Between $H$ and $T$ (the ample cone), we have $\overline{M}_{0,0}(\mathbb{P}^3, 3)$.
- Between $H$ and $F$, we have $\mathcal{H}$, and between $F$ and $D_{\text{deg}}$, we have $\mathcal{H}(2)$.
- Between $T$ and $\Delta$, we have $\overline{M}_{0,0}(\mathbb{P}^3, 3, 2)$. 
• At the ends $D_{\text{deg}}$ and $\Delta$, the space is a point. At $H$, we get $C^\nu$. Moving across $H$ gives a flip, while $\mathcal{H} \to \mathcal{H}(2)$ and $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3, 2)$ are divisorial contractions.

Here is a description of each of the elementary maps:

• The map $\mathcal{H} \to \mathcal{H}(2)$ is a blow-up along the locus where the curve is the intersection of a line and a conic. The preimage in $\mathcal{H}$ describes the embedded data at the intersection point.

• In general, there is a map $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, k + 1) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, k)$ contracting tails of degree $d - k$. In our case, the locus of contraction is $\Delta$.

• The maps $\mathcal{H} \to C^\nu$ and $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3) \to C^\nu$ contract $N$ and $M$, respectively. Observe that $M$ and $N$ have codimension at least 2.

To see that $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3, 2)$ is a model for the chamber $[T, \Delta)$, it is enough to consider $D = T$.

Our goal is to show that

$$\text{Proj} \left( \bigoplus H^0(mT) \right) = \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3, 2).$$

(10.14)

Suppose $R$ is a curve such that $R \cdot T = 0$. We must have $R \subseteq \Delta$. Also in situations where the image curve has a node, the node and the conic must be fixed in $\mathbb{P}^3$. The same is true of the conic if the image curve is the intersection of a line and a conic, and any ramification points of a map where one branch maps 2:1 or 3:1.

So in $R$, only components which map with degree 1 can vary.

Now $\text{Proj} \left( \bigoplus H^0(mT) \right)$ contracts the union of all such $R$ (forgetting the tails of degree 1 but remembering the attaching point). This exactly describes the contraction map to $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3, 2)$.

11 Stable Quotients

We define the spaces $\overline{\mathcal{Q}}_{g,m}(\mathbb{P}^r)$ (except for $g = 0, m = 0, 1$) parameterizing curves $C$ along with sequences of sheaves

$$0 \to S \to \mathcal{O}^{r+1} \to Q \to 0$$

(11.1)

such that $C$ is genus $g$ with $m$ marked points, $\deg S^* = d$, and rank $S = 1$, with the stability conditions $(S^*)^c \otimes \omega(p_1, \ldots, p_m)$ being ample and the torsion $Q_p$ are being away from the nodes.

12 Localization

Suppose $G \acts M$. The goal is to get global information about $M$ from local information at the fixed loci of the $G$-action. As an example, recall:

**Theorem 12.1** (Poincare-Hopf). If $V$ is a vector field on $M$ with isolated zeros, then

$$\chi(M) = \sum F \text{ind}(F).$$

(12.1)
12.1 Equivariant Cohomology

Given $G \circlearrowleft X$, we want a cohomology theory such that:

1. If $G$ acts freely, then $H_G^\bullet(X) = H^\bullet(X/G)$.

2. $H_G^\bullet(pt)$ should depend on $G$.

3. $H_G^\bullet$ should be functorial.

To determine $H_G^\bullet(pt)$, the space $EG$ (uniquely determined up to homotopy) is a contractible space with a free action of $G$. Then we want $H_G^\bullet(pt) = H^\bullet(BG)$, where $BG = EG/G$.

In the case $G = \mathbb{C}^\times$, we may take $EG = \mathbb{C}^\infty \setminus \{0\}$ and $BG = \mathbb{C}P^\infty$. Thus (taking $\mathbb{Q}$-coefficients for our cohomology), $H_{\mathbb{C}^\times}^\bullet(pt) = \mathbb{Q}[t]$, where $t$ is the hyperplane class on $\mathbb{C}P^\infty$. $t$ can also be thought of as the dual class to $c_1(S)$ where $S$ is the tautological bundle on $\mathbb{C}P^\infty$. Similarly, $H_{(\mathbb{C}^\times)^n}^\bullet(pt) = \mathbb{Q}[t_1, \ldots, t_n]$.

Now for $G \circlearrowleft M$ in general, define $M_G = EG \times_G M$, where $G$ acts on $EG$ from the right and $M$ from the left. Here

$$EG \times_G M = (EG \times M)/((eg, m) \sim (e, gm)). \quad (12.2)$$

Then the $G$-equivariant cohomology is given by

$$H_G^\bullet(M) = H^\bullet((EG \times M)/G) = H^\bullet(M_G). \quad (12.3)$$

The fiber of $M_G \rightarrow M/G$ at $[m]$ is given by $\{(e, m)\}/ \sim$ for $m$ a fixed lift of $[m]$. Taking $G_m = \{g \in G | gm = m\}$, this fiber is then $EG/G_m$, homotopy equivalent to $BG_m$. Meanwhile, the fiber of $M_G \rightarrow BG$ at $[e]$ is $M$.

By choosing a fiber, we get an inclusion $i : M \hookrightarrow M_G$, and therefore, by pulling back, a map $H_G^\bullet(M) \rightarrow H^\bullet(M)$.

Examples:

1. If $G \circlearrowleft M$ trivially, then $M_G = BG \times M$. Hence $H_G^\bullet(M) = H_G^\bullet(pt) \otimes H^\bullet(M)$ by Kunneth.

2. If $G \circlearrowleft M$ freely, then $M_G$ is homotopy equivalent to $M/G$, and so $H_G^\bullet(M) = H^\bullet(M/G)$.

Pulling back the map $M \rightarrow pt$ gives a map $H_G^\bullet(pt) \rightarrow H_G^\bullet(M)$, and therefore makes $H_G^\bullet(M)$ a module over $H_G^\bullet(pt)$. For the case $G = \mathbb{C}^\times$, $H_G^\bullet(pt) = \mathbb{Q}[t]$ is a principal ideal domain.

For the case where $G$ acts trivially, this module is free of rank $\dim \mathbb{Q} H^\bullet(M)$. At the other extreme, if the $G$-action is free, then $H_G^\bullet(M)$ is finite dimensional, so the $\mathbb{Q}[t]$-module structure is necessarily torsion. (From now on, we will assume $G = \mathbb{C}^\times$ unless explicitly stated otherwise.)

If $M$ is smooth, then the fixed locus $F$ of the $\mathbb{C}^\times$-action is smooth (possibly disconnected). The inclusion $i : F \hookrightarrow M$ gives a pullback map

$$\mu : H_{\mathbb{C}^\times}^\bullet(M) \rightarrow H_{\mathbb{C}^\times}^\bullet(F) = H^\bullet(F) \otimes \mathbb{Q}[t]. \quad (12.4)$$
Theorem 12.2 (Localization). \( \mu \) is an isomorphism after tensoring with \( \mathbb{Q}(t) \).

**Theorem 12.3.** For any \( \phi \in H^\bullet_G(M) \),

\[
\mu(\phi) = \sum_{F_i} \frac{i_* i^* \phi}{e(N_{F_i/M})}
\]

where \( e \) is the Euler class.

As an example, we can compute \( \chi(M) \) for \( M = \mathbb{P}^m \). In this case, if the \( F_i \) are points,

\[
\chi(\mathbb{P}^m) = \int_M e(T_M)
\]

\[
= \sum_{F_i} \int_{F_i} i^* e(T_M)
\]

\[
= \sum_{F_i} \int_{F_i} e(N_{F_i/M})
\]

\[
= \sum_{F_i} \int_{F_i} 1.
\]

(In general, we’d get \( \sum_{F_i} \chi(F_i) \).) In the case of \( \mathbb{P}^m \), with

\[
\lambda \cdot [x_0 : \cdots : x_m] = [\lambda^{\alpha_0} x_0 : \cdots : \lambda^{\alpha_m} x_m]
\]

with the \( \alpha_i \) distinct integers, this number is \( m + 1 \).

As another example, take \( M = G(k,m) \). The \( \mathbb{C}^\times \)-action on the Grassmannian is induced from that on \( \mathbb{C}^m \), where

\[
\lambda(x_1, \ldots, x_m) = (\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_m} x_m).
\]

If again the \( \alpha_i \) are generic, the fixed loci are the coordinate planes. So \( \chi(G(k,m)) \) is the number of fixed points, equal to \( \binom{m}{k} \).

For \( F_m \), the complete flag variety in \( \mathbb{C}^m \), the fixed loci are coordinate complete flags, so \( \chi(F_m) = m! \).

The \( \mathbb{C}^\times \)-action also gives a cell decomposition of \( M \) when the fixed loci are points. In our cases, the cells are even-dimensional, so the Betti numbers can be determined.

For \( G = \mathbb{C}^\times \), \( BG \) is not finite dimensional, so we’ll approximate. If \( E_m \) is connected with free \( G \)-action and \( H^1(E_m) = 0 \) for \( 0 < i < k(m) \), then

\[
H^i(E_m \times_G X) \cong H^i(EG \times_G X) = H^i_G(X)
\]

for \( i < k(m) \). Hence any computation of \( H^\bullet_G(X) \) can be done with \( \mathbb{C}^m \) for some \( n \).

Suppose we have \( \varphi : G \to G' \) and \( f : X \to X' \) such that \( f(g \cdot x) = \varphi(g) \cdot f(x) \). Then there is a pullback map \( f^* : H^\bullet_{G'}(X') \to H^\bullet_G(X) \) coming from \( E \times_G X \to E' \times_{G'} X' \).
If $W \to X$ is an equivariant vector bundle, then we get a map $E_m \times_G W \to E_m \times_G X$. Define the equivariant Chern classes

$$c^G_i(W) = c_i(E_m \times_G W) \in H^{2i}(E_m \times_G X) = H^{2i}_G(X). \quad (12.13)$$

For $V \subseteq X$ a $G$-invariant subvariety of codimension $d$, $E_m \times_G GV \subseteq E_m \times_G X$ will also be of codimension $d$, and so we may associate

$$[V]^G = [E_m \times_G V] \in H^{2d}(E_m \times_G X) = H^{2d}_G(X). \quad (12.14)$$

For the case where $X$ is a point, an equivariant vector bundle $W \to \text{pt}$ is a representation of $G$. So to any representation of $G$, we can associate $c^G_i(W) \in H^{2i}_G(\text{pt})$.

When $G = \mathbb{C}^\times$, let $L_\alpha$ be the representation $\mathbb{C}^\times \to GL_1(\mathbb{C})$ by $\lambda : x \mapsto \lambda^ax$. The associated map is then

$$(\mathbb{C}^m \setminus \{0\}) \times_{\mathbb{C}^\times} L_\alpha \to (\mathbb{C}^m \setminus \{0\}) \times_{\mathbb{C}^\times} \text{pt} = \mathbb{P}^{m-1}. \quad (12.15)$$

We’ll determine the first Chern class in $H^2(\mathbb{P}^{m-1})$. The above line bundle is $\mathcal{O}_{\mathbb{P}^{m-1}}(-\alpha)$. Letting $\alpha = -t = c_1(\mathcal{O}_{\mathbb{P}^{m-1}}(1))$, our Chern class is then $at$.

We can similarly consider $(\mathbb{C}^\times)^n$ acting on $\mathbb{C}^n$ by the standard action. Call this representation $L_T$. We have a fiber bundle $W = (\mathbb{C}^m \setminus 0)^n \times (\mathbb{C}^\times)^n L_T \to (\mathbb{P}^{m-1})^n$.

Claim.

$W \cong \mathcal{O}_{\mathbb{P}^{m-1}}(-1) \boxplus \cdots \boxplus \mathcal{O}_{\mathbb{P}^{m-1}}(-1). \quad (12.16)$

This implies $c_i((\mathbb{C}^\times)^n(L_T))$ is the $i$th elementary symmetric polynomial in the $t_i$.

Now our goal is to determine

$$H^*_T(\mathbb{P}^{n-1}) = H^*(ET \times_T \mathbb{P}^{n-1}) \quad (12.17)$$

where $T = (\mathbb{C}^\times)^n$. $ET \times_G \mathbb{P}^{n-1} \to BT$ is a fiber bundle with fibers $\mathbb{P}^{n-1}$. Approximating, we get bundles

$$\mathbb{P}(W) = (\mathbb{C}^m \setminus 0)^n \times_T \mathbb{P}^{n-1} \to (\mathbb{C}\mathbb{P}^{m-1})^n. \quad (12.18)$$

The cohomology of this bundle is given by

$$H^*(\mathbb{P}(W)) = H^*((\mathbb{C}\mathbb{P}^{m-1})^n)[H]/(H^n + c_1(W)H^{n-1} + \cdots + c_n(W)). \quad (12.19)$$

Here $H = c^T_1(\mathcal{O}_{\mathbb{P}^{m-1}}(1))$. The above can be expressed as

$$\mathbb{Q}[t_1, \ldots, t_n, H]/\prod_i (H + t_i). \quad (12.20)$$

For $G = T$ acting on $X$, we would like situations where $H^*_G(X) \to H^*(X)$ is surjective and $H^*_G(X) \to H^*_G(F)$ is injective, where $F$ is the $G$-fixed locus. This doesn’t always happen: $H^*_{\mathbb{C}^\times}(\mathbb{C}^\times) = 0$ while $H^1(\mathbb{C}^\times) = \mathbb{Z}$. In addition, there are no fixed points.
Consider the case of \( \mathbb{P}^{n-1} \) and let \( p_0, \ldots, p_{n-1} \) be the fixed points. We have \( t_i|_{p_j} = t_i \) in \( H^*((\mathbb{C}P^n)^n) \). For \( H|_{p_j} \), the tautological bundle restricted to \( p_j \) is a line bundle over \( p_j \) given by scaling by \( \lambda_j \). So \( H|_{p_j} = -t_j \).

In our situation, the map

\[
\mathbb{Q}[t_1, \ldots, t_n, H] / \prod_i (H + t_i) \to \bigoplus_i \mathbb{Q}[t_1, \ldots, t_n]
\] (12.21)

given by \( H \mapsto -t_j \) in the \( j \)th component, is injective. We can also see that the map \( H^*_T(\mathbb{P}^{n-1}) \to H^*_T(\mathbb{P}^{n-1}) \) is surjective (taking \( H \) to \( H \) and \( t_j \) to 0).

As an example, we can verify that two lines in \( \mathbb{P}^2 \) intersect in a point. Equivariantly, \( h = c_1^T(\mathcal{O}_{\mathbb{P}^m}(1)) \) where we are given a \((\mathbb{C}^x)^{m+1}\)-action. Recall that \( \mathcal{O}(-1) \) is the total space of the tautological bundle, with the canonical \((\mathbb{C}^x)^{m+1}\)-action

\[
(t_0, \ldots, t_m)(x_0, \ldots, x_m) \mapsto (t_0x_0 + \cdots + t_mx_m).
\] (12.22)

With \( h \) defined as above, localization shows that

\[
h.h = \sum_{F_i} \frac{i^*(h.h)}{e(N_{F_i}/\mathbb{P}^2)}. \] (12.23)

\( \mathbb{P}^2 \) has three fixed points under the \( \mathbb{C}^x \)-action, \( \phi_0, \phi_1, \phi_2 \) given by the component points. We have \( h|_{\phi_i} = \alpha_i \) by reasoning similar to above. Meanwhile, \( N_{\phi_0/\mathbb{P}^2} \) is the tangent space at \( \phi_0 \), which is \( (\frac{x_1}{x_0}, \frac{x_2}{x_0}) \). So we get \( e(N_{\phi_0/\mathbb{P}^2}) = (t_1 - t_0)(t_2 - t_0) = (\alpha_0 - \alpha_1)(\alpha_0 - \alpha_2) \). We end up with

\[
h.h = \sum_{\text{cyc}} \frac{\alpha_0^2}{(\alpha_0 - \alpha_1)(\alpha_0 - \alpha_2)}. \] (12.24)

Since we have

\[
\sum_{\text{cyc}} \alpha_0^2 \frac{(x - \alpha_1)(x - \alpha_2)}{(\alpha_0 - \alpha_1)(\alpha_0 - \alpha_2)} = x^2
\] (12.25)

this sum is 1. (Alternatively, \( \int_{\mathbb{P}^1} \frac{\alpha_0^2}{(x - \alpha_1)(x - \alpha_2)} dx = 0 \) and compute residues.) But there is an easier way to perform the localization!

Now suppose \( G \cap X \) and \( V \subseteq X \) is a subvariety fixed by \( G \); then associate \([V]^G\). As an example, we may consider \((\mathbb{C}^x)^{m+1}\)-fixed hyperplanes in \( \mathbb{P}^m \). For \( \mathbb{P}^2 \), let \( H_{01} \) be the line joining \( \phi_0 \) and \( \phi_1 \). Since \( H_{01} \times_G \mathbb{P}^n \cap \phi_2 \times_G \mathbb{P}^n = 0 \) in normal cohomology, we have \( H_{01}|_{\phi_2} = 0 \). For \( H_{01}|_{\phi_0} \), the excess intersection formula gives \( e(N_{H_{01}/\mathbb{P}^2}) \). In coordinates, the normal bundle is \([x_0 : 0 : x_2]\) so the \((\mathbb{C}^x)^3\)-action has a single weight \( \alpha_0 - \alpha_2 \). Hence this is also the Euler class, so \( H_{01}|_{\phi_0} = \alpha_0 - \alpha_2 \). More generally \( (H - \alpha_j)|_{\alpha_i} = \alpha_i - \alpha_j \).

Then we have
\[ h.h = \sum_{\phi_i} \frac{(H - \alpha_1)(H - \alpha_2)}{e(N_{\phi_i}/\mathbb{P}^m)} \]  
\[ = \sum \frac{(\alpha_0 - \alpha_1)(\alpha_0 - \alpha_2)}{(\alpha_i - \alpha_j)(\alpha_i - \alpha_k)} \]  
\[ = 1. \] (12.26)  
\[ (12.27) \]

As another example, consider the number of lines in \( \mathbb{P}^3 \) through four general lines. We perform localization in \( Gr(2,4) \). The \((\mathbb{C}^*)^4\)-fixed loci in \( Gr(2,4) \) are six points \( F_{ij} \), corresponding to the lines between the four fixed points of \( \mathbb{P}^3 \). To determine \( e(N_{F_{ij}/Gr(2,4)}) \), the tangent space is \( \text{Hom}(\mathcal{O}, \Omega)^4 \) with weights \( t_2 - t_0, t_3 - t_0, t_2 - t_1, t_3 - t_1 \). Here \( \mathcal{O} \) and \( \Omega \) come from the tautological sequence

\[ 0 \rightarrow \mathcal{O}^4 \xrightarrow{\delta} \Omega \rightarrow 0 \] (12.29)

associated to the Grassmannian. We have

\[ e(N_{F_{ij}/Gr(2,4)}) = (\alpha_i - \alpha_k)(\alpha_i - \alpha_\ell)(\alpha_j - \alpha_k)(\alpha_j - \alpha_\ell). \] (12.30)

Let \( \sigma \) be the class of lines in \( \mathbb{P}^3 \) intersecting a fixed line. Then \( \sigma = c_1(Q) \), the zero locus of a generic section of \( \Lambda^2 Q \). A generic section is \( q(x_1) \land q(x_2) \), which is zero if and only if \( \langle q(x_1), q(x_2) \rangle \) spans a space of dimension at most 1, in which case \( S \cap \langle x_1, x_2 \rangle \) has dimension at least 1.

The number of lines in \( \mathbb{P}^3 \) intersecting 4 general lines is then

\[ \int_{Gr(2,4)} c_1(Q)^4 = \sum_{F_{ij}} \frac{(c_1(Q)|_{F_{ij}})^4}{e(N_{F_{ij}/Gr(2,4)})} \] (12.31)

\[ = \sum_{F_{ij}} \frac{(-\alpha_k - \alpha_\ell)^4}{(\alpha_i - \alpha_k)(\alpha_i - \alpha_\ell)(\alpha_j - \alpha_k)(\alpha_j - \alpha_\ell)}. \] (12.32)

We’ll use indices 1 through 4 now. The above sum may be written as

\[ \sum_{(12)(34)} \frac{(\alpha_1 + \alpha_2)^4 + (\alpha_3 + \alpha_4)^4}{(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_4)}. \] (12.33)

We can alternatively consider \( \mathbb{C}^* \cap \mathbb{P}^3 \) by

\[ t[x_1 : x_2 : x_3 : x_4] = [tx_1 : t^2x_2 : t^3x_3 : t^4x_4]. \] (12.34)

Taking \( \alpha = -t \) and specializing, we get \( \alpha_i \mapsto i\alpha \). We may then cancel out the factors of \( \alpha \) in the above sum, and end up with

\[ \frac{3^4 + 7^4 + (-4)(4^4 + 6^4) + 3(5^4 + 5^4)}{12} = 2. \] (12.35)
As a last example, we will compute the number of lines on a cubic hypersurface in \( \mathbb{P}^3 \). Again look inside \( \text{Gr}(2, 4) \). Consider the universal curve

\[
\Sigma \xrightarrow{f} \mathbb{P}^3 \\
\downarrow \pi \\
\text{Gr}(2, 4)
\]

Our desired class is \( \pi_\ast f_\ast \mathcal{O}(3) \). \( H^0(f_\ast \mathcal{O}(3)|_{F_{ij}}) \) has eigensections \( x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3 \), with weights \( 3\alpha_0, 2\alpha_0 + \alpha_1, \alpha_0 + 2\alpha_1, 3\alpha_1 \). The number of lines is then

\[
\int_{\text{Gr}(2, 4)} e(\pi_\ast f_\ast \mathcal{O}(3)) = \sum_{i<j} \frac{e(\pi_\ast f_\ast \mathcal{O}(3)|_{F_{ij}})}{e(N_{F_{ij}/\text{Gr}(2, 4)})} \\
= 9 \sum_{i<j} \frac{\alpha_i \alpha_j (2\alpha_i + \alpha_j)(\alpha_i + 2\alpha_j)}{\alpha_i - \alpha_k)(\alpha_i - \alpha_l)(\alpha_j - \alpha_k)(\alpha_j - \alpha_l)} \\
= 27.
\]

13 Localization on \( \overline{M}_{g,n}(\mathbb{P}^m, d) \)

The \((\mathbb{C})^{m+1}\)-action on \( \mathbb{P}^m \) induces one on the moduli space of stable maps. On \( \mathbb{P}^m \), the fixed loci are points \( \phi_0, \ldots, \phi_m \), and the preserved subvarieties are the coordinate subspaces. So if a genus \( g \) curve \( C \) mapped into \( \mathbb{P}^m \) under \( f \) is fixed by the action, \( f(C) \) must be contained in the union of the coordinate lines. Also all nodes, marked points, and ramification points must be mapped to the \( \phi_i \). In particular, each noncontracted component has at most two ramification points, so must be rational.

The fixed loci are labelled by decorative graphs. A decorative graph consists of vertices (given numbers \( g \) and \( \mu \)), edges (given numbers), and tails (half-edges, given numbers). The vertices correspond to contracted genus \( g \) curves, where \( \mu \) indicates which fixed point the component is contracted to, while edges are noncontracted genus 0 curves, with number indicating degree. The graph (without the tails) then looks like the image curve. The tails are numbered with the marked points.

\[\begin{array}{c}
\begin{array}{c}
1 \\
\end{array} \\
\begin{array}{c}
2 \\
\end{array} \\
\end{array} \quad \begin{array}{c}
\begin{array}{c}
3 \\
\end{array} \\
\begin{array}{c}
2 \\
\end{array} \\
\end{array} \quad \begin{array}{c}
\begin{array}{c}
4 \\
\end{array} \\
\begin{array}{c}
2 \\
\end{array} \\
\end{array} \end{array}
\]

The moduli space \( \overline{M}_\Gamma \subseteq \overline{M}_{0,2}(\mathbb{P}^2, 9) \) consisting of stable maps associated to \( \Gamma \) is given by

\[
\overline{M}_\Gamma \subseteq \overline{M}_{2,6} \times \overline{M}_{1,2} \times \overline{M}_{1,2} \times (\overline{M}_{0,2}(\mathbb{P}^1, 2))^3 \times \overline{M}_{0,2}(\mathbb{P}^1, 3).
\]
The moduli spaces have total ramification over 0 and \( \infty \), and there is a unique such map of degree \( d \), having \( d \) automorphisms, so we actually have

\[
\tilde{M}_\Gamma = \left( \overline{M}_{2,6} \times \overline{M}_{1,2} \times \overline{M}_{1,2} \times \left( \frac{\text{pt}}{2} \right)^3 \times \frac{\text{pt}}{3} \right) / \text{Aut}(\Gamma).
\] (13.3)

There is an exact sequence of groups

\[
0 \to \prod_{e \text{ edge}} \mathbb{Z}/d(e) \to A_{\Gamma} \to \text{Aut}(\Gamma) \to 1.
\] (13.4)

\( \tilde{M}_\Gamma \) has the form \( M_\Gamma/A_{\Gamma} \). In our example, we will have

\[
M_\Gamma = \overline{M}_{2,6} \times \overline{M}_{1,2} \times \overline{M}_{1,2}.
\] (13.5)

Localization will be performed on \( M_\Gamma \), and then we must divide our answers by \( \# A_{\Gamma} \) to obtain the correct results.

To determine \( e(N_\Gamma) \), consider a point \( \zeta \in \tilde{M}_\Gamma \). The normal bundle at this point is the piece of \( T_{\overline{M}_0,0}(\mathbb{P}^r,d) \) on which the \( \mathbb{C}^\times \)-action is nontrivial. To handle this, consider the deformation exact sequence.

Given a map \( \mathbb{P}^1 \to \mathbb{P}^1 \) of degree \( d \) ramified only at 0 and \( \infty \), let the domain coordinates be \([z_0 : z_1]\) and the image coordinates by \([x_0 : x_1]\). Then \( x_i = z_i^d \). We have a natural action of \( (\mathbb{C}^\times)^2 \) on the image \( \mathbb{P}^1 \) by \((t_0,t_1)[x_0 : x_1] = [t_0x_0 : t_1x_1]\). This induces an action of the domain \( \mathbb{P}^1 \): let \( t_i = \lambda_i^d \). Then the action is \((t_0,t_1)[z_0 : z_1] = [\lambda_0z_0 : \lambda_1z_1]\).

Our goal is to determine \( H^0(\Sigma, f^*\mathcal{O}_{\mathbb{P}^1}(1)) \), where \( \Sigma \) is the domain \( \mathbb{P}^1 \). For starters, \( H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \) has a basis of sections \( x_0 \mapsto t_0x_0 \) and \( x_1 \mapsto t_1x_1 \), having weights \( \alpha_0 \) and \( \alpha_1 \). \( H^0(\Sigma, f^*\mathcal{O}_{\mathbb{P}^1}(1)) \) has as a basis of sections the degree \( d \) monomials in \( z_0 \) and \( z_1 \); the eigenvalues are \( \lambda_0^d, \lambda_0^{d-1}\lambda_1, \ldots, \lambda_1^d \), so the weights are

\[
\alpha_0, \frac{(d-1)\alpha_0 + \alpha_1}{d}, \ldots, \alpha_1. \tag{13.6}
\]

Next, \( H^1(\Sigma, f^*\mathcal{O}_{\mathbb{P}^1}(-1)) = H^0(\Sigma, \omega_\Sigma \otimes f^*\mathcal{O}_{\mathbb{P}^1}(1)) \) by Serre duality. In our case, this is \( H^0(\Sigma, \mathcal{O}_{\mathbb{P}^1}(d-2)) \). The weights are \( \frac{d}{d}(\alpha_0 - \alpha_1) \) for \( 1 \leq i \leq d-1 \).

Here is the result: the Euler class \( e(N_\Gamma) \) satisfies

\[
\frac{1}{e(N_\Gamma)} = \prod_{F \text{ flag}} \frac{1}{w_F - \psi_F} \prod_{\nu \neq \mu(F)} \frac{(\alpha_\mu(F) - \alpha_\nu)}{\omega_F}
\] (13.7)

\[
\prod_{v \text{ vertex}} \frac{1}{\omega_v} \frac{1}{\alpha_{\mu(F)} - \alpha_{\nu}} \prod_{\text{val}(v) = 2} \frac{1}{w_{F,v,1} + w_{F,v,2}} \prod_{\text{val}(v) = 1} \omega_F
\] (13.8)

\[
\prod_{e \text{ edge}} \frac{(-1)^{d(e)}(d(e))^{2d(e)}}{(d(e))![d(e)]^2(\alpha_i - \alpha_j)^{2d(e)}} \prod_{\alpha_{k \neq i,j}} \frac{1}{\alpha_i + \frac{b}{d(e)} \alpha_j - \alpha_k}
\] (13.9)

38
where \( w_F = \frac{\alpha(F) - \alpha(F)}{d(e)} \) for \( F = (v, e) \).

As an example, we will verify that there is exactly one line through two points in \( \mathbb{P}^2 \). Our integral is

\[
\int_{\mathcal{M}_{0,2}(\mathbb{P}^2, 1)} \ev_1^*(\text{pt}) \cap \ev_2^*(\text{pt}) = \sum_{F \in \mathcal{F}} \int_{\mathcal{F}} \frac{\ev_1^*(\text{pt}) \cap \ev_2^*(\text{pt})}{e(N_{\mathcal{F}})} \tag{13.10}
\]

The fixed loci in \( \mathcal{M}_{0,2}(\mathbb{P}^2, 1) \) consists of twelve points: the line passes through two of the three coordinate points in \( \mathbb{P}^2 \) and there are four choices of where to send the two marked points. For \( H_a = H - \alpha_a, \phi_a = (H - \alpha_b)(H - \alpha_c) \). Then \( \ev_1^*(\phi_a)|_{\mathcal{F}} = (H - \alpha_b)(H - \alpha_c)|_{\phi_i} = (\alpha_i - \alpha_a)(\alpha_i - \alpha_c) \).

Choose an equivariant lift given by \( \ev_1^*(\phi_a) \cap \ev_2^*(\phi_b) \). The unique fixed point on which the numerator does not vanish is the graph

\[
\begin{array}{c}
2 \leftarrow b \\
\downarrow \\
1 \leftarrow a
\end{array}
\tag{13.11}
\]

Our integral is then equal to

\[
(\alpha_a - \alpha_b)(\alpha_a - \alpha_c)(\alpha_b - \alpha_a)(\alpha_b - \alpha_c)
\tag{13.12}
\]

\[
\cdot \left( \frac{1}{\alpha_a - \alpha_b - \psi_{ab}} \frac{1}{\alpha_b - \alpha_a - \psi_{ba}} \cdot (\alpha_a - \alpha_b)(\alpha_a - \alpha_c)(\alpha_b - \alpha_a)(\alpha_b - \alpha_c) \right)
\tag{13.13}
\]

\[
\cdot \frac{1}{\alpha_a - \alpha_b} \frac{1}{\alpha_a - \alpha_c} \frac{1}{\alpha_b - \alpha_a} \frac{1}{\alpha_b - \alpha_c} (\alpha_a - \alpha_b)(\alpha_b - \alpha_a)
\tag{13.14}
\]

\[
\cdot \frac{(-1)}{(\alpha_a - \alpha_b)^2} \frac{1}{\alpha_a - \alpha_c} \frac{1}{\alpha_b - \alpha_c}
\tag{13.15}
\]

Since the fixed loci are 0-dimensional, there will be no contribution from the \( \psi \)-classes (if we expanded as a power series in \( \psi \), only the \( \psi^0 \) term would produce terms whose eventual degree in the \( \alpha \)'s is zero). So we can replace \( \alpha_a - \alpha_b - \psi_{ab} \) with \( \alpha_a - \alpha_b \), etc. After enough inspection, the result is 1.

### 14 Calculation of the Euler Class

Now we will compute part of \( e(N_{\mathcal{F}}) \). Consider \( H^0(\omega_{\mathbb{P}^1} \otimes \mathcal{O}(d)) \). Locally, sections of \( \omega_{\mathbb{P}^1} \) are of the form \( r(\frac{z_0}{z_1})d(\frac{z_0}{z_1}) \) where \( r \) is of degree 0, so that \( r(\frac{z_0}{z_1}) \) is expressible as a quotient of homogeneous polynomials of equal degree. Meanwhile, sections of \( \mathcal{O}(d) \) are locally quotients of homogeneous polynomials whose difference of degrees is \( d \).

The bundle \( \omega_{\mathbb{P}^1} \otimes \mathcal{O}(d) \) has as its global sections \( z_2^2 \tilde{p}(z_0, z_1)d(\frac{z_0}{z_1}) \) where \( \tilde{p} \) is homogeneous of degree \( d - 2 \). Take a basis consisting of monomial \( \tilde{p} \). The weights are
Next, suppose \((f, \Sigma, p_1, \ldots, p_n)\) is a point in \(\mathcal{M}_T/\mathcal{A}_T\). Consider the deformation exact sequence. The fixed locus in \(\text{Aut}(\Sigma, p_1, \ldots, p_n)\), modulo \(\mathcal{A}_T\), consists of \(\mathbb{C}^\times\) automorphisms on each noncontracted rational curve. The fixed locus in \(\text{Def}(\Sigma, p_1, \ldots, p_n)\) consists of deformations of positive genus components (giving the tangent space to \(\mathcal{M}_T/\mathcal{A}_T\)). The fixed locus in \(\text{Def}(f)\) will be described later. We’ll see that there is one dimension of weight 0 for each noncontracted rational curve.

Now we will compute the moving parts of these. First, the moving part of \(\text{Aut}(\Sigma, p_1, \ldots, p_n)\) is as follows: for each vertex of valence 1 and no tails, there is a contribution \(w_F\). Next, the moving part of \(\text{Def}(\Sigma, p_1, \ldots, p_n)\) consists of deformations which smooth nodes, with deformation spaces given as the product of the tangent spaces. In the case of a node connecting a genus 0 curve with a positive genus curve, there is a contribution \(w_F - \psi_F\) over flags having vertices corresponding to contracted components. In the case of a node connecting two \(\mathbb{P}^1\)'s, this corresponds to flags at vertices of hairiness 2 (hairiness is the sum of the valence and the number of tails). The contribution is \(w_{F_1} + w_{F_2}\).

Finally, consider \(\text{Def}(f)\), given as \(H^0(\Sigma, f^*(T_{\mathbb{P}^m}))\). If \(\mu : \tilde{\Sigma} \to \Sigma\) is the normalization, there is an exact sequence

\[
0 \to \mathcal{O}_\Sigma \to \mu_* \mathcal{O}_{\tilde{\Sigma}} \to \mathcal{Q} \to 0
\]

(14.2)

where \(\mathcal{Q}\) is a skyscraper sheaf supported on the nodes. In general, for our graph, we will have an exact sequence

\[
0 \to \mathcal{O}_\Sigma \to \bigoplus_v \mathcal{O}_{\Sigma_v} \oplus \bigoplus_e \mathcal{O}_{\Sigma_e} \to \bigoplus_F \mathcal{O}_{x_F} \to 0.
\]

(14.3)

Now tensoring with \(f^*T_{\mathbb{P}^m}\) and taking cohomology, we get an exact sequence

\[
0 \to H^0(\Sigma, f^*T_{\mathbb{P}^m}) \to \bigoplus_v H^0(\Sigma_v, f^*T_{\mathbb{P}^m}) \oplus \bigoplus_e H^0(\Sigma_e, f^*T_{\mathbb{P}^m}) \to \bigoplus_F H^0(x_F, f^*T_{\mathbb{P}^m}) \to 0
\]

(14.4)

since the \(H^1\) term is zero (a property of \(\mathbb{P}^m\)). We can express this as

\[
0 \to H^0(\Sigma, f^*T_{\mathbb{P}^m}) \to \bigoplus_v T_{\mu(v)}(\mathbb{P}^m) \oplus \bigoplus_e H^0(\mathbb{P}^1, f^*\mathcal{O}(m + 1)) \to \bigoplus_F T_{\mu(v)}(\mathbb{P}^m) \to 0
\]

(14.5)

The Euler class we want is

\[
e(\text{Def}(\Sigma, p_1, \ldots, p_n, f)^{\text{mov}}) = \frac{e(\text{Def}(f)^{\text{mov}} e(\text{Def}(\Sigma, p_1, \ldots, p_n)^{\text{mov}})}{e(\text{Aut}(\Sigma, p_1, \ldots, p_n)^{\text{mov}})}
\]

(14.6)

Using the above, we can obtain the formula that was claimed earlier.

As an example, we will try to compute the number of nodal cubics through 8 points in \(\mathbb{P}^2\) (the answer is 12). We need to integrate...
The fixed loci correspond to graphs $\Gamma$ with either an edge of label 3, two edges of labels 2 and 1, or a chain of three lines with each line having label 1. It’s left as exercises to count the fixed loci and to compute the whole integral.

We’ll work out one piece of the calculation. If $\phi_a, \phi_b, \phi_c$ are the fixed points, choose the equivariant lift

$$ev_1^*(\phi_a) \cap \cdots \cap ev_3^*(\phi_a) \cap ev_4^*(\phi_b) \cap ev_5^*(\phi_b) \cap ev_6^*(\phi_c) \cap \cdots \cap ev_8^*(\phi_c).$$

(14.8)

The contribution from the graphs with a single edge of label 3 (which corresponds to a complicated moduli space) will be zero. Meanwhile, for the decorated graph

$$\begin{array}{c|c|c}
a & 2 & b \\
\hline & 1 & \\
\end{array}$$

the numerator is

$$\left((\alpha_a - \alpha_b)(\alpha_a - \alpha_c)\right)^3 \left((\alpha_b - \alpha_a)(\alpha_b - \alpha_c)\right)^2 \left((\alpha_c - \alpha_a)(\alpha_c - \alpha_b)\right)^3.$$  

(14.10)

The Euler class satisfies

$$\frac{1}{\mathcal{N}_\Gamma} = \frac{1}{\alpha_a - \alpha_b - \psi_{F_1}} \frac{1}{\alpha_b - \alpha_c - \psi_{F_2}} \frac{1}{\alpha_c - \alpha_b - \psi_{F_3}} \frac{1}{\alpha_b - \psi_{F_4}} (\alpha_b - \alpha_a)(\alpha_b - \alpha_a)$$  

$$\cdot \frac{1}{(-1)^2 2^4 (\alpha_b - \alpha_a)^4} \cdot \frac{1}{(\alpha_a - \alpha_c)(\alpha_b + \frac{\alpha_c}{2} - \alpha_c)(\alpha_b - \alpha_c)}$$  

$$\cdot \frac{1}{(-1)(\alpha_c - \alpha_b)^2 (\alpha_b - \alpha_a)(\alpha_c - \alpha_a)}.$$  

(14.11)

Hence the contribution to the integral is

$$\frac{1}{2} \int_{\mathcal{M}_{0,4} \times \mathcal{M}_{0,4} \times \mathcal{M}_{0,4}} \frac{\text{num}}{\text{denom}}.$$  

(14.14)

To compute this integral, expand the formula for $\frac{1}{\mathcal{N}_\Gamma}$ as power series in the $\psi_{F_j}$, and the relevant terms will be $\psi_{F_1} \psi_{F_2} \psi_{F_3}$ and $\psi_{F_1} \psi_{F_3} \psi_{F_4}$. (The classes $\psi_{F_2}$ and $\psi_{F_3}$ both lie on the middle $\mathcal{M}_{0,4}$, while $\psi_{F_1}$ and $\psi_{F_4}$ lie on the outer ones.)
such that $N_{p1} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

15.1 Proof of Localization

This is only a sketch. Details can be found in Atiyah-Bott.

Suppose $T \subseteq X$. Then $H_T^*(X)$ is a module over $H_T^*(pt) = \mathbb{C}[\alpha_1, \ldots, \alpha_r]$. Let

$$
\widetilde{H}_T^*(X) = H_T^*(X) \otimes_{H_T^*(pt)} \mathbb{C}(\alpha_1, \ldots, \alpha_r).
$$

Let $X^T = \bigsqcup F_j$, $i : F \hookrightarrow X$ the inclusions. Then consider $e(N_F X) \in H_T^*(F)$. The claim is that

$$
\sum_F i^* e(N_F X) : \widetilde{H}_T^*(X) \xrightarrow{\sim} \bigoplus_F \widetilde{H}_T(F)
$$

is an isomorphism.

1. Suppose $X = T/K$ for $K \subseteq T$ a subtorus. Then $H_T^*(X) = H_K^*(pt)$ since $T/K \times ET/T = ET/K$ which is a classifying space for $K$. Now $H_K^*(pt) = \mathcal{O}(t)$ and similarly $H_T^*(pt) = \mathcal{O}(t)$. The module structure arises from $\mathcal{O}(t) \twoheadrightarrow \mathcal{O}(t)$, which becomes zero after tensoring with $\mathbb{C}(\alpha_1, \ldots, \alpha_n)$. Hence $\widetilde{H}_T^*(T/K) = 0$.

2. $X = D^1 \times T/K$. Then again $\widetilde{H}_T(X) = 0$.

3. For general compact $X$, let $U = X \setminus X^T$. $U$ has no $T$-fixed points, so is a union of torus orbits

$$
U = \bigsqcup_{K,i} (T/K)_i.
$$

Claim. $U$ can be expressed as a finite union of spaces as in (2). (Actually not entirely true.)

Then Mayer-Vietoris show that $\widetilde{H}_T^*(U) = 0$.

15.2 Aspimall-Morisson

Let $X$ be a Calabi-Yau threefold, and choose an embedding $\mathbb{P}^1 \hookrightarrow X$. Then deg $N_{p1} = -2$. The only case in which $N_{p1}$ is rigid is $N_{p1} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The goal is to show $\langle \rangle_{0,d,\beta}^X = \frac{1}{d^3}$ for $\beta = [\mathbb{P}^1]$ and all $d$. We have to determine

$$
\int_{\mathbb{M}_{0,0}(N_{p1},d)} e(H^1(\mathbb{P}^1, f^* (\mathcal{O}(-1)_{p1} \oplus \mathcal{O}(-1)_{p1}))).
$$

The only decorated graph that contributes to localization is

$$
0 \quad d \quad \infty
$$

For if $f : \Sigma \rightarrow \mathbb{P}^1$ is nodal, the long exact sequence associated to the normalization exact sequence gives an inclusion
\[ \bigoplus H^0(\mathcal{O}_b \otimes f^*(\mathcal{O}(-1))) \to H^1(\Sigma, f^*(\mathcal{O}(-1))). \] (15.6)

Choose linearizations \( \alpha_0 - H \) and \( \alpha_1 - H \) for \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \). Then at least one factor in the Euler class will vanish.

For our decorated graph, the formula for the Euler class simplifies to

\[ \frac{1}{e(N_{\Gamma})} = \frac{(-1)^{d-1}d^{2d-2}}{(d!)^2(\alpha_0 - \alpha_\infty)^{2d-2}}. \] (15.7)

For the numerator of the localization formula, \( H^1(\mathbb{P}^1, f^*\mathcal{O}(-1)) \) has weights \( \frac{i(\alpha_0 - \alpha_\infty)}{d} \) for \( i = 1, \ldots, d - 1 \), so the Euler class is \( \frac{(d-1)!(\alpha_0 - \alpha_\infty)^{d-1}}{d^{d-1}} \). Hence we end up with the \( \frac{1}{d^r} \) result.

Faber-Pandharipande have shown the contribution is \( \frac{1}{12d} \) for \( g = 1 \), and in general \( \frac{|b_{2g}|d^{2g-3}}{2g(2g-2)!} \), for \( b_{2g} \) the Bernoulli numbers.

16 Virtual Fundamental Classes

Although \( \mathcal{M}_{0,n}(\mathbb{P}^r, d) \) is smooth, in general we would like to consider \( \mathcal{M}_{g,n}(X, \beta) \) for \( X \) general. Often the dimensions do not match up with the expected dimension, even in the case of \( \mathcal{M}_{0,0}(X, d) \) for \( X \) a Calabi-Yau. In this case, we need to consider virtual fundamental classes. If \( \mathcal{M} \) is smooth, but of greater than the expected dimension, we can hope that we can take \( e(E) \) for some suitable bundle \( E \). If \( \mathcal{M} \) is not smooth, this is more difficult.

Alternatively, we may use virtual localization. Then it’s enough to compute the virtual fundamental classes of the fixed loci.

As an example, \( \mathcal{M}_{g,n}(X, 0) \cong \mathcal{M}_{g,n} \times X \), but the expected dimension of the left hand side is \( (\dim X - 3)(1 - g) + n \), which is smaller.

**Lemma 16.1** (Behrend). *If there is a locally free obstruction bundle \( E \) of rank \( e \), then \( \mathcal{M} \) is smooth of dimension \( e \) more than its expected dimension, and

\[ [\mathcal{M}]^{\text{vir}} = e(E) \cap [\mathcal{M}]. \] (16.1)

At a point in the moduli space corresponding to \( f : \Sigma \to Q \in X \), the deformation exact sequence splits into two exact sequences

\[ 0 \to T_QX \to H^1(\Sigma, T_\Sigma) \oplus T_QX \to H^1(\Sigma, T_\Sigma) \to 0 \] (16.2)

and

\[ 0 \to H^1(\Sigma, f^*T_X) \to H^1(\Sigma, N_{\Sigma/X}) \to 0. \] (16.3)

Now we have
\[ H^1(\Sigma, f^*T_X) \cong H^0(\Sigma, \omega_\Sigma \otimes f^*T_X^\vee) \]  
\[ = H^0(\Sigma, \omega_\Sigma)^\vee \otimes T_0 X \]  
(16.4)

or \( \mathbb{E}^\vee \otimes T_X \), where \( \mathbb{E} \) is the Hodge bundle. We find that

\[ [\mathcal{M}_{g,n}(X, 0)]_{\text{vir}} = e(\mathbb{E}^\vee \otimes T_X) \cap [\mathcal{M}_{g,n}(X, 0)]. \]  
(16.6)

### 17 Counting Rational Curves

Consider \( X \) a quintic threefold, and let

\[ N_d = \int_{[\mathcal{M}_{0,0}(X,d)]_{\text{vir}}} 1. \]  
(17.1)

The moduli space has expected dimension 0, so the integral actually yields a number.

Consider \( X \) as a subvariety of \( \mathbb{P}^4 \), and so \( \mathcal{M}_{0,0}(X,d) \) can be thought of as a subvariety of \( \mathcal{M}_{0,0}(\mathbb{P}^4, d) \). \( N_1 = 2875 \), and can be computed in a similar way to determining the number of lines on a cubic. \( N_2 \) is more difficult, and equals \( 609609 \frac{3}{8} \), and \( N_3 = 317206481 \frac{13}{27} \).

The fractions arise since, for example, \( \mathcal{M}_{0,0}(X,2) \) also consists of double covers of lines, and each such component contributes \( \frac{1}{27} \). This suggests defining numbers \( n_d \), where

\[ N_1 = n_1 \]  
(17.2)

\[ N_2 = n_2 + \frac{1}{8} n_1 \]  
(17.3)

\[ N_3 = n_3 + \frac{1}{27} n_1 \]  
(17.4)

\[ N_4 = n_4 + \frac{1}{8} n_2 + \frac{1}{64} n_1 \]  
(17.5)

and so on.

**Conjecture 17.1** (Clemens). There are finitely many rational curves of degree \( d \) on a general Calabi-Yau threefold.

**Conjecture 17.2** (Gopakuma-Vafa). The \( n_d \) are integers.

This last conjecture was proven recently by Ionel-Parker.

### 18 Mirror Symmetry

Consider now the generating function \( F(q) = \sum N_d q^d \). Associated to \( X \), there is supposed to be a mirror \( X^{\text{mirror}} \), where genus 0 Gromov-Witten invariants for \( X \) should correspond to solutions of the Picard-Fuchs equation for \( X^{\text{mirror}} \), which are encoded in a hypergeometric function. If such a function is \( G(Q) \), then we expect \( f(q) = G(Q) \) after a change of variables.
19 Stable Quotients (In More Detail)

20 Birational Geometry of Stable Quotients
To do...

☐ 1 (p. 6): Fill in Tuesday’s gap.

☐ 2 (p. 14): Work this out.