Topics in Automorphic Forms
Fall 2013, taught by Jack Thorne.

References:

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• Bushnell, Henniart: Local Langlands conjecture for \( GL(2) \).
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1 Motivation

One key question in number theory is to understand the absolute Galois group $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For example, for primes $p$ is $GL_2(\mathbb{F}_p)$ a quotient? We would like to construct number fields with prescribed local behavior at all primes $p$.

Miracle 1: we can use automorphic forms to do this.

Miracle 2: W can use the Langlands $L$-group to understand these.

2 Classical Theory

Let $\mathfrak{h}$ be the upper half-plane, on which $GL_2(\mathbb{R})^+$ acts transitively. Fix an integer $k \geq 1$. If $f : \mathfrak{h} \to \mathbb{C}$ and $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL_2(\mathbb{R})^+$, we define $f|_k[\gamma] : \mathfrak{h} \to \mathbb{C}$ by $(f|_k[\gamma])(\tau) = \det(\gamma)^{k-1} \cdot (c\tau + d)^{-k} \cdot f(\gamma \tau)$.

**Lemma 2.1.** This is an action: for $\gamma, \gamma' \in GL_2(\mathbb{R})^+$,

\[(f|_k[\gamma])[\gamma'] = f|_k[\gamma\gamma'].\]  (2.1)

**Proof.** If $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$, define $j(\gamma, \tau) = (c\tau + d)$ the “modular cocycle”. It satisfies $j(\gamma\gamma', \tau) = j(\gamma', \tau)j(\gamma, \gamma'\tau)$. □
Define \( \gamma(1) = \text{SL}_2(\mathbb{Z}) \subseteq \text{GL}_2(\mathbb{R})^+ \). More generally, for \( N \geq 1 \), define

\[
\Gamma(N) = \ker(\text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/N\mathbb{Z})).
\]  

(2.2)

If \( \Gamma \subseteq \Gamma(1) \) is any subgroup, we say that \( \Gamma \) is a congruence subgroup if there exists \( N \geq 1 \) such that \( \Gamma(N) \subseteq \Gamma \). For \( k \geq 1 \) and \( \Gamma \) a congruence subgroup, a function \( f : \mathfrak{h} \to \mathbb{C} \) is weakly modular if:

1. \( f \) is holomorphic.
2. For every \( \gamma \in \Gamma \), \( f|_k[\gamma] = f \).

Given \( \Gamma \), we let \( h \in \mathbb{Z}_{\geq 1} \) be the smallest integer such that \( \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma \). If \( f \) is weakly modular, then \( f(\tau + h) = f(\tau) \), implying \( f \) admits a Laurent expansion

\[
f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n, \quad q^n = \exp \left( \frac{2\pi i \tau}{h} \right).\]  

(2.3)

We say that \( f \) is “holomorphic at \( \infty \)” if \( a_n = 0 \) for every \( n < 0 \) and “vanishing at \( \infty \)” if \( a_n = 0 \) for \( n \leq 0 \).

Let \( k \) and \( \Gamma \) be as above. Then \( f : \mathfrak{h} \to \mathbb{C} \) which is weakly modular on \( \Gamma \) of weight \( k \) is said to be a modular form if it also satisfies:

3. For every \( \alpha \in \text{SL}_2(\mathbb{Z}) \), \( f|_k[\alpha] \) is holomorphic at \( \infty \).

We say that a modular form is cuspidal if in addition, for every \( \alpha \in \text{SL}_2(\mathbb{Z}) \), \( f|_k[\alpha] \) vanishes at \( \infty \).

Two important congruence subgroups are given by

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \left| c \equiv 0 \pmod{N} \right. \right\} \]  

(2.4)

\[
\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \left| c \equiv 0 \pmod{N}, a \equiv d \equiv 1 \pmod{N} \right. \right\}. \]  

(2.5)

2.1 Examples

For \( k \geq 4 \), we define the weight \( k \) Eisenstein series

\[
G_k(\tau) = \sum_{(m,n) \neq (0,0)} (m\tau + n)^{-k}.
\]  

(2.6)

**Proposition 2.2.** \( G_k(\tau) \) is a modular form on \( \Gamma(1) \) of weight \( k \).

**Proof.** \( G_k(\tau) \) converges absolutely, and uniformly on compact subsets. We can write

\[
G_k(\tau) = \sum_{0 \neq \lambda \in \Lambda} \lambda^{-k}.
\]  

(2.7)
where $\Lambda = \mathbb{Z} \oplus \mathbb{Z} \tau \subseteq \mathbb{C}$.

$\Gamma(1)$ acts on $\mathbb{Z}^2$ on the right; we set $\Gamma_\infty = \text{stab}_{\Gamma(1)}(0, 1) = \{(\frac{1}{0}, 1)\}$. We can then write

$$G_k(\tau) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma(1)} j(\gamma, \tau)^{-k}. \quad (2.8)$$

Hence

$$G_k(\gamma' \tau) = \sum_{\gamma} j(\gamma, \gamma' \tau)^{-k} \quad (2.9)$$

$$= j(\gamma', \tau)^k \sum_{\gamma} j(\gamma, \tau)^{-k}. \quad (2.10)$$

It remains to show that $G_k(\tau)$ is bounded as $\tau \to \infty$. Since $G_k(\tau)$ is uniformly convergent in $[-\frac{1}{2}, \frac{1}{2}] \times [1, \infty)$, we have

$$\lim_{\tau \to \infty} G_k(\tau) = \sum_{(m,n) \neq (0,0)} \lim_{\tau \to \infty} (m\tau + n)^{-k} = \sum_{n \neq 0} n^{-k} \quad (2.11)$$

$$= 2\zeta(k). \quad (2.12)$$

In fact, we can compute the entire Fourier expansion of $G_k$: it is given as

$$G_k(\tau) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n \quad (2.14)$$

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}. \quad (2.15)$$

The normalized Eisenstein series is $E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)}$, so that

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n \quad (2.16)$$

where $B_k \in \mathbb{Q}$ is the $k$th Bernoulli number.

We define

$$\Delta(\tau) = \frac{E_4^3 - E_6^2}{1728} = q - 24q^2 + 252q^3 + \cdots \quad (2.17)$$

This is called the Ramanujan $\Delta$ function, and is a cusp form of weight 12 on $\Gamma$. $\Delta$ is (up to scalar) the unique cusp form on $\Gamma(1)$ of minimal weight.
If $\Gamma$ is a congruence subgroup, set $M_k(\Gamma)$ to be the $\mathbb{C}$-vector space of modular forms on $\Gamma$ of weight $k$, and $S_k(\Gamma) \subseteq M_k(\Gamma)$ the subspace of cusp forms. Then $S_k(\Gamma(1)) = 0$ for $k \leq 11$, and $S_{12}(\Gamma(1))$ has dimension 1.

**Proposition 2.3.** If $\Gamma$ is a congruence subgroup and $k \geq 1$, then $M_k(\Gamma)$ and $S_k(\Gamma)$ are finite dimensional.

**Sketch of Proof.** Define $Y(\Gamma) = \Gamma \backslash \mathfrak{h}$ and $X(\Gamma) = Y(\Gamma) \amalg \left( \Gamma \backslash \mathbb{P}^1(\mathbb{Q}) \right)$ as a set. (Here finitely many points were added.) We can show that $X(\Gamma)$ admits the structure of a compact Riemann surface, with $Y(\Gamma) \subseteq X(\Gamma)$ an open subset. In fact, $X(\Gamma(1)) \cong \mathbb{P}^1(\mathbb{C})$ by $\tau \mapsto j(\tau)$. Then the following hold (not quite true if $\Gamma$ has torsion, but can be easily modified):

- A weakly modular function $f$ is equivalent to giving an element of $H^0(Y(\Gamma), \mathcal{L}_k)$ for some line bundle $\mathcal{L}_k$ on $X(\Gamma)$.
- $f$ is a modular form of weight $k$ if and only if it comes from an element of $H^0(X(\Gamma), \mathcal{L}_k)$.

$\square$

### 2.2 Hecke Operators

$M_k(\Gamma)$ and $S_k(\Gamma)$ have Hecke operators. For now, assume $\Gamma = \Gamma(1)$.

**Lemma 2.4.** If $\alpha \in GL_2(\mathbb{Q})^+$, then $\Gamma(1)\alpha\Gamma(1) \subseteq GL_2(\mathbb{Q})^+$ contains finitely many (left) $\Gamma(1)$-orbits.

**Proof.** Let $\Gamma_\alpha = \alpha^{-1}\Gamma(1)\alpha \cap \Gamma(1)$. Then we have

$$\begin{align*}
\Gamma(1)\backslash \Gamma(1)\alpha\Gamma(1) &\longrightarrow \Gamma_\alpha \backslash \Gamma(1) \\
\Gamma(1)\alpha\beta &\longmapsto \Gamma_\alpha \beta \\
\Gamma(1)\alpha\beta &\longmapsto \Gamma_\alpha \beta
\end{align*} \quad (2.18)$$

Check that these maps are well-defined, so this map is a bijection.

It suffices to prove that $\Gamma_\alpha \subseteq \Gamma(1)$ is of finite index. Choose $N \geq 1$ such that $N\alpha \in M_2(\mathbb{Z})$ and $N\alpha^{-1} \in M_2(\mathbb{Z})$. Then

$$\begin{align*}
\alpha \Gamma(N^2)\alpha^{-1} &\subseteq \alpha(1 + N^2M_2(\mathbb{Z}))\alpha^{-1} \\
&= 1 + (N\alpha)M_2(\mathbb{Z})(N\alpha^{-1}) \\
&\subseteq 1 + M_2(\mathbb{Z}) \\
\alpha \Gamma(N^2)\alpha^{-1} &\subseteq \Gamma(1) \\
\Gamma(N^2) &\subseteq \alpha^{-1}\Gamma(1)\alpha \cap \Gamma(1). \quad (2.23)
\end{align*}$$

$\square$
Given \( \alpha \) and \( f \) weakly modular on \( \Gamma(1) \) of weight \( k \), we define

\[
f_k[\Gamma(1)|\Gamma(1)] = \sum_{\beta_j} f_k[\beta_j]
\]

(2.24)

\[
\Gamma(1)|\Gamma(1) = \prod_{j=1}^{n} \Gamma(1)\beta_j.
\]

(2.25)

This is well-defined.

For \( p \) prime, we write \( T_p \) for the endomorphism of \( M_k(\Gamma(1)) \) or \( S_k(\Gamma(1)) \) induced by \( f \mapsto f|_k[\Gamma(1)(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}) \Gamma(1)] \).

**Lemma 2.5.** If \( f \) is a modular form (respectively cusp form), then so is \( T_p(f) \).

**Proof.** \( T_p(f) \) is weakly modular: if \( \gamma \in \Gamma(1) \), then

\[
f|_k[\Gamma(1)(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}) \Gamma(1)]|_k[\gamma] = \sum_{\beta_j} f|_k[\beta_j \gamma].
\]

(2.26)

Now \( \Gamma(1) \setminus \Gamma(1)|\Gamma(1) \) admits a right action of \( \Gamma(1) \). If \( \beta_j \) are left \( \Gamma(1) \)-orbit representatives in \( \Gamma(1)|\Gamma(1) \), then so are \( \beta_j \gamma \).

To show that \( T_p(f) \) is holomorphic at \( \infty \), we compute the effect on \( q \)-expansions. Suppose \( f(\tau) = \sum_{n \geq 0} a_n q^n \). \( \Gamma(1) \) acts on \( \mathbb{Z}^2 \) on the right, and

\[
\Gamma(1)(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}) \Gamma(1) = \{ x \in M_2(\mathbb{Z}) | \det(x) = p \}.
\]

(2.27)

Representatives of \( \Gamma(1) \setminus \Gamma(1)(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}) \Gamma(1) \) correspond to \( M \subseteq \mathbb{Z}^2 \) of index \( p \) by \( \Gamma(1)\delta \mapsto \mathbb{Z}^2\delta \). So we can write

\[
\Gamma(1)(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}) \Gamma(1) = \left[ \prod_{a=0}^{p-1} \Gamma(1)(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}) \right] \Gamma(1)(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}).
\]

(2.28)

Now we have

\[
f|_k[\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}](\tau) = p^{k-1} p^{-k} f(\frac{\tau + a}{p})
\]

(2.29)

\[
f|_k[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}](\tau) = p^{k-1} f(p\tau).
\]

(2.30)

We can write
\[
\sum_{a=0}^{p-1} f\left(\frac{\tau + a}{p}\right) = \sum_{n \geq 0} a_n e^{2\pi i n \tau} \sum_{a=0}^{p-1} e^{\frac{2\pi i a n}{p}} = p \sum_{n \geq 0} a_n p q^n. \tag{2.31}
\]

Hence

\[
T_p(f) = \sum_{n \geq 0} a_n p q^n + p^{k-1} \sum_{n \geq 0} a_n q^{np}. \tag{2.32}
\]

This may also be written by

\[
a_n(T_p(f)) = a_n p q^n + p^{k-1} a_0. \tag{2.33}
\]

**Corollary 2.6.** $T_p$ and $T_q$ commute.

**Proof.** The expression for $a_n(T_p(T_q(f)))$ is symmetric in $p$ and $q$. \qed

### 2.3 The Petersson Inner Product

Fix $k \geq 1$. If $f, g : \mathfrak{h} \to \mathbb{C}$ are holomorphic, we define the 2-form

\[
\omega(f, g) = f(\tau)\overline{g(\tau)}y^{k-2} dx dy. \tag{2.35}
\]

If $\gamma \in GL_2(\mathbb{R})^+$, then we have

\[
\Im(\gamma\tau) = |j(\gamma, \tau)|^{-2} \det(\gamma) \Im(\tau) \tag{2.36}
\]

\[
\gamma^*(dx dy) = \det(\gamma)^2 |j(\gamma, \tau)|^{-4} dx dy. \tag{2.37}
\]

Hence if $\gamma' = \det(\gamma)\gamma^{-1}$, then

\[
\omega(f_{k[\gamma]}, g) = \gamma^* \omega(f, g_{k[\gamma']}). \tag{2.38}
\]

For $f, g \in M_k(\Gamma(1))$, the $\omega(f, g)$ is $\Gamma(1)$-invariant.

If $f, g \in S_k(\Gamma(1))$, define

\[
\langle f, g \rangle = \int_{\Gamma(1) \backslash \mathfrak{h}} \omega(f, g). \tag{2.39}
\]

This is a hermitian, positive definite inner product.
Lemma 2.7. For \( p \) prime, \( T_p : S_k(\Gamma(1)) \to S_k(\Gamma(1)) \) is self-adjoint with respect to this inner product.

Proof. Exercise. \( \square \)

Corollary 2.8. There exists a basis \( f_1, \ldots, f_N \) of \( S_k(\Gamma(1)) \) such that for each \( i \), \( f_i \) is a \( T_p \)-eigenvector with real eigenvalues for all \( p \).

If \( f_i \) is a basis vector, we can associate Galois representations and motives to \( f \).

2.4 Conclusion

For \( p \) prime, \( \mathbb{Q}(\zeta_p) \) is a degree \( p - 1 \) extension on \( \mathbb{Q} \). For \( G = \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \), let \( \omega : G \rightarrow \{\mathbb{Z}/p\}^\times \) be the cyclotomic character. Let \( M \) be the ideal class group of \( \mathbb{Q}(\zeta_p) \). Then \( G \circ C = M/pM = M \otimes \mathbb{F}_p \).

Decompose \( C = \bigoplus_i C_i \) for

\[
C_i = \{ x \in C | \forall g \in G, g(x) = \omega(g)^i x \}. \tag{2.40}
\]

Theorem 2.9 (Herbrand, Ribet). If \( 2 \leq k \leq p - 3 \) is even, then \( C_{1-k} \neq 0 \) if and only if \( p \mid B_k \).

By class field theory, the statement that \( C_{1-k} \neq 0 \) is equivalent to the existence of an unramified extension of \( \mathbb{Q}(\zeta_p) \) with some additional properties.

Ribet constructed this field using modular forms. For example, if \( p = 691 \), then \( p \) divides \( B_{12} \). In fact, \( E_{12} \equiv \Delta \pmod{691} \).

3 The Restricted Direct Product

Let \( I \) be a countable set. Suppose that for all \( i \in I \), we are given a locally compact topological group \( G_i \), and a finite subset \( S \) of \( I \) such that for every \( i \in I \setminus S \), we are given a compact subgroup \( K_i \subseteq G_i \). Then define

\[
\prod_{i \in I} G_i = \lim_{S \subseteq T \subseteq I \text{ finite}} \prod_{i \in T \setminus K_i} G_i \times \prod_{i \in I \setminus K_i} G_T \text{.} \tag{3.1}
\]

- This is a topological group, and is locally compact since each \( G_T \) is (a consequence of Tychonoff).
- This definition does depend on the choice of \( S \) and the \( K_i \). However, changing finitely many \( K_i \) or enlarging \( S \) does not change the \( \prod' G_i \).

For example, let \( F \) be a number field. For every place \( v \) of \( F \), we obtain a local field \( F_v \). If \( v \nmid \infty \), then the valuation ring \( \mathcal{O}_{F_v} \subseteq F_v \) is an open compact subring. Take \( I \) to be the set of places \( v \) of \( F \) and \( S \) to be the subset consisting of those \( v \) dividing \( \infty \). Take \( G_v = F_v \) and \( K_v = \mathcal{O}_{F_v} \). Then \( \prod'_v F_v = \mathbb{A}_F \) and \( \prod'_{v|\infty} F_v = \mathbb{A}_F^\infty \) (the ring of finite adeles).
Here is a more complicated example. With $I$ and $S$ as before, take $G_v = GL_2(F_v)$ and, for $v \notin S$, $K_v = GL_2(O_{F_v})$. Then define $\prod_v' GL_2(F_v) = GL_2(\mathbb{A}_F)$. (This restricted product characterization is necessary to define the topology.) Define $GL_2(\mathbb{A}^\infty_R)$ similarly.

Now consider the case $F = \mathbb{Q}$. Then $GL_2(\mathbb{A}^\infty)$ contains $\prod_p GL_2(\mathbb{Z}_p) = GL_2(\hat{\mathbb{Z}})$.

4 Adelic Modular Forms

Let $\mathcal{H}^\pm$ be the union of the upper and lower half-planes. Equivalently, $\mathcal{H}^\pm = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$. Fix an integer $k \geq 1$ and an open compact subgroup $U$ of $GL_2(\mathbb{A}^\infty)$. A modular form of level $U$ is a function $f : GL_2(\mathbb{A}^\infty) \times \mathcal{H}^\pm \to \mathbb{C}$ satisfying:

1. For every $g \in GL_2(\mathbb{A}^\infty)$, $\tau \mapsto f(g, \tau)$ is holomorphic.
2. For every $\gamma \in GL_2(\mathbb{Q})$,
   \[ f(\gamma g, \tau) = \det(\gamma)^{-1} f(\gamma, \tau)^k f(g, \tau). \] (4.1)
   Here we view $GL_2(\mathbb{Q})$ embedded diagonally in $GL_2(\mathbb{A}^\infty)$ and $\gamma(g, \tau) = (\gamma g, \gamma \tau)$.
3. For every $u \in U$, $f(gu, \tau) = f(g, \tau)$.
4. For every $g \in GL_2(\mathbb{A}^\infty)$, $\tau \mapsto f(g, \tau)$ is holomorphic at $\infty$ (meaning it tends to a finite limit at $\infty$).

If furthermore $\tau \mapsto f(g, \tau)$ vanishes at $\infty$ for every $g \in GL_2(\mathbb{A}^\infty)$, we say that $f$ is a cusp form of level $U$.

**Theorem 4.1.** Fix $k \geq 1$ and $U \subseteq GL_2(\hat{\mathbb{Z}})$ an open subgroup such that $\det U = \hat{\mathbb{Z}}^\times$. Define $\Gamma = U \cap GL_2(\mathbb{Q})^+$, a congruence subgroup of $SL_2(\mathbb{Z})$. Then $M_k(\Gamma) \cong M_k(U)$, the vector space of modular forms of level $U$. Similarly $S_k(\Gamma) \cong S_k(U)$.

**Lemma 4.2.** 1. Let $N \geq 1$ be an integer. Then $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/NZ)$ is surjective.
2. $GL_2(\mathbb{A}^\infty) = GL_2(\mathbb{Q})GL_2(\hat{\mathbb{Z}})$.
3. $SL_2(\mathbb{A}^\infty) = SL_2(\mathbb{Q})SL_2(\hat{\mathbb{Z}})$.

**Proof.** 1. Let $Y \in SL_2(\mathbb{Z}/NZ)$ and choose a lift $Y \in M_2(\mathbb{Z})$, so that $\det Y \equiv 1 \pmod{N}$. We want to find $\gamma \in SL_2(\mathbb{Z})$ such that $\gamma \equiv Y \pmod{N}$.

By multiplying $Y$ on the left and right by $SL_2(\mathbb{Z})$, we can assume that $Y$ is a diagonal matrix in $M_2(\mathbb{Z})$. If $x$ and $y$ are the diagonal entries, then $xy \equiv 1 \pmod{N}$, then we can take

\[ \gamma = \begin{pmatrix} x & -(1-xy) \\ 1-xy & y(2-xy) \end{pmatrix}. \] (4.2)
2. Considering $\mathbb{Q}$ and $\mathbb{Z}$ as embedded in $\mathbb{A}^\infty$, $\mathbb{Q} \cap \mathbb{Z} = \mathbb{Z}$ and $\mathbb{Q} + \mathbb{Z} = \mathbb{A}^\infty$. In fact, if $n \geq 2$ is an integer, there is a bijection

$$\{\text{free } \mathbb{Z}\text{-modules } M \subseteq \mathbb{Q}^n \text{ of rank } n\} \leftrightarrow \{\text{free } \mathbb{Z}\text{-modules } \hat{M} \subseteq (\mathbb{A}^\infty)^n \text{ of rank } n\}$$

$$M \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \mapsto \hat{M} \cap \mathbb{Q}^n$$

(4.3)

Now $GL_n(\mathbb{Q}) \circ \{M \subseteq \mathbb{Q}^n\}$ with $\text{stab}_{GL_n(\mathbb{Q})}Z^n = GL_n(\mathbb{Z})$. Similarly $GL_n(\mathbb{A}^\infty) \circ \{\hat{M} \subseteq (\mathbb{A}^\infty)^n\}$ with $\text{stab}_{GL_n(\mathbb{A}^\infty)} = GL_n(\hat{\mathbb{Z}})$. Thus we have a bijection between $GL_n(\mathbb{Z}) \setminus GL_n(\mathbb{Q})$ and $GL_n(\hat{\mathbb{Z}}) \setminus GL_n(\mathbb{A}^\infty)$.

3. Any $g \in SL_n(\mathbb{A}^\infty)$ can be written $g = \gamma \cdot k$ for $\gamma \in GL_n(\mathbb{Q})$ and $k \in GL_n(\hat{\mathbb{Z}})$. Then

$$1 = \det g = \det \gamma \det(k)$$

(4.4)

so both determinants must be $\pm 1$. If $\det \gamma = 1$, we’re done, otherwise change $\gamma$ by any element of $GL_n(\mathbb{Z})$ of determinant $-1$.

\[\square\]

**Theorem 4.3** (Strong Approximation for $SL_2$). $SL_2(\mathbb{Q})$ is dense in $SL_2(\mathbb{A}^\infty)$.

**Proof.** Let $U \subseteq SL_2(\mathbb{A}^\infty)$; we will then show that $SL_2(\mathbb{Q}) \cdot U = SL_2(\mathbb{A}^\infty)$. We can assume that $U \supseteq \hat{\Gamma}(N)$, where

$$\hat{\Gamma}(N) = \ker(SL_2(\hat{\mathbb{Z}}) \to SL_2(\mathbb{Z}/NZ)).$$

(4.5)

Let $g \in SL_2(\mathbb{A}^\infty)$. Write $g = \delta k$ for $\delta \in SL_2(\mathbb{Q})$ and $k \in SL_2(\hat{\mathbb{Z}})$. Write $k = \gamma \ell$ for $\gamma \in SL_2(\mathbb{Z})$ and $\ell \in \hat{\Gamma}(N)$. Then $g = (\delta \gamma) \ell \in SL_2(\mathbb{Q}) \hat{\Gamma}(N)$. \[\square\]

**Corollary 4.4.** Let $U \subseteq GL_2(\hat{\mathbb{Z}})$ be an open compact subgroup such that $\det(U) = \hat{\mathbb{Z}}^\times$. Then:

1. $GL_2(\mathbb{Q}) \cdot [U \times GL_2(\mathbb{R})^+] = GL_2(\mathbb{A})$.
2. $GL_2(\mathbb{Q}) \cdot [U \times \mathfrak{h}] = GL_2(\mathbb{A}^\infty) \times \mathfrak{h}^\pm$.

**Proof.** (1 $\implies$ 2) follows since $GL_2(\mathbb{R})$ acts transitively on $\mathfrak{h}^\pm$. To prove (1), by strong approximation,

$$SL_2(\mathbb{A}) \subseteq GL_2(\mathbb{Q}) \cdot [U \times GL_2(\mathbb{R})^+] \subseteq GL_2(\mathbb{A})$$

(4.6)

and the image of the determinant on this (a priori just subset) is

$$\mathbb{Q}^\times(\det(U)\mathbb{R}_{\geq 0}) = \mathbb{Q}^\times(\hat{\mathbb{Z}}^\times \cdot \mathbb{R}_{\geq 0}) = \mathbb{A}^\times$$

(4.7)
so it must be all of $GL_2(\mathbb{A})$.

Proof of Theorem 4.1. For $f \in M_k(U)$ we can associate $F \in M_k(\Gamma)$ by $F(\tau) = f(1, \tau)$. Conversely, given $F \in M_k(\Gamma)$, associate $f \in M_k(U)$ by

$$f(g, \tau) = \det(\gamma)^{2-k} F|_k[\gamma](\tau)$$

where $\gamma \in GL_2(\mathbb{Q})$ satisfies $\gamma(g, \tau) \in U \times \mathfrak{h}$ (such $\gamma$ this exists by Corollary 4.4). This is well-defined: if $\gamma'(g, \tau) \in U \times \mathfrak{h}$ as well, then write $\gamma(g, \tau) = (u, \gamma) \tau$ and $\gamma'(g, \tau) = (v, \gamma'/\tau)$ for some $u, v \in U$, namely $u = \gamma g$ and $v = \gamma' g$. Then $\gamma'\gamma^{-1} = vu^{-1} \in U$, so there exists $w \in U$ with $\gamma' = w\gamma$. Now $w \in GL_2(\mathbb{Q})^+ \cap U = \Gamma$, so det $\gamma' = $ det $\gamma$ and

$$F|_k[\gamma'](\tau) = F_k[w\gamma](\tau) = F_k[\gamma](\tau).$$

$f$ satisfies the relation

$$f(\delta(g, \tau)) = \det(\delta)^{-1} j(\delta, \tau)^k f(g, \tau).$$

There exists $\gamma$ with $\gamma(g, \tau) \in U \times \mathfrak{h}$, so $(\gamma\delta^{-1})\delta(g, \tau) \in U \times \mathfrak{h}$. Then

$$f(\delta(g, \tau)) = \det(\gamma\delta)^{-1}F|_k[\gamma\delta^{-1}](\delta \tau).$$

This can be checked to equal $f(g, \tau)j(\delta, \tau)^k \det(\delta)^{-1}$ using the cocycle relation for $j$.

The rest of the proof is left as an exercise.

Remark. The most common choice of $U$ is

$$U_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) \bigg| c \equiv 0 \pmod{N}, d \equiv 1 \pmod{N} \right\}.$$

Then $\det(U_1(N)) = \mathbb{Z}^\times$ and

$$U_1(N) \cap GL_2(\mathbb{Q})^+ = \Gamma_1(N),$$

Define $\mathcal{M}_k = \varprojlim_U M_k(U)$, the limit being over open compact $U$, and $\mathcal{S}_k$ similarly. Then for $g \in GL_2(\mathbb{A}^\infty)$, let $(g \cdot f)(h, \tau) = f(hg, \tau)$. If $f \in M_k(U)$, then $(g \cdot f) \in M_k(g^{-1}Ug) \in \mathcal{M}_k$.

An automorphic representation of weight $k$ is (for now) an irreducible representation $\pi$ of $GL_2(\mathbb{A}^\infty)$ which is isomorphic to a subquotient of $\mathcal{M}_k$. A cuspidal automorphic representation of $GL_2(\mathbb{A}^\infty)$ is an irreducible subquotient of $\mathcal{S}_k$.

5 Locally Profinite Groups

A topological group $G$ is called locally profinite if it has a basis of neighborhoods of $1$ consisting of profinite open subgroups. For example $GL_2(\mathbb{Z})$ and $GL_2(\mathbb{A}^\infty)$ are locally profinite, but not $GL_2(\mathbb{A})$.

A representation of a locally profinite group $G$ is a pair $(\pi, V)$ for $U$ a $\mathbb{C}$-vector space and $\pi : G \to \text{Aut}(V)$ a homomorphism. We say that $\pi$ is:
• smooth if for every \( v \in V \), there exists an open compact subgroup \( K \) of \( G \) such that \( v \in V^K = \{ w \in V | \pi(k)(w) = w \ \forall \ k \in K \} \). \hfill (5.1)

• admissible if it is smooth and for every open compact subgroup \( K \) of \( G \), \( V^K \) is finite-dimensional.

If \( K \) is a profinite group and \( V \) is an admissible representation of \( K \), then \( V = \bigoplus \sigma_i V_i \) where \( V_i \) are finite dimensional irreducible representations of \( K \).

Remark. \( \text{Rep}(G) \), the category of smooth representations of \( G \), is abelian. In particular, the notion of subquotient is defined. And if \( (\pi, U) \) is admissible, so is any subquotient.

Remark. 1. If \( G \) is compact, and \( (\pi, V) \) is an irreducible, smooth representation, then \( V \) is finite dimensional, and \( \ker \pi \) contains an open normal subgroup of \( G \).

2. In general, if \( (\pi, V) \) is smooth and \( K \subseteq G \) is an open compact subgroup, then for any \( v \in V \), \( \mathbb{C}[K] \cdot v \subseteq V \) is finite-dimensional, and semisimple as a representation of \( K \).

If \( \sigma \) is a smooth irreducible representation of \( K \), we write \( V[\sigma] \subseteq V \) for the span of all \( K \)-subspaces of \( V \) which are isomorphic to \( \sigma \). This is called the \( \sigma \)-isotypic piece of \( V \). Then we have

\[
V = \bigoplus \sigma V[\sigma]
\]  \hfill (5.2)

the sum being over isomorphism classes of smooth irreducible representations of \( K \), and \( V \) is admissible if and only if every \( V[\sigma] \) is finite dimensional.

If \( U \subseteq GL_2(\mathbb{A}^\infty) \), then \( \mathcal{M}_k^U = M_k(U) \) and \( \mathcal{S}_k^U = S_k(U) \). So the admissibility of \( \mathcal{M}_k \) and \( \mathcal{S}_k \) is equivalent to the finite dimensionality of \( M_k(U) \) and \( S_k(U) \) for every \( U \). We proved this in the case \( U \subseteq GL_2(\hat{\mathbb{Z}}) \) and \( \det(U) = \hat{\mathbb{Z}}^\times \), and can be proved in general using similar methods.

An automorphic representation \( \pi \) of \( GL_2(\mathbb{A}^\infty) \) is an irreducible subquotient of \( \mathcal{M}_k \) in the category \( \text{Rep}(GL_2(\mathbb{A}^\infty)) \), and is admissible.

### 5.1 Integration

For \( G \) locally profinite, define

\[
C_c^\infty(G) = \{ f : G \to \mathbb{C} | \text{compactly supported, locally constant} \}. \hfill (5.3)
\]

Then \( C_c^\infty(G) \) is a smooth representation of \( G \) by either left translation:

\[
[L(g)(f)](x) = f(g^{-1}x) \hfill (5.4)
\]

or by right translation:

\[
[R(g)(f)](x) = f(xg). \hfill (5.5)
\]

Any \( f \in C_c^\infty(G) \) can be written in one of the forms
\[ f = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{g_i K} \]  
\[ = \sum_{i=1}^{n'} \beta_i \mathbf{1}_{g'_i K} \]  
\[ = \sum_{i=1}^{n''} \gamma_i \mathbf{1}_{g''_i} \]  

for an open compact \( K \subseteq G \) depending on \( f \). Here for \( X \subseteq G \), \( \mathbf{1}_X \) is the characteristic function of \( X \).

As \( G \) is a locally compact topological group, there exists a Haar integral \( C_c^\infty(G) \rightarrow \mathbb{C} \), given by \( f \mapsto \int_{g \in G} f(g) \, dg \) satisfying:

1. If \( f \) takes positive real values, then \( \int_G f \, dg \geq 0 \).
2. For every \( f \in C_c^\infty(G) \) and every \( g \in G \),
\[
\int_{h \in G} f(gh) \, dh = \int_{h \in G} f(h) \, dh. \tag{5.9}
\]

This linear functional is unique up to a positive real multiple; now we will fix one. We will assume that \( G \) is unimodular. This means that for every \( f \in C_c^\infty(G) \) and \( g \in G \),
\[
\int_{h \in G} f(hg) \, dh = \int_{h \in G} f(h) \, dh. \tag{5.10}
\]

This means the Haar integral is also invariant under right translation.

For example \( G = GL_2(\mathbb{Q}_p) \) and \( GL_2(\mathbb{A}_\infty) \) are unimodular. More generally, if \( G \) is any reductive group over \( \mathbb{Q}_p \), then \( G = G(\mathbb{Q}_p) \) is unimodular.

Having fixed a Haar integral, \( C_c^\infty(G) \) becomes an associative algebra under convolution: for \( f_1, f_2 \in C_c^\infty(G) \),
\[
(f_1 \cdot f_2)(g) = \int_{x \in G} f_1(x) f_2(x^{-1} g) \, dx. \tag{5.11}
\]

Remark. If \( f = \sum \alpha_i \mathbf{1}_{g_i K} \), then we have
\[
\int_{g \in G} f(g) \, dg = \sum \alpha_i \mu(K). \tag{5.12}
\]

If \((\pi, V)\) is a smooth representation of \( G \), then \( V \) becomes a \( C_c^\infty(G) \)-module: for \( f \in C_c^\infty(G) \) and \( v \in V \),
\[
\pi(f)(v) = \int_{g \in G} f(g) \pi(g)(v) \, dg. \tag{5.13}
\]
If $K \subseteq G$ is an open compact and $v \in V^K$, with $f = \sum \alpha_i 1_{g_i K}$, then

$$\pi(f)(v) = \sum \alpha_i \mu(K) \pi(g_i)(v).$$

(5.14)

Recall that a general representation of $G$ is a $\mathbb{C}[G]$-module. The smooth representations are the $C_c^\infty(G)$-modules.

Now fix an open compact subgroup $K \subseteq G$, and normalize such that $\mu(K) = 1$. Define $\mathcal{H}(G, K) \subseteq C_c^\infty(G)$ to be the space of locally constant, compactly supported functions which are $K$-biinvariant. This means that for $k_1, k_2 \in G$ and $g \in G$,

$$f(k_1 g k_2) = f(g).$$

(5.15)

Equivalently, $f$ may be expressed in the form

$$f = \sum_{i=1}^n \alpha_i 1_{K g_i K}$$

(5.16)

for some $\alpha_i \in K$ and $g_i \in G$.

Define $e_K = 1_K$. Then:

**Lemma 5.1.** For every $f \in C_c^\infty(G)$, then $f \in \mathcal{H}(G, K)$ if and only if

$$e_K \cdot f = f = f \cdot e_K.$$  

(5.17)

It is also easy to check that $e_K \cdot e_K = e_K$. Therefore $\mathcal{H}(G, K) = e_K \cdot C_c^\infty(G) \cdot e_K$ is a subalgebra of $C_c^\infty(G)$ with unit $e_K$.

**Remark.** $\mathcal{H}(G, K)$ has a vector space basis $1_{K g K}$ over $g \in K \backslash G / K$. However, the multiplication law with respect to this basis is complicated, even for “simple” subgroups $K \subseteq G$.

**Theorem 5.2.** Fix $K \subseteq G$ such that $\mu(K) = 1$.

1. If $(\pi, V)$ is a smooth representation of $G$, then for any $f \in \mathcal{H}(G, K)$, $\pi(f)V \subseteq V^K$. In particular, $V^K$ is a $\mathcal{H}(G, K)$-module.

2. Let $(\pi, V)$ be an irreducible admissible representation. Then either $V^K = 0$ or $V^K$ is an irreducible $\mathcal{H}(G, K)$-module. If $\pi_1$ and $\pi_2$ are irreducible representations of $G$ with $\pi^K_1, \pi^K_2 \neq 0$, then $\pi_1 \cong \pi_2$ if and only if $\pi^K_1 \cong \pi^K_2$ as $\mathcal{H}(G, K)$-modules.

6 **The Case $GL_2(\mathbb{Q}_p)$**

Consider now $G = GL_2(\mathbb{Q}_p)$ and $K = GL_2(\mathbb{Z}_p)$. Then any compact subgroup $L \subseteq G$ is contained in a $G$-conjugate of $K$; $K$ is a maximal compact subgroup.

Let $(\pi, V)$ be an irreducible admissible representation of $G$. We say that $\pi$ is unramified if $\pi^K \neq 0$.  

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Theorem 6.1 (Satake isomorphism for \( GL(2) \)). Let \( T_p = 1_K \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \in \mathcal{H}(G, K) \) and \( S_p = 1_K \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{H}(G, K) \). Then \( \mathcal{H}(G, K) = \mathbb{C}[T_p, S_p, S_p^{-1}] \). In particular, \( \mathcal{H}(G, K) \) is commutative.

Proof. To be given later. \( \square \)

Corollary 6.2. If \( \pi \) is an unramified representation of \( G \), then \( \pi^K \) is 1-dimensional over \( \mathbb{C} \).

Proof. \( \pi^K \) is an irreducible, finite dimensional \( \mathcal{H}(G, K) \)-module. But \( \mathcal{H}(G, K) \) is commutative.

We also find that \( \pi^K \) is determined by its eigenvalues of \( T_p \) and \( S_p \). \( \square \)

However, \( \pi \) itself will generally be infinite dimensional.

Let \( k \geq 1 \) and \( N \geq 1 \) and take \( U_1(N) \subseteq GL_2(\hat{\mathbb{Z}}) \) as before, so that \( \Gamma_1(N) = U_1(N) \cap GL_2(\mathbb{Q})^+ \).

Recall that we have

\[
S_k(\Gamma_1(N)) \cong S_k(U_1(N)) = S_k^{U_1(N)}.
\]  
(6.1)

Now pick \( p \nmid N \), and write \( U_1(N) = U_1^p(N) \times GL_2(\mathbb{Z}_p) \), where \( U_1^p(N) \) is contained in \( \prod_{q \neq p} GL_2(\mathbb{Z}_q) \).

Then we have

\[
S_k^{U_1(N)} = (S_k^{U_1^p(N)})^{GL_2(\mathbb{Z}_p)} \subseteq S_k^{U_1^p(N)}.
\]  
(6.2)

We see that \( S_k^{U_1(N)} \) is an admissible representation of \( GL_2(\mathbb{Q}_p) \), therefore an \( \mathcal{H}(G_p, K_p) \) where \( G_p = GL_2(\mathbb{Q}_p) \) and \( K_p = GL_2(\mathbb{Z}_p) \).

Define the endomorphism \( T_p \) of \( S_k(\Gamma_1(N)) = S_k(U_1(N)) \) to be the one induced by the element \( T_p \) of \( \mathcal{H}(G_p, K_p) \). In the case \( N = 1 \), this agrees with the classical Hecke operator \( T_p \) of \( S_k(SL_2(\mathbb{Z})) \).

Indeed, for \( f \in S_k(GL_2(\hat{\mathbb{Z}})) \), recall we obtain \( F \in S_k(SL_2(\mathbb{Z})) \) by \( F(\tau) = f(1, \tau) \). The inverse is given by

\[
f(g, \tau) = \det(\gamma)j(\gamma, \tau)^4 F(\gamma \tau)
\]  
(6.3)

where \( \gamma(g, \tau) \) is taken to be in \( GL_2(\hat{\mathbb{Z}}) \times \mathfrak{h} \). Now we have a decomposition

\[
K_p \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} K_p = \prod_{i=0}^{p-1} \begin{pmatrix} p & -i \\ 0 & 1 \end{pmatrix} K_p \Pi \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} K_p \subseteq G_p.
\]  
(6.4)

Denote these coset representatives by \( \beta_i \), where \( \beta_p \) is the last representative. Then we have

\[
\beta_i^{-1} = \begin{cases} \begin{pmatrix} 1 & i \\ p & p \end{pmatrix} & i = 0, \ldots, p-1 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & i = p \end{cases}
\]  
(6.5)

We need to show that \( T_p^{\text{adelic}}(F) = T_p(F) \). We have
\[ T_p^{\text{adelic}}(F)(\tau) = (T_p f)(1, \tau) \]  
\[ = \int_{g \in K_p \left( \begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) K_p} f(g, \tau) \, dg \]  
\[ = \sum_{i=0}^{p} f(\beta_i, \tau) \]  
(6.6)  
(6.7)  
(6.8)

where \( \beta_i \in GL_2(\mathbb{Q}_p) \hookrightarrow GL_2(\mathbb{A}^\infty) \). Define \( \gamma_i \) to be the same matrices as \( \beta_i \) in \( GL_2(\mathbb{Q}) \), this time embedded diagonally in \( GL_2(\mathbb{A}^\infty) \). Then \( \gamma_i^{-1} \beta_i \) has \( p \)-entry 1 and \( q \)-entry in \( GL_2(\mathbb{Z}_q) \) for \( q \neq p \).

\[
\sum_{i=0}^{p} f(\beta_i, \tau) = \sum_{i=0}^{p} \det(\gamma_i^{-1}) j(\gamma_i^{-1}, \tau)^{-k} F(\gamma_i^{-1} \tau) 
\]
(6.9)
\[
= \sum_{i=0}^{p-1} p^{-1} F(p^{-1} \tau + p^{-1} i) + p^{-1} p^k F(p \tau) 
\]
(6.10)
\[
= p^{-1} \sum_{i=0}^{p-1} F(\frac{\tau + i}{p}) + p^{k-1} F(p \tau) 
\]
(6.11)
\[
= T_p(F). 
\]
(6.12)

Now we also have endomorphisms \( S_p \) of \( S_k(SL_2(\mathbb{Z})) \) induced by the element \( S_p \) of \( \mathcal{H}(G_p, K_p) \).

Some computation reveals that \( S_p F = p^{k-2} F \) for every \( F \in S_k(SL_2(\mathbb{Z})) \).

Now \( GL_2(\mathbb{Q}_p) \) acts on \( S_k^{GL_2(\mathbb{Z}_p)} \) with fixed vectors \( S_k(GL_2(\mathbb{Z})) \). One can show that \( S_k^{GL_2(\mathbb{Z}_p)} \) is a semisimple representation of \( GL_2(\mathbb{Q}_p) \). A decomposition \( S_k^{GL_2(\mathbb{Z}_p)} = \bigoplus_{i=1}^{\infty} \pi_i \) with \( \pi_i \) irreducible induces

\[
S_k(SL_2(\mathbb{Z})) = \bigoplus_{i=1}^{\infty} \pi_i^{K_p}. 
\]
(6.13)

After reordering, we can assume that \( \pi_i^{K_p} \neq 0 \) if and only if \( 1 \leq i \leq r \) for some appropriate \( r \). This diagonalizes \( T_p \). The isomorphism class of \( \pi_i \) for \( 1 \leq i \leq r \) is determined by the isomorphism class of \( \pi_i^{K_p} \) in \( \mathcal{H}(G_p, K_p) \), and therefore the eigenvalues of \( T_p \) and \( S_p \). The \( S_p \) eigenvalue is \( p^{k-2} \) and the \( T_p \) eigenvalue can be computed classically.

### 7 Unitary Representations

Let \( G \) be a locally profinite group, and \( (\pi, V) \) a smooth representation of \( G \). We say that \( \pi \) is unitary if there exists a Hermitian, positive definite inner product \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{C} \) which is \( G \)-invariant. (Here our convention is that inner products will be linear in the first variable.)

**Remark.** If \( G \) is compact, or in particular finite, every irreducible representation is unitary.
Lemma 7.1. If $(\pi, V)$ is unitary and admissible, then $\pi$ is semisimple (meaning a direct sum of irreducible admissible representations, possibly infinite).

Proof. By a Zorn’s Lemma argument, it suffices to show that any $G$-invariant subspace $W \subseteq V$ has a $G$-invariant complement. In fact $V = W \oplus W^\perp$. The intersection $W \cap W^\perp$ is 0, so we just need to show that $V$ is spanned by $W$ and $W^\perp$.

Choose an open compact subgroup $K$ of $G$, so that $V = \bigoplus \sigma V[\sigma]$ into isotypic components. Admissibility implies every $V[\sigma]$ is finite dimensional, and it will be enough to show $V[\sigma] \subseteq W + W^\perp$ for each $\sigma$. Since $V[\sigma]$ is finite dimensional, $V[\sigma] = W[\sigma] \oplus (V[\sigma] \cap W[\sigma]^\perp)$. (7.1)

For every smooth irreducible representation $\tau$ of $K$, $W[\tau]$ is perpendicular to $V[\sigma] \cap W[\sigma]^\perp$. For if $\tau = \sigma$, $V[\sigma] \cap W[\sigma]^\perp \subseteq W[\sigma]^\perp$; otherwise $V[\sigma] \cap W[\sigma]^\perp \subseteq V[\sigma]$, and $V[\sigma]$ and $V[\tau]$ are perpendicular.

Now $W = \bigoplus \tau W[\tau]$, so $V[\sigma] \cap W[\sigma]^\perp \subseteq W^\perp$, implying $V[\sigma] \subseteq W + W^\perp$. \(\square\)

Corollary 7.2. If $k \geq 1$ is an integer, $S_k$ is a semisimple representation of $GL_2(\mathbb{A}^\infty)$. Hence

\[ S_k = \bigoplus \pi \] (7.2)

where $\pi$ runs over cuspidal automorphic representations of $GL_2(\mathbb{A}^\infty)$.

Proof. There exists a character $\psi : GL_2(\mathbb{A}^\infty) \to \mathbb{C}^\times$ such that if $(R, S_k)$ denotes the representation of $GL_2(\mathbb{A}^\infty)$ on $S_k$, then $(R \otimes \psi, S_k)$ is unitary. The unitary structure is a generalization of the Petersson inner product. \(\square\)

8 The Action Over Finite Adeles

Recall that we have defined $GL_2(\mathbb{A}^\infty)$ as a direct limit of $G_T$ over finite sets of primes $T$, where

\[ G_T = \prod_{p \in T} GL_2(\mathbb{Q}_p) \times \prod_{p \not\in T} GL_2(\mathbb{Z}_p). \] (8.1)

Let $G_p = GL_2(\mathbb{Q}_p)$ and $K_p = GL_2(\mathbb{Z}_p)$. Suppose we are given a finite set of primes $S$, and irreducible admissible representations $\pi_p$ of $G_p$ such that $\pi_p$ is unramified for $p \not\in S$. Then for $p \not\in S$, $\pi_p^{K_p}$ is 1-dimensional. Choose a nonzero $x_p \in \pi_p^{K_p}$ for such $p$.

If $T \supseteq S$ is a finite set of primes, define

\[ V_T = \bigotimes_{p \in T} \pi_p. \] (8.2)

$G_T$ acts on $V_T$, and $GL_2(\mathbb{Z}_p)$ acts trivially if $p \not\in T$. If $T \subseteq T'$, define a map $V_T \to V_{T'}$ by
We obtain a diagram

\[
\begin{array}{c}
V_T \\ \downarrow \\
G_T \\
\end{array}
\begin{array}{c}
\longrightarrow \\
\circ \\
\longrightarrow \\
\end{array}
\begin{array}{c}
V_{T'} \\ \downarrow \\
G_{T'} \\
\end{array}
\]

(8.4)

The restricted tensor product of the \( \pi_p \) is defined to be

\[
\bigotimes_p \pi_p = \lim_{\longrightarrow T} V_T 
\]

as a representation of \( GL_2(\mathbb{A}^\infty) \).

**Proposition 8.1.**

1. \( \bigotimes'_p \pi_p \) is an irreducible admissible representation of \( GL_2(\mathbb{A}^\infty) \), and only depends on \( \pi_p \), not on \( S \) or the choices of \( x_p \).

2. If \( \pi \) is any irreducible admissible representation of \( GL_2(\mathbb{A}^\infty) \), then there exist \( \pi_p \) and an isomorphism \( \pi \cong \bigotimes'_p \pi_p \).

3. The isomorphism classes of the \( \pi_p \) are uniquely determined.

**Proof.** Can be found in Flath in Corvallis I. The same holds if \( GL_2(\mathbb{A}^\infty) \) is replaced by any restricted direct product \( \prod'_{i \in I} T_i \) with respect to data \( (G_i, K_i)_{i \in I} \), such that for almost all \( i \), the algebra \( \mathcal{H}(G_i, K_i) \) is commutative.

Recall that we have defined a cuspidal automorphic representation \( \pi \) of \( GL_2(\mathbb{A}^\infty) \) of weight \( k \) to be an irreducible subquotient of \( S_k \). We now describe these \( \pi \) in classical terms.

**Proposition 8.2.**

1. Let \( N \geq 1 \), and \( p \) a prime with \( p \nmid N \). Then \( T_p \) acts semisimply on \( S_k(\Gamma_1(N)) \) and the \( T_p \) and \( T_q \) commute for \( q \) prime with \( q \nmid N \).

2. If \( f \in S_k(\Gamma_1(N)) \) is an eigenvector for all \( T_p \) with \( p \nmid N \), then the submodule of \( S_k \) generated by \( f \) is irreducible. Conversely, every irreducible submodule \( V \subseteq S_k \) is obtained in this way.

**Proof.**

1. Let \( \pi_1, \ldots, \pi_r \) be the (finitely many) irreducible submodules of \( S_k \) such that \( \pi_i^{U_1(N)} \neq 0 \). Then, as modules for \( \mathbb{C}[T_p] \),

\[
S_k(\Gamma_1(N)) \cong \bigoplus_{i=1}^r \left( \bigotimes_{q \neq p}^t \pi_{i,q} \right) \otimes \pi_i^{GL_2(\mathbb{Z}_p)} \quad \text{(8.6)}
\]

where \( T_p \) acts trivially on the restricted tensor product of the \( \pi_{i,q} \) and in the usual way on \( \pi_{i,p} \). In particular, \( T_p \) acts semisimply. That \( T_p \) and \( T_q \) commute can be shown similarly.
2. We need some nontrivial global information:

**Theorem 8.3 (Multiplicity 1).** If $V, V' \subseteq S_k$ are irreducible $GL_2(\mathbb{A}^\infty)$-submodules such that $V \cong V'$, then $V = V'$ (That is, $S_k$ decomposes with multiplicity one.)

**Theorem 8.4 (Strong Multiplicity 1).** If $V = \bigotimes'_p V_p$ and $V' = \bigotimes'_p V_p'$ are cuspidal automorphic representations of $GL_2(\mathbb{A}^\infty)$ of weight $k$ such that for almost all $p$, $V_p \cong V_p'$, then $V \cong V'$.

Choose $f \in S_k(\Gamma_1(N)) \cong S_k(U_1(N))$ which is an eigenvector of the $T_p$ for all $p \nmid N$, and let $V \subseteq S_k$ be the submodule generated by $f$. Since $S_k$ is semisimple, we may decompose $V = \bigoplus_{i=1}^r V_i$ with each $V_i$ irreducible.

We know that if $V_i = \bigotimes'_p V_{i,p}$, then for every $i, j$ and every $p \nmid N$, $V_{i,p} \cong V_{j,p}$, for these are unramified representations, and their isomorphism class is determined by the eigenvalue of $T_p$ on $V_{i,p}$, and therefore the eigenvalue of $T_p$ on $f$.

The strong multiplicity 1 theorem implies $V_i \cong V_j$ for every $i$ and $j$, and the multiplicity 1 theorem then implies $r = 1$.

For the converse, we need some local information.

**Theorem 8.5.** Let $p$ be a prime and $m \geq 0$ an integer. Define the subgroup

$$K_1(p^m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p) \middle| c \equiv 0 \pmod{p^m}, d \equiv 1 \pmod{p^m} \right\}.$$  (8.7)

If $\pi$ is any irreducible admissible representation of $GL_2(\mathbb{Q}_p)$, then there exists $m \geq 0$ such that $\pi^{K_1(p^m)} \neq 0$. (The smallest such $m$ is called the conductor of $\pi$.)

Furthermore, if $\pi$ is a cuspidal automorphic representation of weight $k$, then writing $\pi \cong \bigotimes'_p \pi_p$, $\pi_p$ is infinite dimensional for every $p$.

**Remark.** If $\pi \otimes GL_2(\mathbb{Q}_p)$ is an irreducible admissible finite dimensional representation, then $\pi = \chi \circ \det$ for $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ a smooth character.

If $\pi$ is an infinite dimensional irreducible admissible representation of $GL_2(\mathbb{A}^\infty)$, then this theorem implies there exists $N \geq 1$ such that $\pi^{U_1(N)} \neq 0$. If $\pi \subseteq S_k$, we can choose

$$f \in \pi^{U_1(N)} \subseteq S_k^{U_1(N)} = S_k(U_1(N)) \cong S_k(\Gamma_1(N)).$$  \hspace{1cm} (8.8)

Then $f$ is a $T_p$-eigenvector for all $p \nmid N$, and $f$ generates $\pi$.

\[\square\]

9 The $GL_2(\mathbb{R})$ Action

Let $G = GL_2(\mathbb{R})$ and $g = \text{Lie } G = M_{2 \times 2}(\mathbb{R})$. Inside $G$, we have a subgroup $K^0 = SO(2)$ with respect to the Euclidean inner product, and $K = O(2) = K^0 \Pi \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) K^0$. Then $K$ is a maximal compact subgroup of $G$. Write $\mathfrak{k} = \text{Lie } K$. Write $C^\infty(G)$ to be the set of smooth functions $f : G \to \mathbb{C}$, and $C^\infty_c(G)$ to be the subset of those functions which are compactly supported. Then $G$ acts on $C^\infty_c(G)$ by right translation: $(g \circ f)(x) = f(xg)$. If $X \in g$, define
\[
(Xf)(x) = \frac{d}{dt}[f(x \exp(tX))]_{t=0}.
\]  

(9.1)

This makes \(C^\infty(G)\) into a representation of the Lie algebra \(\mathfrak{g}\), so extends to a representation of the universal enveloping algebra \(U(\mathfrak{g})\).

Denote \(g_C = \mathfrak{g} \otimes \mathbb{C}\) to be the complexified Lie algebra, and \(U(g_C)\) similarly. These act on \(C^\infty(G)\) by linearity.

We introduce some specific elements of \(g_C\):

\[
Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

(9.2)

\[
W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

(9.3)

\[
X_\pm = \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix}
\]

(9.4)

Then \(Z\) spans the center of \(g_C\). \(W\) spans \(\mathfrak{k}\), and \(X_{\pm}\) are the eigenvectors for the adjoint action of \(\mathfrak{k}\) on \(g_C\) with nonzero eigenvalues. Specifically, \([W, X_{\pm}] = \pm 2iX_{\pm}\). Also \([X_+, X_-] = -4iW\).

Write \(Z(\mathfrak{g}) \subseteq U(\mathfrak{g})\) for the center of the universal enveloping algebra. One can show that \(Z(g_C) = \mathbb{C}[Z, \Delta]\) for \(\Delta\) the Casimir operator

\[
\Delta = -\frac{1}{4}(X_-X_+ - 2iW - W^2).
\]

(9.5)

Proposition 9.1. Let \(f : \mathfrak{h} \to \mathbb{C}\) be a smooth function. Define \(\varphi : G^+ \to \mathbb{C}\) by

\[
\varphi(g) = \det(g)f(gi)j(g,i)^{-k}
\]

(9.6)

for \(k \geq 1\) an integer. Then:

1. \(\varphi((zI)gr_\theta) = z^{2-k}e^{ik\theta}\varphi(g)\), where

\[
r_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}
\]

(9.7)

2. \(f\) is holomorphic if and only if \(X_-\varphi = 0\).

Proof. Recall the Iwasawa decomposition \(GL_2(\mathbb{R})^+ = ZNAK^0\): every \(g \in GL_2(\mathbb{R})^+\) has a unique decomposition of the form

\[
g = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} r_\theta
\]

(9.8)

for \(x \in \mathbb{R}, y, z \in \mathbb{R}_{>0}\), and \(\theta \in \mathbb{R}/2\pi\mathbb{Z}\). In these coordinates, we have

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\[ \varphi(g) = z^{2-k}f(x + iy)y^k e^{ik\theta}. \] (9.9)

This makes the transformation law clear.

For the second part, if \( \psi : GL_2(\mathbb{R})^+ \to \mathbb{C} \) is smooth,

\[ X_- \psi = e^{-2i\theta} \left( -2iy \frac{d\psi}{dx} + 2y \frac{d\psi}{dy} + i \frac{d\psi}{d\theta} \right). \] (9.10)

Hence, for our function \( \varphi \),

\[ X_- \varphi = 2z^{2-k}y^{k+1}e^{i(k-2)\theta} \left( -i \frac{df}{dx} + \frac{df}{dy} \right) \] (9.11)

which is zero if and only if \( f \) is holomorphic. \( \square \)

10 Automorphic Representations

References:

- Gelbart: Automorphic forms on adele groups.
- Deligne: Modular functions of one variable II.

Let \( k \geq 1 \) be an integer. Associated to \( f \in \mathcal{M}_k \) given by a function \( GL_2(\mathbb{A}) \times h^+ \to \mathbb{C} \) is a function \( \varphi : GL_2(\mathbb{A}) \to \mathbb{C} \) by

\[ \varphi(g) = \det(g_\infty)f(g_\infty, g_\infty \cdot i) j(g_\infty, i)^{-k} \] (10.1)

where \( g = g^\infty g_\infty \) with \( g^\infty \in GL_2(\mathbb{A}^\infty) \) and \( g_\infty \in GL_2(\mathbb{R}) \). The function \( \varphi \) satisfies:

1. For every \( g^\infty \in GL_2(\mathbb{A}^\infty) \), \( g_\infty \mapsto \varphi(g^\infty g_\infty) \) is smooth.
2. There exists an open compact subgroup \( U \subseteq GL_2(\mathbb{A}^\infty) \) such that for every \( g \in GL_2(\mathbb{A}) \) and \( u \in U \), \( \varphi(ug) = \varphi(g) \).
3. For every \( g \in GL_2(\mathbb{A}) \) and \( \gamma \in GL_2(\mathbb{Q}) \), \( \varphi(\gamma g) = \varphi(g) \).
4. For every \( g \in GL_2(\mathbb{A}^\infty) \) and \( z \in \mathbb{R}^+ \), \( \varphi(g(zI)) = z^{2-k}\varphi(g) \).
5. For every \( g \in GL_2(\mathbb{A}) \) and \( \theta \in \mathbb{R} \), \( \varphi(gr\theta) = e^{ik\theta}\varphi(g) \).
6. \( X_- \varphi = 0 \).
7. For every \( g^\infty \in GL_2(\mathbb{A}^\infty) \), there exist \( N, C > 0 \) such that

\[ |\varphi(g^\infty g_\infty)| \leq C\|g_\infty\|^N \] (10.2)

for every \( g_\infty \in GL_2(\mathbb{R}) \).
Proposition 10.1. The assignment \( f \sim \varphi \) gives an isomorphism of \( GL_2(\AA^\infty) \)-modules between \( \mathcal{M}_k \) and the space of \( \varphi \) satisfying properties (1) through (7). This restricts to an isomorphism between \( S_k \) and those \( \varphi \) which also satisfy

\[
\int_{\Q \setminus \A} \varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \, dx = 0
\]  

(10.3)

for every \( g \in GL_2(\AA) \).

Proof. We will check the cuspidality condition in a special case: \( f \in \mathcal{M}_k(GL_2(\Z)) \) and \( a \in \R_{>0} \). Then

\[
\int_{\Q \setminus \A} \varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) \, dx = \int_{\Q \setminus \A} \varphi \left( \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \right) \, dx
\]

(10.4)

\[
= \int_{\Q \setminus \A} a^k f \left( \begin{pmatrix} 1 & x^\infty \\ 0 & 1 \end{pmatrix}, x_\infty + a^2 i \right) \, dx.
\]

(10.5)

\( f \) is invariant under \( GL_2(\Z) \), so the integrand is invariant under \( \Z \) embedded as \( \begin{pmatrix} 1 & \frac{x}{a} \\ 0 & 1 \end{pmatrix} \). The inclusion \( \R \hookrightarrow \A \) extends to an isomorphism \( \Z \setminus \R \xrightarrow{\sim} \Q \setminus \A / \Z. \) Hence the integral equals

\[
a^k \int_0^1 f(1, x + a^2 i) \, dx = a^k a_0
\]

(10.6)

where we have

\[
f(1, \tau) = \sum_{n \geq 0} a_n q^n.
\]

(10.7)

This also produces a nice interpretation of the Petersson inner product: for \( \varphi_1, \varphi_2 : GL_2(\Q) \backslash GL_2(\AA) \to \C, \varphi_1 \varphi_2 \) is another such function. We can define the unitary (up to twist) structure on \( S_k \) by integrating \( \varphi_1(g) \varphi_2(g) |\det(g)|^{2-k} \) over \( \AA \times GL_2(\Q) \backslash GL_2(\AA), \) which has finite volume.

The space \( \mathcal{A} \) of automorphic forms on \( GL_2 \) is the space of functions \( \varphi : GL_2(\AA) \to \C \) such that:

1. For every \( \gamma \in GL_2(\Q), \varphi(\gamma g) = \varphi(g). \)
2. For every \( g^\infty \in GL_2(\AA^\infty) \), the function \( g^\infty \mapsto \varphi(g^\infty g_\infty) \) is smooth.
3. The space spanned by the right \( K \)-translates of \( \varphi \) is finite-dimensional (we say \( \varphi \) is right \( K \)-finite). Here \( K = O(2) \subseteq GL_2(\R). \)
4. There exists an open compact subgroup \( U \subseteq GL_2(\AA^\infty) \) such that for every \( g \in GL_2(\AA), \varphi(gu) = \varphi(g). \)
5. \( Z(g) \cdot \varphi \) is finite dimensional.
6. For every \( g^\infty \in GL_2(\mathbb{A}^\infty) \), there exists \( N \) and \( C \) positive such that

\[
|\varphi(g^\infty g_\infty)| \leq C\|g_\infty\|^N
\]

for every \( g_\infty \in GL_2(\mathbb{R}) \).

The space \( \mathcal{A}_0 \subseteq \mathcal{A} \) of cusp forms is defined by the further condition

\[
\int_{\mathbb{Q}\setminus \mathbb{A}} \varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \, dx = 0.
\]  

(10.9)

Remark. 1. \( \mathcal{A} \) is a smooth \( GL_2(\mathbb{A}^\infty) \)-module, with this group acting by right translation.

2. The space \( \mathcal{A} \) depends on \( K \), and \( GL_2(\mathbb{R}) \) does not act by right translation; the conjugates of \( K \) are not commensurable with \( K \).

A \((\mathfrak{g}_C, K)\)-module \( V \) is a \( \mathbb{C} \)-vector space which is a \( K \)-module and a \( \mathfrak{g}_C \)-module such that:

1. Every \( v \in V \) is \( K \)-finite: the space \( W_v = \mathbb{C}[K] \cdot v \subseteq V \) is finite dimensional, and \( K \to GL(W_v) \) is continuous (therefore analytic and semisimple).

2. For every \( x \in \text{Lie } K = \mathfrak{k} \) and \( v \in V \),

\[
X_v = \frac{d}{dt}(\exp(tX)v)|_{t=0}.
\]

(10.10)

For every \( X \in \mathfrak{g}_C, k \in K, \) and \( v \in V \),

\[
k \cdot X \cdot v = (\text{Ad}(k) \cdot X) \cdot k \cdot v.
\]

(10.11)

We say that \( V \) is admissible if for every \( \sigma : K \to GL_n(\mathbb{C}) \) continuous and irreducible, the isotypic subspace \( V[\sigma] \subseteq V \) is finite dimensional.

We say that \( V \) is irreducible if it is algebraically irreducible: that is, there exist no proper \((\mathfrak{g}_C, K)\)-submodules.

Remark. \( \mathcal{A} \) is a \((\mathfrak{g}_C, K)\)-module. There are two things to check:

1. The action of \( \mathfrak{g}_C \) preserves right \( K \)-finiteness.

2. The action of \( \mathfrak{g}_C \) preserves the property of moderate growth at \( \infty \).

Roughly: If \( f \in C^\infty(GL_2(\mathbb{R})) \) is \( K \)-finite and \( Z(\mathfrak{g}) \)-finite, then there exists \( \alpha \in C_c^\infty(GL_2(\mathbb{R})) \) such that \( f \star \alpha = f \). For every \( X \in \mathfrak{g}_C \),

\[
X f = X f \star \alpha = f \star X \alpha.
\]

(10.12)

A \((\mathfrak{g}_C, K) \times GL_2(\mathbb{A}^\infty)\)-module \( V \) is a \( \mathbb{C} \)-vector space endowed with the structure of a \((\mathfrak{g}_C, K)\)-module and a commuting smooth action of \( GL_2(\mathbb{A}^\infty) \). We say that \( V \) is admissible if for every open compact subgroup \( U \subseteq GL_2(\mathbb{A}^\infty) \) and every continuous irreducible representation \( \sigma : U \times K \to GL_n(\mathbb{C}) \), the isotypic subspace \( V[\sigma] \subseteq V \) is finite-dimensional. We say that \( V \) is irreducible if it is algebraically irreducible.
**Proposition 10.2.** Let \( \pi \) be an irreducible admissible \((g_{\mathbb{C}}, K) \times GL_2(\mathbb{A}_\infty)\)-module. Then there exist \( \pi_\infty \), an irreducible admissible \( (g_{\mathbb{C}}, K) \) and for all primes \( p \), an irreducible admissible \( GL_2(\mathbb{Q}_p) \)-module \( \pi_p \), almost all of which are unramified, and an isomorphism

\[
\pi \cong \left( \bigotimes_p \pi_p \right) \otimes_{\mathbb{C}} \pi_\infty
\]

(10.13)

**Proof.** Can be found in Flath, Corvallis I. \( \square \)

**Theorem 10.3.**

1. The spaces \( A \) and \( A_0 \) are naturally \((g_{\mathbb{C}}, K) \times GL_2(\mathbb{A}_\infty)\)-modules.

2. For every \( v \in A \) (or \( A_0 \)), \( v \) generates an admissible \( (g_{\mathbb{C}}, K) \times GL_2(\mathbb{A}_\infty)\)-module.

3. \( A_0 \) is semisimple.

An automorphic representation of \( GL_2(\mathbb{A}) \) is an irreducible \((g_{\mathbb{C}}, K) \times GL_2(\mathbb{A}_\infty)\)-module which is isomorphic to a subquotient of \( A \). It is cuspidal if it is isomorphic to a subquotient of \( A_0 \).

**Remark.** Every automorphic representation is admissible.

### 11 Discrete Series

**Proposition 11.1.** Let \( k \geq 1 \) Define a \((g_{\mathbb{C}}, K)\)-module \( D_{k-1} \) as follows: as a vector space,

\[
D_{k-1} = \bigoplus_{\substack{n = k \pmod{2} \\mid \n \geq k}} \mathbb{C} \cdot e_n.
\]

(11.1)

As a \( K \)-module,

\[
r_\theta \cdot e_n = e^{i\theta} e_n
\]

(11.2)

\[
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot e_n = e_{-n}.
\]

(11.3)

As a \( g_{\mathbb{C}} \)-module,

\[
X_+ \cdot e_n = (k + n) \cdot e_{n+2}
\]

(11.4)

\[
X_- \cdot e_n = (k - n) \cdot e_{n-2}
\]

(11.5)

\[
Z \cdot e_n = (2 - k) \cdot e_n
\]

(11.6)

and \( W \cdot e_n \) is obtained by differentiating the action of \( K^0 \).

Then \( D_{k-1} \) is an irreducible admissible \((g_{\mathbb{C}}, K)\)-module. Conversely, if \( f \in C^\infty(GL_2(\mathbb{R})) \) satisfies \( X_- f = 0 \), and
\[ f((zI)g^{\theta}) = e^{ik\theta}z^{2-k}f(g) \] (11.8)

for every \( z \in \mathbb{R}^\times \), \( g \in GL_2(\mathbb{R}) \), and \( \theta \in \mathbb{R} \), then \( f \) generates a \((\mathfrak{g}_C, K)\)-submodule of \( C^\infty(GL_2(\mathbb{R})) \) which is isomorphic to \( D_{k-1} \).

The \( D_{k-1} \) are called discrete series.

**Corollary 11.2.** There are isomorphisms of smooth \( GL_2(\mathbb{A}^\infty) \)-modules

\[
\Hom_{(\mathfrak{g}_C, K)}(D_{k-1}, \mathcal{A}) \cong \mathcal{M}_k \\
\Hom_{(\mathfrak{g}_C, K)}(D_{k-1}, \mathcal{A}_0) \cong \mathcal{S}_k
\] (11.9) (11.10)

given by \( f \mapsto f(e_k) \). In particular, there is a bijection between irreducible \( GL_2(\mathbb{A}^\infty) \)-submodules of \( \mathcal{S}_k \) and irreducible \((\mathfrak{g}_C, K) \times GL_2(\mathbb{A}^\infty)\)-submodules \( \pi \) of \( \mathcal{A}_0 \) such that \( \pi_\infty \cong D_{k-1} \).

Using principal series at \( \infty \) instead of discrete series yields maass forms instead.

## 12 Reductive Groups

Let \( F \) be a field of characteristic zero. We will take a variety \( X \) over \( F \) to mean a reduced scheme of finite type. We denote the \( F \)-points by \( X(F) \), and more generally the \( S \)-points by \( X(S) \), for any \( F \)-scheme \( S \).

An algebraic group \( G \) over \( F \) is a variety, along with morphisms \( m : G \times G \to G \) and \( i : G \to G \) and a point \( e \in G(F) \) making \( G \) a group object in the category of varieties.

A representation of \( G \) is a pair \((\pi, V)\), where \( V \) is a finite dimensional \( F \)-vector space and \( \pi : G \to GL(V) \) is a homomorphism of algebraic groups. \( G \) is said to be linear if it admits a faithful representation.

A linear algebraic group \( G \) is reductive if it is geometrically connected and every representation of \( G \) is semisimple. A basic example of a reductive group is \( GL(V) \).

If \( F \) is a number field and \( G/F \) is reductive, then automorphic forms live on \( G(F) \setminus G(\mathbb{A}_F) \).

### 12.1 Solvable and Unipotent Groups

For now, we will assume that \( F \) is algebraically closed of characteristic 0.

A torus \( T \) is a linear algebraic group such that \( T \cong \mathbb{G}_m^n \) for some \( n \geq 1 \). In this case, we define the character and cocharacter groups

\[
X^*(T) = \Hom(T, \mathbb{G}_m) \\
X_s(T) = \Hom(\mathbb{G}_m, T)
\] (12.1) (12.2)

free abelian groups of rank \( n \).
Lemma 12.1.  

1. The natural pairing

\[ X^*(T) \times X_*(T) \to \text{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z} \quad (12.3) \]

is a perfect pairing. The assignment \( T \mapsto X^*(T) \) is an equivalence of categories between the category of \( F \)-tori and the category of finitely generated free abelian groups.

2. Every irreducible representation of a torus \( T \) is 1-dimensional, and every representation of \( T \) is semisimple (so \( T \) is reductive).

If \( G \) is any linear algebraic group, let \( DG \) be the derived group of \( G \), the intersection of all closed normal subgroups \( N \) such that \( G/N \) is abelian. \( DG \) is a closed algebraic subgroup of \( G \). The derived central series is then

\[ G \supseteq DG \supseteq DDG \supseteq \cdots \quad (12.4) \]

We say that \( G \) is soluble (or solvable) if this series terminates in the trivial group.

To define unipotent groups, recall the Jordan decomposition: if \( V \) is a finite dimensional vector space and \( g \in GL(V)(F) \), then there exist unique commuting elements \( g_s, g_u \in GL(V)(F) \), with \( g_s \) diagonalizable and \( g_u \) unipotent, such that \( g = g_s g_u \).

Theorem 12.2. Let \( G \) be a linear algebraic group and \( g \in G(F) \). Then there exist unique commuting elements \( g_s \) and \( g_u \) of \( G(F) \) such that:

1. \( g = g_s g_u \).
2. For every representation \( (\pi, V) \) of \( G \), \( \pi(g_s) \) is semisimple and \( \pi(g_u) \) is unipotent.

A linear algebraic group \( G \) is unipotent if every \( g \in G(F) \) is unipotent.

Proposition 12.3.  

1. Every connected soluble group \( G \) admits a faithful representation \( \pi : G \to GL(V) \) with image contained in the subgroup of upper triangular matrices.

2. Every unipotent group \( U \) admits a faithful representation \( \pi : G \to GL(V) \) with image contained the subgroup of unipotent upper triangular matrices.

3. Every connected soluble group \( G \) contains a unique normal unipotent group \( U \) such that \( G/U \) is a torus. In other words, there is a torus \( T \) such that

\[ 1 \to U \to G \to T \to 1 \quad (12.5) \]

is exact on \( F \)-points.

Let \( G \) be a linear algebraic group. The radical \( RG \) is the maximal connected normal solvable subgroup. The unipotent radical \( RuG \) is the unipotent part of \( RG \).

Fact. If \( G \) is a connected linear algebraic group, then \( G \) is reductive if and only if \( RuG \) is trivial. (This fails if \( F \) has characteristic \( p \). In this case, reductive is defined as \( RuG \) being trivial.)
12.2 Structure Theory

Let $G$ be a reductive group. We say that a torus $T \subseteq G$ (as a closed subgroup) is maximal if it is not contained in any strictly larger torus.

**Proposition 12.4.** Let $T \subseteq G$ be a torus. Then:

1. The centralizer $Z_G(T)$ is reductive (and connected).
2. $N_G(T)^0 = Z_G(T)$, so $N_G(T)/Z_G(T)$ is finite.
3. $T$ is maximal if and only if $Z_G(T) = T$.

The group $W(G,T) = N_G(T)/Z_G(T)$ is the Weyl group of $(G,T)$.

Fix a maximal torus $T \subseteq G$. $G$ has a natural representation: if $g = \text{Lie } G = T_e G$, then we obtain a representation $\text{Ad} : G \to GL(\mathfrak{g})$ given by conjugation and differentiation. Restricting the adjoint representation to $T$, we obtain the Cartan decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha$$  \hspace{1cm} (12.6)

into eigenspaces with eigenvalues characters of $T$.

**Proposition 12.5.** $\mathfrak{g}_0 = \text{Lie } T$, and each $\mathfrak{g}_\alpha$ has dimension either 0 or 1.

The elements $\alpha \in X^*(T)$ such that $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$ are called roots of $(G,T)$. We write $\Phi \subseteq X^*(T)$ for the set of roots.

If $\alpha$ is a root, we define

$$T_\alpha = \ker(\alpha)^0 \subseteq T.$$  \hspace{1cm} (12.7)

Then $G_\alpha = Z_G(T_\alpha)$ is a reductive subgroup of $G$ such that $T \subseteq G_\alpha$ is a maximal torus.

**Proposition 12.6.** $W(G_\alpha,T)$ contains a unique nontrivial element $s_\alpha$.

$W = W(G,T)$ acts on $T$ by conjugation, and therefore on $X^*(T)$ and $X_*(T)$ by functoriality. There exists a unique $\alpha^\vee \in X_*(T)$ such that

$$s_\alpha(x) = x - \langle \alpha^\vee, x \rangle \alpha$$  \hspace{1cm} (12.8)

for every $x \in X^*(T)$.

$\alpha^\vee$ is called the coroot associated to the root $\alpha$. We write $\Phi^\vee \subseteq X_*(T)$ for the set of coroots. We have a canonical bijection between $\Phi$ and $\Phi^\vee$.

We now have a tuple

$$(X^*(T), \Phi, X_*(T), \Phi^\vee)$$  \hspace{1cm} (12.9)

which is a root datum.
A root datum is a tuple \((M, \Psi, M^\vee, \Psi^\vee)\) with a bijection between \(\Psi\) and \(\Psi^\vee\) and a perfect pairing \(\langle , \rangle :: M \times M^\vee \to \mathbb{Z}\), where \(M\) and \(M^\vee\) and finitely generated free abelian groups, and \(\Psi \subseteq M \setminus \{0\}\) and \(\Psi^\vee \subseteq M^\vee \setminus \{0\}\) are finite subsets, such that:

1. For every \(\alpha \in \Psi\), \(\langle \alpha, \alpha^\vee \rangle = 2\).

2. For every \(\alpha \in \Psi\), the automorphism \(s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha\) of \(M\) leaves \(\Psi\) invariant.

3. The subgroup of \(\text{Aut}(M)\) generated by the \(s_\alpha\) for \(\alpha \in \Psi\) is finite.

A simple example is \(G = SL(2)\). Then \(\mathfrak{g} = \text{sl}(2)\) consists of the traceless \(2 \times 2\) matrices, and the adjoint representation is conjugation. We can take

\[T = \left\{ s(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right\}\] (12.10)

Since \(Z_G(T) = T\), \(T\) is a maximal torus. \(X^*(T)\) and \(X_*(T)\) are then isomorphic to \(\mathbb{Z}\).

\(\mathfrak{g}\) is spanned by

\[H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\] (12.11)
\[E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\] (12.12)
\[F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\] (12.13)

Then \(H\) spans \(\text{Lie} T\), and we have

\[\text{Ad}(s(t))(E) = t^2 E\] (12.14)
\[\text{Ad}(s(t))(F) = t^{-2} F.\] (12.15)

The Cartan decomposition is

\[\mathfrak{g} = \langle H \rangle \oplus \left( \bigoplus_{\alpha \neq 0} \langle E \rangle \oplus \langle F \rangle \right).\] (12.16)

Using additive notation for the group \(X^*(T)\), we then have

\[\alpha(s(t)) = t^2\] (12.17)
\[(-\alpha)(s(t)) = t^{-2}.\] (12.18)

We have \(\Phi = \{\alpha, -\alpha\} \subseteq X^*(T) \cong \mathbb{Z}\).
To find the coroots, take \( G_\alpha = Z_G(T_\alpha) \) and \( T_\alpha = \ker(\alpha)^0 \). Since \( \ker \alpha = \{ (\pm 1) \} \), \( T_\alpha = 1 \) and \( G_\alpha = G \).

A representative for the nontrivial element of \( W(G_\alpha, T) \) is \( w = (1 - 1) \), and we have

\[
ws(t)w^{-1} = s(t)^{-1}.
\]

(12.19)

So \( s_\alpha \) acts as \(-1\) on \( X^*(T) \), implying \( s_\alpha(x) = -x \). Check that \( \langle \alpha, \alpha^\vee \rangle = 2 \). The root datum of \( SL(2) \) is then

\[
(\mathbb{Z}, \{ 2, -2 \}, \mathbb{Z}, \{ 1, -1 \}).
\]

(12.20)

Now consider \( G = GL(n) \), \( T \) the subgroup of diagonal matrices (a maximal torus). \( N_G(T) \) consists of the monomial matrices (matrices with exactly one nonzero entry in each row and column). \( W(G, T) \) is isomorphic to \( S_n \).

Let \( e_i \in X^*(T) \) be the character which extracts the \( i \)th diagonal entry. Then the \( e_i \) form a basis of \( X^*(T) \). The Cartan decomposition is

\[
g = \text{Lie} T \oplus \bigoplus E_{ij}
\]

where if \( 1 \leq i, j \leq n \) and \( i \neq j \), then \( E_{ij} \) is the matrix which has a 1 on the \( (i \)th row, \( j \)th column) entry and 0 everywhere else. We have

\[
\text{Ad}(t)E_{ij} = (e_i - e_j)(t)(E_{ij})
\]

(12.22)

so that \( \Phi = \{ e_i - e_j | i \neq j \} \).

To find the coroots, consider \( \alpha = e_1 - e_2 \). Then \( \ker \alpha \) consists of diagonal matrices with first two entries equal (so \( T_\alpha \) is also this subgroup). Then \( G_\alpha \) consists of block diagonal matrices with first block of size 2 and the rest of size 1. \( W(G_\alpha, T) \) has a single nontrivial element, represented by \( (1 - 1) \) on the \( 2 \times 2 \) block and the identity elsewhere. Conjugation by \( w \) switches the first two diagonal entries, so \( s_\alpha \in W(G, T) \cong S_n \) is the transposition \( (12) \). More generally, if \( \alpha = e_i - e_j \), then \( s_\alpha = (ij) \). Check that the associated coroot is the cocharacter mapping \( t \) to the diagonal matrix which is \( t \) in the \( i \)th diagonal entry, \( t^{-1} \) in the \( j \)th, and 1 in the others.

The root datum of \( GL(n) \) is then

\[
(\mathbb{Z}^n = \langle e_i \rangle, \Phi, \mathbb{Z}^n = \langle e_i \rangle, \Phi^\vee)
\]

(12.23)

where we have

\[
\Phi = \{ e_i - e_j | i \neq j \}
\]

(12.24)

\[
\Phi^\vee = \{ e_i - e_j | i \neq j \}
\]

(12.25)

**Theorem 12.7.** For \( G \) a reductive group:

1. \( G \) contains a unique \( G \)-conjugacy class of maximal tori \( T \). So the root datum of \( (G, T) \) is independent of the choice of \( T \).
2. **The root datum of** $G**$ **determines** $G**$ **up to isomorphism.**

3. **Every abstract root datum is equivalent to the root datum of a reductive group.**

Here are some more examples of reductive groups:

- $A_n = SL(n + 1)$.
- $B_n = SO(2n + 1)$.
- $C_n = Sp(2n)$.
- $D_n = SO(2n)$.
- The exceptional groups $E_6, E_7, E_8, F_4, G_2$.

These are the simple algebraic groups. Every reductive group is built from these.

Let $G$ be a reductive group.

1. A **Borel subgroup** $B \subseteq G$ is a maximal connected solvable subgroup.

2. A **parabolic subgroup** $P \subseteq G$ is a subgroup such that $G/P$ is projective as an algebraic variety.

**Theorem 12.8.**

1. $G$ contains a unique conjugacy class of Borel subgroups.

2. A subgroup $P \subseteq G$ is parabolic if and only if it contains a Borel subgroup.

As an example, take $G = GL_n(F) = GL(F^n)$. Fix a partition $n = n_1 + \cdots + n_r$ into positive integers, and let $P \subseteq G$ be the block upper triangular subgroup with blocks of size $n_1, \ldots, n_r$. Then $G/P$ is the Grassmannian of filtrations

$$0 \subseteq \text{Fil}^1 \subseteq \text{Fil}^2 \subseteq \cdots \subseteq \text{Fil}^r = F^n$$

such that $\dim_F \text{Fil}^i/\text{Fil}^{i-1} = n_i$. For $G$ acts on this space, and $P$ is the stabilizer of the standard filtration.

The situation $n_i = 1$ for every $i$ gives the subgroup of upper triangular matrices, which is solvable (therefore a Borel subgroup).

Fix a maximal torus $T \subseteq G$. Let $(M, \Phi, M^\vee, \Phi^\vee)$ be the root datum of $G$. A **root basis** of $(M, \Phi, M^\vee, \Phi^\vee)$ is a subset $S \subseteq \Phi$ such that every root $\beta \in \Phi$ can be expressed uniquely as a sum

$$\beta = \sum_{\alpha \in S} n_\alpha \alpha$$

where the $n_\alpha \in \mathbb{Z}$ and are either all nonnegative or all nonpositive.

**Remark.** 1. Root bases always exist.
2. Upon fixing a root basis $S$, we get a decomposition $\Phi = \Phi^+ \uplus \Phi^-$ into positive and negative roots. Here

$$\Phi^+ = \left\{ \beta = \sum_{\alpha \in S} n_\alpha \alpha, n_\alpha \geq 0 \right\}$$  \hspace{1cm} (12.28)

$S$ is called the set of simple roots.

**Proposition 12.9.** Given $T$ a maximal torus of $T$, there is a canonical bijection between the set of root bases and the set of Borel subgroups of $G$ containing $T$, such that

$$\text{Lie } B = \text{Lie } T \oplus \bigoplus_{\alpha \in \Phi^+} g_\alpha.$$  \hspace{1cm} (12.29)

Under this bijection, $W = W(G,T)$ acts simply transitively on either side.

In the case where $G = GL_n$ and $T$ is the subgroup of diagonal matrices, if $e_i \in X^*(T)$ maps a diagonal matrix to its $i$th diagonal entry, then $\Phi = \{ e_i - e_j | i \neq j \}$.

**Fact.** Borel subgroups $B \subseteq G$ containing $T$ are in bijection with orderings of the standard basis $v_1, \ldots, v_n$ of $F^n$.

If $\sigma \in S_n$, we can define a filtration $\text{Fil}_{\sigma}^i$ by

$$\text{Fil}_{\sigma}^i = \langle v_{\sigma(1)}, \ldots, v_{\sigma(i)} \rangle.$$  \hspace{1cm} (12.30)

The stabilizer $B_\sigma$ of this flag in $G$ is then a Borel subgroup.

Let $B = \text{stab}(\text{Fil}_{1}^i)$, the subgroup of upper triangular matrices. Let

$$S = \{ e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n \}.$$  \hspace{1cm} (12.31)

Then $S$ is a root basis, with $\Phi^+ = \{ e_i - e_j | i < j \}$. Lie $B$ is the subalgebra of upper triangular matrices, so decomposes as

$$\text{Lie } B = \text{Lie } T \oplus \bigoplus_{i < j} (E_{ij}).$$  \hspace{1cm} (12.32)

Thus $B$ corresponds to $S$ under the bijection.

A based root datum is a tuple $(M, \Phi, S, M^\vee, \Phi^\vee, S^\vee)$, where $(M, \Phi, M^\vee, \Phi^\vee)$, where $S \subseteq \Phi$ is a root basis and $S^\vee \subseteq \Phi^\vee$ is the image of $S$ under the bijection between $\Phi$ and $\Phi^\vee$.

If $G$ is reductive and $T \subseteq B \subseteq G$ are a maximal torus and a Borel subgroup, then we get a based root datum.

If $P \subseteq G$ is a proper parabolic subgroup of $G$, then it is not reductive, but there exists an exact sequence

$$1 \rightarrow R_u P \rightarrow P \rightarrow L \rightarrow 1$$  \hspace{1cm} (12.33)
where \( L \) is reductive. In fact, this sequence is always split. There exist closed reductive subgroups \( L' \subseteq P \) such that \( R_u P \times L' \to P \) is an isomorphism mapping \( L' \) to \( L \). This \( L' \) is unique up to conjugation by \( R_u P \). Any such \( L' \) is called a Levi subgroup, and the decomposition \( P = R_u P \times L' \) is called a Levi decomposition.

12.3 Classification

A reductive group \( G \) is semisimple if \( RG \) is trivial. A semisimple group \( G \) is almost simple if it has no normal subgroup of dimension at least 1, and simple if it has no nontrivial normal subgroup.

In general, if \( G \) is a reductive group, then \( RG = Z(G)^0 \) is a torus, and the derived group \( DG \) is semisimple. Moreover, the natural homomorphism \( DG \times RG \to G \) is surjective with finite kernel. This reduces the classification problem to that of semisimple groups along with finite central subgroups.

An isogeny of semisimple groups is a surjective homomorphism \( f : G \to G' \) with finite (central) kernel.

If \( G \) is semisimple, we say that \( G \) is:

- simply connected if there is no proper isogeny \( f : G' \to G \).
- adjoint if there is no proper isogeny \( f : G \to G' \). Equivalently, \( G \) has trivial center.

If \( G \) is any semisimple group, then there exist normal, pairwise commuting almost simple subgroups \( G_1, \ldots, G_r \subseteq G \) such that the homomorphism \( G_1 \times \cdots \times G_r \to G \) is an isogeny.

The relation of isogeny generates an equivalence relation on the set of semisimple groups \( G \); the equivalence classes are called isogeny classes.

Let \( G \) be a semisimple group, \( T \subseteq G \) a maximal torus, and \( (M, \Phi, M^\vee, \Phi^\vee) \) the root datum of \( (G, T) \). Let \( f : G' \to G \) be an isogeny, and \( T' = f^{-1}(T)^0 \subseteq G' \). Then \( T' \) is a maximal torus, and the map \( f|_{T'} : T' \to T \) is an isogeny of tori. This gives a map between character groups \( f|_{T'} : X^*(T) \to X^*(T') \), which becomes an isomorphism after tensoring with \( \mathbb{Q} \). We have

\[
Z\Phi \subseteq M \subseteq (Z\Phi^\vee)^* \subseteq M \otimes \mathbb{Q}.
\] (12.34)

Here \( Z\Phi \) is the span of the roots in \( M \), which, for \( G \) semisimple, is a finite index submodule of \( M \). Also

\[
(Z\Phi^\vee)^* = \{ m \in M \otimes \mathbb{Q} | \forall \alpha \in \Phi^\vee, \langle \alpha, m \rangle \in \mathbb{Z} \}.
\] (12.35)

Then \( M \subseteq (Z\Phi^\vee)^* \) has finite index.

If we define

\[
N = (f|_{T'})^{-1} X^*(T') \subseteq X^*(T) \otimes \mathbb{Q} = M \otimes \mathbb{Q}
\] (12.36)

then \( (N, \Phi, N^\vee, \Phi^\vee) \) is a root datum, so \( Z\Phi \subseteq N \subseteq (Z\Phi^\vee)^* \) inside \( M \otimes \mathbb{Q} \).

The group \((Z\Phi^\vee)^*/(Z\Phi)\) is finite, and given an isogeny \( f : G' \to G \), we get a subgroup \( N/(Z\Phi) \). More generally, if \( G'' \) is any group in the isogeny class of \( G \), we get a subgroup of \((Z\Phi^\vee)^*/(Z\Phi)\).
Theorem 12.10. If $G$ is a semisimple group, then there is an order-preserving bijection between isogeny classes of $G$ and subgroups $(\mathbb{Z}\Phi^\vee)/\mathbb{Z}\Phi)$, by $G' \mapsto N/(\mathbb{Z}\Phi)$.

It remains to understand the isogeny classes of almost simple groups.

If $G$ is an almost simple group, we associate to it a graph, called the Dynkin diagram of $G$. Let $(M, \Phi, M^\vee, \Phi^\vee)$ be the root datum of $G$. Fix a root basis $S \subseteq \Phi$. The Dynkin diagram is defined as follows:

- The vertex set is given by $S$.
- If $\alpha, \beta \in S$, define $n(\beta, \alpha) = \langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$. One can show that
  \[ n(\beta, \alpha)n(\alpha, \beta) \in \{0, 1, 2, 3\}. \]  
(12.37)

Join $\alpha$ and $\beta$ by $n(\beta, \alpha)n(\alpha, \beta)$ edges. If the number of edges is 2 or 3, then WLOG let $|n(\beta, \alpha)| > |n(\alpha, \beta)|$ (the absolute values will be different), and then draw an arrow pointing from $\beta$ to $\alpha$.

Theorem 12.11. For $G$ almost simple:

1. The Dynkin diagram of $G$ is connected.
2. If $G'$ is another almost simple group, then $G$ and $G'$ are in the same isogeny class if and only if they have the same Dynkin diagram.

12.4 Automorphisms

Let $G$ be a reductive group. We have an exact sequence

\[ 1 \to \text{Inn}(G) \to \text{Aut}(G) \to \text{Out}(G) \to 1 \]  
(12.38)

We consider $\text{Aut}(G)$ as the group of automorphisms of $G$ as an algebraic group over $F$.

A pinning of $G$ is a tuple $(T, B, \{X_\alpha\}_{\alpha \in S})$ where $T$ is a maximal torus, $B$ is a Borel subgroup, $S \subseteq \Phi(G, T)$ is the root basis of $B$, and for $\alpha \in S$, $X_\alpha$ is a basis of $g_\alpha$.

Let $\mathcal{P}$ be a pinning, and $\mathcal{R}$ be the based root datum of $\mathcal{P}$. Define $\text{Aut}(G, \mathcal{P}) \subseteq \text{Aut}(G)$ consisting of those $\varphi : G \to G$ such that $\varphi$ preserves $T$, $B$, and the set $\{X_\alpha\}$ (up to permutation).

If $R = (M, \Phi, S, M^\vee, \Phi^\vee, S^\vee)$, define $\text{Aut}(R)$ to be the set of $f \in GL(M)$ such that $f(\Phi) = \Phi$ and $f(S) = S$. Then we have maps

\[ \text{Aut}(G, \mathcal{P}) \quad \text{Out}(G) \quad \text{Aut}(\mathcal{R}) \]  
(12.39)
Theorem 12.12.  1. These maps are isomorphisms, so that

\[
\text{Aut}(G) \cong \text{Inn}(G) \rtimes \text{Aut}(G, \mathcal{P}) \quad (12.40)
\]

\[
\cong \text{Inn}(G) \rtimes \text{Aut}(\mathcal{R}). \quad (12.41)
\]

2. All pinnings of $G$ are $\text{Inn}(G)$-conjugate.

13 The Non-algebraically Closed Case

References: Linear Algebraic Groups (Borel, Springer, Humphreys - three different books), and Springer - Corvallis I.

Let $F$ be an arbitrary field of characteristic $0$ and $\overline{F}$ an algebraic closure. Let $G$ be a linear algebraic group over $F$. We say that $G$ is reductive, semisimple, unipotent, soluble, or a torus if $G$ satisfies the same property over $\overline{F}$.

The Galois group $\Gamma_F = \text{Gal}(\overline{F}/F)$ acts on $G(\overline{F})$ with fixed points $G(F)$.

A torus $T$ is a group over $F$ which becomes isomorphic to $\mathbb{G}_m^n$ over $\overline{F}$ for some $n \geq 0$. $T$ is said to be split if there exists an isomorphism $T \cong \mathbb{G}_m^n$ defined over $F$. If $T$ is any torus over $F$, define

\[
X^*(T) = \text{Hom}(T/F, \mathbb{G}_m/F). \quad (13.1)
\]

This is a finitely generated free abelian group with an action of $\Gamma_F$.

**Theorem 13.1.** The assignment $T \mapsto X^*(T)$ defines an equivalence of categories between the category of tori over $F$ and the category of finitely generated free abelian groups with smooth $\Gamma_F$-action.

Hence isomorphism classes of tori over $F$ are in bijection with conjugacy classes of homomorphisms $\Gamma_F \to GL_n(\mathbb{Z})$. If $\rho : \Gamma_F \to GL_n(\mathbb{Z})$ is the associated representation, then we may take

\[
T(F) \cong (\overline{F}^\times \otimes_{\mathbb{Z}} \text{Hom}(\mathbb{Z}^n, \mathbb{Z}))^{\Gamma_F} \quad (13.2)
\]

where $\Gamma_F$ acts diagonally.

As an example, consider the case $n = 1$, so $GL_1(\mathbb{Z}) = \{\pm 1\}$. Homomorphisms $\rho$ then consist of quadratic extensions $K/F$, along with $F \times F$. If $K/F$ is a quadratic extension with associated torus $T$, and the bar denotes the nontrivial element of the Galois group, then

\[
T(F) = (\overline{F}^\times \otimes \mathbb{Z})^{\Gamma_F} \quad (13.3)
\]

\[
= \{\lambda \in K^\times | \overline{\lambda} = \lambda^{-1}\} \quad (13.4)
\]

\[
= \{\lambda \in K^\times | N_{K/F}(\lambda) = 1\}. \quad (13.5)
\]
13.1 Split Groups

Let $G$ be a reductive group over $F$. We say that a torus $T \subseteq G$ is maximal if $T_\mathbb{F} \subseteq G_\mathbb{F}$ is maximal.

**Fact.** $G$ always contains a maximal torus.

A maximal split torus $T \subseteq G$ is a split torus, maximal with respect to being a split torus. We say $G$ is split if it contains a split maximal torus (meaning a maximal torus which happens to be split).

**Fact.** There is always a unique $G(F)$-conjugacy class of maximal split tori.

**Theorem 13.2.** 1. Let $\mathcal{R}$ be a root datum. Then there exists a reductive group $G$ over $F$ and a split maximal torus $T \subseteq G$ such that $\mathcal{R}$ is the root datum of $(G,T)$.

2. If $G$ and $G'$ are split reductive groups over $F$ with the same root datum, then $G$ and $G'$ are isomorphic over $F$.

Let $G$ be any linear algebraic group over $F$. A form of $G$ is a linear algebraic group $G'$ over $F$ such that there exists an isomorphism $G_\mathbb{F} \overset{\sim}{\to} G'_\mathbb{F}$.

**Remark.** Any reductive group $G'$ over $F$ is a form of a split reductive group over $F$. (Indeed, we can find a split group $G$ having the same root datum as $G'$.)

Let $G$ be split over $F$, and $G'$ a form of $G$. Fix an isomorphism $f : G_\mathbb{F} \overset{\sim}{\to} G'_\mathbb{F}$. For any $\sigma \in \Gamma_F$, we obtain another isomorphism $\sigma f$. Then $f^{-1} \circ \sigma f$ is an automorphism of $G_\mathbb{F}$.

Let $A$ be a group on which $\Gamma_F$ acts (continuously, when $A$ has the discrete topology). A 1-cocycle is a map $\Gamma_F \to A$, denoted $\sigma \mapsto a_\sigma$, such that for every $\sigma, \tau \in \Gamma_F$, $a_{\sigma \tau} = a_\sigma \sigma a_\tau$. Two 1-cocycles are 1-cohomologous if there exists $c \in A$ such that $b_\sigma = c^{-1} a_\sigma c$ for every $\sigma \in \Gamma_F$. This defines an equivalence relation on the set of 1-cocycles, and we write $H^1(F,A)$ for the pointed set of equivalence classes (with distinguished element the identity cocycle).

**Remark.** 1. If $H$ is a linear algebraic group over $F$, then $H^1(F,H(\mathbb{F}))$ and $H^1(F,\text{Aut}(H(\mathbb{F})))$ are defined.

2. If $A, B$ are groups on which $\Gamma_F$ acts, and there is a $\Gamma_F$-equivariant homomorphism $A \to B$, then there is a natural map $H^1(F,A) \to H^1(F,B)$.

3. If $\Gamma_F$ acts trivially on $A$, then $H^1(F,A)$ is the set of homomorphisms $\Gamma_F \to A$, up to $A$-conjugacy.

In our above construction, $a_\sigma = f^{-1} \circ \sigma f$ is a 1-cocycle, and replacing $f$ by $f \circ h$ for $h \in \text{Aut}(G_\mathbb{F})$ does not change the cohomology class of $a_\sigma$. We obtain a map from the set of isomorphism classes of forms of $G$ to $H^1(F,\text{Aut}(G_\mathbb{F}))$, where $G$ is mapped to the distinguished point.

**Proposition 13.3.** This map is bijective.

Let $G$ be a split reductive group, and fix a pinning $\mathcal{P} = (T,B,\{X_\alpha\}_{\alpha \in S})$ where $T \subseteq G$ is split, $B \subseteq G$ is defined over $F$, and $X_\alpha \in g_\alpha$ are defined over $F$. Let $\mathcal{R}$ be the based root datum associated to $\mathcal{P}$. $\Gamma_F$ acts on the short exact sequence determining $\text{Aut}(G_\mathbb{F})$. Because $G$ is split and $\mathcal{P}$ is defined over $F$, $\Gamma_F$ acts trivially on $\text{Out}(G_\mathbb{F}) \cong \text{Aut}(\mathcal{R})$, and the splitting is $\Gamma_F$-equivariant. Hence there exists a map
and the second cohomology set is the set of homomorphisms \( \mu : \Gamma_F \to \text{Aut}(\mathcal{R}) \) up to conjugacy. In particular, if \( G' \) is a form of \( G \), we get a map \( \mu_{G'} \), unique up to conjugacy.

13.2 Quasi-Split Groups

A reductive group \( G \) over \( F \) is \textit{quasi-split} if it contains a Borel subgroup; that is, a subgroup \( B \subseteq G \) such that \( B \mathcal{F} \subseteq G \mathcal{F} \) is a Borel subgroup.

Remark. 1. Split groups are quasi-split.

2. In general, being quasi-split is a restrictive condition.

Let \( G, G' \) be reductive groups over \( F \). We say that \( G' \) is an inner form of \( G \) if there exists an isomorphism \( f : G \mathcal{F} \to G' \mathcal{F} \) such that for every \( \sigma \in \Gamma_F, f^{-1} \circ \sigma f \in \text{Inn}(G \mathcal{F}) \). Equivalently, the class \( c_{G,G'} \in H^1(F, \text{Aut}(G \mathcal{F})) \) which classifies \( G' \) lies in the image of the map \( H^1(F, \text{Inn}(G \mathcal{F})) \to H^1(F, \text{Aut}(G \mathcal{F})). \)

Lemma 13.4. If \( G \) is a reductive group over \( F \), then \( G \) has a unique quasi-split inner form.

If \( G', G'' \) are forms of the same split group \( G \), then they have the same quasi-split inner form if and only if \( \mu_{G'} \) and \( \mu_{G''} \) differ by conjugacy.

Suppose \( G \) is split. Take \( \mathcal{P} \) an \( F \)-pinning and \( \mathcal{R} \) its based root datum. We obtain \( \mu : \Gamma_F \to \text{Aut}(\mathcal{R}) \). Because the map \( \text{Aut}(G \mathcal{F}) \to \text{Out}(G \mathcal{F}) \) has a section, \( \mu \) gives an element of \( H^1(F, \text{Aut}(G \mathcal{F})) \). The corresponding form \( G_{\mu} \) of \( G \) is a quasi-split group.

“How to classify reductive groups over \( F \):”

1. From a root datum \( R \), associate a split group \( G \).

2. For \( \mu : \Gamma_F \to \text{Aut}(\mathcal{R}) \), get a quasi-split group \( G_{\mu} \).

3. Think of a general group \( G' \) as an inner form of \( G_{\mu_{G'}} \).

13.3 The Case of \( GL_n \)

Now assume that \( G = GL_n \).

A central simple algebra \( A \) over \( F \) of rank \( n \) is a unital associative algebra \( A \) with center \( F \), \( \dim_F A = n^2 \), and no nontrivial two-sided ideals.

If \( F = \overline{F} \), every central simple algebra of rank \( n \) over \( F \) is isomorphic to \( M_n(F) \). In general, there can be many more.

In the case \( n = 2 \), the central simple algebras are quaternion algebras. Choose \( a, b \in F^\times \). Then \((a, b; F)\) is the algebra over \( F \) with basis \( 1, i, j, k \) such that \( i^2 = a, j^2 = b, \) and \( k = ij = -ji \). The algebra \((-1, 1; \mathbb{R})\) is the algebra \( \mathbb{H} \) of Hamiltonian quaternions.

If \( A \) is a central simple algebra of rank \( n \), define a group \( G_A \) over \( F \) by its functor of points: for \( R \) an \( F \)-algebra, take \( G_A(R) = (R \otimes_F A)^\times \). (This functor is representable.) In particular, \( G_A(F) = A^\times \). \( G_A \) depends only on \( A \) up to isomorphism, and so
\[ G_{A,F} \cong G_{A,F} \cong G_{M_n(F)} = GL_n,F. \]  

(13.7)

Every inner form of \( GL_n \) is isomorphic to \( G_A \) for some \( A \). For example, if \( F = \mathbb{R} \) and \( n = 2 \), the inner forms are \( GL_2(\mathbb{R}) \) and \( \mathbb{H}^x \).

Now we consider the quasi-split forms. Fix a standard pinning \( \mathcal{P} \): \( T \) is the set of invertible diagonal matrices, \( B \) is the set of invertible upper triangular matrices, and \( \{ X_\alpha \} = \{ E_{i,i+1} \} \). We get a based root datum \( \mathcal{R} \).

An exercise is to show that \( \text{Aut}(\mathcal{R}) \cong \mathbb{Z}/2\mathbb{Z} \). The nontrivial element of \( \text{Aut}(\mathcal{R}) \) acts as a reflection on the Dynkin diagram (a line with \( n - 1 \) nodes).

So quasi-split forms of \( GL_n \), besides \( GL_n \) itself, correspond to quadratic extensions \( K/F \). Fix the \( n \times n \) matrix

\[
J = \begin{pmatrix}
1 & & & 1 \\
& -1 & & \\
& & 1 & \\
(-1)^{n-1} & & & \\
\end{pmatrix}
\]  

(13.8)

Define a Hermitian form \( \langle \cdot, \cdot \rangle \) on \( K^n \) by \( \langle x, y \rangle = t^x J y \), where the bar is the nontrivial element of \( \text{Gal}(K/F) \). We define a group \( G^K \) by its functor of points: if \( R \) is an \( F \)-algebra,

\[
G^K(R) = \{ g \in GL_n(R \otimes_F K) | t^g J g = J \}. 
\]  

(13.9)

In particular,

\[
G^K(F) = \{ g \in GL_n(K) | t^g J g = J \}. 
\]  

(13.10)

\( G^K \) becomes isomorphic to \( GL_n \) over \( K \) via \( K \otimes_F K \cong K \times K \). If \( R \) is a \( K \)-algebra, then \( K \otimes_F T \cong R \times R \). Moreover, the automorphism \( x \otimes y \mapsto x \otimes \overline{y} \) of \( K \otimes_F K \) induces \( (x,y) \mapsto (y,x) \) of \( R \times R \). Then

\[
G^K(R) = \{ (g_1, g_2) \in GL_n(R) \times GL_n(R) | t^{g_1} J g_2 = J \} 
\]  

(13.11)

shows that \( G^K_R \cong GL_{n,K} \). Also, with this choice, the upper triangular subgroup of \( G^K \) is a Borel subgroup, so \( G^K \) is quasi-split.

Now we want to know the inner forms of \( G^K \). Take a pair \( (B, \ast) \) where \( B \) is a central simple algebra over \( K \) of rank \( n \) and \( \ast \) is an anti-involution of \( B \) that acts on \( K \subseteq B \) as \( x \mapsto \bar{x} \). For example, we could take \( B = M_n(K) \) and \( g^\ast = \overline{J^{-1}} J g \).

Define a group \( G_{(B, \ast)} \) by its functor of points:

\[
G_{(B, \ast)}(R) = \{ g \in (R \otimes_F B)^\times | g^\ast g = 1 \}. 
\]  

(13.12)

It is an exercise to show that \( G_{(B, \ast)} \) is a form of \( GL_n \), and in fact an inner form of \( G^K \).

Fact. Every inner form of \( G^K \) arises in this way.
14 Measures on Groups

Reference: Borel-Jacquet, Corvallis I.

Let \( F \) be a number field and \( G \) be a reductive group over \( F \), with a given embedding \( G \hookrightarrow GL_{n,F} \). For any place \( v \) of \( F \), give \( G(F_v) \) the subspace topology from \( GL_n(F_v) \). Here \( GL_{n,F} \) is topologized by embedding it into \( M_{n,F} \times \mathbb{A}^1_F \) by

\[
g \mapsto (g, \det(g^{-1})). \tag{14.1}
\]

For \( v \) a finite place, define \( K_v = G(F_v) \cap GL_n(O_{F_v}) \), an open compact subgroup of \( G(F_v) \). A different choice of embedding \( G \hookrightarrow GL_{n',F} \) will have \( K_v = K'_v \) for almost all \( v \). Now we define

\[
G(\mathbb{A}_F) = \prod'_v G(F_v) \tag{14.2}
\]

the restricted product being with respect to the \( K_v \). This is also the subspace topology from \( GL_n(\mathbb{A}_F) \).

If \( H \) is a Hausdorff locally compact topological group, then it has a Haar measure: a functional \( \mu : C_c(H) \to \mathbb{C} \) satisfying:

1. If \( f \in C_c(H) \) and \( f \geq 0 \), then

\[
\mu(f) = \int_{h \in H} f(h) \, dh \geq 0. \tag{14.3}
\]

2. \( \int_{h \in H} f(gh) \, dh = \int_{h \in H} f(h) \, dh. \)

Haar measures always exist, and are unique up to a positive real multiple.

There is a continuous homomorphism \( \delta_H : \to \mathbb{R}_{>0} \), called the modular character, such that for all \( g \in H \),

\[
\delta_H(g) \int_H f(h) \, dh = \int_H f(h) \, dh. \tag{14.4}
\]

If \( \delta_H \) is trivial, we say that \( H \) is unimodular. If \( G \) is reductive over \( F \) or unipotent over \( F \), then \( G(F_v) \) and \( G(\mathbb{A}_F) \) are unimodular groups.

If \( H \) is a unimodular locally compact group and \( Z \subseteq H \) is a closed unimodular subgroup, then there exists a Haar integral \( \mu \) on \( Z \setminus H \) which is right \( H \)-invariant. That is to say, there is a functional \( \mu : C_c(Z \setminus H) \to \mathbb{C} \) such that:

1. \( \mu \) is nonnegative (and not trivial).

2. For every \( h \in H \),

\[
\int_{x \in Z \setminus H} f(xh) \, dx = \int_{Z \setminus H} f(x) \, dx. \tag{14.5}
\]

Again, \( \mu \) is unique up to a positive real multiple.
15 Automorphic Forms on Reductive Groups

Let $F$ be a number field, $G$ a reductive group over $F$, and $G_\infty = G(F \otimes_\mathbb{Q} \mathbb{R})$, a real Lie group. Choose $K_\infty \subseteq G_\infty$ a maximal compact subgroup. $K_\infty$ is unique up to conjugacy.

For example, we have $O_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$ and $U_{n_1}(\mathbb{R}) \subseteq GL_{n_1}(\mathbb{C})$.

Also, let $U(\mathfrak{g})$ be the universal enveloping algebra and $Z(\mathfrak{g})$ be its center. We have an isomorphism $Z(\mathfrak{g}) \cong \mathbb{C}[T_1, \ldots, T_n]$ for suitable $T_i$.

An automorphic form on $G$ is a function $\varphi : G(\mathbb{A}_F) \to \mathbb{C}$ satisfying:

1. For every $\gamma \in G(F)$ and $g \in G(\mathbb{A}_F)$, $\varphi(\gamma g) = \varphi(g)$.
2. For every $g^\infty \in G(\mathbb{A}_F^\infty)$ the function $g_\infty \mapsto \varphi(g^\infty g_\infty)$ is smooth on $G_\infty$.
3. $\varphi$ is $Z(\mathfrak{g})$-finite, meaning $Z(\mathfrak{g}) \cdot \varphi$ is finite dimensional.
4. $\varphi$ is $K_\infty$-finite; $\mathbb{C}[K_\infty] \cdot \varphi$ is finite dimensional.
5. There exists $K_\infty \subseteq G(\mathbb{A}_F^\infty)$ an open compact subgroup such that for every $g \in G(\mathbb{A}_F)$ and $k^\infty \in K_\infty$, $\varphi(g k^\infty) = \varphi(g)$.
6. For every $g^\infty \in G(\mathbb{A}_F^\infty)$, the function $g_\infty \mapsto \varphi(g^\infty g_\infty)$ is slowly increasing. This means that for one (equivalently all) embeddings $\pi : G_\infty \hookrightarrow GL_{n, \mathbb{R}}$ of real algebraic groups, there exist $C, N > 0$ such that

$$|f(g)| \leq C|\text{tr} \ f^\pi(g)\pi(g) + \text{tr} \ f^\pi(g^{-1})\pi(g^{-1})|^N. \quad (15.1)$$

We say that an automorphic form $\varphi$ is cuspidal if, in addition, for every proper parabolic subgroup $P \subset G$ defined over $F$, and every $g \in G(\mathbb{A}_F)$,

$$\int_{N(F) \setminus N(\mathbb{A}_F)} f(ng) \ dn = 0 \quad (15.2)$$

where $N$ is the unipotent radical of $P$.

We write $\mathcal{A}$ for the space of automorphic forms on $G$, and $\mathcal{A}_0 \subseteq \mathcal{A}$ for the subspace of cusp forms. As for $GL_n$, these spaces are $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$-modules. Here $G(\mathbb{A}_F^\infty)$ acts by right translation, $K_\infty$ acts by right translation, and $\mathfrak{g}$, and more generally $U(\mathfrak{g})$, acts by differentiation.

**Remark.** If $f : G_\infty \to \mathbb{C}$ is smooth, we say that $f$ has uniform moderate growth if for every $D \in U(\mathfrak{g})$, $Df$ is slowly increasing.

**Theorem 15.1** (Harish-Chandra). If $f$ is $Z(\mathfrak{g})$-finite and $K_\infty$-finite then $f$ is analytic. If $f$ is also slowly increasing, then $f$ is of uniform moderate growth.

If $f$ satisfies these hypotheses, then there will exist $\alpha \in C_c^\infty(G_\infty)$ such that $f \ast \alpha = f$. Then for $D \in U(\mathfrak{g})$,

$$Df = D(f \ast \alpha) = f \ast D\alpha \quad (15.3)$$

which is slowly increasing.
Theorem 15.2. If \( \varphi \in A \), then it generates an admissible \((g, K_{\infty}) \times G(\mathbb{A}_{F}^\infty)\)-submodule of \( A \).

An automorphic representation \( \pi \) is an irreducible \((g, K_{\infty}) \times G(\mathbb{A}_{F}^\infty)\)-module which is isomorphic to a subquotient of \( A \). We say that \( \pi \) is cuspidal if it is isomorphic to a subquotient of \( A_0 \).

**Remark.** 1. Every automorphic representation is admissible.

2. An automorphic representation \( \pi \) always has a factorization \( \pi \cong \pi_\infty \otimes \bigotimes_{v \mid \infty} \pi_v \) where \( \pi_\infty \) is a \((g, K_{\infty})\)-module, just as for \( GL_2(\mathbb{A}) \). We will say more on this later.

3. If \( F \) is an arbitrary field and \( G \) is reductive over \( F \), we say that \( G \) is anisotropic if every maximal split torus of \( G \) is trivial. If \( G \) is anisotropic, then it contains no nontrivial parabolic subgroups. So if \( F \) is a number field and \( G \) is anisotropic, then \( A_0 = A \).

For a semisimple group \( G \), \( G(F) \backslash G(\mathbb{A}_F) \) will be compact if and only if \( G \) is anisotropic. In general, the quotient will be noncompact with cusps corresponding to classes of parabolic subgroups of \( G \).

4. The cuspidality condition is equivalent to \( \varphi \) being “rapidly decreasing”.

### 15.1 Examples

An example is given by Hilbert modular forms. Let \( F \) be a totally real number field of degree \( d \), \( v_1, \ldots, v_d \) be the real places, and \( \sigma_1, \ldots, \sigma_d \) the corresponding embeddings of \( F \) into \( \mathbb{R} \).

Let \( G = GL_{2,F} \). Then \( G_\infty = \prod_{i=1}^{d} GL_2(\mathbb{R}) \). Choose \( K_{\infty} = \prod_{i=1}^{d} O_2(\mathbb{R}) \). Hence we have

\[
K_{\infty}^0 = \{(r_{\theta_1}, \ldots, r_{\theta_d}) | \theta_i \in [0, 2\pi)\}
\]  

(15.4)

where \( r_{\theta} \) is the matrix of clockwise rotation by \( \theta \).

Recall that for every integer \( k \geq 1 \), there exists a unique \((gl_2, O_2(\mathbb{R}))\)-module \( D_{k-1} \) which is irreducible, admissible, and satisfies:

1. There exists \( v \in D_{k-1} \) such that \( r_{\theta}v = e^{ik\theta}v \) and \( X_- v = 0 \).

2. \( Z \), the identity matrix, acts as \( 2 - k \in \mathbb{R} \).

Choose integers \( k_1, \ldots, k_d \geq 1 \). Consider cuspidal automorphic representations \( \pi \) of \( GL_2(k_F) \) such that \( \pi_\infty \cong D_{k_1-1} \otimes \cdots \otimes D_{k_d-1} \) as a representation of \( G_\infty \). These are the cuspidal automorphic representations which correspond to classical Hilbert modular forms of weight \((k_1,\ldots,k_d)\). These automorphic representations are in bijection with the \( GL_2(\mathbb{A}_F^\infty)\)-constituents of a space of functions \( \varphi : GL_2(\mathbb{A}_F) \rightarrow \mathbb{C} \) satisfying:

1. For every \( \gamma \in G(F) \), \( \varphi(\gamma g) = \varphi(g) \).

2. For every \( g^\infty \in GL_2(\mathbb{A}_F^\infty) \), let \( f(g^\infty) = \varphi(g^\infty g^\infty) \). Then \( X_- f = 0 \) and

\[
f(g^\infty k^\infty) = \prod_{i=1}^{d} e^{ik_i \theta_i} f(g^\infty)
\]  

(15.5)

where \( k^\infty = (r_{\theta_1}, \ldots, r_{\theta_d}) \in K_{\infty}^0 \).
3. There exists $K^\infty \subseteq GL_2(\mathbb{A}_F^\infty)$ such that $\varphi(gk^\infty) = \varphi(g)$ for every $k^\infty \in K^\infty$.

4. Growth and cuspidality conditions.

Given such $\varphi$, define $F : \mathfrak{h}^d \to \mathbb{C}$ by

$$F(\tau_1, \ldots, \tau_d) = \varphi(1^\infty, g_1, \ldots, g_d) \cdot \prod_{i=1}^{d} j(g_i, i)^{k_i} \det(g_i)^{-1} \quad (15.6)$$

where $g_j \in GL_2(\mathbb{R})^+$ satisfies $g_j \cdot i = \tau_j$. One can check that there exists $\Gamma \subseteq GL_2(\mathcal{O}_F)_{>0}$ (those matrices having positive determinant over every real embedding) such that for every $\gamma \in \Gamma$,

$$F(\sigma_1(\gamma)\tau_1, \ldots, \sigma_d(\gamma)\tau_d) \cdot \prod_{i=1}^{d} j(\sigma_i(\gamma), \tau_i)^{-k_i} = F(\tau_1, \ldots, \tau_d). \quad (15.7)$$

Now we look at another example: quaternion algebras over $\mathbb{Q}$.

Recall that if $A$ is a quaternion algebra over $\mathbb{Q}$, we defined $G = G_A$, an inner form of $GL_2$. Q

**Theorem 15.3.**

Let $F = \mathbb{Q}_p$ or $\mathbb{R}$. Then there exist exactly two isomorphism classes of quaternion algebras over $F$, and the one other than $M_2(F)$ is nonsplit.

Over $\mathbb{Q}$, the isomorphism classes of quaternion algebras are in bijection with finite sets of places of $\mathbb{Q}$ of even cardinality, by mapping a quaternion algebra $A$ to the set of places $v$ where $A_v$ is nonsplit. The inverse map is $A_S \leftrightarrow S$.

Let $S$ be a nonempty set of finite places and $G = G_{A_S}$. Then $G$ is anisotropic modulo center, since $G_{\mathbb{Q}_p}$ is anisotropic modulo center if $p \in S$. Moreover, $G(\mathbb{R}) \cong GL_2(\mathbb{R})$. One can check that cuspidal automorphic representations $\pi$ of $G(\mathbb{A}_\mathbb{Q})$ such that $\pi_\infty \cong D_{k-1}$ can be described in terms of holomorphic functions on compact quotients of $\mathfrak{h}$.

Now suppose $S$ contains $\infty$, and again let $G = G_{A_S}$. In this case, $G$ is anisotropic modulo center and $G(\mathbb{R}) \cong \mathbb{H}_\infty^\times$. We will describe the automorphic representations $\pi$ such that $\pi_\infty$ is the trivial $(g, K_\infty)$-representation. These are in bijection with the $GL_2(\mathbb{A}_\mathbb{Q})$-constituents of the space of $\varphi : G(\mathbb{A}) \to \mathbb{C}$ such that:

1. For every $\gamma \in G(\mathbb{Q})$, $\varphi(\gamma g) = \varphi(g)$.
2. There exists $K^\infty \subseteq G(\mathbb{A}_\mathbb{Q}^\infty)$ which fixes $\varphi$ on the right.
3. For every $g_\infty \in G(\mathbb{R})$ and $g \in G(\mathbb{A}_\mathbb{Q})$, $\varphi(gg_\infty) = \varphi(g)$.

In other words, these are the functions $\varphi : G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}^\infty) \to \mathbb{C}$ which are locally constant. If $\varphi$ is fixed by $K^\infty$, then $\varphi$ descends to a map on $G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}^\infty)/K^\infty$, a finite set.

**Theorem 15.4** (Jacquet-Langlands). The following are in bijection:

1. Infinite dimensional automorphic representations $\pi$ of $G(\mathbb{A})$ such that $\pi_\infty$ is trivial.
2. Cuspidal automorphic representations $\pi$ of $GL_2(\mathbb{A})$ such that $\pi_{\infty} \cong D_1$, and for every $p \in S \setminus \{\infty\}$, $\pi_p$ is square-integrable.

As an example, if $E/\mathbb{Q}$ is an elliptic curve, and $E$ has multiplicative reduction at $p$, then $\pi_p$ is square-integrable.

16 Unramified Groups

References:

- Bushnell-Henniart: Local Langlands Conjecture for $GL(2)$.
- Cartier, Corvallis I.

Let $F$ be a finite extension of $\mathbb{Q}_p$ and $G$ a reductive group over $F$. We will also write $G$ for for the topological group $G(F)$.

The group $G$ is unramified if it is quasi-split, and split over an unramified extension of $F$.

An example (other than split groups) is the quasi-split outer form of $GL_n$ associated to the extension $\mathbb{Q}_p^2/\mathbb{Q}_p$.

**Proposition 16.1.** Let $G^{\text{ad}}$ be the quotient of $G$ modulo its center (equal to $\text{Inn}(G)$). Then $G(F)$ contains a unique $G^{\text{ad}}(F)$-conjugacy class of open compact subgroups $K$, called hyperspecial maximal compacts, satisfying:

1. There exists a smooth group scheme $\mathcal{G}$ over the ring $A$ of integers of $F$ such that $\mathcal{G}_F = G$ and $\mathcal{G}_A/(\varpi)$ is a connected algebraic group with trivial unipotent radical, and $K = G(A)$.
2. If $\mu$ is a Haar measure on $G$, and $K' \subseteq G$ is another open compact subgroup, then $\mu(K) \geq \mu(K')$.

Here are some examples:

- If $G = GL_n(F)$, we may take $K = GL_n(\mathcal{O}_F)$. Every open compact subgroup is contained in a $G$-conjugate of $K$.
- If $G = SL_n(F)$ and $K = SL_n(\mathcal{O}_F)$, then there are $n$ $G$-conjugacy classes of hyperspecial maximal compacts, which form a single $G^{\text{ad}} = PGL_n(F)$-conjugacy class. We have $PGL_n(F)/\text{Inn}(SL_n(F)) \cong F^\times/(F^\times)^n$, and conjugation by $PGL_n(\mathcal{O}_F)$ fixes $K$.

Let $G$ be unramified. Fix $S \subseteq G$ a maximal split torus. Let $T = Z_G(S)$, a maximal torus of $G$ and $T^c$ be the maximal compact subgroup of $T$.

For example, if $G = GL_n(F)$ and $S$ is the diagonal torus, then $T = S \cong F^\times \times \cdots \times F^\times$, and $T^c$ will be $A^\times \times \cdots \times A^\times$.

Let $W_d = N_G(S)/Z_G(S)$, a constant group over $F$ which acts faithfully on $S$ and $T$. 

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**Theorem 16.2** (Satake Isomorphism). There exists a canonical isomorphism
\[ \mathcal{H}(G, K) \cong \mathcal{H}(T, T^c)^{W_d} \]  
for \( K \subseteq G \) a hyperspecial compact.

*Sketch of proof.* We can assume \( T^c = T \cap K \). We will describe the map \( \mathcal{H}(G, K) \to \mathcal{H}(T, T^c) \), called the Satake transform.

Fix \( B \subseteq G \) a Borel subgroup containing \( T \) and \( \delta_B \) its modular character. Given \( f \in \mathcal{H}(G, K) \), we let
\[ S(f)(t) = \frac{1}{\delta_B(t)} \int_{n \in N} f(tn) \, dn \]  
where \( N \) is the unipotent radical of \( B \). Check that \( S(f) \) has image in \( \mathcal{H}(T, T^c)^{W_d} \).

Again take \( G \) unramified, and \( K \subseteq G \) a hyperspecial maximal compact. If \( \pi \) is an irreducible admissible representation of \( G \), we say that \( \pi \) is \( K \)-unramified if \( \pi^K \neq 0 \). We say that a character \( \chi : T \to \mathbb{C}^\times \) is unramified if \( \chi \) is trivial on \( T^c \).

**Remark.** The notion of \( K \)-unramified representations depends on the choice of \( K \). For instance, if \( SL_2(F) \), there exist nonconjugate hyperspecial maximal compacts \( K, K' \) and a representation \( \pi \) such that \( \pi^K \neq 0 \) but \( \pi^{K'} \neq 0 \). (This is related to \( L \)-packets. It doesn’t happen for \( GL_n \).)

**Corollary 16.3.** If \( \pi \) is \( K \)-unramified, then \( \dim \pi^K = 1 \).

*Proof.* \( \mathcal{H}(G, K) \) is commutative.

**Corollary 16.4.** The following sets are in canonical bijection:

1. The set of isomorphism classes of \( K \)-unramified representations \( \pi \).
2. The set of \( W_d \)-conjugacy classes of unramified characters \( \chi : T \to \mathbb{C}^\times \).
3. The set of \( W_d \)-conjugacy classes of homomorphisms \( X_*(S) \to \mathbb{C}^\times \).
4. The set of algebra homomorphisms \( \mathbb{C}[X_*(S)]^{W_d} \to \mathbb{C} \).

*Proof.* The bijection between (1) and (2) follows from the Satake isomorphism. The equivalence of (2) and (3) is clear. To show that (3) and (4) are equivalent, the map \( X_*(S) \to T/T^c \) given by \( \lambda \mapsto \lambda(\bar{\pi}) \) is a group isomorphism.

17 Parabolic Induction

Let \( G \) be any reductive group over \( F \). Fix \( P \subseteq G \) a parabolic subgroup and consider its Levi decomposition \( P = MN \), where \( M \) is reductive and \( N \) is the unipotent radical of \( P \).

Let \((\sigma, W)\) be an admissible representation of \( M \). As \( M \) is a quotient of \( P \), we may inflate \( \sigma \) to \( P \). Define
\[
\text{Ind}^G_P \sigma = \{ f : G \to W \text{ locally constant} | \forall p \in P, g \in G, f(pg) = \sigma(p)f(g) \}.
\] (17.1)

Then \( G \) acts on \( \text{Ind}^G_P \sigma \) by right translation.

**Proposition 17.1.** \( \text{Ind}^G_P \sigma \) is admissible.

**Proof.** Fix \( K \subseteq G \) an open compact. We must show that \( (\text{Ind}^G_P \sigma)^K \) is finite dimensional. \( G/P \) is projective, hence compact, so there exists \( X \subseteq G \) finite such that \( PXK = G \).

Now any \( f \in (\text{Ind}^G_P \sigma)^K \) is determined by \( f(x) \) for \( x \in X \). One can check that we must have \( f(x) \in W^{\delta(x^{-1}P)} \), a finite dimensional vector space since \( \sigma \) is admissible.

Let \( \delta_P \) be the modular character of \( P \). We have the expression

\[
\delta_P(p) = | \det \text{Ad}_n(p) |\]

(17.2)

where \( n = \text{Lie} N \) for \( N \) the unipotent radical of \( P \), and \( | \bullet | \) is the normalized absolute value, satisfying \( |\varpi| = |A/\langle \varpi \rangle|^{-1} \).

If \( (\sigma, W) \) is an admissible representation of \( M \), we define

\[
\text{i}^G_P \sigma = \text{Ind}^G_P \sigma \otimes \delta_P^{\frac{1}{2}}
\] (17.3)

\[
= \{ f : G \to W | f(pg) = \delta_P^{\frac{1}{2}}(p)\sigma(p)f(g) \}
\] (17.4)

called the normalized or unitary induction.

**Proposition 17.2.** If \( \sigma \) is unitary, then so is \( \text{i}^G_P \sigma \).

**Sketch of proof.** We consider the case where \( \sigma : M \to \mathbb{C}^\times \) is a character. Then \( \sigma \) being unitary means that \( \sigma \bar{\sigma} = 1 \). Choose \( f_1, f_2 \in \text{i}^G_P \sigma \) and define \( F(g) = f_1(g)f_2(g) \). The idea is to take \( \langle f_1, f_2 \rangle = \int F \), but there does not exist a \( G \)-invariant integral on \( P \backslash G \) since \( P \) is not unimodular. But we do have a right \( G \)-invariant integral

\[
f \mapsto \int_{P \backslash G} f(x) \, dx
\] (17.5)

on the space of locally constant \( f : G \to \mathbb{C} \) satisfying \( f(pg) = \delta_P(p)f(g) \). Since \( F(pg) = \delta_P(p)F(g) \), we can define \( \langle f_1, f_2 \rangle = \int_{P \backslash G} F(x) \, dx \).

Now suppose that \( G \) is unramified and let \( K \subseteq G \) be a hyperspecial maximal compact, \( S \subseteq G \) a maximal split torus, \( T = \text{Z}_G(S) \), and \( B \subseteq G \) a Borel subgroup containing \( T \). Let \( \chi : T \to \mathbb{C}^\times \) be a smooth character. Define \( I(\chi) = \text{i}^G_B \chi \), the principal series representation associated to \( \chi \).

**Fact.** If \( \chi \) is unramified, then \( I(\chi)^K \) is 1-dimensional.

As a consequence, \( I(\chi) \) has exactly one \( K \)-unramified irreducible subquotient, which we call \( \pi_\chi \). Conversely, if \( \pi \) is \( K \)-unramified, we get an unramified character \( \chi : T \to \mathbb{C}^\times \), determined up to \( W^d \)-conjugacy, via the Satake isomorphism.
Theorem 17.3. If \( \pi \) is unramified and \( \chi : T \to \mathbb{C}^\times \) is a representation of the associated \( W_d \)-conjugacy class of characters, then \( \pi \cong \pi_{\chi} \).

Remark. If \( \chi : T \to \mathbb{C}^\times \) and \( w \in W_d \), then \( I(\chi) \) and \( I(\chi^w) \) need not be isomorphic, but they both have finite length and the same Jordan-Holder factors.

As an example, if \( G = GL_n(F) \), take \( S = T \) the diagonal torus and \( B \) the upper triangular subgroup. Then \( W_d = S_n \) and we may take \( K = GL_n(O_F) \). Then

\[
\mathcal{H}(G, K) \cong \mathbb{C}[e_1^\pm, \ldots, e_m^\pm]_{S_n} \\
= \mathbb{C}[T, \ldots, T_n, T_n^{-1}]
\]

where \( e_i \in X_*(S) \) maps \( t \) onto the matrix with \( t \) on the \( i \)th diagonal entry and 1 on the others. This forms the standard basis of \( X_*(S) \). Then \( T_i \) is the \( i \)th elementary symmetric polynomial in the \( e_j \).

Fix \( k, N \geq 1 \) integers, and choose \( f \in S_k(\Gamma_0(N)) \) a classical holomorphic modular form, which is an eigenvector for all \( T_p \) with \( p \nmid N \). Then \( f \) generates an automorphic representation \( \pi = \pi_\infty \otimes \otimes_p \pi_p \) of \( GL_2(\mathbb{A}) \), and for \( p \nmid N \), \( \pi_p \) is \( K_p \)-unramified, where \( K_p = GL_2(\mathbb{Z}_p) \).

Let \( a_p \) be the eigenvalue of \( T_p \) on \( f \), and factor the polynomial

\[
X^2 - a_pX + p^{k-1} = (X - p^{\frac{1}{2}}\alpha_1)(X - p^{\frac{1}{2}}\alpha_2).
\]

We then have an unramified character \( \chi : T(\mathbb{Q}_p) \to \mathbb{C}^\times \) which maps \( \left( \begin{array}{cc} p^a \\ p^b \end{array} \right) \) to \( \alpha_1^a\alpha_2^{-b} \).

Fact. The representation \( I(\chi) \) is irreducible, and \( \pi_p \cong I(\chi) \), which is explicitly given as

\[
I(\chi) = \left\{ f : GL_2(\mathbb{Q}_p) \to \mathbb{C} \mid f \left( \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right) g \right) = \alpha_1^{\text{ord}_p(a)}\alpha_2^{\text{ord}_p(d)} a^{\frac{1}{2}} f(g) \right\}.
\]

18 Global Consequences

Suppose now that \( F \) is a number field, and \( G \) is a reductive group over \( F \). In order to define \( G(\mathbb{A}_F^\infty) \), we must choose open compact subgroups \( K_v \) of \( G(F_v) \) for each \( v \). To make such a choice, we choose an embedding \( G \hookrightarrow GL_n,F \) and let \( K_v = G(F_v) \cap GL_n(O_{F_v}) \). Away from finitely many places, the groups \( K_v \) are independent of choice of embedding.

Lemma 18.1. For all but finitely many finite places \( v \) of \( F \), \( G_{F_v} \) is unramified and \( K_v \) is a hyperspecial maximal compact subgroup.

Sketch of proof. There is an \( \text{Aut}(G) \)-torsor \( \text{Isom}(G, G^{qs}) \), where \( G^{qs} \) is the quasi-split inner form of \( G \). This torsor has a distinguished connected component \( \text{Im} \text{Isom}(G, G^{qs}) \), which is a torsor for \( \text{Inn}(G) = G^{ad} \). The assertion that \( G_{F_v} \) is quasi-split almost everywhere is then equivalent to the assertion that \( \text{Im} \text{Isom}(G, G^{qs})(F_v) \) is nonempty for almost all \( v \).

Now if \( U \subseteq G(\mathbb{A}_F^\infty) \) is an open compact subgroup, then there exists \( V \subseteq U \) of the form \( V = \prod_v V_v \) with \( V_v = K_v \) for almost all \( v \).

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If \( \pi \) is an irreducible admissible representation of \( G(\mathbb{A}_F^\infty) \), then there exists \( V = \prod_v V_v \) of this form such that \( \pi^V \neq 0 \).

If \( K_v \) is hyperspecial, then \( \mathcal{H}(G(F_v), K_v) \) is commutative.

The above two facts then imply:

**Proposition 18.2.** If \( \pi \) is an irreducible admissible representation of \( G(\mathbb{A}_F^\infty) \), then for \( v \nmid \infty \), there exists an irreducible admissible representation \( \pi_v \) of \( G(F_v) \) and an isomorphism \( \pi \cong \bigotimes_v \pi_v \). In particular, \( \pi_v^{K_v} \neq 0 \) for almost all \( v \) (so then \( \dim_{\mathbb{C}} \pi_v^{K_v} = 1 \) for almost all \( v \)).

For reference, see Flath, Corvallis I.

### 19 Classes of Representations

Let \( F \) be a finite extension of \( \mathbb{Q}_p \) for some prime \( p \), and let \( G \) be a reductive group over \( F \). Let \( Z \subseteq G \) be its center as an algebraic group; we also have \( Z(F) \subseteq G(F) \) as groups of \( F \)-points.

Let \((\pi, V)\) be an irreducible admissible representation of \( G(F) \).

**Lemma 19.1 (Schur).** There exists a smooth character \( \omega : Z(F) \rightarrow \mathbb{C}^\times \) such that for every \( z \in Z(F) \) and \( v \in V \), \( \pi(z)v = \omega(z)v \).

We define \( \pi^\vee \), the contragredient of \( \pi \), as the space of smooth vectors in the algebraic dual \( \text{Hom}_{\mathbb{C}}(V, \mathbb{C}) \). This definition makes sense for arbitrary smooth representations, but if \( \pi \) is admissible, the natural map \( \pi \rightarrow \pi^\vee \) is an isomorphism.

Let \( \pi \) be an irreducible admissible representation of \( G(F) \) with central character \( \omega \). Choose \( v \in V \) and \( v^\vee \in V^\vee \). Then for \( g \in G(F) \), define

\[
c_{v,v^\vee}(g) = \langle \pi(g)v, v^\vee \rangle
\]

where \( \langle \ , \ \rangle : V \times V^\vee \rightarrow \mathbb{C} \) is the natural pairing. This is a matrix coefficient of \( \pi \).

We say that \( \pi \) is square-integrable if \( \omega \) is unitary (\( \omega \omega = 1 \)) and for every \( v \in V \) and \( v^\vee \in V^\vee \),

\[
\int_{Z(F)\backslash G(F)} |c_{v,v^\vee}(g)|^2 \, dg < \infty. \tag{19.2}
\]

**Remark.** This makes sense as the integral is defined and the function \( g \mapsto |c_{v,v^\vee}(g)|^2 \) descends to the quotient (since \( \omega \) is unitary).

We say that \( \pi \) is supercuspidal if for every \( v \in V \) and \( v^\vee \in V^\vee \), the function \( c_{v,v^\vee}(g) \) is compactly supported modulo \( Z(F) \).

**Proposition 19.2.** 1. If \( \pi \) is irreducible and admissible, then \( \pi \) is square-integrable (respectively supercuspidal) if and only if there exists a choice of nonzero \( v \in V \) and \( v^\vee \in V^\vee \) such that \( c_{v,v^\vee} \) is square-integrable modulo \( Z(F) \) (respectively compactly supported modulo \( Z(F) \)) and \( \omega \) is unitary (respectively no condition on \( \omega \)).

2. If \( \omega \) is unitary and \( \pi \) is supercuspidal, then \( \pi \) is square-integrable.

3. If \( \pi \) is square-integrable, then \( \pi \) is unitary.
Proof. 1. Fix \( v^\vee \). Let \( W \subseteq V \) be the subspace of \( v \in V \) such that \( c_{v,v^\vee}(g) \) is compactly supported modulo \( Z(F) \). Check that \( W \subseteq V \) is a nonzero \( G \)-invariant subspace; irreducibility then forces \( W = V \). The same argument in reverse then shows that \( c_{v,v^\vee}(g) \) is compactly supported modulo \( Z(F) \) for all \( v \in V \) and \( v^\vee \in V^\vee \).

2. Easy to see.

3. We define an inner product on \( V \) as follows: fix a nonzero \( v^\vee \in V^\vee \). Then for \( v,w \in V \), define

\[
(v,w) = \int_{Z(F)\backslash G(F)} c_{v,v^\vee}(g) \overline{c_{w,w^\vee}(g)} \, dg. \tag{19.3}
\]

Check that this gives a unitary structure.

Let \( \pi \) be irreducible and admissible with \( \pi \) unitary. We say that \( \pi \) is tempered if for every \( v \in V \) and \( v^\vee \in V^\vee \), the function \( c_{v,v^\vee}(g) \) lies in \( L^{2+\epsilon}(Z(F)\backslash G(F)) \) for every \( \epsilon > 0 \).

There is a hierarchy of irreducible admissible representations of \( G(F) \) with unitary central character:

\[
\{ \text{all representations} \} \supseteq \{ \text{unitary} \} \supseteq \{ \text{tempered} \} \supseteq \{ \text{square-integrable} \} \supseteq \{ \text{supercuspidal} \}. \tag{19.4}
\]

**Theorem 19.3.** 1. An irreducible admissible representation \((\pi,V)\) of \( G(F) \) is supercuspidal if and only if there does not exist a proper parabolic subgroup \( P = MN \subsetneq G \) and an admissible representation \( \sigma \) of \( M \), together with an embedding \( \pi \hookrightarrow i_G^P \sigma \).

2. Any irreducible admissible representation \((\pi,V)\) admits an embedding \( \pi \hookrightarrow i_G^P \sigma \) for some parabolic subgroup \( P = MN \subsetneq G \) and supercuspidal representation \( \sigma \) of \( M \).

Reference: Bill Casselman (unpublished notes on \( p \)-adic groups).

Now consider \( G = GL_n,F \). The standard parabolic subgroups then correspond to partitions \( n = n_1 + \cdots + n_r \) where the \( n_i \) are positive integers. The corresponding Levi subgroup \( M \) is then the subgroup of block diagonal matrices with blocks of size \( n_i \). So every irreducible admissible representation \( \pi \) of \( GL_n(F) \) embeds in a representation

\[
i_G^P \sigma_1 \otimes \cdots \otimes \sigma_r \tag{19.5}
\]

where for each \( i \), \( \sigma_i \) is a supercuspidal representation of \( GL_{n_i}(F) \). We’ve seen this for unramified representations; if \( \pi \) is unramified, it’s a subquotient of \( i_B^G \chi_1 \otimes \cdots \otimes \chi_n \) for \( B \) the standard Borel and \( \chi_i \) unramified characters of \( GL_1(F) \).

We can think of this presentation of \( \pi \) in terms of the local Langlands correspondence for \( GL_n(F) \). (The LLC for \( GL_n \) was proven by Harris-Taylor and Henniart.)

Recall the Weil group \( W_F \): fix an algebraic closure \( \overline{F} \) of \( F \). Then there are exact sequences
If $F^\text{ur} \subseteq F$ is the maximal unramified extension of $F$, then $I_F = \text{Gal}(F/F^\text{ur})$ and $\widehat{\mathbb{Z}} = \text{Gal}(F^\text{ur}/F)$, topologically generated by the Frobenius element. The LLC for $GL_n(F)$ is a bijection $\text{rec}_F$ between the following two sets:

1. Isomorphism classes of irreducible admissible representations.
2. Conjugacy classes of semisimple homomorphisms $\varphi : W_F \times SL_2(\mathbb{C}) \to GL_n(\mathbb{C})$ such that $\varphi|_{SL_2(\mathbb{C})}$ is algebraic.

$\text{rec}_F$ is characterized abstractly by some identities relating to $L$-functions.

For example, if $\pi$ is the unramified subquotient of $i_B^G \chi_1 \otimes \cdots \otimes \chi_n$ where the $\chi_i$ are unramified characters, let $\varphi = \text{rec}_F(\pi)$. Then $\varphi$ is trivial on the $SL_2(\mathbb{C})$ factor, unramified on the $W_F$ factor, and satisfies

$$\text{Frob} \mapsto \begin{pmatrix} \chi_1(\varpi) \\ \vdots \\ \chi_n(\varpi) \end{pmatrix}$$

where $\varpi \in F$ is a uniformizer.

The function $\text{rec}_F$ restricts to a bijection between:

1. Classes of supercuspidal representations.
2. Classes of irreducible representations $\varphi : W_F \to GL_n(\mathbb{C})$ (trivial on $SL_2(\mathbb{C})$).

Suppose we write $n = n_1 + \cdots + n_r$ with $n_i \geq 1$. Then for each $i$, $\varphi_i : W_F \to GL_{n_i}(\mathbb{C})$ is irreducible Let $\text{rec}_F^{-1}(\varphi_i) = \pi_i$, a supercuspidal representation of $GL_{n_i}(F)$. Let $\varphi = \varphi_1 \oplus \cdots \oplus \varphi_r$. Then if $\pi = \text{rec}_F^{-1}(\varphi)$, then $\pi$ is a subquotient of $i_P^G \pi_1 \otimes \cdots \otimes \pi_r$.

The moral is that building irreducible admissible representations of $GL_n(F)$ from supercuspidal representations of Levi subgroups mirrors the process of building semisimple representations $W_F \to GL_n(\mathbb{C})$ by taking sums of irreducible representations.

As another example, write $n = ab$, and let $\varphi : W_F \to GL_a(\mathbb{C})$ and $\psi : SL_2(\mathbb{C}) \to GL_b(F)$ be irreducible representations. Let $\Phi = \varphi \otimes \psi : W_F \times SL_2(\mathbb{C}) \to GL_n(\mathbb{C})$. Then $\text{rec}_F^{-1}(\Phi)$ is a subquotient of

$$i_P^G \sigma| \cdot |_{\frac{b+1}{2}} \otimes \sigma| \cdot |_{\frac{b-3}{2}} \otimes \cdots \otimes \sigma| \cdot |_{\frac{b+1}{2}}.$$

Here $\sigma| \cdot |_{\alpha} = \sigma \otimes | \det|_{F}$. 49
20 The Steinberg Representation

Let $G$ be a reductive group. There is a canonical square integrable representation of $G$, called the Steinberg representation. Choose a minimal parabolic subgroup $P_0 \subseteq G$, and define

$$\pi_0 = i^G_{P_0} \delta_{P_0}^{-1} = \text{Ind}^G_{P_0} 1 = C^\infty_c(G(F)/P_0(F)).$$

(20.1)

**Proposition 20.1.**

1. The representations $\pi \subseteq \pi_0$ are in bijections with parabolic subgroups $P \subseteq G$ containing $P_0$. Then $\pi_P = C^\infty_c(G(F)/G(P(F)))$.

2. Let $\text{St}$ be the quotient of $\pi$ by the span of the $\pi_P$ over $P \supseteq P_0$. Then $\text{St}$ is irreducible and square-integrable.

If $G = GL_{n,F}$, then $\text{St}$ corresponds under LLC to a homomorphism $\varphi : W_F \times SL_2(\mathbb{C}) \to GL_n(\mathbb{C})$. $\varphi$ is trivial on $W_F$, and on $SL_2(\mathbb{C})$, it’s the unique irreducible representation of dimension $n$. So $\text{St}$ is a subquotient of $\delta_B^\frac{1}{2} i^G_B | \cdot |^{\frac{n-1}{2}} \otimes \cdots \otimes | \cdot |^{\frac{1-n}{2}}$.

(20.2)

If $G = GL_{2,Q_p}$, then we have an exact sequence

$$1 \to Q_p \to C^\infty_c(\mathbb{P}^1(Q_p)) \to \text{St} \to 1$$

(20.3)

where $\mathbb{P}^1(Q_p) = GL_{2,Q_p}/B(Q_p)$, so the middle term is $i^G_B \delta_B^{-\frac{1}{2}}$. This sequence is nonsplit. It gives an example of a reducible unramified principal series representation (as $\delta_B$ is unramified).

If $E/Q$ is an elliptic curve, then $E$ corresponds to a cuspidal automorphic representation $\pi_\bullet$ of $GL_2(A_Q)$. $E$ has multiplicative reduction at $p$ if and only if $\pi_p$ is an unramified twist of $\text{St}$.

Another example is a simple supercuspidal representation. Take $F = Q_2$ and $G = SL_2(Q_2)$. Let $I$ be the Iwahori subgroup:

$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}_2) \middle| c \in 2\mathbb{Z}_2 \right\}.$$  

(20.4)

Let $\chi : I \to \mathbb{C}^\times$ be defined by

$$\begin{pmatrix} a & b \\ 2c & d \end{pmatrix} \mapsto (-1)^{b+c}.$$  

(20.5)

Check that this is a character. Let $\pi = \text{Ind}_I^G \chi$, given as

$$\pi = \{ f : G \to \mathbb{C}, \text{ locally constant, compactly supported} \mid f(ig) = \chi(i)f(g) \forall i \in I, g \in G \}.  

(20.6)$$

Then $\pi$ is a supercuspidal representation. In general, unitary and supercuspidal representations are nontrivial.
21 Representations Over the Reals

References:

- Knapp, Representation Theory of Semisimple Groups.
- Wallach, Representation Theory of Reductive Groups.
- Corvallis I.

Let $G$ be a reductive group over $\mathbb{R}$, $\mathfrak{g}_0 = \text{Lie} G$, $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$, $K \subseteq G(\mathbb{R})$ a maximal compact subgroup, and define $\mathfrak{t}_0$ and $\mathfrak{t}$ similarly.

A representation $(\pi, H)$ of $G(\mathbb{R})$ is a (separable) Hilbert space $H$ and a homomorphism $\pi : G(\mathbb{R}) \rightarrow \text{GL}(H)$ (where $\text{GL}(H)$ is the group of bounded invertible linear operators) such that:

- The map $G(\mathbb{R}) \times H \rightarrow H$ is continuous.
- For every $k \in K$, $\pi(k)$ is unitary (an isometry of $H$).

We say that $\pi$ is:

- irreducible if there is no nontrivial closed $G(\mathbb{R})$-invariant subspace of $H$.
- unitary if for every $g \in G(\mathbb{R})$, $\pi(g)$ is unitary.

21.1 The Compact Case

Suppose that $G(\mathbb{R})$ is compact. For example, we could have $G = U(n)$, the unitary group of the hermitian form

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w_i}. \quad (21.1)$$

**Theorem 21.1** (Peter-Weyl).

- Any irreducible representation of $G(\mathbb{R})$ is finite-dimensional.
- Any unitary representation $(\pi, H)$ of $G(\mathbb{R})$ decomposes as

$$H \cong \bigoplus_{i \in I} H_i \quad (21.2)$$

where $(\pi_i, H_i)_{i \in I}$ is a countable set of irreducible subrepresentations $H_i \subseteq H$.

**Fact.**

- There exists a unique $G(\mathbb{R})$-conjugacy class of maximal tori $T \subseteq G$. Then $T(\mathbb{R})$ is also compact, of the form $(S^1)^r$ for some $r$.
- If $T \subseteq G$ is a maximal torus and $R = (X^*(T_\mathbb{C}), \Phi, X_* (T_\mathbb{C}), \Phi^\vee)$, then the map

$$N_{G(\mathbb{R})} T(\mathbb{R}) / T(\mathbb{R}) \hookrightarrow W(G_\mathbb{C}, T_\mathbb{C}) \quad (21.3)$$

is bijective.
If \((\pi, H)\) is an irreducible representation, we decompose

\[
\pi|_{T(\mathbb{R})} = \bigoplus_{\lambda} H_\lambda
\]  

(21.4)
as a sum of weight spaces. The map

\[
X^*(T_\mathbb{C}) \to \text{Hom}(T(\mathbb{R}), \mathbb{C}^*)
\]  

(21.5)
is an isomorphism, so we can view this sum as being over \(\lambda \in X^*(T_\mathbb{C})\).

As \(\pi|_{T(\mathbb{R})}\) arises from a representation of \(G(\mathbb{R})\), the set \(\{\lambda \in X^*(T_\mathbb{C}) | H_\lambda \neq 0\}\) is invariant under \(N_{G(\mathbb{R})}T(\mathbb{R})\), hence under \(W(G_\mathbb{C}, T_\mathbb{C})\).

**Theorem 21.2** (Theorem of the Highest Weight). Fix a root basis \(S \subseteq \Phi\). Then:

1. There exists a unique weight \(\lambda = \lambda_H \in X^*(T_\mathbb{C})\) such that \(H_\lambda \neq 0\), and for any \(\mu \in X^*(T_\mathbb{C})\) such that \(H_\mu \neq 0\), we can write

\[
\mu = \lambda - \sum_{\alpha \in S} n_\alpha \alpha
\]  

(21.6)
for \(n_\alpha\) nonnegative integers. \(\lambda\) is called the “highest weight” of \(\pi\).

2. The assignment

\[
(\pi, H) \mapsto \lambda_H \in X^*(T_\mathbb{C})
\]  

(21.7)
defines a bijection between the set of isomorphism classes of irreducible representations and the set of “dominant weights”; that is, the elements \(\lambda \in X^*(T_\mathbb{C})\) such that \(\langle \lambda, \alpha^\vee \rangle \geq 0\) for every \(\alpha \in S\).

**Remark.** If \(G\) is any reductive group over \(\mathbb{R}\) and \((\pi_1, H_1)\) and \((\pi_2, H_2)\) are representations of \(G(\mathbb{R})\), we say that \(\pi_1, \pi_2\) are equivalent if there is a bijective continuous homomorphism \(T : H_1 \to H_2\) which intertwines \(\pi_1\) and \(\pi_2\). If \(\pi_1\) and \(\pi_2\) are unitary, we say that they are unitarily equivalent if \(T\) can be chosen to be unitary.

### 21.2 General Groups

Now take \(G\) to be any reductive group over \(\mathbb{R}\). Fix \(K \subseteq G(\mathbb{R})\) a maximal compact subgroup. \(K\) is uniquely determined up to \(G(\mathbb{R})\)-conjugacy.

Examples include \(G = SL_n\), in which case \(K = SO_n\) (the definite special orthogonal group), and \(G = U(p, q)\) with \(K = U(p) \times U(q)\) embedded as block diagonal matrices.

Recall that a \((\mathfrak{g}, K)\)-module is a \(\mathbb{C}\)-vector space \(V\) with the structure of a \(\mathfrak{g}\)-module and a \(\mathbb{C}[K]\)-module such that:

1. For every \(v \in V\), \(v\) is \(K\)-finite: \(\mathbb{C}[K] \cdot v = W_v\) is finite dimensional, and the map \(K \to GL(W_v)\) is continuous.
2. For $X \in \mathfrak{t}$ and $v \in V$, then

$$Xv = \left[ \frac{d}{dt} (\exp(tX)v) \right]_{t=0}. \quad (21.8)$$

3. If $k \in K$, $X \in \mathfrak{g}$, and $v \in V$, then $kXv = \text{Ad}(k)(X)kv$.

We say that $V$ is admissible if for every irreducible representation $\tau$ of $K$, $\text{Hom}_K(\tau, V)$ is finite dimensional. We say that $V$ is unitary if there is a positive definite hermitian inner product $\langle , \rangle : V \times V \to \mathbb{C}$ such that:

1. For every $X \in \mathfrak{g}$ and $v, w \in V$, $\langle Xv, w \rangle + \langle v, Xw \rangle = 0$.
2. For every $k \in K$ and $v, w \in V$, $\langle kv, kw \rangle = \langle v, w \rangle$.

If $(\pi, H)$ is a representation of $G(\mathbb{R})$, then by Peter-Weyl applied to $K$, there is a decomposition

$$\pi|_K = \bigoplus_{i \in I} \pi_i \quad (21.9)$$

for $\pi_i \subseteq \pi$ irreducible subrepresentations. We say that $\pi$ is admissible if each isomorphism class of irreducible representations of $K$ appears only finitely many times.

We say that $v \in H$ is $K$-finite if the space $\mathbb{C}[K] \cdot v = W_v$ is finite dimensional. If $\pi$ is admissible, then the subspace $H^{K\text{-finite}} \subseteq H$ of $K$-finite vectors is given by

$$H^{K\text{-finite}} = \bigoplus_{i \in I} \pi_i. \quad (21.10)$$

If $v \in H$ and $X \in \mathfrak{g}$, we can define

$$Xv = \left[ \frac{d}{dt} (\exp(tX)v) \right]_{t=0} \quad (21.11)$$

If this limit exists, we say that $v$ is differentiable. We say that $v$ is smooth if for every $r \geq 1$ and $X_1, \ldots, X_r \in \mathfrak{g}$, $X_1(\cdots(X_r v)\cdots)$ exists.

**Theorem 21.3.** Let $(\pi, H)$ be an admissible representation of $G(\mathbb{R})$.

1. Every $v \in H^{K\text{-fin}}$ is smooth. Let $V = H^{K\text{-fin}}$ with induced actions of $\mathfrak{g}$ and $K$; then $V$ is an admissible $(\mathfrak{g}, K)$-module. Moreover, $H$ is irreducible as a representation of $G(\mathbb{R})$ if and only if $V$ is algebraically irreducible as a $(\mathfrak{g}, K)$-module.

2. Every irreducible admissible $(\mathfrak{g}, K)$-module $V$ arises in this way from some irreducible admissible representation of $G(\mathbb{R})$.

Let $(\pi_1, H_1)$ and $(\pi_2, H_2)$ be admissible representations of $G(\mathbb{R})$. We say that $\pi_1$ and $\pi_2$ are infinitesimally equivalent if their associated $(\mathfrak{g}, K)$-modules are algebraically isomorphic.
Remark. There do exist infinitesimally equivalent representations which are not equivalent. Consider \( G = SL_2(\mathbb{R}) \) and \( B \) the upper triangular subgroup. There is a notion of parabolic induction \( i_B^G \chi \) for \( \chi : B(\mathbb{R}) \to \mathbb{C}^\times \) a character. Different ways of defining the parabolic induction yield non-equivalent representations.

**Theorem 21.4.** Let \( V \) be an admissible \((g,K)\)-module. Then:

1. \( V \) is unitary if and only if there exists a unitary admissible representation \((\pi,H)\) and an isomorphism \( V \cong H^{K_{-}\text{fin}} \) of \((g,K)\)-modules.

2. If \((\pi_1,H_1)\) and \((\pi_2,H_2)\) are irreducible admissible unitary representations of \( G(\mathbb{R}) \), then they are unitarily equivalent if and only if there is an algebraic isomorphism \( f : H_1^{K_{-}\text{fin}} \to H_2^{K_{-}\text{fin}} \) preserving unitary structures.

Suppose \((\pi,H)\) is any admissible representation of \( G(\mathbb{R}) \). A matrix coefficient of \( \pi \) is a function of the form

\[
g \mapsto c_{v,w}(g) = \langle \pi(g)v,w \rangle
\]

(21.12)

for some \( v, w \in H \). A \( K \)-finite matrix coefficient is one of the form \( c_{v,w} \) for \( v, w \in H^{K_{-}\text{fin}} \).

We say that \( \pi \) is:

- square-integrable if it is irreducible, of unitary central character, and its \( K \)-finite matrix coefficients are square-integrable modulo \( Z(\mathbb{R}) \), for \( Z \subset G \) the center as an algebraic subgroup.
- tempered if it is irreducible of unitary central character, and its \( K \)-finite matrix coefficients are in \( L^{2+\epsilon}(Z(\mathbb{R})\backslash G(\mathbb{R})) \) for every \( \epsilon > 0 \).

**Fact.** Every tempered or square integrable representation of \( G(\mathbb{R}) \) is infinitesimally equivalent to a unitary one.

If \( \pi \) is an irreducible admissible representation, the set of \( K \)-finite matrix coefficients of \( \pi \) only depends on the infinitesimal equivalence class of \( \pi \).

**Remark.** 1. There are no supercuspidal representations (representations with compactly supported matrix coefficients).

2. The irreducible admissible representations of any real reductive group have been classified.

3. The local Langlands correspondence is known for any real reductive group. Recall that the Weil group satisfies

\[
1 \to \mathbb{C}^\times \to W_{\mathbb{R}} \to \{1,j\} \to 1.
\]

(21.13)

### 21.3 The Harish-Chandra Isomorphism

We used the notion of \( Z(g) \)-finiteness, for \( Z(g) \) the center of the universal enveloping algebra \( U(g) \). In fact, \( Z(g) \) is isomorphic to a polynomial algebra in \( r \) generators, where \( r = \text{rank} G_{\mathbb{C}} \) (the dimension of a maximal torus).
We change our notation now: Let $G$ be a reductive group over $\mathbb{C}$, and fix a pinning $\mathcal{P} = (T, B, \{X_\alpha\}_{\alpha \in S})$ of $G$. Write $t = \text{Lie } T$ and $b = \text{Lie } B$. Let

$$
\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \in X^* (T) \otimes \mathbb{Q}.
$$

(21.14)

Then $\langle \rho, \alpha^\vee \rangle = 1$ for every $\alpha \in S$. The differential $d\rho \in \text{Hom} (\text{Lie } T, \mathbb{C}) = t^*$ is defined.

**Proposition 21.5.** Let $\mathcal{H} = U(t) \subseteq U(g)$ and $J = \sum_{\alpha \in S} U(g)X_\alpha$. Then $\mathcal{H} \subseteq U(g)$ is a subalgebra and $J \subseteq U(g)$ is a left ideal. Then:

1. $\mathcal{H} \cap J = 0$ and $Z(g) \subseteq \mathcal{H} \oplus J$.
2. There exists a unique algebra automorphism $\sigma_S : \mathcal{H} \to \mathcal{H}$ such that for every $H \in t$, we have $\sigma_S (H) = H - d\rho (H) \cdot 1$.

We define $p_S : Z(g) \to \mathcal{H}$ to be the projection along $J$. We define the Harish-Chandra transform $\gamma : Z(g) \to \mathcal{H}$ as the composite $\gamma = \sigma_S \circ p_S$.

**Theorem 21.6.** The map $\gamma$ is an algebra homomorphism, independent of $S$, and defines an isomorphism

$$
\gamma : Z(g) \xrightarrow{\sim} \mathcal{H}^W \subseteq \mathcal{H}
$$

(21.15)

where $W = W(G, T)$ acts on $\mathcal{H}$.

**Remark.** Since $W$ is a reflection group, a theorem of Chevalley implies $\mathcal{H}^W$ is actually a polynomial algebra in $r$ generators, where $r = \text{rank } G = \text{dim } T$.

For example, in the case $G = SL_2$, we claim $Z(g) = \mathbb{C}[\Delta]$, where $\Delta = \frac{H^2}{2} + EF + FE$. We can rewrite $\Delta = \frac{H^2}{2} + H + 2FE$, and, under an appropriate choice of positive root $\alpha$, $E$ is a basis of the $\alpha$-root space. We have $p_S (\Delta) = \frac{H^2}{2} + H$. Also $\rho = \frac{\alpha}{2}$ with $d\rho (H) = 1$, so $\sigma_S (H) = H - 1$, implying

$$
\Gamma (\Delta) = \frac{(H - 1)^2}{2} + (H - 1)
$$

(21.16)

$$
= \frac{1}{2} (H^2 - 1).
$$

(21.17)

Meanwhile, the nontrivial element $w$ of $W(G, T)$ has $H^w = -H$, so the invariants are the polynomials in $H^2$.

Take $G$ to be a reductive group over $\mathbb{R}$ and $T \subseteq G$ a maximal torus. Write $g$ and $t$ for the complexified Lie algebras. The theorem implies $Z(g) = U(t)^W$, for $W = W(G_C, T_C)$. If $K \subseteq G(\mathbb{R})$ is a maximal compact subgroup and $V$ is an irreducible admissible $(g, K)$-module, a version of Schur’s Lemma says that every element of $Z(g)$ acts as a scalar on $V$. Hence there exists a homomorphism $\chi_V : Z(g) \to \mathbb{C}$, called the infinitesimal character of $V$.

If $\lambda \in t^*$, we define $\chi_\lambda : Z(g) \to \mathbb{C}$ to be the composite

$$
Z(g) \xrightarrow{\sim} U(t)^W \subseteq U(t) \xrightarrow{\lambda} \mathbb{C}.
$$

(21.18)
Fact. Every homomorphism \( \chi : \mathbb{Z}(g) \to \mathbb{C} \) arises in this manner, and \( \chi_\lambda = \chi_{\lambda'} \) if and only if \( \lambda \) and \( \lambda' \) are in the same \( W \)-orbit.

Now suppose \( G(\mathbb{R}) \) is compact. Take \( S \subseteq \Phi \subseteq X^*(T_\mathbb{C}) \) a root basis, then we define a bijection between irreducible representations \((\pi, H)\) of \( G(\mathbb{R}) \) and elements \( \lambda \in X^*(T_\mathbb{C}) \) such that \( \langle \lambda, \alpha^\vee \rangle \geq 0 \) for every \( \alpha \in S \).

Fact. If \( \pi \) has highest weight \( \lambda \), then the infinitesimal character is \( \chi_\lambda + \rho \), where \( \rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \in X^*(T_\mathbb{C}) \otimes \mathbb{Q} \).

22 Square-Integrable Representations

Let \( G \) be reductive over \( \mathbb{R} \). Assume that \( G \) is semisimple (so has finite center).

**Theorem 22.1.** The following are equivalent:

1. \( G \) has a maximal torus \( T \) such that \( T(\mathbb{R}) \) is compact.
2. \( G \) has an inner form \( G' \) such that \( G'(\mathbb{R}) \) is compact.
3. \( G(\mathbb{R}) \) has square-integrable representations.

For example, \( SL_2 \) has the inner form \( SU_2 \), so \( SL_2(\mathbb{R}) \) has discrete series representations. But for \( n \geq 3 \), \( SL_n \) has no compact inner form.

Write \( \hat{G} \) for the set of unitary equivalence classes of unitary irreducible admissible representations of \( G(\mathbb{R}) \). Write \( \hat{G}_d \) for the subset of square-integrable representations.

Assume that \( \hat{G}_d \neq \emptyset \). We can then choose a maximal torus \( T \subseteq G \) and a maximal compact subgroup \( K \subseteq G(\mathbb{R}) \) such that \( T(\mathbb{R}) \subseteq K \). Let \( W \) be the complex Weyl group and \( W_\mathbb{R} = N_{G(\mathbb{R})}(T(\mathbb{R}))/T(\mathbb{R}) \) be the real Weyl group. Then there is a natural injection \( W_\mathbb{R} \hookrightarrow W \).

We say \( \mu \in X^*(T_\mathbb{C}) \) is regular if \( \langle \mu, \alpha^\vee \rangle \neq 0 \) for every \( \alpha \in \Phi(G_\mathbb{C}, T_\mathbb{C}) \). This condition is invariant under \( W \).

**Theorem 22.2.**
1. If \( \pi \) is a unitary square-integrable representation of \( G(\mathbb{R}) \), then \( \pi \) has infinitesimal character \( \chi_\mu \) for some \( \mu \in X^*(T_\mathbb{C}) \) regular.
2. For \( \mu \in X^*(T_\mathbb{C}) \) regular, there is a bijection between:
   
   (a) The isomorphism classes of square-integrable representations of \( G(\mathbb{R}) \) with infinitesimal character \( \chi_\mu \).
   
   (b) The set of \( W_\mathbb{R} \)-orbits inside \( W \cdot \mu \).

   In particular, there are \( \#(W_\mathbb{R} \backslash W) \) such representations.

If \( G(\mathbb{R}) \) is compact, then \( W = W_\mathbb{R} \). So the irreducible representations of \( G(\mathbb{R}) \) are in bijection of \( W \)-orbits of regular elements in \( X^*(T_\mathbb{C}) \). If \( S \subseteq X^*(T_\mathbb{C}) \) is a root basis, then the **theorem of the highest weight** says that irreducible representations of \( G(\mathbb{R}) \) are in bijection with dominant weights \( \lambda \).
Recall that if \((\pi, H)\) is a representation of \(G(\mathbb{R})\) with highest weight \(\lambda\), then \(\pi\) has infinitesimal character \(\chi_{\mu}\), where \(\mu = \lambda + \rho\). This shows the two bijections are equivalent. Observe, however, that the set of \(W\)-orbits of regular elements is independent of root basis \(S\).

Consider the groups \(SU(p, q)\), the special unitary group of the hermitian form

\[
(z, w) = \sum_{i=1}^{p} z_{i} \bar{w}_{i} - \sum_{i=p+1}^{n} p + q z_{i} \bar{w}_{i}.
\]

\(SU(p, q)\) is a semisimple group over \(\mathbb{R}\) which is a form of \(SL_{n}\) and an inner form of \(SU(n)\), a compact group. A compact maximal torus in \(SU(p, q)\) is the diagonal torus. A maximal compact subgroup is \(S(U(p) \times U(q))(\mathbb{R}) \subseteq SU(p, q)\). We have \(W = S_{n}\) and \(W_{\mathbb{R}} = S_{p} \times S_{q}\). So every packet of square-integrable representations has \(\binom{n}{p}\) elements.

### 23 The case \(SL_{2}(\mathbb{R})\)

Take \(G = SL_{2}\) and \(K = SO_{2}(\mathbb{R}) \subseteq G(\mathbb{R})\). In this case, there exists a maximal torus \(T \subseteq G\) such that \(T(\mathbb{R}) = K\). We have \(W = S_{2}\) and \(W_{\mathbb{R}}\) trivial. The discrete series fall into packets of \(#(W_{\mathbb{R}} \setminus W) = 2\) elements, indexed by \(W\)-orbits on \(X^{+}(T_{\mathbb{C}})_{\text{reg}} \cong \mathbb{Z} \setminus \{0\}\).

Write \(S \subseteq G\) for the diagonal (split) maximal torus, and \(M = \{\pm 1\} \subseteq G(\mathbb{R}), A = \{\left(t t^{-1}\right) | t \in \mathbb{R}_{>0}\}, B \subseteq G\) the upper triangular subgroup, and \(N \subseteq B\) the unipotent radical. We then have \(S(\mathbb{R}) = M \times A\) and \(B(\mathbb{R}) = S(\mathbb{R}) \times N(\mathbb{R}) = M \times A \times N(\mathbb{R})\), the “Langlands decomposition for \(B(\mathbb{R})\)”. The characters of \(S(\mathbb{R})\) are indexed by \((p, s) \in \{\pm 1\} \times \mathbb{C}, where

\[
\chi_{p,s}\left(\left(\begin{array}{cc} t & \varepsilon \varepsilon^{-1} \\ \varepsilon & t^{-1} \end{array}\right)\right) = \text{sgn}(t)^{p} \cdot |t|^{s}.
\]

\(23.1\)

In terms of a pair \((\pm, s)\), we define a representation \(P^{\pm, s}\) of \(G(\mathbb{R})\). Let \(P^{\pm, s}\) denote the pair of smooth functions \(f : G(\mathbb{R}) \to \mathbb{C}\) such that for every \(b = tn \in B(\mathbb{R})\), \(f(bg) = \chi_{p,s+1}(t)f(g)\).

Iwasawa decomposition: the product map \(S(\mathbb{R}) \times N(\mathbb{R}) \times K \to G(\mathbb{R})\) is a diffeomorphism. So \(f \in P^{\pm, s}\) is uniquely determined by its restriction to \(K\). We then define an inner product on \(P^{\pm, s}\) by

\[
\langle f, g \rangle = \int_{K} f(k) \overline{g(k)} dk
\]

\(23.2\)

where \(dk\) is a Haar measure on \(K\). Define \(P^{\pm, s}\) to be the Hilbert space completion of \(P^{\pm, s}\); then the action of \(G(\mathbb{R})\) on \(P^{\pm, s}\) extends to \(P^{\pm, s}\), making this a representation. If \(\chi_{p,s}\) is unitary, then \(P^{\pm, s}\) is unitary.

**Fact.** Every irreducible admissible representation of \(G(\mathbb{R})\) is infinitesimally equivalent to a subquotient of some \(P^{\pm, s}\).

The element \(\Delta \in Z(g)\) acts on \(P^{\pm, s}\) by the scalar \(\frac{1}{2}(s^{2} - 1)\).

We now give a list of all irreducible admissible representations of \(G(\mathbb{R})\) up to infinitesimal equivalence:

1. For every \(n \geq 1\), there is a pair \(D_{n}^{\pm}\) of discrete series representations.
2. There is a pair $D_0^\pm$ of "limits of discrete series representations"; in fact, $\mathcal{P}^{-,0} = D_0^+ \oplus D_0^-$. 
3. $\mathcal{P}^{+,it}$ for $t \geq 0$ and $\mathcal{P}^{-,it}$ for $t > 0$.
4. If $n \geq 1$, there is a finite dimensional representation of dimension $n$. 
5. The representations $\mathcal{P}^{\pm,s}$ with $\Re(s) > 0$, and either $s / \in \mathbb{Z}$, or $s \in \mathbb{Z}$ and $(\pm 1)^p \neq (\pm 1)^{s+1}$.

The unitary representations are those of type (1) - (3), the trivial representation of type (4), and the $\mathcal{P}^{\pm,s}$ of type (5) with $s \in (0, 1)$. The tempered representations are those of type (1) - (3). The square-integrable representations are those of type (1).

### 24 Trace Formula

Let $H$ be a (separable) Hilbert space. We say that a linear operator $T : H \to H$ is Hilbert-Schmidt if for some (equivalently every) orthonormal basis $(e_i)_{i=1}^\infty$,

$$\sum_{i=1}^\infty \|Te_i\|^2 < \infty. \tag{24.1}$$

$T$ is of trace class if there exist Hilbert-Schmidt operators $A, B$ such that $T = A^*B$. In this case, we may define

$$\text{tr } T = \sum_{i=1}^\infty \langle Te_i, e_i \rangle \tag{24.2}$$

$$= \sum_{i=1}^\infty \langle Be_i, Ae_i \rangle \tag{24.3}$$

which is absolutely convergent and independent of the choice of orthonormal basis $(e_i)$.

If $(X, \mu)$ is a Hausdorff compact measure space, and $k : X \times X \to \mathbb{C}$ is a continuous function, then we can define $T : L^2(X) \to L^2(X)$ by

$$T(f)(x) = \int_X k(x, y)f(y) \, d\mu(y). \tag{24.4}$$

Then $T$ is Hilbert-Schmidt.

Choose now $G$ a unimodular locally compact Hausdorff topological group. Fix $\Gamma \subseteq G$ a discrete, cocompact subgroup. Examples include $G = \mathbb{R}$ and $\Gamma = \mathbb{Z}$, or $G = SL_2(\mathbb{R})$ and $\Gamma$ a discrete subgroup arising from a quaternion algebra over $\mathbb{Q}$. If $D$ is a nonsplit quaternion algebra which is split at $\infty$, fix an order $O_D \subseteq D$ (a subring, free of rank 4 as a $\mathbb{Z}$-module, such that $O_D \otimes \mathbb{Q} = D$). Then we may take

$$\Gamma = \{ \gamma \in O_D^\times | N(\gamma) = 1 \} \tag{24.5}$$

$\Gamma$ is a subgroup of $(D \otimes \mathbb{Q})^\times, N=1 = SL_2(\mathbb{R})$. 

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For such a $G$ and $\Gamma$, take $X = \Gamma \setminus G$. Let $H = L^2(\Gamma \setminus G)$, a unitary representation of $G$ under right translation. (Check that the action map $G \times H \to H$ is continuous.) Choose $\varphi \in C_c(G)$, continuous of compact support. Define $T_\varphi : H \to H$ by

$$T_\varphi(f)(x) = \int_G \varphi(g)f(xg)\,dg.$$  \hfill (24.6)

**Claim.** $T_\varphi$ is given by a kernel $k$ on $X \times X$.

**Proof.** We have

$$T_\varphi(f)(x) = \int_G \varphi(x^{-1}g)f(g)\,dg \hfill (24.7)$$

$$= \int_{\Gamma \setminus G} \sum_{\gamma \in \Gamma} \varphi(x^{-1}\gamma y)f(y)\,dy. \hfill (24.8)$$

The function $k_\varphi(x, y) = \sum_{\gamma \in \Gamma} \varphi(x^{-1}\gamma y)$ does descend to a function on $X \times X$. \hfill \QED\hfill \Box

**Remark.** If $x, y \in G$ and $U_x, U_y \subseteq G$ are compact neighborhoods, then the sum defining $k_\varphi(x', y')$ can be taken, for $x' \in U_x, y' \in U_y$, over $\gamma \in \Gamma \cap [U_{x^{-1}}\text{supp}(\varphi)U_y]$, a finite set.

**Lemma 24.1.**

1. For every $\varphi \in C_c(G)$, $T_\varphi$ is Hilbert-Schmidt.

2. Define $\varphi^*(g) = \overline{\varphi(g^{-1})}$. Then $T_\varphi^* = T_{\varphi^*}$.

3. Define convolution on $C_c(G)$ by

$$(\varphi \ast \psi)(g) = \int_G \varphi(x)\psi(x^{-1}g)\,dx.$$  \hfill (24.9)

Then $T_\varphi \circ T_\psi = T_{\varphi \ast \psi}$.

**Proof.** (1) is immediate, and (2) and (3) follow from calculation. \hfill \QED\hfill \Box

**Theorem 24.2.** We can decompose

$$L^2(\Gamma \setminus G) \cong \bigoplus_{\pi \in \hat{G}} m(\pi)\pi$$  \hfill (24.10)

with each $m(\pi)$ finite. Recall that $\hat{G}$ is the set of unitary equivalence classes of irreducible unitary representations of $G$.

**Sketch of proof.** Choose $\varphi \in C_c(G)$ such that $\varphi^* = \varphi$. Then $T_\varphi$ is self-adjoint. But $T_\varphi$ is also Hilbert-Schmidt; therefore it is compact. The spectral theorem then allows us to decompose

$$H = H_{\varphi,0} \oplus \bigoplus_{\lambda \neq 0} H_{\varphi,\lambda}$$  \hfill (24.11)

where $H_{\varphi,\lambda}$ is the $\lambda$-eigenspace of $\varphi$, and each $H_{\varphi,\lambda}$ is finite dimensional.
If $V \subseteq H$ is the closed linear span of all $H_{\varphi,\lambda}$ as $\varphi = \varphi^*$ varies and $\lambda \neq 0$ varies, then $V = H$. For if $f \in V^\perp$, then for all $\varphi = \varphi^*$, $T_\varphi(f) = 0$. Now use that you can approximate the identity by functions of this type.

We now claim that $H$ has a closed $G$-invariant subspace. Choose $\varphi$ and $\lambda \neq 0$ such that $H_{\varphi,\lambda} \neq 0$. Then choose $W' \subseteq V$ a closed, $G$-invariant subspace, with the property that $W' \cap H_{\varphi,\lambda}$ is nonzero, and minimal with respect to this property. Now let $W$ be the intersection of all closed, $G$-invariant subspaces $W''$ of $H$ such that $W'' \cap H_{\varphi,\lambda} = W' \cap H_{\varphi,\lambda}$.

Claim. $W$ is irreducible.

Otherwise, we can write $W = W_1 \oplus W_2$, an orthogonal direct sum of closed $G$-invariant subspaces. It follows from the definition of $W$ that one of $W_1$ and $W_2$ must be zero.

In particular, we now know that $W$ is a closed, irreducible $G$-invariant subspace of $H$. Apply the same argument to $W^\perp \subseteq H$ and Zorn’s lemma to get $H = \bigoplus_i H_i$.

Another consequence of Lemma 24.1 is that if $\varphi \in C_c(G)$ is a convolution, or a sum of convolutions, then $T_\varphi$ is of trace class.

**Lemma 24.3.** If $T_\varphi$ is of trace class, then

$$\text{tr} T_\varphi = \int_{\Gamma \backslash G} k(x, x) \, dx. \quad (24.12)$$

**Proof.** Assume first that $\varphi = \varphi^*$. We can choose orthonormal eigenvectors $e_1, \ldots$ of $T_\varphi$ forming an orthonormal basis of $\bigoplus_{\lambda \neq 0} H_{\varphi,\lambda} \subseteq H$. The functions $e_i \in L^2(\Gamma \backslash G)$ are continuous, as they are in the image of $T_\varphi$. We can write

$$k(x, y) = \sum_{n \geq 1} \lambda_n e_n(x) \overline{e_n(y)} \quad (24.13)$$

for $\lambda_n \in \mathbb{C}$, where this sum converges in $L^2(\Gamma \backslash G \times \Gamma \backslash G)$. Now let, for each $N \geq 1$,

$$k_N(x, y) = \sum_{n \leq N} \lambda_n e_n(x) \overline{e_n(y)}. \quad (24.14)$$

Then $\lim_{N \to \infty} k_n(x, x) = k(x, x)$ in $L^1(\Gamma \backslash G)$. Indeed,

$$\|k_n - k\|_1 \leq \sum_{n > N} |\lambda_n| \to 0 \quad (24.15)$$

since $T_\varphi$ is of trace class. We have

$$\int_{\Gamma \backslash G} k_N(x, x) \, dx = \sum_{n \leq N} \lambda \quad (24.16)$$

and then a limiting argument shows that

$$\text{tr} T_\varphi = \int_{\Gamma \backslash G} k(x, x) \, dx \quad (24.17)$$
as \( k_n \to k \) in \( L^1 \).

In general, we can write

\[
\varphi = \frac{\varphi + \varphi^*}{2} + \frac{i \varphi - \varphi^*}{2i}
\]  

(24.18)

as a linear combination of two self-adjoint operators.

The trace formula gives two different expressions for \( \text{tr} T \varphi \) whenever \( T \varphi \) is of trace class.

The geometric side is

\[
\text{tr} T \varphi = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} \varphi(x^{-1} \gamma x) \, dx
\]

(24.19)

\[
= \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} \sum_{\delta \in \Gamma \gamma \Gamma} \varphi(x^{-1} \delta^{-1} \gamma x) \, dx
\]

(24.20)

for \( \{\Gamma\} \) the set of conjugacy classes in \( \Gamma \), and \( \Gamma_\gamma = Z_\gamma(\Gamma) \) the centralizer of \( \gamma \). Now rewrite as

\[
\sum_{\gamma \in \{\Gamma\}} \int_{\Gamma_\gamma \backslash G} \varphi(x^{-1} \gamma x) \, dx = \sum_{\gamma \in \{\Gamma\}} \int_{G_\gamma \backslash G} \int_{\Gamma_\gamma \backslash G_\gamma} \varphi(x^{-1} u^{-1} \gamma u x) \, du \, dx
\]

(24.21)

for \( G_\gamma = Z_\gamma(G) \). We assume \( G_\gamma \) is a unimodular group, so that the quotient measure is defined. Finally, we may rewrite as

\[
\sum_{\gamma \in \{\Gamma\}} \text{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} \varphi(x^{-1} \gamma x) \, dx.
\]

(24.22)

We also have the spectral side:

\[
\text{tr} T \varphi = \sum_{\pi \in \hat{G}} m(\pi) \text{tr} \pi(\varphi)
\]

(24.23)

where \( H = \hat{\bigoplus} m(\pi) \pi \).

**Theorem 24.4.** If \( T \varphi \) is of trace class, then

\[
\sum_{\pi} m(\pi) \text{tr} \pi(\varphi) = \sum_{\gamma \in \{\Gamma\}} a(\gamma) \mathcal{O}_\gamma(\varphi)
\]

(24.24)

where \( a(\gamma) = \text{vol}(\Gamma_\gamma \backslash G_\gamma) \) and the orbital integral \( \mathcal{O}_\gamma(\varphi) \) is

\[
\mathcal{O}_\gamma(\varphi) = \int_{G_\gamma \backslash G} \varphi(x^{-1} \gamma x) \, dx.
\]

(24.25)

**Remark.** Both \( a(\gamma) \) and \( \mathcal{O}_\gamma(\varphi) \) depend of choices of measures on \( G, \Gamma_\gamma \backslash G_\gamma, G_\gamma \backslash G \). We assume that these are chosen compatibly, and then the product \( a(\gamma) \mathcal{O}_\gamma(\varphi) \) is independent of choices.
Assume $F$ is a number field and $G$ is a semisimple group which is anisotropic over $F$. (Semisimplicity is for convenience; it is easy to extend results to the case where $G$ is reductive.) In this case, the quotient $G(F) \backslash G(\mathbb{A}_F)$ is compact. Define

$$C_c^\infty(G(\mathbb{A}_F)) = \left( \frac{C_c^\infty(G(\mathbb{A}_F))}{\text{locally constant, compact support}} \right) \otimes_{\mathbb{C}} \left( \frac{C_c^\infty(G(F \otimes_{\mathbb{Q}} \mathbb{R}))}{\text{smooth, compact support}} \right)$$

(Dixmier-Malliavin: every $\varphi \in C_c^\infty(\mathbb{A}_F)$ is a convolution, so $T_\varphi$ is of trace class.)

**Proposition 24.5.**

1. Let $K_\infty \subseteq G_\infty = G(F \otimes_{\mathbb{Q}} \mathbb{R})$ be a maximal compact subgroup. If $(\pi, H)$ is an irreducible unitary representation of $G(\mathbb{A}_F)$, then the submodule $V \subseteq H$ of $K$-finite vectors for $K = K_\infty K_\infty$, for any $K_\infty \subseteq G(\mathbb{A}_F)$, is an algebraically irreducible and admissible $(\mathfrak{g}, K) \times G(\mathbb{A}_F)$-module.

2. The assignment $H \mapsto V$ defines a bijection between isomorphism classes of irreducible unitary representations of $G(\mathbb{A}_F)$ and the set of isomorphism classes of unitary, irreducible admissible $(\mathfrak{g}, K) \times G(\mathbb{A}_F)$-modules.

3. The assignment restricts to a bijection between the set of isomorphism classes of irreducible representations $\pi$ which appear in

$$L^2(G(F) \backslash G(\mathbb{A}_F)) = \bigoplus m(\pi) \pi$$

and the set of classes of automorphic representations of $G(\mathbb{A}_F)$.

(See Flath, Corvallis I.)

**Remark.** (3) of the prop really depends on the fact that $G$ is anisotropic. In particular, $G(F) \backslash G(\mathbb{A}_F)$ is compact.

References for the following:

- Henniart, MSMF ‘84.
- Arthur, CJM ‘86.
- Arthur, JAMS ‘88.

Such a trace formula cannot exist if $G$ is not anisotropic. The two main problems in this situation are:

1. $L^2(G(F) \backslash G(\mathbb{A}_F))$ does not decompose discretely (as a countable Hilbert direct sum).

2. The function $k(x) = \sum_{\gamma \in G(F)} \varphi(x^{-1} \gamma x)$ on $G(F) \backslash G(\mathbb{A}_F)$ is not integrable in general. And in general, the operator $T_\varphi$ is not of trace class.

Here are two possible resolutions:

1. Describe the continuous part of the $L^2$ spectrum, and truncate to $k$ to get a well-defined expression. (This leads to the Arthur trace formula.)
2. Adopt some simplifying assumptions on the possible \( \varphi \).

We define the cuspidal subspace \( L^2_0(G(F) \setminus G(\mathbb{A}_F)) \) of \( L^2 \) as the subspace of functions \( f \) such that for all parabolic subgroups \( P = MN \subseteq G \) defined over \( F \),

\[
\int_{N(F) \setminus N(\mathbb{A}_F)} f(gn) \, dn = 0
\]

for almost every \( g \in G(\mathbb{A}_F) \).

**Theorem 24.6.**

1. The subspace \( L^2_0 \subseteq L^2 \) is closed, \( G(\mathbb{A}_F) \)-invariant, and decomposes discretely as a countable Hilbert direct sum of subrepresentations, each appearing with finite multiplicity. For all \( \varphi \in C^\infty_c(G(\mathbb{A}_F)) \), the restriction of \( T_\varphi \) to \( L^2_0 \) is of trace class.

2. The assignment \( (\pi, H) \mapsto V \), the space of \( K \)-finite vectors in \( H \), gives rise to a bijection between the set of unitary equivalence classes of irreducible subrepresentations of \( L^2_0 \) and the set of cuspidal automorphic representations of \( G(\mathbb{A}_F) \).

**Remark.** For the situation where \( G \) is merely reductive, fix a unitary central character \( \psi : Z(\mathbb{A}) \to \mathbb{C}^\times \). Then define

\[
L^2_\psi = \{ f : G(F) \setminus G(\mathbb{A}_F) \to \mathbb{C} | f(zg) = \psi(z)f(g) \}
\]

and the cuspidal subspace \( L^2_{0,\psi} \subseteq L^2_\psi \).

We write \( T_{0,\varphi} \) for the restriction of \( T_\varphi \) to \( L^2_0 \); then \( T_{0,\varphi} \) is of trace class.

If \( K \) is any field of characteristic 0, \( H \) is a reductive group over \( K \), and \( \gamma \in H(K) \), then we say:

- \( \gamma \) is elliptic if \( \gamma \) is contained in an elliptic maximal torus of \( H \) (a torus for which \( X^*(T_K)^{F_K} = 0 \); essentially totally nonsplit.)

- \( \gamma \) is regular semisimple if \( Z_H(\gamma)^0 \) is a maximal torus.

For example, if \( H = SL_n \), then \( \gamma \in H(K) \) is regular semisimple if and only if its characteristic polynomial has distinct roots, and is elliptic regular semisimple if and only if the characteristic polynomial is also irreducible over \( K \).

**Remark.** Many authors use a different definition of “elliptic”.

**Theorem 24.7.** Given \( G/F \), suppose the function \( \varphi = \prod_v \varphi_v \in C^\infty_c(G(\mathbb{A}_F)) \) is such that for some places \( v_1 \neq v_2 \) of \( F \):

1. \( \varphi_{v_1} \) is a matrix coefficient of a supercuspidal representation of \( G(F_{v_1}) \).

2. For every \( \gamma \in G(F_{v_2}) \), \( O_\gamma(\varphi_{v_2}) = 0 \) unless \( \gamma \) is \( F_{v_2} \)-elliptic.

Then we have the formula
\[
\text{tr } R_0(\varphi) = \sum_{\pi} m_0(\pi) \text{tr } \pi(\varphi) = \sum_{\gamma \in \{G(F)\}} a(\gamma) O_\gamma(\varphi)
\]

where \( R_0 \) is the restriction of the right regular representation of \( G(\mathbb{A}_F) \) to the cuspidal subspace \( L^2_0(G(F) \backslash G(\mathbb{A}_F)) \) of \( L^2 \); the first sum runs over cuspidal automorphic representations of \( G(\mathbb{A}_F) \), and \( m_0(\pi) \) is the multiplicity of \( \pi \) in \( L^2_0(G(F) \backslash G(\mathbb{A}_F)) \).

**Theorem 24.8.** Given \( G/F \), suppose the function \( \varphi = \prod_v \varphi_v \in C_\infty^c(G(\mathbb{A}_F)) \) is such that for some places \( v_1 \neq v_2 \) of \( F \):

1. \( \varphi_{v_1} \) is a matrix coefficient of a supercuspidal representation of \( G(F_{v_1}) \).
2. For every \( \gamma \in G(F_{v_2}) \), \( \varphi(\gamma) = 0 \) unless \( \gamma \) is regular \( F_{v_2} \)-elliptic.

Then we have the formula

\[
\sum_{\pi} m_0(\pi) \text{tr } \pi(\varphi) = \sum_{\gamma \in \{G(F)\}} a(\gamma) O_\gamma(\varphi).
\]

**Remark.**
1. Theorem 24.7 is due to Clozel, Kottwitz, Arthur; see Arthur’s JAMS paper, §7. Theorem 24.8, proven earlier, is due to Deligne and Kazhdan. See Henniart MSMF ’84 for one possible proof.
2. Any formula of this type is called a simple trace formula.
3. Theorem 24.7 is much stronger than Theorem 24.8. There are many cases where we are given a function \( \varphi \) and only know about \( O_\gamma(\varphi) \) and not about \( \varphi \) itself.
4. Theorem 24.7 allows contribution from singular elliptic elements of \( G(F) \), for example the identity element in \( G(F) \). Here \( a(1)O_1(\varphi) = \text{vol}(G(F) \backslash G(\mathbb{A}_F))\varphi(1) \).

\( G(F) \backslash G(\mathbb{A}_F) \) always has finite volume if \( G \) is semisimple. It has a canonical measure, therefore a canonical volume \( \tau(G) \), called the Tamagawa number of \( G \). Kottwitz (Annals ’86) proved that if \( G \) is simply connected, then \( \tau(G) = 1 \); this used previous results along with Theorem 24.7.

**Sketch of proof of Theorem 24.8.** Recall \( \varphi = \prod_v \varphi_v \) where \( \varphi_{v_1} \) is a matrix coefficient of a supercuspidal representation and \( \varphi_{v_2} \) is supported on the regular elliptic set of \( G(F_{v_2}) \).

**Lemma 24.9.** The image of \( T_\varphi \) is contained in \( L^2_0(G(F) \backslash G(\mathbb{A}_F)) \).

**Proof.** We need to use the following

**Fact.** If \( f \in C_\infty^c(G(F_{v_1})) \) is a matrix coefficient of a supercuspidal representation, then for every \( P = MN \subseteq G_{F_{v_1}} \), defined over \( F_{v_1} \), then
\[ \int_{N(F_{v_1})} f(xny) \, dn = 0 \quad (24.33) \]

for every \( x, y \in G(F_{v_1}) \).

We take \( f \in L^2(G(F) \backslash G(C_F)) \). Then

\[ (T_{\varphi} f)(g) = \int_{G(C_F)} \varphi(h) f(gh) \, dh. \quad (24.34) \]

Choose \( P = MN \subseteq G \) defined over \( F \). Then

\[ \int_{N(F) \backslash N(C_F)} (T_{\varphi} f)(ng) \, dn = \int_{N(F) \backslash N(C_F)} \varphi(h) f(ngh) \, dh \, dn \]

\[ = \int_{N(F) \backslash N(C_F)} \int_{N(F) \backslash G(C_F)} \sum_{\gamma \in N(F)} \varphi(g^{-1}n^{-1}\gamma h) f(h) \, dh \, dn. \quad (24.35) \]

\( N(F) \backslash N(C_F) \) is compact and \( \varphi \) is compactly supported, so we can reverse the order of integration:

\[ \int_{N(F) \backslash G(C_F)} \int_{N(F) \backslash N(C_F)} \sum_{\gamma \in N(F)} \varphi(g^{-1}n^{-1}\gamma h) f(h) \, dn \, dh = \int_{N(F) \backslash G(C_F)} \int_{N(C_F)} \varphi(g^{-1}n^{-1}h) \, dn \, f(h) \, dh. \quad (24.36) \]

Choosing the measure on \( N(C_F) \) to be a product \( dn = \prod_v dn_v \), we have

\[ \int_{N(C_F)} \varphi(g^{-1}n^{-1}h) \, dn = \prod_v \left[ \int_{N(F_v)} \varphi(g_{v}^{-1}n_{v}^{-1}h_{v}) \, dn_v \right]. \quad (24.37) \]

Observe that because of the assumption on \( \varphi_{v_1} \), the contribution to the product from the \( v_1 \) factor is 0.

This implies that \( T_\varphi \) is of trace class, and \( \mathrm{tr} \, T_\varphi = \mathrm{tr} \, T_{0_\varphi} \). One can show that the function

\[ k_\varphi(x, x) = \sum_{\gamma \in G(F)} \varphi(x^{-1}\gamma x) \quad (24.39) \]

is integrable along \( G(F) \backslash G(C_F) \), and this integral equals \( \mathrm{tr} \, T_\varphi \).

**Lemma 24.10.** The function

\[ x \mapsto \sum_{\gamma \in G(F)} |\varphi(x^{-1}\gamma x)| \quad (24.40) \]

has compact support in \( G(F) \backslash G(C_F) \).
This uses the assumption at $v_2$; the proof uses reduction theory.

We now have

$$\text{tr} T_\varphi = \int_{G(F) \backslash G(\mathbb{A}_F)} \sum_{\gamma \in G(F)} \varphi(x - 1\gamma x) \, dx \tag{24.41}$$

along with

$$\int_{G(F) \backslash G(\mathbb{A}_F)} \sum_{\gamma \in G(F)} |\varphi(x^{-1}\gamma x)| \, dx < \infty. \tag{24.42}$$

The same manipulations as in the case of the compact quotient are now valid, and lead to the statement of Theorem 24.8.

Remark. In both Theorem 24.7 and Theorem 24.8, the sum $\sum_{\gamma} a(\gamma) \mathcal{O}_\gamma(\varphi)$ has only finitely many terms. In fact, there are only finitely many elliptic classes $\gamma \in \{G(F)\}$ whose $G(\mathbb{A}_F)$-conjugacy class meets the support of $\varphi$. See Arthur CJM ’86.

Here is one possible application of Theorem 24.8:

**Theorem 24.11.** Let $F$ be a number field and $G$ a semisimple group over $F$. Let $v_1$ be a place of $F$ and $\pi$ a supercuspidal representation of $G(F_{v_1})$. Then there exists a cuspidal automorphic representation $\Pi$ of $G(\mathbb{A}_F)$ such that $\Pi_{v_1} \cong \pi_{v_1}$.

This theorem was used in Deligne-Kazhdan’s proof of the local Jacquet-Langlands correspondence between $GL_n(F_{v_1})$ and $D_{v_1}^\times$.

**Idea of proof.** We will use Theorem 24.8 with $\varphi = \prod_v \varphi$ such that:

- $\varphi_{v_1}$ is a matrix coefficient of $\pi_{v_1}^\vee$.
- $\varphi_v$ for $v \neq v_1$ is arbitrary (for now).

To prove our theorem, it’s enough to show that

$$\sum_{\Pi} m_0(\Pi) \text{tr} \Pi(\varphi) \neq 0 \tag{24.43}$$

for then there exists a cuspidal automorphic representation $\Pi$ such that $\text{tr} \Pi(\varphi) \neq 0$. If $\sigma$ is any irreducible admissible automorphic representation of $G(F_{v_1})$ and $\text{tr} \sigma(\varphi_{v_1}) \neq 0$, then $\sigma \cong \pi_{v_1}$. (We can write down an explicit map from $\sigma$ to $\pi_{v_1}$.)

By Theorem 24.8, we have

$$\sum_{\Pi} m_0(\Pi) \text{tr} \Pi(\varphi) = \sum_{\gamma \in \{G(F)\}} a(\gamma) \mathcal{O}_\gamma(\varphi) \tag{24.44}$$

and the right-hand sum has only finitely many nonzero terms. Now we have
\[ \mathcal{O}_\gamma(\varphi) = \int_{G(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} \varphi(x^{-1} \gamma x) \, dx. \]  

(24.45)

If the measures are chosen appropriately, then

\[ \mathcal{O}_\gamma(\varphi) = \prod_v \mathcal{O}_{\gamma}(\varphi_v) \]  

(24.46)

\[ = \prod_v \int_{G(\mathbb{F}_v) \backslash G(\mathbb{F}_v)} \varphi(x_v^{-1} \gamma x_v) \, dx_v. \]  

(24.47)

Now choose \( \varphi_v \) for \( v \neq v_1 \) so that exactly one class \( \gamma \in \{ G(F) \} \) contributes to the trace formula and satisfies \( \mathcal{O}_\gamma(\varphi) \neq 0 \).

\[ \square \]

### 25 The L-Group

To each reductive group \( G \) over \( k \), we can associate a complex Lie group \( L^G \).

Suppose first that \( G \) is split. Let \( T \) be a maximal split torus, \( M = X^*(T) \), and \( M^\vee = X_*(T) \). Inside these sets, let \( \Phi \) and \( \Phi^\vee \) be the subsets of roots and co-roots.

To obtain the co-roots, suppose \( X_\alpha \) is a basis of \( g_\alpha \). We then will have some embedding of Lie algebras \( \langle X_\alpha, Y_\alpha, H_\alpha \rangle \cong \mathfrak{sl}_2 \) into \( g \), and therefore a homomorphism \( SL_2 \rightarrow G \), mapping unipotent elements to unipotent elements. Restricting to the torus in \( SL_2 \) gives a map \( \alpha^\vee: G_m \rightarrow T \).

Recall \( s_\alpha \) given by \( s_\alpha(m) = m - \langle m, \alpha^\vee \rangle \alpha \) is an involution preserving roots.

A root datum \( R = (M, \Phi, M^\vee, \Phi^\vee) \) determines the split group \( G \) up to isomorphism over \( k \). However, \( R \) has nontrivial automorphisms. We have a map \( Aut(G \supseteq T) \rightarrow Aut(R) \). \( N_G(T) \) acts on \( G \) by conjugation, and yields \( W \subseteq Aut(R) \), the Weyl group.

Consider the case \( G = GL_n \). \( M \) has a basis \( \{ e_i \} \), where \( e_i \) maps a diagonal matrix to its \( i \)th entry. The roots are \( \{ e_i - e_j | i \neq j \} \) and the coroots are \( \{ e_i^\vee - e_j^\vee | i \neq j \} \). The Weyl group is \( S_n \), with \( \sigma_{e_i-e_j} = (ij) \). But these are not all of the automorphisms of \( R \)! Indeed, \( -1 \in GL(M) \) is always an automorphism of \( R \). It turns out that \( Aut(R) = S_n \times \langle -1 \rangle \).

As another case, if \( G = G^d_m \), then the Weyl group is trivial while \( Aut(R) = GL_d(\mathbb{Z}) \).

Choose a Borel subgroup \( B \supseteq T \) in \( G \). The choices of Borels are permuted simply transitively by the Weyl group. Then we have a decomposition

\[ b = t \oplus \bigoplus_{\Phi_+ \subseteq \Phi} g_\alpha. \]  

(25.1)

for some subset \( \Phi_+ \) of \( \Phi \). Let \( \Delta \subseteq \Phi_+ \) be a root basis. The based root datum is then \( BR = (M, \Delta, M^\vee, \Delta^\vee) \). We have a decomposition \( Aut(T) = W \rtimes Aut(BR) \).

In the case \( G = GL_n \) and \( B \) the standard Borel, we have

\[ \Delta = \{ e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n \}. \]  

(25.2)
BR is isomorphic to $\mathbb{Z}/2$ generated by the matrix

$$
\begin{pmatrix}
-1 & & \\
& \ddots & \\
& & -1
\end{pmatrix}
$$

(25.3)

This matrix can be expressed as $(1n)(2, n - 1) \cdots \times (-1)$.

We want to lift automorphisms of $BR$ to automorphisms of $G$ preserving $B$ and $T$. To avoid the action of $T/ZG$ on $G$, we also preserve a pinning: a basis $X_\alpha$ for each $g_\alpha$, over $\alpha \in \Delta$. Equivalently, we have isomorphisms $e_\alpha : G_\alpha \to U_\alpha \subseteq G$. $T$ acts transitively on the set of pinnings, and the stabilizer of each is $Z_G$.

We now consider $\text{Aut}(G, B, T, \{e_\alpha\}_\alpha \in \Delta)$. There are no nontrivial inner automorphisms. Chevalley proved that this group is isomorphic to $\text{Aut}(BR)$, and that $\text{Aut}(G) \cong G/Z \rtimes \text{Aut}(BR)$.

We can define a based root datum with additional action of $\text{Gal}(k^s/k)$ when $G$ is quasi-split over $k$. Take $G \supseteq B \supseteq T$ defined over $k$. Over some separable extension $E$, $T$ splits over $E$. Then $\text{Gal}(E/k)$ acts on $M$ and $M^\vee$, and because $B$ is defined over $k$, it preserves the subsets $\Delta$ and $\Delta^\vee$.

Given a based root datum $BR = (M, \Delta, M^\vee, \Delta^\vee)$, we have a dual based root datum $BR^\vee = (M^\vee, \Delta^\vee, M, \Delta)$, with $\text{Aut}(BR^\vee) = \text{Aut}(BR)$. The component group $\pi_0(\text{Aut}(BR))$ is isomorphic to $G/Z$. There are no nontrivial inner automorphisms. Chevalley proved that this group is isomorphic to $\text{Aut}(BR)$, and that $\text{Aut}(G) \cong G/Z \rtimes \text{Aut}(BR)$.

In the case of $SL_2$, $\Delta = \{2\}$ while $\Delta^\vee = \{1\}$, where $M$ and $M^\vee$ are associated with $\mathbb{Z}$, so the dual root datum corresponds to $PGL_2$. We also have $BR^\vee(SL_2) = BR(SO_{2n+1})$.

We now consider $\text{Aut}(G, B, T, \{e_\alpha\}_\alpha \in \Delta)$. There are no nontrivial inner automorphisms. Chevalley proved that this group is isomorphic to $\text{Aut}(BR)$, and that $\text{Aut}(G) \cong G/Z \rtimes \text{Aut}(BR)$. Now define the $L$-group

$$L_G = \widehat{G} \rtimes \text{Gal}(E/k)$$

why the semidirect product action is through pinned automorphisms of $\widehat{G}$.

As an example, if $G = GL_n$, then $^*G = GL_n(\mathbb{C})$. If $G = U_n$, split over $E$, then $L_G = GL_n(\mathbb{C}) \rtimes \text{Gal}(E/k)$. If $n$ is even, it is a subgroup of $Sp_{2n}(\mathbb{C})$, namely the normalizer of the Levi subgroup of the Siegel parabolic. If $n$ is odd, it is contained in $O_{2n}(\mathbb{C})$.

A Langlands parameter for a local field $k$ is a homomorphism

$$\varphi : \text{Gal}(k^s/k) \to \widehat{G} \rtimes \text{Gal}(E/k)$$

such that the projection of the image to $\text{Gal}(E/k)$ is restriction to $E$, up to conjugation by $\widehat{G}$.

Here is a problem with the semidirect product: suppose $G = GL_2$. Then we have $\varphi : \text{Gal}(k^s/k) \to SL_2(\mathbb{C})$. Suppose $\varphi$ factors through $N(\widehat{T})$, an extension of $\mathbb{C}^\times$ by $\mathbb{Z}/2$. The projection down to $\mathbb{Z}/2$ then gives a quadratic extension $E/k$. Let $T = U_1(E/k)$, so $\widehat{T} = \mathbb{C}^\times$ But $N(\widehat{T})$ cannot be $L_T$ because is it not a semidirect product!

We have $\text{stab}_\varphi = Z_{\widehat{G}}(\varphi) \subseteq \widehat{G}$. The component group $\pi_0(Z_{\widehat{G}}(\varphi))$ is denoted by $A_\varphi$, a finite group.

Consider $SL_2$ over $\mathbb{Q}_p$ for $p > 2$. Then $L_G = \widehat{G}$, which can be given either as $PGL_2(\mathbb{C})$ or $SO_3(\mathbb{C})$. There exists a Klein 4-group $\Gamma \subseteq SO_3(\mathbb{R})$, with $Z(\Gamma) = \Gamma$ and $N(\Gamma) = S_4$. For $p$ odd, we
have $\Gamma \cong \mathbb{Q}_p^\wedge/(\mathbb{Q}_p^\wedge)^2$, giving a unique extension $E$ of $\mathbb{Q}_p$ corresponding to $\Gamma$. All automorphisms of $\Gamma$ are given by conjugation in $SO_3(\mathbb{C})$, so there is a unique Langlands parameter $\varphi$, with $A_\varphi \cong \Gamma$. Thus we should obtain four different representations of $SL_2$ over $\mathbb{Q}_p$!

Inside $SL_2(\mathbb{Q}_p)$, we have maximal compacts $K = SL_2(\mathbb{Z}_p)$ and $K'$, conjugation of $K$ by $(p_1)$. There are two representations of $SL_2(\mathbb{F}_p)$ of dimension $\frac{p-1}{2}$, which can then be lifted to $K$ and $K'$.

$L^G$ comes with a natural (principal) $SL_2$. Recall $L^G$ is a pinned group. Take a basis $X_\alpha$ of each $e_\alpha$, and let $X = \sum_{\alpha \in \Delta} X_\alpha$ in $\mathfrak{b}$, a principal nilpotent element. Over $\mathbb{C}$, we have $\langle X, Y, H \rangle \cong sl_2 \hookrightarrow \hat{\mathfrak{g}}$, and then exponentiating gives $SL_2 \to \hat{G}$. Also $X$ is fixed by $\text{Gal}(E/k)$, so the $SL_2$ is fixed by the Galois group. Thus we get a homomorphism

$$SL_2(\mathbb{C}) \times \text{Gal}(E/k) \to \hat{G} \times \text{Gal}(E/k) = L^G.$$  \hspace{1cm} (25.6)

We modify our definition of the Langlands parameter. For $k$ $p$-adic, a Langlands parameter is a homomorphism

$$\varphi : W(k^s/k) \times SL_2(\mathbb{C}) \to L^G$$  \hspace{1cm} (25.7)

for $W$ the Weil group, a subgroup of $\text{Gal}(k^s/k)$. Pairs $(\varphi, \rho)$ for $\rho$ irreducible representations of $A_\varphi$ should correspond to irreducible representations of $G(k)$.

26 Local Langlands

References:

- Borel, Corvallis I.
- Langlands, Problems in the theory of automorphic forms.
- Gross-Reeder, Laplace to Langlands.

Let $F$ be a field of characteristic $0$, $\overline{F}$ an algebraic closure, and $\Gamma_F = \text{Gal}(\overline{F}/F)$. Also let $G$ be a reductive group over $F$. Take $\mathcal{R}$ a based root datum of $G_{\overline{F}}$. Let $\hat{G}$ be the split reductive group over $\mathbb{C}$ with based root datum $\mathcal{R}^\vee$, equipped with a pinning $\mathcal{P}$ giving rise to $\mathcal{R}^\vee$. The map

$$\mu_G : \Gamma_F \to \text{Aut}(\mathcal{R}) \cong \text{Aut}(\mathcal{R}^\vee) \cong \text{Aut}(\hat{G}, \mathcal{P}).$$  \hspace{1cm} (26.1)

We will now define $L^G = \hat{G}(\mathbb{C}) \rtimes \Gamma_F$. There are possible variants of this construction:

- If $E/F$ is a Galois extension such that $\mu_G$ is trivial over $\Gamma_E$, we might take $L^G = \hat{G}(\mathbb{C}) \rtimes \text{Gal}(E/F)$.

- $\overline{\mathbb{Q}}_p$ may be used instead of $\mathbb{C}$.

Now assume $F$ is a local field. This means a locally compact topological field; for example, a finite extension of $\mathbb{Q}_p$ or $\mathbb{R}$. Let $W_F$ be the Weil group of $F$. In the $p$-adic case, $W_F$ comes from the exact sequence
0 → I_F → W_F → Z → 0. \hspace{1cm} (26.2)

For \( F = \mathbb{R} \), \( W_F = \mathbb{C}^\times \Pi \mathbb{C}^\times j \) where \( j^2 = -1 \) and \( jzj^{-1} = \overline{z} \) for \( z \in \mathbb{C}^\times \). And for \( F = \mathbb{C} \), \( W_F = \mathbb{C}^\times \).

Take \( G \) a reductive group over \( F \). If \( F \) is \( p \)-adic, define a Langlands parameter to be a continuous homomorphism

\[
\varphi : W_F \times SL_2(\mathbb{C}) \to ^LG
\]

satisfying:

1. The composition \( W_F \to ^LG \to \Gamma_F \) is the usual inclusion.
2. For every \( g \in W_F \), \( \varphi(g) \) is semisimple.
3. The induced homomorphism

\[
\varphi|_{SL_2(\mathbb{C})} : SL_2(\mathbb{C}) \to \widehat{G}(\mathbb{C})
\]

is induced by an algebraic homomorphism \( SL_2 \to \widehat{G} \).

If \( \varphi \) and \( \varphi' \) are Langlands parameters, we say that \( \varphi \) and \( \varphi' \) are equivalent if they are \( \widehat{G}(\mathbb{C}) \)-conjugate. Write \( \Phi(G) \) for the set of equivalence classes of parameters \( \varphi \), and \( \text{Irr}_G \) for the set of classes of irreducible admissible representations of \( G(F) \).

If \( F \) is \( \mathbb{R} \) or \( \mathbb{C} \), a Langlands parameter is a semisimple continuous homomorphism \( \varphi : W_F \to ^LG \) such that the composition \( W_F \to ^LG \to \Gamma_F \) is the usual inclusion. We say that \( \varphi \) and \( \varphi' \) are equivalent if they are \( \widehat{G}(\mathbb{C}) \)-conjugate. Again write \( \Phi(G) \) for the set of classes of parameters, and this time \( \text{Irr}_G \) is the set of isomorphism classes of irreducible admissible \((\mathfrak{g}, K)\)-modules. Here we view \( G(F) \) as a real Lie group, \( \mathfrak{g} \) its complexified Lie algebra, and \( K \subseteq G(F) \) any choice of maximal compact subgroup.

**Conjecture 26.1 (Local Langlands).** Let \( F \) be a local field and \( G \) a reductive group over \( F \). The there is a natural partition

\[
\text{Irr}_G = \bigsqcup_{\varphi \in \Phi_G} \Pi_{\varphi}
\]

into finite disjoint sets \( \Pi_{\varphi} \), called \( L \)-packets. If \( G \) is quasi-split, then every \( \Pi_{\varphi} \) is nonempty.

For this conjecture to be meaningful, there are extra conditions which characterize the sets \( \Pi_{\varphi} \).

**Known cases:**

1. If \( F \) is \( \mathbb{R} \) or \( \mathbb{C} \), then the conjecture is known, and the sets \( \Pi_{\varphi} \) can be written down explicitly using the parameterization of \( \widehat{G}_d \) as a starting point. (This is due to Langlands.)
2. If $F$ is $p$-adic and $G = GL_n$, then alls $\Pi_\varphi$ have cardinality 1, giving a natural bijection between $\text{Irr}_G$ and $\Phi(G)$. (Due to Harris-Taylor and Henniart.)

The correspondence is characterized by some compatibility relations from the theory of $L$-functions (observed by Henniart).

3. If $F$ is $p$-adic and $G$ is a quasi-split classical group (so of the form $SO$, $Sp$, or $U$), then the correspondence exists and is characterized by comparison with $GL_n$. (This is recent work of Arthur and Mok.)

4. If $F$ is $p$-adic and $G$ is unramified, then there’s a natural correspondence between unramified elements of $\text{Irr}_G$ and unramified Langlands parameters (those which factor through the unramified quotient $\mathbb{Z}$ of $W_F$ and are trivial on the $SL_2(\mathbb{C})$ factor).

5. If $F$ is $p$-adic and $G$ is a torus, then the correspondence exists, due to local class field theory. (It is LCFT if $G = GL_1$.)

27 The Unramified Correspondence

Now assume that $F$ is a finite extension of $\mathbb{Q}_p$, and $G/F$ is an unramified reductive group. Fix $K \subseteq G(F)$ a hyperspecial maximal compact.

**Theorem 27.1.** There is a canonical bijection

$$\{ \pi \in \text{Irr}_G | \pi^K \neq 0 \} \leftrightarrow \{ \text{unramified parameters } \varphi \in \Phi(G) \}.$$  \hspace{1cm} (27.1)

**Sketch of proof.** Let $S \subseteq G$ be a maximal split torus and $T = Z_G(S)$, a maximal torus of $G$. Let $W_d = N_G(S)/T$, a constant group acting faithfully on $S$. Let $E/F$ be the minimal extension splitting $G$. Then $E/F$ is unramified. Let Frob $\in \mathbb{Z}$ be the (arithmetic) Frobenius, and $\phi \in \text{Gal}(E/F)$ be the image of Frob. Think of $L^G$ as $\widehat{G}(\mathbb{C}) \times \text{Gal}(E/F)$. This contains the subset $\widehat{G}(\mathbb{C}) \times \{ \phi \}$ normalized by $G(\mathbb{C})$. If $\varphi : W_F \times SL_2(\mathbb{C}) \to L^G$ is an unramified Langlands parameter, then it’s determined by $\varphi(\text{Frob}) \in \widehat{G}(\mathbb{C}) \times \{ \phi \}$. Thus unramified classes in $\Phi(G)$ are in bijection with semisimple $\widehat{G}(\mathbb{C})$-conjugacy classes in $\widehat{G}(\mathbb{C}) \times \{ \phi \} \subseteq L^G$.

Write $\widehat{T}$ for the torus in the pinning $\mathcal{P}$ of $\widehat{G}$. We can assume that $X^*(\widehat{T}) = X_*(T_\mathcal{P})$. Then there’s a quotient $\widehat{T} \to \widehat{S}$, corresponding to inclusion $S \subseteq T$. Moreover, the group $W_d$ acts on $\widehat{T}$ by functoriality.

The Satake isomorphism produces a bijection between $k$-unramified representations of $\text{Irr}_G$ and $W_d$-conjugacy classes of unramified homomorphisms $T(F) \to \mathbb{C}^\times$.

Recall that if $T_c \subseteq T(F)$ is the maximal compact subgroup, then $X_*(S) \cong T/T_c$ by $\lambda \mapsto \lambda(\varpi)$, for $\varpi \in F^\times$ a uniformizer. The set $\text{Hom}(T/T_c, \mathbb{C}^\times)$ is then identified with the complex torus

$$\text{Hom}(X_*(S), \mathbb{C}^\times) = X^*(S) \otimes_{\mathbb{Z}} \mathbb{C}^\times$$  \hspace{1cm} (27.2)

$$= X_*(\widehat{S}) \otimes_{\mathbb{Z}} \mathbb{C}^\times$$  \hspace{1cm} (27.3)

$$= \widehat{S}(\mathbb{C}).$$  \hspace{1cm} (27.4)

So $k$-unramified elements of $\text{Irr}_G$ are in bijection with $\widehat{S}(\mathbb{C})/W_d$. 

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We finally claim that there are natural bijections between both \( \hat{S}(\mathbb{C})/W_d \) and \{ semisimple \( \hat{G}(\mathbb{C}) \)-conjugacy classes in \( \hat{G}(\mathbb{C}) \times \{ \phi \} \) \} with \( (\hat{T}(\mathbb{C}) \times \{ \phi \})/N_d \). Here \( N_d \) is the inverse image of \( W_d \) in \( N_{\hat{G}}(\hat{T}) \).

(See Borel, Automorphic L-functions, Corvallis II, §6.1.)

If \( G \) is split, then \( S = T, \hat{S} = \hat{T} \), and \( W_d = W(\hat{G}, \hat{T}) = W(G, T) \). If \( G = GL_n \), then \( \Phi(G) \) is the set of \( GL_n(\mathbb{C}) \)-classes of parameters \( \varphi : W_F \times SL_2(\mathbb{C}) \to GL_n(\mathbb{C}) \), and the unramified elements in \( \Phi(G) \) are in bijection with the semisimple conjugacy classes in \( GL_n(\mathbb{C}) \), or with the \( S_n \)-orbits in the set of invertible diagonal matrices. Every unramified representation \( \pi \in \text{Irr}_{GL_n(F)} \) is a subquotient of an induction \( i^G \chi_1 \otimes \cdots \otimes \chi_n \), where \( \chi_i : F^\times \to \mathbb{C}^\times \) are unramified characters, determined up to \( S_n \)-conjugacy. In this instance, the bijection of Theorem 27.1 is

\[
\pi \mapsto \begin{pmatrix}
\chi_1(\varpi) \\
\vdots \\
\chi_n(\varpi)
\end{pmatrix}
\] (27.5)

28 Global Langlands

Now assume that \( F \) is a number field and \( G \) is a reductive group over \( F \). Recall:

- For all but finitely many places \( v \) of \( F \), the group \( G_{F_v} \) is unramified.
- If \( \pi \) is an irreducible admissible representation of \( G(\mathbb{A}_F^\infty) \), the for almost every \( v \), \( G_{F_v} \) is unramified and \( \pi_v \) is unramified.

Choose for every place \( v \) of \( F \) an algebraic closure \( F_v \) and an embedding \( F \hookrightarrow F_v \) extending \( F \hookrightarrow F_v \). This gives a map \( \Gamma_{F_v} \hookrightarrow \Gamma_F \), hence a map

\[
L_{G_{F_v}} = \hat{G}(\mathbb{C}) \rtimes \Gamma_{F_v} \to \hat{G}(\mathbb{C}) \rtimes \Gamma_F = L_G.
\] (28.1)

If \( \pi \) is any irreducible admissible representation of \( G(\mathbb{A}_F^\infty) \), the unramified LLC gives, for almost all \( v \), a \( \hat{G}(\mathbb{C}) \)-conjugacy class \( t_{\pi_v} \subseteq L_{G_{F_v}} \subseteq L_G \). If \( v \) is a place at which \( G_{F_v} \) and \( \pi_v \) are unramified, we take \( t_{\pi_v} \) to be \( \varphi_v(\text{Frob}) \), where \( \varphi_v \) is the unramified parameter associated to \( \pi_v \). We call the collection of elements \( t_{\pi_v} \), defined for almost all \( v \), the Satake parameters of the representation \( \pi \).

If \( G \) and \( H \) are reductive groups over \( F \), then an admissible homomorphism \( \rho : L_G \to L_H \) is a continuous homomorphism satisfying:

1. The diagram

\[
\begin{array}{ccc}
L_G & \xrightarrow{\rho} & L_H \\
\downarrow & & \downarrow \\
\Gamma_F & \xrightarrow{} & \Gamma_F
\end{array}
\] (28.2)

commutes.
2. The induced homomorphism \( \hat{G}(\mathbb{C}) \to \hat{H}(\mathbb{C}) \) is induced by a homomorphism \( \hat{G} \to \hat{H} \) of algebraic groups over \( \mathbb{C} \).

**Conjecture 28.1** (Global Langlands). Let \( G \) and \( H \) be reductive groups over \( F \), \( \rho : {}^L G \to {}^L H \) admissible, and \( \pi \) an automorphic representation of \( G(\mathbb{A}_F) \). Assume \( H \) is quasi-split. Then there exists an automorphic representation \( \sigma \) of \( H(\mathbb{A}_F) \) such that \( t_{\sigma_v} = \rho(t_{\pi_v}) \) for almost all places \( v \) of \( F \).

Examples:

- \( H \) is the quasi-split inner form of \( G \), and \( \rho : {}^L G \to {}^L H \) is the identity, the conjecture says that if \( \pi \) is an automorphic representation of \( G(\mathbb{A}_F) \), then there exists an automorphic representation \( \sigma \) of \( H(\mathbb{A}_F) \) such that for almost all \( v \), \( G(F_v) \cong H(F_v) \) and \( \pi_v \cong \sigma_v \).

  If \( G = G_A \) for \( A \) a central simple algebra of rank \( n \) and \( H = GL_n \), this is the global Jacquet-Langlands correspondence.

- If \( G = \{1\} \) and \( H = GL_n \), choose \( \tau : \Gamma_F \to GL_n(\mathbb{C}) \) any continuous irreducible representation. Then \( \tau \) may be viewed as an admissible homomorphism

  \[
  {}^L G = \{1\} \times \Gamma_F \to {}^L H = GL_n(\mathbb{C}) \times \Gamma_F
  \]  

  (28.3)

  by \( \{1\} \times \sigma \mapsto \tau(\sigma) \times \sigma \). The conjecture then says that there exists an automorphic representation \( \pi \) of \( GL_n(\mathbb{A}_F) \) such that \( t_{\pi_v} \) is the conjugacy class of \( \tau(\text{Frob}_v) \) for almost all places (called the Strong Artin Conjecture).

29 Algebraic Representations

References:

- Clozel, Milne (Eds.), Ann Arbor Volumes.
- Buzzard, Gee, “On the conjectural relations between automorphic forms and...”.

The global Langlands correspondence aims to produce a bijection between algebraic automorphic representations and algebraic Galois representations, for some suitable definition of “algebraic” which we will describe.

29.1 Galois Representations

For \( F \) a number field, fix:

- \( \overline{F} \) an algebraic closure of \( F \).
- \( \overline{F}_v \) an algebraic closure of \( F_v \) for every place \( v \) of \( F \).
- For each \( v \), an embedding \( F \hookrightarrow \overline{F}_v \) extending \( F \hookrightarrow F_v \).
These choices give maps \( \Gamma_{F_v} \to \Gamma_F \). The image of this map is a decomposition group at \( v \).

If \( S \) is a finite set of finite places of \( F \), write \( F^S \subseteq \overline{F} \) for the maximal extension of \( F \) unramified away from \( S \). Write \( \Gamma_{F,S} = \text{Gal}(F^S/F) \).

If \( v \notin S \) is a finite place, then the map \( \Gamma_{F_v} \to \Gamma_F \to \Gamma_{F,S} \) factors through the unramified quotient \( \Gamma_{F_v} \to \widehat{\mathbb{Z}} \). Write \( \text{Frob}_v \in \Gamma_{F,S} \) to be the image of the geometric Frobenius (the inverse of the arithmetic Frobenius \( x \mapsto x^{q_v} \), for \( q_v = \# \mathcal{O}_F/\mathfrak{p}_v \)).

Let \( G \) be a reductive group over \( F \), \( \ell \) a prime, and \( \overline{\mathbb{Q}}_\ell \) an algebraic closure of \( \mathbb{Q}_\ell \). Let \( K/F \) be a finite Galois extension which splits \( G \). Now view \( \hat{G} \) as being defined over \( \overline{\mathbb{Q}}_\ell \). View \( L_G = \hat{G} \times \text{Gal}(K/F) \) as a linear algebraic group over \( \overline{\mathbb{Q}}_\ell \) with connected component \( \hat{G} \). We endow \( L_G(\overline{\mathbb{Q}}_\ell) \) with its natural \( \ell \)-adic topology (induced from an algebraic embedding into \( GL_N(\mathbb{Q}_\ell) \) for some \( N \)).

We will consider continuous homomorphisms \( \rho : \Gamma_F \to L_G(\overline{\mathbb{Q}}_\ell) \). We say that \( \rho \) is admissible if the composition \( \Gamma_F \to L_G(\overline{\mathbb{Q}}_\ell) \to \text{Gal}(K/F) \) is the restriction to \( K \).

1. We say that a continuous homomorphism \( \rho : \Gamma_F \to GL_n(\overline{\mathbb{Q}}_\ell) \) is algebraic if:
   
   • There is a finite set \( S \) such that \( \rho \) factors through \( \Gamma_F \to \Gamma_{F,S} \).
   
   • For all places \( v|\ell \) of \( F \), the restriction \( \rho|_{\Gamma_{F_v}} \) is de Rham.

2. We say that a continuous homomorphism \( \rho : \Gamma_F \to L_G(\overline{\mathbb{Q}}_\ell) \) is algebraic if for every algebraic representation \( R : L_G \to GL_N \), the composition \( R \circ \rho \) is algebraic. Equivalently, this holds for a single faithful \( R \).

For example, suppose \( X \) is a smooth, geometrically connected, projective variety over \( F \). Then the \( \ell \)-adic étale cohomology groups \( H^i(X_\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell) \) are finite dimensional \( \overline{\mathbb{Q}}_\ell \)-vector spaces on which \( \Gamma_F \) acts. The associated representations \( \rho_i : \Gamma_F \to GL(H^i(X_\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell)) \) are algebraic. It factors through \( \Gamma_{F,S} \) by the proper smooth base change theorem in étale cohomology. The de Rham condition follows from Faltings’ comparison theorem.

In the special case where \( X = E \) is an elliptic curve, \( H^1(E_\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell) \) is dual to \( T_1(E) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell \). In particular, after choosing a basis, we get \( \rho : \Gamma_F \to GL_2(\overline{\mathbb{Q}}_\ell) \). If \( v \nmid \ell \) is a place of good reduction for \( E \), then \( \rho \) is unramified at \( v \) and \( \text{tr} \rho(\text{Frob}_v) \) is an integer which is determined by the number of points of \( E(\mathcal{O}_F/\mathfrak{p}_v) \).

### 29.2 Automorphic Representations

Let \( F \) be a number field and \( G \) a reductive group over \( F \). Fix \( v|\infty \) of \( F \) induced by an embedding \( F \subseteq \mathbb{C} \). Consider \( G(F_v) \) as a real Lie group and \( \mathfrak{g} \) its complexified Lie algebra. Choose \( K_v \subseteq G(F_v) \) a maximal compact subgroup. Choose \( T \subseteq G_{\mathbb{C}} \) a maximal torus, \( t = \text{Lie}(T) \), and \( W = W(G_{\mathbb{C}}, T_{\mathbb{C}}) \).

Recall the Harish-Chandra isomorphism \( Z(\mathfrak{g}) \cong U(t)^W \). Also if \( V \) is an irreducible \((\mathfrak{g}, K)\)-module, then there exists \( \lambda \in t^* \) such that \( V \) has infinitesimal character \( \chi_{\lambda} \), where \( \chi_{\lambda} \) is the homomorphism

\[
Z(\mathfrak{g}) \overset{\sim}{\rightarrow} U(t)^W \subseteq U(t) \overset{\lambda}{\rightarrow} \mathbb{C}.
\]  

(29.1)

This determines \( \lambda \in \text{Hom}_{\mathbb{C}}(t, \mathbb{C}) = t^* \) up to \( W \)-conjugacy.
t* has an integral lattice $X^*(T) \hookrightarrow t^*$ by $\alpha \mapsto d\alpha$, and induces an isomorphism $X^*(T) \otimes_{\mathbb{Z}} \mathbb{C} \cong t^*$. View $\lambda_V$ as being in $X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$.

We say that $V$ is $L$-algebraic if $\lambda_V$ lies in $X^*(T)$. $V$ is $C$-algebraic if $\lambda_V + \rho \in X^*(T)$, where $\rho$ is half the sum of the positive roots, for some root basis.

If $\pi$ is an automorphic representation of $G(\mathbb{A}_F)$, then for every $v|\infty$, we get a $(\mathfrak{g}_v, K_v)$-module $\pi_v$. We say that $\pi$ is $L$-algebraic if $\pi_v$ is $L$-algebraic for every $v$. $\pi$ is $C$-algebraic if $\pi_v$ is $C$-algebraic for every $v$.

**Remark.** • In general, $\rho \in X^*(T) \otimes \mathbb{Q}$. If $\rho \in X^*(T)$, then being $L$-algebraic and $C$-algebraic are equivalent. In the case $GL_n$, this occurs if and only if $n$ is odd.

• The $C$ either stands for Clozel or cohomological. Clozel (in Ann Arbor) defined a notion of algebraic for $GL_n$ which coincides with $C$-algebraic. But Buzzard and Gee observed that if $\pi$ is $C$-algebraic, then there isn’t always a Galois representation $\rho: \Gamma_F \rightarrow LG(\overline{\mathbb{Q}}_{{\ell}})$, even for $G = PGL_2$ (as $^L G = SL_2$).

If $\pi$ is an automorphic representation of $GL_n$, we get $\chi_{\pi_v} \in X^*(T) \otimes \mathbb{Z} \subseteq \mathbb{Z}^n \otimes \mathbb{Z} \subseteq \mathbb{C}$. If $\rho: \Gamma_F \rightarrow GL_n(\mathbb{C})$ is a continuous representation, we get Hodge-Tate-Sen weights, living in $X^*(T) \otimes \mathbb{Z} \overline{\mathbb{Q}}_{{\ell}}$. If $\rho$ is de Rham, these weights live in $X^*(T)$.

For example, consider holomorphic modular forms of weight $k \geq 2$. Fix $k$ and $N \geq 1$. Suppose we are given $f \in S_k(\Gamma(N), \mathbb{C})$ which is an eigenfunction for all $T_p$ over $p \nmid N$. Associate to $f$ an automorphic representation $\pi$ of $GL_2(\mathbb{A}_F)$. For $p \nmid N$, $\pi_p = \chi_{B(p)}(\chi_1 \otimes \chi_2)$ (the normalized induction), where $\chi_1$ and $\chi_2$ are unramified characters and $\chi_1(p) + \chi_2(p) = \frac{a_p}{\sqrt{p}}$ where $T_p f = a_p f$.

There is a continuous character $| \cdot |: \mathbb{Q}^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$ given by $(x_p, x_{\infty}) \mapsto \prod_{p \leq \infty} |x_p|_p$. We can also consider $\pi_s = \pi \otimes | \det(g)|^s$ for every $s \in \mathbb{C}$. $\pi_s$ is also a cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$, which we can view as being associated to $f$. $\pi_s$ is unitary precisely when $\Re s = \frac{k-2}{2}$.

Let $T \subseteq GL_2(\mathbb{C})$ be the diagonal maximal torus and $\mathfrak{g} = gl_2(\mathbb{C})$. Then $Z(\mathfrak{g}) = \mathbb{C}[Z, \Delta] \cong U(\mathfrak{t})^W$, where $\Delta = \frac{1}{2}(H^2 + 2EF + FE)$. We have $\gamma(Z) = Z$ and $\gamma(\Delta) = \frac{1}{2}(H^2 - 1)$. An algebra homomorphism $U(\mathfrak{t})^W \rightarrow \mathbb{C}$ is induced by an element of $X^*(T)$ if and only if there exist $a, b \in \mathbb{Z}$ such that $Z \mapsto a + b$ and $H \mapsto a - b$, in which case $\Delta \mapsto \frac{1}{2}((a-b)^2 - 1)$.

On the other hand, we can show that $\pi_s$ has infinitesimal character $Z \mapsto 2s + k - 2$ and $\Delta \mapsto \frac{1}{2}((a-b)^2 - 1)$. This is $L$-algebraic if and only if $s \in \frac{1}{2} + \mathbb{Z}$ and $C$-algebraic if and only if $s \in \mathbb{Z}$. For instance, $\pi$ has a twist which is both $L$-algebraic and unitary if and only if $k$ is odd.

Let $F/\mathbb{Q}$ be a totally real field of degree $d$, and fix $\sigma_1, \ldots, \sigma_d$ real embeddings. Let $f$ be a cuspidal Hilbert modular form of weight $(k_1, \ldots, k_d)$, where each $k_i \geq 2$. We can lift $f$ to a cuspidal automorphic representation $\pi$, which is defined only up to a twist by some character.

Write $v_i$ for the place of $F$ induced by $\sigma_i$. Then $GL_2(F_{v_i}) = GL_2(\mathbb{R})$, so the local theory at $\infty$ is the same as that for $F = \mathbb{Q}$. To each $\pi_{v_i}$, we may associate $k_i \in \mathbb{Z}$ and $s_i \in \mathbb{C}$ such that $\pi_{v_i}$ has infinitesimal character $Z \mapsto 2s_i + k_i - 2$ and $\Delta \mapsto \frac{1}{2}((k_i - 1)^2 - 1)$.

Suppose $\pi_i$ is $L$-algebraic. Then $s_i \in \frac{1}{2} + \mathbb{Z}$ so $2s_i + k_i - 2 \in \mathbb{Z}$. As $\pi$ is cuspidal, there exists $w \in \mathbb{R}$ such that $\pi \otimes | \cdot |^w$ is unitary. Then $w = 2s_i + k_i - 2$, independently if $i = 1, \ldots, d$. In particular, $k_i \equiv 1 + w \pmod{2}$ is independent of $i$ modulo 2. This means we can lift $f$ to an $L$-algebraic $\pi$ only if the parity of the $k_i$ is independent of $i$. (One can show that this condition is also sufficient; see Clozel.)
30 Conjectured Relations

Fix a prime $\ell$ and an isomorphism $\iota : \mathbb{Q}_\ell \cong \mathbb{C}$. (We would expect that it would be enough to fix embeddings of $\mathbb{Q}$ into $\mathbb{C}$ and $\mathbb{Q}_\ell$.)

Suppose $\pi$ is an automorphic representation of $G$. Then there exists a finite set $S$ of finite places of $F$ such that for all finite places $v \notin S$, the Satake parameter $t_{\pi_v} \subseteq L(G(\mathbb{C}))$ is defined. (This means that $G_{F_v}$ and $\pi_v$ are unramified; then $t_{\pi_v}$ is a semisimple $\widehat{G}(\mathbb{C})$-conjugacy class.) Now consider $\iota^{-1} t_{\pi_v} \subseteq L(G(\mathbb{Q}_\ell))$.

**Conjecture 30.1** (Buzzard-Gee). Suppose $\pi$ is $L$-algebraic. Then there exists a finite set $T$ of places of $F$, containing the infinite places, the places dividing $\ell$, and the places at which either $G$ or $\pi_v$ is ramified, and an algebraic representation $r_{\iota}(\pi) : \Gamma_F \to L(G(\mathbb{Q}_\ell))$ satisfying:

1. $r_{\iota}(\pi)$ is unramified outside $T$.
2. For $v \notin T$, $r_{\iota}(\pi)(\text{Frob}_v) \in \iota^{-1} t_{\pi_v}$.

**Remark.** This conjecture is not optimal, since it doesn’t determine $r_{\iota}(\pi)$ uniquely. However, if $G = GL_n$, these conditions do determine $r_{\iota}(\pi)$ uniquely up to semisimplification and $\widehat{G}(\mathbb{Q}_\ell)$-conjugacy. For Cebotarev implies $\{\text{Frob}_v\}_{v \notin T} \subseteq \Gamma_{F,T}$ is a dense subset, and representations in $GL_n$ are determined by characters.

**Conjecture 30.2.** Let $S$ be a finite set of places of $F$, containing the places dividing $\infty$ and $\ell$. Then there is a bijection between:

1. Irreducible algebraic Galois representations $\rho : \Gamma_F \to GL_n(\mathbb{Q}_\ell)$ unramified outside $S$, up to isomorphism.
2. Cuspidal $L$-algebraic automorphic representations $\pi$ of $GL_n(A_F)$ which are unramified outside $S$.

This correspondence should be characterized by $\rho = r_{\iota}(\pi)$.

This conjecture is more or less implied by conjectures of Clozel and Fontaine-Mazur.

30.1 $G = GL_1$

If $v$ is a finite place of $F$, write $\text{Art}_{F_v} : F_v^\times \to \Gamma_{F_v}^{\text{ab}}$, normalized so that $\text{Art}_{F_v}(\varpi_v)|_{F_v^\times} = \text{Frob}_v$, the geometric Frobenius. Write $\text{Art}_F : A_F^\times \to \Gamma_F^{\text{ab}}$ for $\text{Art}_F = \prod_v \text{Art}_{F_v}$.

An automorphic representation of $GL_1(A_F) = A_F^\times$ is a Hecke character; that is, a continuous homomorphism

$$\chi = \prod_v \chi_v : F_v^\times \backslash A_F^\times \to \mathbb{C}^\times. \quad (30.1)$$

We would like to know when $\chi$ is $L$-algebraic (or equivalently $C$-algebraic, since $\rho = 0$).
If \( v \) is a real place, then \( \chi_v \) has the form
\[
\chi_v(x) = |x|^s \text{sgn}(x)^\epsilon
\]
for some \( s \in \mathbb{C} \) and \( \epsilon \in \{0,1\} \).

\( L \)-algebraicity means that the differential of \( \chi_v \) agrees with the differential of an algebraic character of \( GL_1 \). This occurs if and only if \( s \in \mathbb{Z} \).

If \( v \) is a complex place, we choose an embedding \( \tau : F \hookrightarrow \mathbb{C} \) corresponding to \( v \). Then \( \chi_v \) has the form
\[
\chi_v(z) = (\frac{\bar{z}}{|z|})^p |z|^s z^{\frac{s+p}{2}} z^{\frac{s-p}{2}}
\]
for some \( p \in \mathbb{Z} \) and \( s \in \mathbb{C} \). Let \( S = \text{Res}_{\mathbb{R}}^\mathbb{C} GL_1 \). Then \( S \) is a torus over \( \mathbb{R} \) of rank 2, having functor of points
\[
S(R) = (R \otimes_{\mathbb{R}} \mathbb{C})^\times
\]
for any \( \mathbb{R} \)-algebra \( R \). In particular, \( S(\mathbb{R}) = \mathbb{C}^\times \). The corresponding condition for \( \chi_v \) to be \( L \)-algebraic is that the differential of \( \chi_v \) agrees with the differential of an algebraic character of \( S \). This occurs if and only if the map has the form \( z \mapsto z^a \bar{z}^b \) for integers \( a,b \in \mathbb{Z} \). Equivalently, \( s + p \in 2\mathbb{Z} \).

Remark. These are exactly the “characters of type \( A_0 \)” described by Weil.

Here is a compact way to describe an \( L \)-algebraic character \( \chi \): \( \chi \) is \( L \)-algebraic if there exist integers \( n_\tau \) indexed by embeddings \( \tau : F \hookrightarrow \mathbb{C} \) such that for all \( x = (x_v) \in [(F \otimes \mathbb{Q})^\times]_0 \subseteq \mathbb{A}_F^\times \), we have
\[
\chi(x) = \prod_\tau x_\tau^{n_\tau}
\]
where \( x_\tau = \tau(x_v) \) for \( v \) the place of \( F \) determined by \( \tau \).

**Theorem 30.3.** Fix a prime \( \ell \) and an isomorphism \( \iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C} \), and an \( L \)-algebraic character \( \chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times \). Then there exists a unique representation \( r_\chi(x) : \Gamma_F \to \overline{\mathbb{Q}}_\ell^\times \) which satisfies the conclusion of the conjecture for \( GL_n \). \( r_\chi \) is given by the formula
\[
\iota \left( (r_\chi(x) \circ \text{Art}_F)(x) \prod_{\tau:F \hookrightarrow \overline{\mathbb{Q}}_\ell} x_\tau^{-n_\tau} \right) = \chi(x) \prod_{\tau:F \hookrightarrow \mathbb{C}} x_\tau^{-n_\tau}
\]
for every \( x \in \mathbb{A}_F^\times \).

Conversely, every algebraic representation \( \rho : \Gamma_F \to \overline{\mathbb{Q}}_\ell^\times \) arises in this way from a unique \( L \)-algebraic \( \chi \).

Remark. The integers \( n_\tau \) are the Hodge-Tate weights of \( r_\chi(x) \), up to sign.

Proof. This uses class field theory, and (for characters) being de Rham, Hodge-Tate, and “locally algebraic” are equivalent. (See Serre, Abelian \( \ell \)-adic representations.)
30.2 \( G = GL_n \)

A number field \( E \) is said to be CM if there exists \( c \in \text{Aut}(E) \) such that for all \( \tau : E \hookrightarrow \mathbb{C} \), \( \tau(x^c) = \overline{\tau(x)} \) for every \( x \in E \). If \( E \) is CM, let \( F = E^{c=1} \). Then either we have:

1. \( F = E \) is totally real.
2. \( F \) is totally real and \( E/F \) is a totally complex quadratic extension.

**Theorem 30.4.** Let \( E \) be a CM field, and \( \pi \) a cuspidal \( L \)-algebraic automorphic representation of \( GL_n(\mathbb{A}_E) \). Suppose that in addition:

1. If \( E \) is totally real, then \( \pi = \pi^\vee \).
2. If \( E \) is totally complex, then \( \pi^c = \pi^\vee \), where \( c \) acts on \( GL_n(\mathbb{A}_E) \).
3. \( \pi \) is regular. That is, if \( \tau : E \hookrightarrow \mathbb{C} \), and \( \lambda_\tau \in X^*(\hat{T}) \) is associated to the infinitesimal character, then \( \lambda_\tau \) is regular.

Then for every prime \( \ell \) and \( \iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C} \), there exists an algebraic representation \( r_\iota(\pi) : \Gamma_E \rightarrow GL_n(\overline{\mathbb{Q}}_\ell) \) such that \( r_\iota(\pi)(\text{Frob}_v)s \in \iota^{-1}t_{\pi_v} \) when \( \pi_v \) is unramified.

(Recall that any element \( g \in GL_n(\overline{\mathbb{Q}}_\ell) \) decomposes as \( g = gs gu \), where \( gs \) is semisimple and \( gu \) is unipotent.

This theorem relies on work of (among others) Chenevier, Clozel, Harris, Kottwitz, Labesse, Shin, and Taylor. Most \( r_\iota(\pi) \) can be found in the étale cohomology of Shimura varieties, but some of them need \( \ell \)-adic interpolation. See Chenevier-Harris in the Cambridge Mathematics Journal.

**Theorem 30.5.** Theorem 30.4 holds without hypothesis (1) or (2), with the provisor that \( r_\iota(\pi) \) are not yet known to be de Rham at the places dividing \( \ell \).

This is a result of Harris-Lan-Taylor-T. Again these are results of \( \ell \)-adic interpolation of \( r_\iota(\pi) \) of Theorem 30.4. Scholze has given another proof, which gives much more.

We also know the correspondence \( \pi \mapsto r_\iota(\pi) \) in some cases where \( \pi \) is not regular (Goldring) and in some cases where \( G \) is a classical group (\( Sp_{2n} \) or \( SO_n \)).

30.3 A Refinement of Theorem 30.4

Let \( F \) be a number field, and \( \pi \circ GL_n(\mathbb{A}_F) \) cuspidal and \( L \)-algebraic. Suppose \( \rho = r_\iota(\pi) \) is known to exist. This means that \( \rho(\text{Frob}_v)s \in \iota^{-1}t_{\pi_v} \) whenever \( \pi_v \) is unramified. Recall the local Langlands correspondence: for \( v \) a finite place of \( F \), there is a bijection \( \text{rec}_{F_v} \) between:

1. Irreducible admissible representations \( \sigma \) of \( GL_n(F_v) \), up to isomorphism.
2. Parameters \( \varphi : W_{F_v} \times SL_2(\mathbb{C}) \rightarrow GL_n(\mathbb{C}) \).

Local-global compatibility should hold, so it should be possible to calculate \( \rho|_{G_{F_v}} \) at ramified places in terms of \( \pi_v \), using \( \text{rec}_{F_v}(\pi_v) \).

You can say exactly what \( \rho_{|G_{F_v}} \) should be (when \( v \nmid \ell \)) using Grothendieck’s \( \ell \)-adic monodromy theorem: there is an equivalence of categories between:
1. Frobenius semisimple representations $\sigma : W_{F_v} \to GL_n(\overline{\mathbb{Q}}_\ell)$.

2. Parameters $\varphi : W_{F_v} \times SL_2(\mathbb{C}) \to GL_n(\mathbb{C})$.

This is worked out in Tate’s article “Number-theoretic background” in Corvallis II.

Local-global compatibility is known to hold in the setting of Theorem 30.4, by work of Taylor-Yoshida and Caraiani.

Suppose now that $E$ is a CM field, and $\pi \otimes GL_n(\mathbb{A}_E)$ is an automorphic representation satisfying the hypotheses of Theorem 30.4. Fix a prime $\ell$ and $\iota : \overline{\mathbb{Q}}_\ell \sim \mathbb{C}$, and fix a finite place $u_0$ of $E$ which is coprime to $\ell$.

**Theorem 30.6.** Suppose $\pi_v$ is unramified if $v \neq u_0$. Moreover, suppose that $\pi_{u_0}$ is a supercuspidal representation. Then $\varphi = rec_{E_{u_0}}(\pi_{u_0}) : W_{E_{u_0}} \times SL_2(\mathbb{C}) \to W_{E_{u_0}} \to GL_n(\mathbb{C})$ is an irreducible representation of $W_{E_{u_0}}$. The representation $\rho = r_{\iota}(\pi)$ is unramified at all $v \nmid \ell u_0$, and

$$\rho|_{W_{E_{u_0}}} \cong rec_{E_{u_0}}(\pi_{u_0}) \otimes_{\ell -1, \mathbb{C}} \overline{\mathbb{Q}}_\ell. \quad (30.6)$$

## 31 Fields With Limited Ramification

**References:**

- Chenevier, Number fields with given ramification. (Compositio)
- Chenevier, Clozel, Corps de nombres peu ramifiés. (JAMS)

**Theorem 31.1.** Let $p$ be a prime, $K/\mathbb{Q}_p$ a finite extension. Then there exists an extension $L/\mathbb{Q}$ such that $[L : \mathbb{Q}] = [K : \mathbb{Q}_p]$ and $L_v \cong K$ as $\mathbb{Q}_p$-algebras, where $v$ is the unique place of $L$ above $p$.

**Proof.** Choose $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$, and let $f_p(x) \in \mathbb{Q}_p[x]$ be its minimal polynomial. Write $f_p(x) = \sum_{i=0}^n a_i x^i$ with $a_n = 1$, and choose $f(x) \in \mathbb{Q}[x]$ such that $f(x) = \sum_{i=0}^b b_i x^i$ with $b_n = 1$, and $|a_i - b_i|_p < \epsilon$ for every $i$, for some fixed $\epsilon > 0$.

Krasner’s lemma states that if $\epsilon$ is small enough, $f(x)$ has a root $\beta \in K$ and $K = \mathbb{Q}_p(\beta)$. Now set $L = \mathbb{Q}[x]/f(x)$. \qed

**Corollary 31.2.** Choosing algebraic closures $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and an embedding $j : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, the map

$$j^* : \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \quad (31.1)$$

is injective.

The proof of Theorem 31.1 does not give control of $L$ at primes away from $p$. Weak approximation (Artin-Whaples) allows you to control $f$ at finitely many places.

Consider the following property of a finite set $S$ of primes and $p \in S P_{S,\beta}$: if $\mathbb{Q}_S$ denotes the maximal subfield of $\overline{\mathbb{Q}}$ unramified outside $S$, then the map
is injective. Equivalently, if $K/Q_p$ is a finite extension, then there exists a number field $L/Q$ unramified outside $S$, a place $v$ of $L$ above $p$, and an embedding $K \to L_v$ of $\mathbb{Q}_p$-algebras.

**Theorem 31.3** (Chenevier-Clozel). $P_{\{p,\ell\},p}$ holds if $p$ and $\ell$ are distinct primes.

If $E$ is a number field, $S$ is a finite set of finite places of $E$, and $v \in S$, we write $P_{E,S,v}$ for the property that the map

$$\text{Gal}(\overline{E}_v/E_v) \to \text{Gal}(E_S/E)$$

is injective.

To verify $P_{S,p}$, we want to construct various number fields. These can arise by constructing Galois representations with desirable properties, and we can obtain them by constructing certain automorphic representations.

**Lemma 31.4.** Let $m, r > 1$ be integers and $p$ be a prime. Then the set of primes $\ell$ such that the order of $m$ in $\mathbb{F}_\ell^*$ is divisible by $p^r$ is infinite.

**Proof.** It suffices to show the set is nonempty. Define the integer

$$t = \frac{m^{p^r} - 1}{m^{p^r-1} - 1} = (m^{p^r-1} - 1)^{p-1} + p(m^{p^r-1} - 1)^{p-2} + \cdots + p.$$  

Let $\ell$ be a prime dividing $t$. If $\ell \nmid (m^{p^r-1} - 1)$, then we’re done. Otherwise, $\ell | p$, so $\ell = p$.

If $p > 2$, then $p - 1 > 1$, so $p^2 | (m^{p^r-1} - 1)$, implying $p^2 \nmid t$. But also $t > p$, so we can choose $\ell$ such that $\ell \neq p$. If $p = 2$, then $t = (m^{2^{r-1} - 1})^2 + 2$, and checking residues modulo 4 shows that $4 \nmid t$. Again we can choose $\ell \neq 2$. $\square$

**Lemma 31.5.** Let $M$ be a finite extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}_p}$. Let $\Gamma_M$ be the absolute Galois group of $M$, $I_M$ the inertia subgroup, and $I^t_M$ be its tame quotient. Fix $\sigma \in \Gamma_M$. Suppose $\sigma$ acts trivially on $I^t_M$ by conjugation. Then $\sigma \in I_M$.

**Proof.** There is an isomorphism

$$I^t_M \cong \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$$

arising from

$$M^{tame} = M^{ur} (\sqrt{\overline{\varpi}_M} : p | N).$$

Let $\mu : \Gamma_M \to \hat{\mathbb{Z}} = \text{Gal}(M^{ur}/M)$ be the unramified quotient (where 1 corresponds to the arithmetic Frobenius), and $q = \#(\mathcal{O}_M/\varpi_M \mathcal{O}_M)$. If $\sigma \in \Gamma_M$, then conjugation by $\sigma$ acts on $I^t_M$ as
multiplication by \(q^{\mu(\sigma)} \in \prod_{\ell \neq p} \mathbb{Z}^\times_{\ell}\). If this action is trivial, then the image of \(q^{\mu(\sigma)}\) in \(\mathbb{F}^\times_{\ell}\) is trivial for all primes \(\ell\).

Now Lemma 31.4 shows that for all primes \(b\), \(\mu(\sigma)\) has trivial image in \(\mathbb{Z}/b^r\mathbb{Z} = \mathbb{Z}/b^r\mathbb{Z}\) for all \(r > 1\). Hence \(\mu(\sigma) = 0\), so \(\sigma \in I_M\).

**Corollary 31.6.** Let \(M \subseteq \overline{\mathbb{Q}}_p\) be finite over \(\mathbb{Q}_p\) and \(L/M\) be a (possibly infinite) Galois extension inside \(\overline{\mathbb{Q}}_p\). If \(LM^ur = \overline{\mathbb{Q}}_p\), then \(L = \overline{\mathbb{Q}}_p\).

**Proof.** Let \(H = \text{Gal}(\overline{\mathbb{Q}}_p/L)\) and \(I_M = \text{Gal}(\overline{\mathbb{Q}}_p/M^ur)\). These are closed, normal subgroups of \(\Gamma_M\). By hypothesis, \(H \cap I_M = \{1\}\). Hence \(HI_M = H \times I_M \subseteq \Gamma_M\). In particular, \(H\) commutes with \(I_M\). Lemma 31.5 then implies \(H \subseteq I_M\), so we must have \(H = \{1\}\).

**Lemma 31.7.** Let \(E\) be a number field, \(S\) a finite set of finite places of \(E\), and \(v \in S\). Choose \(\ell\) a prime, \(\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}\), and an embedding \(j : E \hookrightarrow \overline{E}_v\) extending the canonical map \(E \to E_v\). Suppose that for every irreducible continuous representation \(\rho : \text{Gal}(\overline{E}_v/E_v) \to GL_n(\mathbb{C})\), there exists a continuous representation \(R : \text{Gal}(E_S/E) \to GL_n(\overline{\mathbb{Q}}_\ell)\) such that

\[
R|_{I_{E_v}} \cong (\rho \otimes \mathbb{C}, \iota^{-1}, \overline{\mathbb{Q}}_\ell)|_{I_{E_v}}. 
\]

Then the property \(P_{E,S,v}\) holds.

**Remark.** One can check that such a \(\rho\) always has finite image. This is referred to as “\(GL_n(\mathbb{C})\) having no small subgroups”.

**Proof.** By Corollary 31.6, it’s enough to show that \(j(E_S)E_v^{ur} = \overline{E}_v\). Fix \(K/E_v\) a finite Galois extension inside \(\overline{E}_v\); we’ll show that \(K \subseteq j(E_S)E_v^{ur}\).

Let \(T = \overline{\mathbb{Q}}_\ell[\text{Gal}(K/E_v)]\), the regular representation of \(\text{Gal}(K/E_v)\). \(T\) decomposes as

\[
T = \bigoplus_\tau \tau^{\dim(\tau)}. 
\]

Applying the hypothesis of Lemma 31.7 to each representation \(\tau\), we obtain representations

\[
R_\tau : \text{Gal}(E_S/E) \to GL_n(\overline{\mathbb{Q}}_\ell)
\]

such that \(R_\tau|_{I_{E_v}} \cong \tau|_{I_{E_v}}\). Let \(R = \bigoplus_\tau R_\tau^{\dim(\tau)}\). Then \(R|_{I_{E_v}} \cong T|_{I_{E_v}}\). But \(T\) is a faithful representation of \(\text{Gal}(K/E_v)\), so \(T|_{I_{E_v}}\) is faithful on its inertia subgroup. As \(R\) descends to \(E_S\), we must have \(K \subseteq j(E_S)E_v^{ur}\).

**Corollary 31.8.** Let \(E\) be a totally complex CM field, and \(F \subseteq E\) its maximal totally real subfield. Let \(c \in \text{Gal}(E/F)\) be the unique nontrivial element. Fix a prime \(\ell\), a finite place \(w_0\) of \(E\) not dividing \(\ell\), and a finite set \(S\) of finite places of \(E\) such that \(\{w|\ell\} \cup \{w_0\} \subseteq S\).

Suppose that for all integers \(n \geq 1\) and for every supercuspidal representation \(\pi_0\) of \(GL_n(E_{w_0})\), there exists a cuspidal conjugate self-dual regular \(C\)-algebraic automorphic representation \(\Pi\) such that:

1. For every place \(w \notin S\) of \(E\), \(\Pi_w\) is unramified.
2. There exists an unramified character $\psi : E_{w_0}^\times \to \mathbb{C}^\times$ such that $\Pi_{w_0} \cong \pi_0 \otimes (\psi \circ \det)$.

Then $P_{E,S,w_0}$ holds.

Proof. We show that the conditions of Lemma 31.7 hold. Fix $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ and an irreducible representation $\rho : \Gamma_{E_{w_0}} \to GL_n(\mathbb{C})$. Recall the local Langlands correspondence for $GL_n(E_{w_0})$ gives a bijection $\operatorname{rec}_{E_{w_0}}^{-1}$. Let $\pi_0 = \operatorname{rec}_{E_{w_0}}^{-1}(\rho|_{W_{E_{w_0}}})$, a supercuspidal representation of $GL_n(E_{w_0})$. Let $\Pi$ be an automorphic representation of $GL_n(A_{E})$ as in the statement of Corollary 31.8. Then $\Pi_1 = \Pi \otimes \chi_{\frac{1}{2} - \frac{1}{2}n}$ is a cuspidal, regular, $L$-algebraic automorphic representation of $GL_n(A_{E})$, and has an associated Galois representation $r_{\iota}(\Pi_1) : \Gamma_E \to GL_n(\overline{\mathbb{Q}}_\ell)$ satisfying:

1. $r_{\iota}(\Pi_1)$ is unramified at all places $w \notin S$ of $E$ (as $\Pi_{1,w}$ is unramified and $w \nmid \ell$).

2. $r_{\iota}(\Pi_1)|_{I_{E_{w_0}}} \otimes \chi_{\frac{1}{2} - \frac{1}{2}n} \cong \rho|_{I_{E_{w_0}}}$. This follows from local-global compatibility and compatibility of the local Langlands correspondence with twisting by characters.

\[ \square \]

32 Chenevier’s Result

References:

- Mok, Endoscopic classification of representations of quasi-split unitary groups.

We won’t prove $P_{Q,(p,\ell),p}$; instead we’ll prove an earlier result of Chenevier:

**Theorem 32.1.**

1. Let $E$ be a totally complex CM field, and $w_0$ be a place of $E$ split over $F$, the maximal totally real subfield. Let $S = \{w|\ell\} \cup \{w_0, w_0^c\}$ for some prime $\ell$ with $w_0 \nmid \ell$. Then $P_{E,S,w_0}$ holds.

2. Let $p$ be a prime and $N$ an integer such that $-N$ is the discriminant of a quadratic imaginary field in which $p$ splits. Let $S$ be the set of places dividing $pN$. Then $P_{S,p} = P_{Q,S,p}$ holds.

Remark. 1. (2) follows from (1) immediately by taking $E$ the quadratic imaginary field of discriminant $-N$.

2. Examples of $(p, N)$ satisfying these conditions are $(2, 7), (2, 15), (3, 8), (3, 11), \ldots$. We deduce, for example, that $Q_{(2,7)}$ is dense in $\overline{Q}_2$.

We would like to apply the trace formula to construct an appropriate $\Pi$. However, $C$-algebraic representations in $GL_n(\mathbb{C})$ are not isolated in the unitary dual of $GL_n(\mathbb{C})$. They are, meanwhile, isolated in the subset of conjugate self-dual representations of $GL_n(\mathbb{C})$.

This suggests using the twisted trace formula, which picks out the conjugate self-dual representation. This method of approach was used by Clozel and Chenevier to verify $P_{Q,(p,\ell),p}$. We’ll use a different approach.
Let $F$ be a number field, $E/F$ a quadratic extension, and $c \in \text{Gal}(E/F)$ the nontrivial element. Let $G$ be a unitary group associated to a nondegenerate Hermitian form $\langle \ , \ \rangle$ on $E^n$, given by $\langle v, w \rangle = t^v c J w$ for some $J \in M_n(E)$. Recall:

- $G$ is the reductive group over $F$ having the functor of points
  \[ G(R) = \{ g \in GL_n(E \otimes_F R) \mid t^g c J g = J \} \tag{32.1} \]

- $G$ is an outer form of $GL_n$, which becomes split over $E$.

If $w$ is a place of $E$ above the place $v$ of $F$, consider the reductive group $G_{F_w}$. If $v$ is split, then $E \otimes_F F_v = E_w \times E_w^\mathrm{c}$ and so $G_{F_v} \cong GL_n(E_w)$. Write $\iota_w : G(F_v) \to GL_n(E_w)$ for such an isomorphism. On the other hand, if $v$ is inert or ramified, then $G_{F_v}$ is a “true unitary group” in the sense that it only becomes split over $E_w$.

The $L$-group of $G$ is $L^G = GL_n(C) \times \text{Gal}(E/F)$, where $c$ acts by the involution $g \mapsto \alpha(g) = \Psi_n \cdot g^{-1} \Psi_n^{-1}$. Here

\[ \Psi_n = \begin{pmatrix} \ddots & 1 \\ -1 \\ \vdots \end{pmatrix} \tag{32.2} \]

so that $\alpha$ is the unique nontrivial automorphism preserving the standard pinning of $GL_n(C)$.

Set $H = \text{Res}_{E/F}^L GL_{n,E}$, a reductive group over $F$ with its functor of points

\[ H(R) = GL_n(E \otimes_F R). \tag{32.3} \]

**Lemma 32.2.**

1. $H$ is a form of $GL_n \times GL_n$.

2. $L^H = (GL_n(C) \times GL_n(C)) \times \text{Gal}(E/F)$, where $c(h, h)c^{-1} = (h, g)$.

3. If $w$ is a place of $E$ above $v$ of $F$, there is an injection $\Phi(G_{F_v}) \hookrightarrow \Phi(H_{F_w})$ and a bijection $\Phi(H_{F_v}) \leftrightarrow \Phi(GL_{n,E_w})$.

**Proof.** (See Mok for details.)

If $R$ is an $E$-algebra, then $H(R) = GL_n(E \otimes_E R) \times GL_n(E \otimes_{(E,c)} R)$, so $H_E \cong GL_n \times GL_n$. This gives the form of $L^H$.

Recall that if $M$ is a reductive over $F_v$, then $\Phi(M)$ is the set of admissible $L$-parameters $\varphi : W_{F_v} \times SL_2(C) \to L^M$ (or $W_{F_v} \to L^M$), modulo conjugation by elements of $\tilde{M}(C)$. Also $\text{Irr}_{M(F_v)}$ is the set of irreducible admissible representations of $M(F_v)$ up to isomorphism.

To get a map $\Phi(G_{F_v}) \hookrightarrow \Phi(H_{F_w})$, we write down an admissible map $f : L^G \to L^H$ by

\[ f(g \times 1) = (g, t^g c^{-1}) \times 1 \tag{32.4} \]

\[ f(1 \times c) = (\Psi_n, \Psi_n^{-1}) \times c \tag{32.5} \]
One can check that this defines an admissible homomorphism.

Then to get a map $\Phi(H_{F_v}) \to \Phi(GL_{n,E_w})$, start with $\varphi : W_{F_v} \times SL_2(\mathbb{C}) \to L_H$. Then upon restriction to $E_w$, we obtain two maps $\varphi_1, \varphi_2 : W_{E_w} \times SL_2(\mathbb{C}) \to GL_n(\mathbb{C})$. The desired map is then $\varphi \mapsto \varphi_1$.

See Mok for the proof of injectivity and bijectivity.

Remark. The bijection $\Phi(\text{Res}^{E_w}_{F_v}GL_n) \leftrightarrow \Phi(GL_{n,E_w})$ is a special case of what Langlands calls “Shapiro’s lemma for $L$-groups”.

Combining the maps of Lemma 32.2, we get an injection $\Phi(G_{F_v}) \hookrightarrow \Phi(GL_{n,E_w})$. If we are in a situation where we know the local Langlands correspondence of $G(F_v)$, we get a map $\text{Irr}_{G(F_v)} \to \text{Irr}_{GL_n(E_w)}$. Such a map is called the local base change map. We know this map exists when:

1. $v$ is split, for then we simply take $\sigma_v \mapsto \sigma_v \circ \iota_w^{-1}$.
2. $v$ is inert in $E$ and $\varphi \in \Phi(G_{F_v})$ is unramified. The map $\sigma_v \mapsto \pi_w$ is described explicitly by Munguez in the Paris book project.
3. $v$ is archimedean.

**Conjecture 32.3** (Functoriality for $f : L_G \to L_H$). Let $\sigma$ be an automorphic representation of $G(\mathbb{A}_F)$. Then there should exist an automorphic representation $\Pi$ of $GL_n(\mathbb{A}_E)$ such that if $w$ is a place of $E$ over $v$ in $F$, and one of the following hold:

1. $v$ splits in $E$,
2. $v$ is inert and $\sigma_v$ is unramified,
3. $v$ is archimedean,

then $\Pi_w$ is given by the local base change applied to $\sigma_v$.

If $\Pi$ exists, we call it the base change of $\sigma$.

**Remark.** Some people call $\Pi$ the weak base change, as $\Pi_w$ is unspecified at finitely many places. They say that $\Pi$ is the strong base change if $\Pi_w$ is specified everywhere.

**Lemma 32.4.** Suppose $\sigma$ is an automorphic representation of $G(\mathbb{A}_F)$ such that:

1. There exists a place $w_0$ of $E$ which is split over $v_0$ of $F$ such that $\sigma_{w_0}$ is supercuspidal.
2. The base change $\Pi$ of $\sigma$ exists.

Then $\Pi$ is cuspidal and conjugate self-dual.

**Proof.** By definition, $\Pi_{w_0} \simeq \sigma_{w_0} \circ \iota_{w_0}^{-1}$ is supercuspidal. By the theory of Eisenstein series, if $\Pi$ is not cuspidal, then at every place $w$, $\Pi_w$ is a subquotient of a parabolic induction from a proper Levi subgroup. (See Langlands’ supplement in Corvallis I.) This is not the case at $w_0$, so $\Pi$ is cuspidal. This also implies that $\Pi \circ c$ and $\Pi^\vee$ are also cuspidal.
Now we use the strong multiplicity one theorem. We need to check that the cuspidal representations \( \Pi \circ c \) and \( \Pi^\vee \) are such that \( \Pi(\circ c)_w \cong \Pi^\vee_w \) are isomorphic for almost all \( w \). It’s enough to show that for all finite places \( w \) of \( E \) over \( v \) in \( F \) such that either \( w \) splits over \( F \), or \( v \) is inert in \( F \) and \( \sigma_v \) is unramified, \( \Pi(\circ c)_w \) and \( \Pi^\vee_w \) have the same Langlands parameter.

We’ll show this in the second case. Then \( c \in \text{Gal}(E_w/F_v) \) is nontrivial, and \( \sigma_v \) has parameter 
\[
\varphi : W_{F_v} \to GL_n(\mathbb{C}) \times \text{Gal}(E/F). \tag{32.6}
\]

The parameter of \( \Pi_w \) is \( \varphi|_{W_{E_w}} : W_{E_w} \to GL_n(\mathbb{C}) \). The parameter of \( \Pi(\circ c)_w \) is \( (\varphi|_{W_{E_w}})^c \), while the parameter of \( \Pi^\vee_w \) is \( \ell(\varphi|_{W_{E_w}})^{-1} \) by compatibility of the local Langlands correspondence for \( GL_n(E_w) \) with passage to contragredient. But we have
\[
(\varphi|_{W_{E_w}})^c = \varphi(\tilde{c})\varphi|_{W_E}\varphi(\tilde{c})^{-1}
\]
where \( \tilde{c} \) is a lift of \( c \). We have \( \varphi(\tilde{c}) = g \times c \) for some \( g \in GL_n(\mathbb{C}) \). The conjugation action of \( 1 \times c \) on \( GL_n(\mathbb{C}) \) is, by definition, just \( h \mapsto \Psi_n^{-1}h^{-1}\Psi_n^{-1} \). Then calculate to get the desired result.

**Theorem 32.5.** Suppose that \( E \) is a totally complex CM field, with maximal totally real subfield \( F \). Let \( F \) be the quasi-split unitary group in \( n \) variables associated to \( E/F \). Let \( \sigma \) be an automorphic representation of \( G(\mathbb{A}_F) \) such that:

1. There exists a place \( w_0 \) of \( E \) which is split over \( v_0 \) of \( F \) such that \( \sigma_{v_0} \) is supercuspidal.
2. For every place \( v|\infty \) of \( F \), \( \sigma_v \) is square-integrable.

Then the base change \( \Pi \) of \( \sigma \) exists, and is a cuspidal, conjugate self-dual, and regular \( C \)-algebraic automorphic representation of \( GL_n(\mathbb{A}_E) \).

**Proof.** For the existence of \( \Pi \), see Mok. We know that such a \( \Pi \) must be cuspidal and conjugate self-dual, so it remains to see that for \( w|\infty \), \( \Pi_w \) is regular and \( C \)-algebraic. To do this, recall that the square-integrable representations of \( G(F_v) \) for \( v|\infty \) are parameterized by their infinitesimal characters, which come from regular elements \( \lambda \in X^*(T_\mathbb{C}) \). In fact, any representation \( \tau \in \hat{G(F_v)}_d \) is regular and \( C \)-algebraic. A calculation using the behavior of infinitesimal characters under base change shows that the base change of \( \tau \) is also regular and \( C \)-algebraic.

We’ve reduced Theorem 32.1 to the following result:

**Theorem 32.6.** Let \( F \) be a number field and \( G \) a reductive group over \( F \). Suppose that for every place \( v|\infty \) of \( F \), \( G(F_v) \) has compact center and has square-integrable representations. (This forces \( F \) to be totally real.) Fix a finite place \( v_0 \) of \( F \), a supercuspidal representation \( \pi_0 \) of \( G(F_{v_0}) \), and an open compact subgroup \( K^{v_0} \) of
\[
G(\mathbb{A}_F^{\infty|v_0}) = \prod_{v|v_0|\infty} G(F_v). \tag{32.7}
\]

Then there exists a cuspidal automorphic representation \( \sigma \) of \( G(\mathbb{A}_F) \) such that:

1. For every \( v|\infty \), \( \sigma_v \) is square-integrable.
2. There exists an unramified character \( \psi : G(F_{v_0}) \to \mathbb{C}^\times \) such that \( \sigma_{v_0} \cong \pi_0 \otimes \psi \).
3. \( \sigma^{K_{v_0}} \neq 0 \).

We’ll prove this result using the simple trace formula. For simplicity, we’ll assume that \( F = \mathbb{Q} \), so \( v_0 \) is a prime \( p \), and that \( G \) has trivial center.

33 Producing the Representation

33.1 The Compact Case


For now, we’ll also assume \( G(\mathbb{R}) \) is compact. In this case, \( G \) is anisotropic, and we’re in the compact quotient case for \( G(\mathbb{Q}) \setminus G(\mathbb{A}) \). This means that for every \( \varphi \in C^\infty_c(G(\mathbb{A}_Q)) \),

\[
\text{tr } R(\gamma) = \sum_{\{\gamma\}} a(\gamma) O_\gamma(\varphi) = \sum_\Pi m(\Pi) \text{tr } \Pi(\varphi). \tag{33.1}
\]

Here we have:

- \( m(\Pi) \) is the multiplicity of the automorphic representation \( \Pi \) in \( L^2(G(\mathbb{Q}) \setminus G(\mathbb{A})) \).
- \( \{\gamma\} \) is a set of representatives for the conjugacy classes in \( G(\mathbb{Q}) \).
- \( a(\gamma) \) is a positive volume term.
- \( O_\gamma(\varphi) = \int_{G_\gamma(\mathbb{A}) \setminus G(\mathbb{A})} \varphi(x^{-1} \gamma x) \, dx \) where \( G_\gamma = Z_G(\gamma) \subseteq G \).

(Both \( a(\gamma) \) and \( O_\gamma(\varphi) \) depend on choices of measures, but the product \( a(\gamma) O_\gamma(\varphi) \) is independent of this choice.)

If \( C \subseteq G(\mathbb{A}_Q) \) is a compact subset, then we can find a finite subset \( S_C \subseteq \{\gamma\} \) such that for \( \varphi \) with \( \text{supp}(\varphi) \subseteq C \), \( O_\gamma(\varphi) \neq 0 \) only if \( \gamma \in S_C \).

We choose \( \varphi \) to be of the form

\[
\varphi = \varphi^{p,\infty} \otimes \varphi_p \otimes \varphi_\infty \tag{33.3}
\]

for \( \varphi^{p,\infty} \in C^\infty_c(G(\mathbb{A}^{p,\infty}_Q)) \), \( \varphi_p \in C^\infty_c(G(\mathbb{Q}_p)) \), and \( \varphi_\infty \in C^\infty_c(G(\mathbb{R})) \). Take \( \varphi^{p,\infty} \) to be the characteristic function of \( K^p \), \( \varphi_p \) the matrix coefficient of \( \pi_0^p \) (compactly supported as \( \pi_0 \) is supercuspidal), and \( \varphi_\infty = \varphi^{\lambda}_\infty \) is allowed to vary depending a parameter \( \lambda \).

Fix \( T \subseteq G_{\mathbb{R}} \) a maximal torus, \( W = W(G_{\mathbb{C}}, T_{\mathbb{C}}) \), \( \Phi \subseteq X^*(T_{\mathbb{C}}) \) the set of roots in \( G_{\mathbb{C}} \), \( S \subseteq \Phi \) a root basis, and \( \Phi^+ \) the set of \( S \)-positive roots. Recall that the isomorphism classes of irreducible representations of \( G(\mathbb{R}) \) are in bijection with the \( S \)-dominant weights \( \lambda \in X^+(T_{\mathbb{C}}) \). Let \( (\pi_\lambda, V_\lambda) \) be the representation associated to \( \lambda \). We’ll take

\[
\varphi^{\lambda}_\infty(\gamma) = \text{tr } \pi_\lambda(\gamma) \tag{33.4}
\]
for $S$-dominant weights $\lambda$.

Let $x_1, \ldots, x_n$ be a basis of the free $\mathbb{Z}$-module $X^*(T_C) \cong \text{Hom}(T(\mathbb{R}), \mathbb{C}^\times)$. We say that a function $Q : X^*(T_C) \to \mathbb{Z}$ is polynomial if it’s given by a polynomial in (the dual basis of) $x_1, \ldots, x_n$. We say that a function $f : T(\mathbb{R}) \to \mathbb{C}$ is rational if it’s given by a rational function in $x_1, \ldots, x_n$.

**Theorem 33.1** (Weyl Character Formula). 1. Fix $(\pi_\lambda, V_\lambda)$ and let $\gamma \in T(\mathbb{R})$ be a regular semisimple element. (Thus, for every $\alpha \in \Phi$, $\alpha(\gamma) \neq 1$.) Then

$$\text{tr} \pi_\lambda(\gamma) = \sum_{w \in W} \epsilon(w) \lambda^w(\gamma) \rho^w(\gamma) \prod_{\alpha \in \Phi^+} (1 - \alpha(\gamma)^{-1}).$$

(33.5)

Note that although we only have $\rho \in X^*(T_C) \otimes \mathbb{Q}$, when expanding out the above, the possible denominator of $\frac{1}{2}$ will go away. Also $\epsilon : W \to \{\pm 1\}$ is the sign character of the Weyl group.

2. Again fix $(\pi_\lambda, V_\lambda)$. Then

$$\text{tr} \pi_\lambda(1) = \dim V_\lambda$$

(33.6)

$$= \prod_{\alpha \in \Phi^+} (\lambda + \rho, \alpha^\vee) / \prod_{\alpha \in \Phi^+} \langle \rho, \alpha^\vee \rangle.$$  

(33.7)

Also $P(\lambda) = \dim C V_\lambda$ is a polynomial function.

For example, consider the case $G = SU_2$, the special unitary group of $z_1 \bar{z}_1 + z_2 \bar{z}_2$ on $\mathbb{C}^2$. Take $T \subseteq G$ the diagonal torus, $\gamma = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$. Identify $X^*(T_C)$ with $\mathbb{Z}$ by $\chi_n(\gamma) = e^{in\theta}$. We have $S = \{\chi_2\} \subseteq \Phi^+$. The dominant weights $\lambda \in X^*(T_C)$ are the nonnegative $n$. The representation $V_n$ of highest weight $\chi_n$ is the $n$th symmetric power of the standard representation $V_1$. We have

$$\text{tr} \pi_n(\gamma) = e^{in\theta} + e^{i(n-2)\theta} + \cdots + e^{-in\theta}$$

(33.8)

$$= \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}}$$

(33.9)

as long as $\chi_2(\gamma) \neq 0$. Also $\dim C V_n = n + 1$.

**Proposition 33.2.** Fix $\gamma \in T(\mathbb{R})$, and let $\lambda \in X^*(T_C)$ be an $S$-dominant weight which is allowed to vary. Then there is an expression

$$\text{tr} \pi_\lambda(\gamma) = \sum_{i=1}^{N_\lambda} E_i(\gamma, \lambda) P_i(\lambda)$$

(33.10)

a finite sum, where $E_i(\gamma, \lambda)$ is a rational function of degree depending on $\lambda$ such that:

- $E_i$ is uniformly bounded as $\lambda$ varies.
- The denominator is nonzero and independent of $\lambda$.  

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$P_i(\lambda)$ is a polynomial function $X^*(T_C) \to \mathbb{Z}$ of degree $\deg P_i < \deg P$ (for $P(\lambda) = \dim_{\mathbb{C}} V_\lambda$), except in the case $\gamma = 1$.

**Proof.** We'll consider the two extreme cases: either $\gamma$ is regular semisimple, or $\gamma = 1$. In the first case, take $P_i(\lambda) = 1$, $N_\gamma = 1$, and

$$E_1(\gamma) = \text{tr} \pi_\lambda(\gamma).$$  \hfill (33.11)

By the Weyl character formula, this is a rational function whose denominator does not depend of $\lambda$, and the numerator is a sum of $\#W$ roots of unity. If $\gamma$ is trivial, $N_\gamma = 1$, $E_i(\gamma, \lambda) = 1$, and $P_i(\lambda) = P(\lambda)$.

For a general $\gamma \in T(\mathbb{R})$, there is an argument of “descent to subgroups”. A proof can be found in Chenevier-Clozel.

**Corollary 33.3.** Let $\lambda_1, \lambda_2, \ldots$ be a sequence of $S$-dominant weights such that for every $\alpha \in S$, $\langle \lambda_i, \alpha^\vee \rangle \to \infty$ as $i \to \infty$. (We say that $\lambda_i \to \infty$ far from the walls.) Then for every $\gamma \in G(\mathbb{R})$ with $\gamma \neq 1$, we have

$$\lim_{i \to \infty} \frac{\text{tr} \pi_{\lambda_i}(\gamma)}{P(\lambda_i)} = 0.$$  \hfill (33.12)

**Proof.** Every $\gamma \in G(\mathbb{R})$ is conjugate to an element of $T(\mathbb{R})$, so we may WLOG let $\gamma \in T(\mathbb{R})$. Then we have

$$\frac{\text{tr} \pi_{\lambda_i}(\gamma)}{P(\lambda_i)} = \sum_{j=1}^{N_\gamma} E_j(\gamma, \lambda_i) \frac{P_j(\lambda_i)}{P(\lambda_i)}. \hfill (33.13)$$

Because $\gamma \neq 1$, we have $\deg P_j < \deg P$ for every $j$. $E_j(\gamma, \lambda_i)$ is uniformly bounded as $i$ varies, and $\frac{P_j(\lambda_i)}{P(\lambda_i)} \to 0$ as $\lambda_i \to \infty$ far from the walls.

We now return to the trace formula. Fix a sequence $\lambda_1, \lambda_2, \ldots$ of $S$-dominant weights such that $\lambda_i \to \infty$ far from the walls. We define $\varphi_i \in C_c^\infty(G(A_\mathbb{Q}))$ by

$$\varphi = \varphi^{p,\infty} \otimes \varphi_p \otimes \varphi^\lambda_\infty.$$  \hfill (33.14)

Observe that:

1. If $\Pi$ is an automorphic representation of $G(A_\mathbb{Q})$ such that $\text{tr} \Pi(\varphi_i) \neq 0$ for some $i$, then $\Pi$ has the desired properties of Theorem 32.6. Indeed, we can factor $\Pi$ as

$$\Pi = \Pi^{p,\infty} \otimes \Pi_p \otimes \Pi_\infty.$$  \hfill (33.15)

$\Pi^{p,\infty}(\varphi^{p,\infty})$ is the $K^p$-equivariant projection $\Pi^{p,\infty} \to (\Pi_p, \infty)^{K^p}$, so the trace is nonzero if and only if $(\Pi^{p,\infty})^{K^p} \neq 0$. Similarly, $\text{tr} \Pi_p(\varphi_p) \neq 0$ if and only if $\Pi_p \cong \pi_0$. (The key here is that a matrix coefficient of $\pi_0^\beta$ is a coefficient for $\pi_0$.) Finally,
\[ \text{tr } \Pi(\varphi_i) = (\text{tr } \Pi_p^{p,\infty}(\varphi_p^{p,\infty}))(\text{tr } \Pi_p(\varphi_p))(\text{tr } \Pi_\infty(\varphi_\infty)). \] (33.16)

So to prove Theorem 32.6, it’s enough to show that

\[ \text{tr } R(\varphi_i) = \sum_{\Pi} m(\pi) \text{tr } \Pi(\varphi_i) \] (33.17)

is nonzero.

2. The \( \mathcal{O}_\gamma(\varphi_i) \) split up, provided our global measure is a product measure:

\[ \mathcal{O}_\gamma(\varphi_i) = \mathcal{O}_\gamma(\varphi_p^{p,\infty})\mathcal{O}_\gamma(\varphi_p)^{\lambda_i}\mathcal{O}_\gamma(\varphi_\infty). \] (33.18)

By definition,

\[ \mathcal{O}_\gamma(\varphi_p^{p,\infty}) = \int_{G_\gamma(\mathbb{A}_p^{p,\infty}) \setminus G(\mathbb{A}_Q)} \varphi_p^{p,\infty}(x^{-1} \gamma x) \, dx \] (33.19)

and so on. (In general, any constants would not affect the proof anyway.)

We now apply the trace formula:

\[ \text{tr } R(\varphi_i) = \sum_{\{\gamma\}} a(\gamma)\mathcal{O}_\gamma(\varphi_i) \] (33.20)

\[ = a(1)\mathcal{O}_1(\varphi_i) + \sum_{\{\gamma\} \setminus \{1\}} a(\gamma)\mathcal{O}_\gamma(\varphi_p^{p,\infty})\mathcal{O}_\gamma(\varphi_p)^{\lambda_i}\mathcal{O}_\gamma(\varphi_\infty). \] (33.21)

Observe that \( \text{supp}(\varphi_i) \subseteq K^p \text{supp}(\varphi_p)G(\mathbb{R}) \), a compact subset of \( G(\mathbb{A}_Q) \) which does not depend on \( i \). By an earlier remark, the sum over \( \{\gamma\} \setminus \{1\} \) can be replaced by a sum over a finite subset \( S_C \subseteq \{\gamma\} \setminus \{1\} \).

We have \( \mathcal{O}_1(\varphi_i) = \varphi_i(1) \) as \( G_1 = G \). Assume in addition that \( \varphi_p(1) \neq 0 \). (We can always assume this, as \( \varphi_p(g) = \langle gv, w \rangle \) for \( v \in \pi_v^0 e c, w \in \pi_0^0 \).) Then we have

\[ \text{tr } R(\varphi_i) = a(1)\varphi_p(1)\varphi_\infty^{\lambda_i}(1) + \sum_{\gamma \in S_C} a(\gamma)\mathcal{O}_\gamma(\varphi_p^{p,\infty})\mathcal{O}_\gamma(\varphi_p)^{\lambda_i}\mathcal{O}_\gamma(\varphi_\infty) \] (33.22)

\[ = a(1)\varphi_p(1)P(\lambda_i) + \sum_{\gamma \in S_C} a(\gamma)\mathcal{O}_\gamma(\varphi_p^{p,\infty})\mathcal{O}_\gamma(\varphi_p)^{\lambda_i}. \] (33.23)

We divide by \( P(\lambda_i) \) to get

\[ \frac{\text{tr } R(\varphi_i)}{P(\lambda_i)} = a(1)\varphi_p(1) + \sum_{\gamma \in S_C} a(\gamma)\mathcal{O}_\gamma(\varphi_p^{p,\infty})\mathcal{O}_\gamma(\varphi_p)^{\lambda_i}. \] (33.24)
Hence if we can show that \( \frac{O_\gamma(\varphi_{\lambda_i})}{P(\lambda_i)} \to 0 \) as \( i \to \infty \), then we will know that \( \text{tr} R(\varphi_i) \neq 0 \) for \( i \) sufficiently large.

Fix \( \gamma \in S_C \). We choose a measure on \( G_\gamma(\mathbb{R}) \backslash G(\mathbb{R}) \) so that for every \( f \in C^\infty_c(G_\gamma(\mathbb{R}) \backslash G(\mathbb{R})) \), we have
\[
\int_{G_\gamma(\mathbb{R}) \backslash G(\mathbb{R})} f(x^{-1} \gamma x) \, dx = \int_{G(\mathbb{R})} f(x^{-1} \gamma x) \, dx. \tag{33.25}
\]

This is possible since \( G_\gamma(\mathbb{R}) \) is compact. We obtain
\[
O_\gamma(\varphi_{\lambda_i}) = \int_{G(\mathbb{R})} \text{tr} \pi_{\lambda_i}(x^{-1} \gamma x) \, dx \tag{33.26}
\]
\[
= \text{vol}(G(\mathbb{R})) \text{tr} \pi_{\lambda_i}(\gamma) \tag{33.27}
\]
as \( \text{tr} \pi_{\lambda_i} \) is conjugation-invariant. Hence
\[
\frac{O_\gamma(\varphi_{\lambda_i})}{P(\lambda_i)} = \frac{\text{vol}(G(\mathbb{R}))}{P(\lambda_i)} \frac{\text{tr} \pi_{\lambda_i}(\gamma)}{P(\lambda_i)} \tag{33.28}
\]
which tends to 0 as \( i \to \infty \).

### 33.2 The General Case

References:

- Chenevier-Clozel, JAMS.
- Clozel-Delorme, “Le théorème de Paley-Wiener invariant pour les groupes réductifs, I, II”.

We now assume that \( F = \mathbb{Q} \), \( v_0 = p \), and \( G \) has trivial center (for simplicity), but we will no longer assume \( G(\mathbb{R}) \) is compact.

Recall Arthur’s simple trace formula, Theorem 24.7.

The identity \( e \in G(\mathbb{R}) \) is \( \mathbb{R} \)-elliptic if and only if \( G_\mathbb{R} \) contains a \( \mathbb{R} \)-elliptic torus if and only if \( G(\mathbb{R}) \) has discrete series.

Recall that if \( C \subseteq G(\mathbb{A}_\mathbb{Q}) \) is a compact subset, then there exists a finite set \( S_C \subseteq \{ \gamma \} \), such that if \( \varphi \) satisfies the above conditions and \( \text{supp}(\varphi) \subseteq C \), then \( O_\gamma(\varphi) = 0 \) unless \( \gamma \in S_C \).

We will apply Theorem 24.7 to \( \varphi \) of the form
\[
\varphi = \varphi_p \otimes \varphi^{p,\infty} \otimes \varphi_{\infty,\lambda} \tag{33.29}
\]
where \( \varphi_p \) is a matrix coefficient of \( \pi_0^\gamma \), \( \varphi^{p,\infty} \) is the characteristic function of \( K_0^p \), and \( \varphi_{\infty,\lambda} \in C^\infty_c(G(\mathbb{R})) \) is a function which will vary depending on a parameter \( \lambda \). We’ll assume that \( O_\gamma(\varphi_{\infty,\lambda}) = 0 \) if \( \gamma \in G(\mathbb{R}) \) is not semisimple \( \mathbb{R} \)-elliptic. We assume that \( \varphi_p(1) = 1 \) and that \( \text{vol}(K^p) = 1 \).

Fix \( T \subseteq G_\mathbb{R} \) an \( \mathbb{R} \)-elliptic maximal torus. The existence of \( T \) is equivalent to the existence of a compact inner form \( H \) of \( G(\mathbb{R}) \). We fix such an \( H \) and an isomorphism \( f : G_C \xrightarrow{\sim} H_C \) such that \( f^{-1} \circ \mathcal{F} \in \text{Aut}(G_C) \) is an inner automorphism. Define \( T_{H,C} = f(T_C) \).
Fact. We can choose \( f \) so that \( T_{H,\mathbb{C}} \subseteq H_{\mathbb{C}} \) is defined over \( \mathbb{R} \), and the restriction \( f|_{T_{\mathbb{C}}} : T_{\mathbb{C}} \rightarrow T_{H,\mathbb{C}} \) is also defined over \( \mathbb{R} \).

Choose such an \( f \), and let \( T_H \subseteq H \) be the real torus which recovers \( T_{H,\mathbb{C}} \) after extending scalars.

For example, if \( p + q = n \), let \( G = U(p, q) \) be the group over \( \mathbb{R} \) defined by the hermitian form

\[
z_1\overline{z}_1 + \cdots + z_p\overline{z}_p - z_{p+1}\overline{z}_{p+1} - \cdots - z_n\overline{z}_n.
\] (33.30)

We may take \( T \subseteq G \) the diagonal torus, and \( T_H \) the diagonal torus of \( H = U(n) \), the definite unitary group. It’s an exercise to find such an \( f \) in this case.

Let \( W = W(G_\mathbb{C}, T_\mathbb{C}) = W(H_\mathbb{C}, T_{H,\mathbb{C}}) \) and \( W_\mathbb{R} = N_{G(\mathbb{R})}(T(\mathbb{R}))/T(\mathbb{R}) \leftrightarrow W \). Recall that the square-integrable representations \( \pi \) fill into packets \( \Pi_\lambda \) indexed by \( W \)-orbits of regular elements \( \lambda \in X^*(T_\mathbb{C}) \). Each packet \( \Pi_\lambda \) contains exactly \( \#(W/W_\mathbb{R}) \) elements. The elements \( \pi \in \Pi_\lambda \) are exactly the square-integrable representations of \( G(\mathbb{R}) \) with infinitesimal character

\[
\chi_\lambda : Z(\mathfrak{g}) \rightarrow U(t)^W \rightarrow U(t) \overset{\sigma_\lambda}{\rightarrow} \mathbb{C}.
\] (33.31)

Similarly, the irreducible representations \( \sigma \) of \( H(\mathbb{R}) \) are in bijection with the \( W \)-orbits of regular elements \( \lambda \in X^*(T_{H,\mathbb{C}}) \). We have \( \sigma_\lambda \leftrightarrow \lambda \) if \( \sigma_\lambda \) has infinitesimal character \( \chi_\lambda \). This differs from the highest weight parameterization by a shift by \( \rho \).

Another way to phrase this parameterization of discrete series representations of \( G(\mathbb{R}) \) is to put them in packets \( \Pi_\sigma \) where \( \sigma \) ranges over irreducible representations of \( H(\mathbb{R}) \). The assignment \( \sigma \mapsto \Pi_\sigma \) does not depend on the choice of \( f \). The assignment \( \sigma \mapsto \Pi_\sigma \) is a special case of (local) Langlands functoriality, since the \( L \)-groups of \( G_\mathbb{R} \) and \( H_\mathbb{R} \) are the same.

Dual to the transfer \( \sigma \mapsto \Pi_\sigma \) is a transfer of conjugacy classes. Let \( \gamma \in G(\mathbb{R}) \) be semisimple \( \mathbb{R} \)-elliptic. Then \( \gamma \) is \( G(\mathbb{R}) \)-conjugate to some element of \( T(\mathbb{R}) \). Without loss of generality, let \( \gamma \in T(\mathbb{R}) \), and then take \( \gamma_H = f(\gamma) \in T_H(\mathbb{R}) \), well-defined up to \( H(\mathbb{R}) \)-conjugacy.

**Theorem 33.4.** Let \( \lambda \in X^*(T_\mathbb{C}) \) be sufficiently regular. (For example, \( \langle \lambda, \alpha^\vee \rangle \gg 0 \) for all \( \alpha \in S \) a root basis.) Then there exists a function \( \varphi_{\infty, \lambda} \in C^\infty_c(G(\mathbb{R})) \) satisfying:

1. If \( \pi \) is a unitary irreducible representation of \( G(\mathbb{R}) \), then

\[
\text{tr} \pi(\varphi_{\infty, \lambda}) = \begin{cases} 1 & \pi \in \Pi_\lambda \\ 0 & \text{otherwise.} \end{cases}
\] (33.32)

2. If \( \gamma \in G(\mathbb{R}) \) is not semisimple elliptic, then \( \mathcal{O}_\gamma(\varphi_{\infty, \lambda}) = 0 \).

3. If \( \gamma \) is elliptic, then there is a sign \( e(\gamma) \) such that \( \mathcal{O}_\gamma(\varphi_{\infty, \lambda}) = e(\gamma) \text{tr} \sigma_\lambda(\gamma_H^{-1}). \)

**Remark.**

1. In the case \( G_\mathbb{R} = H_\mathbb{R} \), we could take \( \varphi_{\infty, \lambda}(\gamma) = \text{tr} \sigma_\lambda(\gamma) \).

2. (1) of the theorem actually implies (2) and (3).

3. The proof of (1) can be found in Clozel-Delorme.

4. We can choose all of the \( \varphi_{\infty, \lambda} \) to have support in a fixed compact subset of \( G(\mathbb{R}) \).
Write $\widehat{G}$ for the unitary dual of $G(\mathbb{R})$; that is, the set of isomorphism classes of unitary irreducible representations, and $\widehat{G}_t \subseteq \widehat{G}$ for the subset of tempered representations. Then $\widehat{G}$ has a natural topology, and the classification of tempered representations allows you to describe $\widehat{G}_t$ explicitly. There’s a map $C_c^\infty(G(\mathbb{R})) \to \text{Fun}(\widehat{G}_t, \mathbb{C})$ given by $\varphi \mapsto (\pi \mapsto \text{tr}(\varphi))$. The theorem of Clozel-Delorme characterizes the image of this map.

We now apply the simple trace formula with $\varphi_\lambda = \varphi_p \otimes \varphi^{p,\infty} = \varphi_{\infty,\lambda}$. We obtain

$$\text{tr} R_0(\varphi) = \sum_{\{\gamma\}} a(\gamma) O_\gamma(\varphi_p) O_\gamma(\varphi^{p,\infty}) O_\gamma(\varphi_{\infty,\lambda}). \quad (33.33)$$

Assume, as we may, that as $\lambda$ varies, all $\varphi_\lambda$ have support in a fixed compact $C \subseteq G(\mathbb{A}_\mathbb{Q})$. Then the formula equals

$$a(1) O_1(\varphi_{\infty,\lambda}) + \sum_{\gamma \in S_C} a(\gamma) O_\gamma(\varphi_p) O_\gamma(\varphi^{p,\infty}) O_\gamma(\varphi_{\infty,\lambda}) \quad (33.34)$$

where $S_C \subseteq \{\gamma\} \setminus \{1\}$ is a finite set of $\mathbb{R}$-elliptic elements of $G(\mathbb{Q})$. We now use the identity $O_\gamma(\varphi_{\infty,\lambda}) = e(\gamma) \text{tr} \sigma_\lambda(\gamma_H^{-1})$ to get

$$\text{tr} R_0(\varphi) = a(1) e(1) \dim C \sigma_\lambda + \sum_{\gamma \in S_C} a(\gamma) O_\gamma(\varphi_p) O_\gamma(\varphi^{p,\infty}) e(\gamma) \text{tr} \sigma_\lambda(\gamma_H^{-1}) \quad (33.35)$$

$$\frac{\text{tr} R_0(\varphi)}{\dim C \sigma_\lambda} = a(1) e(1) + \sum_{\gamma \in S_C} a(\gamma) O_\gamma(\varphi_p) O_\gamma(\varphi^{p,\infty}) e(\gamma) \frac{\text{tr} \sigma_\lambda(\gamma_H^{-1})}{\dim C \sigma_\lambda}. \quad (33.36)$$

Let $\lambda \to \infty$ far from the walls. Then the above expression tends to $a(1) e(1) \neq 0$. In particular, if $\lambda$ is sufficiently far from the walls, then there exists a cuspidal $\Pi$ such that $\text{tr} \Pi(\varphi_\lambda) \neq 0$. Equivalently, $\text{tr} \Pi_p(\varphi_p) \neq 0$, $\text{tr} \Pi^{p,\infty}(\varphi^{p,\infty}) \neq 0$, and $\text{tr} \Pi_\infty(\varphi_{\infty,\lambda}) \neq 0$. This means $\Pi_p \cong \pi_0$, $\Pi^{Kp} \neq 0$, and $\Pi_\infty \in \Pi_\lambda$. So $\Pi$ has the desired properties.