1 Overview

In the first half, we’ll construct algebraic invariants and go over the classification of high dimensional simple knots. For reference, see Farber, Classification of simple knots.

In the second half, we’ll compute invariants and discuss connections to number theory. Time permitting, we’ll go over applications of these invariants.

In its most generality, knot theory studies the embeddings of manifolds inside other manifolds. Classically, people considered knotted circles inside $S^3$. We consider an equivalence relation on embeddings $S^1 \hookrightarrow S^3$.

We might restrict to either smooth embeddings or topological embeddings. However, topological embeddings may have “wild” embeddings such as and infinite shrinking sequence of knots converging on a point. We will want our knots to be locally flat, meaning locally isomorphic to $\mathbb{R}^1$ inside $\mathbb{R}^3$. Another way of avoiding pathological examples would be to consider piecewise linear embeddings. It turns out that for $S^1 \hookrightarrow S^3$, equivalence classes of smooth embeddings, locally flat topological embeddings, and piecewise linear embeddings are the same.

An isotopy of embeddings $f_0, f_1 : M \hookrightarrow N$ for manifolds $M$ and $N$ is a map $F : [0,1] \times M \to N$ which is smooth, continuous, etc. depending on which class of embeddings we’re using, such that
\( F_0 = f_0, F_1 = f_1, \) and \( F_t \) is an embedding for every \( t \). However, we can make any knot isotopic to the standard circle! This can be done by preserving one flat piece while shrinking the rest of the knot to a point.

To fix this, we will consider ambient isotopies. An ambient isotopy between \( f_0 \) and \( f_1 \) is a map \( F : [0,1] \times N \to N \) such that for each \( t \), \( F_t \) is a self (diffeomorphism, homeomorphism) of \( N \), and such that \( F_0 \) is the identity and \( F_1 \circ f_0 = f_1 \). Alternatively, we have a commutative diagram

\[
\begin{array}{ccc}
N & \xrightarrow{F_1} & N \\
\downarrow{f_0} & & \downarrow{F_1} \\
M & \xrightarrow{f_1} & N
\end{array}
\]

for \( F_1 \) some self-homeomorphism in the connected component of the identity in \( \text{Homeo}(N,N) \) (or the set of diffeomorphisms).

**Fact.**
- \( \text{Diffeo}(S^3) \) has two connected components, distinguished by their orientation (the action on \( H_3(S^3) \) as an automorphism of \( \mathbb{Z} \)).
- \( \text{Homeo}(S^n) \) also has two connected components.

A (possibly) weaker form of equivalence when the ambient manifold is \( S^N \) is induced by orientation-preserving homeomorphisms or diffeomorphisms. The homeomorphism version coincides with the relation of (ambient) isotopy.

In the case \( S^n \to S^N \), if \( n \leq N - 3 \), we can always unknot topological and piecewise linear knots, and the smooth knots are finitely generated. Meanwhile for \( n = N - 1 \), the Schoenflies theorem shows the problem is not interesting.

We’ll consider codimension 2. An \( n \)-knot is an equivalence class of embeddings \( S^n \to S^{n+2} \). We might also consider \( K \to S^{n+2} \) for \( K \) homeomorphic to \( S^n \) but not diffeomorphic, even in the category of smooth embeddings.

Here locally flat means the knot locally looks like \( \mathbb{R}^n \) inside \( \mathbb{R}^{n+2} \). In particular, smooth implies locally flat. We also want to have a tubular neighborhood \( N \) of \( S^n \) inside \( S^{n+2} \), where \( N \) is the image of a map \( D^2 \times S^n \to S^{n+2} \) restricting to our given embedding. We can always accomplish this in the smooth case.

We summarize the possible equivalences below:

<table>
<thead>
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<th></th>
<th>smooth</th>
<th>locally flat</th>
<th>piecewise linear</th>
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<tbody>
<tr>
<td>(ambient) isotopy</td>
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<tr>
<td>induced by orientation-preserving</td>
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A complement of a knot \( K \) inside \( S^{n+2} \) is the noncompact manifold \( S^{n+2} \setminus K \). Equivalent knots of homeomorphic (or diffeomorphic) complements. If \( N \) is a tubular neighborhood of \( K \), then \( S^{n+2} \setminus \text{int}(N) \) is a compact manifold with boundary, of the same homotopy type as the knot complement. This manifold is called the knot exterior.

Classical knots are determined by the homeomorphism class of their complements, along with the class of a meridian (loop around the knot) inside the complement. This isn’t always true in higher dimensions.
As a first attempt to say something about the homotopy type of $S^{n+2} \setminus K$, we use Alexander duality. The form we use here is

$$\tilde{H}^i(K) \cong \tilde{H}_{(n+2) - 1 - i}(S^{n+2} \setminus K).$$

(1.2)

which is canonical once orientations for $K$ and $S^{n+2}$ have been chosen. Hence we find

$$\tilde{H}_j(S^{n+2} \setminus K) = \begin{cases} \mathbb{Z} & j = 1 \\ 0 & \text{otherwise} \end{cases}.$$

(1.3)

The homology is independent of the class of the knot $K$. However, we could look at its fundamental group and its covering spaces. In particular, there is a unique infinite cyclic cover of $S^{n+2} \setminus K$, and we can look at the homology groups of the covering space. This leads to construction of Alexander modules.

Another thing we could consider is $V \subseteq S^{n+2}$ of codimension 1, such that $\partial V = K$. Such a $V$ is called a Seifert manifold, and has interesting topological structure. In both situations, we’ll be looking at duality pairings.

2 Classical Knot Theory

Let $K : S^1 \hookrightarrow S^3$. We know that $H_1(\text{ext}(K)) \cong \mathbb{Z}$ by Alexander duality. Let $C_\infty \rightarrow \text{ext}(K)$ be the infinite cyclic cover, with covering group generated by $t$. The choice of $t$ will depend on the orientation of $K$.

We will look at $H_*(C_\infty)$. We know that $H_1$ can be expressed in terms of $G = \pi_1(\text{ext}(K))$. We have $G/[G, G] \cong \mathbb{Z}$. Letting $H = [G, G]$, we have $H_1(C_\infty) = H/[H, H]$. This abelian group has an action by $\langle t \rangle$, making it a module over $\Lambda = \mathbb{Z}[t, t^{-1}]$. The module $H_1(C_\infty)$ is called the Alexander module.

We would like to understand the $\Lambda$-module structure. There are techniques for doing this:

- Fox calculus, or free differential calculus. We’ll give an example of this method, done geometrically.
- We could calculate $H_1(C_\infty)$ directly using a nice topological decomposition of $C_\infty$.

Fix a point $x \in \text{ext}(K)$. We’ll first compute the relative homology $H_1(C_\infty, \pi^{-1}(x); \mathbb{Z})$ and then use the exact sequence

$$0 = H_1(\pi^{-1}(x); \mathbb{Z}) \rightarrow H_1(C_\infty; \mathbb{Z}) \rightarrow H_1(C_\infty, \pi^{-1}(x); \mathbb{Z}) \rightarrow H_0(\pi^{-1}(x); \mathbb{Z}) \cong \Lambda$$

(2.1)

A choice of isomorphism for $H_0$ depends on choosing $\bar{x} \in \pi^{-1}(x)$. The other points are then $t^k\bar{x}$.

We’ll give an example of the computation of the relative homology group. Let $K$ be the trefoil knot, also known as a $(2, 3)$ torus knot. We have

$$\pi_1(\text{ext}(K)) = \langle \alpha, \beta | \alpha^2 = \beta^3 \rangle.$$  

(2.2)
(One generator goes around the torus, while the other goes through the central circle.) We can take $t$ to be the image of $\alpha \beta^{-1}$ in $H_1(\text{ext}(K))$.

Claim. The classes $[\overline{\alpha}]$ and $[\overline{\beta}]$ of $H^1(C_\infty, \pi^{-1}(x))$ generate $H_1(C_\infty, \pi^{-1}(x))$ as a $\Lambda$-module.

Because $\alpha^{-2} \beta^3$ is trivial in $\pi_1$, it will lift to a trivial loop arising from two $\overline{\alpha}^{-1}$ pieces and three $\overline{\beta}$ pieces, shifted appropriately. Specifically, we have

$$
(1 + t^2 + t^4)[\overline{\beta}] - (1 + t^3)[\overline{\alpha}] = 0
$$

in $H_1(C_\infty, \pi^{-1}(x); \mathbb{Z})$.

Claim. This relation generates the space of relations over $\Lambda$.

That is,

$$
M = H_1(C_\infty, \pi^{-1}(x)) = (\Lambda[\overline{\alpha}] + \Lambda[\overline{\beta}])/((t^4 + t^2 + 1)[\overline{\alpha}] - (t^3 + 1)[\overline{\beta}])
$$

We have a map $\varphi : M \to \Lambda \overline{x}$ and wish to determine the kernel. We have $\varphi([\overline{\alpha}]) = (t^3 - 1)\overline{x}$ and $\varphi([\overline{\beta}]) = (t^2 - 1)\overline{x}$. An arbitrary element in $M$ will be in $\ker \varphi$ if

$$
p(t, t^{-1})(t^3 - 1) + q(t, t^{-1})(t^2 - 1) = 0
$$

and

$$
p(t, t^{-1})(1 + t + t^2) = q(t, t^{-1})(-1 - t).
$$

Hence $\ker \varphi$ is generated by

$$
[\zeta] = (-1 - t)[\overline{\alpha}] + (1 + t + t^2)[\overline{\beta}].
$$

$[\zeta]$ is annihilated by $t^2 - t + 1$, as $(t^4 + t^2 + 1)[\overline{\beta}] - (t^3 + 1)[\overline{\alpha}] = (t^2 - t + 1)[\zeta]$. We find

$$
H_1(C_\infty; \mathbb{Z}) \cong \Lambda/(t^2 - t + 1).
$$

3 Alexander Duality

[We are now following the Farber article, The Classification of Simple Knots.]

**Theorem 3.1** (Alexander Duality). If $X$ is a compact, locally contractible, nonempty, proper sub-space of $S^N$, then

$$
\tilde{H}_i(S^N \setminus X; \mathbb{Z}) \cong \tilde{H}^{N-i-1}(X, \mathbb{Z}).
$$

**Proof.** Given $[\alpha] \in H_i(S^N \setminus X; \mathbb{Z})$ and a cycle $\beta \in Z_{N-i-1}(X; \mathbb{Z})$, we must have $\alpha = \partial \gamma$ for some $\gamma \in Z_{i+1}(S^N, S^N \setminus X)$. Then $[\gamma] \in H_{i+1}(S^N, S^N \setminus X)$. Now take the element of $H^{N-i-1}(X; \mathbb{Z})$ represented by the cocycle $\beta \mapsto \beta \sim \alpha$. This correspondence can be shown to be an isomorphism. □
We apply this to a tubular neighborhood of a knot and its exterior. The associated perfect pairing

\[ \text{lk} : H_i(K; \mathbb{Z}) \times H_{n+i-1}(\text{ext}(K); \mathbb{Z}) \to \mathbb{Z} \]  

(3.2)

is called the linking number.

For classical knots, we can compute the linking number from diagrams. Suppose \( \alpha \) and \( \beta \) are disjoint oriented loops in \( S^3 \). Draw diagrams so that all of the crossings are visible. Crossings where the bottom strand moves left relative to the top strand are called positive; those where the bottom strand moves right are called negative. The positive crossings between \( \alpha \) and \( \beta \) contribute \( \frac{1}{2} \) to the linking number, and the negative crossings contribute \(-\frac{1}{2}\).

To do (1)

The algorithm tells us that for loops \( \alpha \) and \( \beta \) in \( S^3 \),

\[ \text{lk}(\alpha, \beta) = \text{lk}(\beta, \alpha). \]

**Theorem 3.2.** If \( X_1, X_2 \subseteq S^{m+n+1} \) are such that \( X_1 \approx S^m \), \( X_2 \approx S^n \), and \( X_1 \) and \( X_2 \) are disjoint, then

\[ \text{lk}(X_1, X_2) = (-1)^{mn+1} \text{lk}(X_2, X_1). \]  

(3.3)

**Sketch of proof.** Let \( X = X_1 \sqcup X_2 \) and look at the cohomology ring of \( X \). We have \( \widetilde{H}^i(S^{m+n+1} \setminus X) = \widetilde{H}_{m+n-i}(X) \). Meanwhile,

\[ \widetilde{H}_j(X) = \begin{cases} ([X_1] - [X_2])\mathbb{Z} & j = 0 \\ [X_1]\mathbb{Z} & j = n \\ [X_2]\mathbb{Z} & j = m \\ 0 & \text{otherwise}. \end{cases} \]  

(3.4)

(If \( m = n \), we get \( \mathbb{Z} \oplus \mathbb{Z} \) for \( j = m \).) Let \([\alpha_{X_1, X_2}] \) be the generating class in \( \widetilde{H}^0 \) We get classes \([X_1]^\vee\), \([X_2]^\vee\), and \([\alpha_{X_1, X_2}]^\vee \) in \( \widetilde{H}^*(S^{m+n+1} \setminus X) \) of dimensions \( n \), \( m \), and \( m+n \). We now have

\[ [X_1]^\vee \smile [X_2]^\vee = \text{lk}(X_1, X_2)[\alpha_{X_1, X_2}]^\vee \]  

(3.5)

\[ [X_2]^\vee \smile [X_1]^\vee = \text{lk}(X_2, X_1)[\alpha_{X_2, X_1}]^\vee \]  

(3.6)

But \([\alpha_{X_2, X_1}] = -[\alpha_{X_1, X_2}] \) and cup product is anticommutative. The result follows.

\[ \square \]

4 Seifert Surfaces and Hypersurfaces

We can consider oriented manifolds whose boundary is a knot. For example, this can be done for the trefoil.
A **Seifert surface** (or hypersurface) for a knot $K$ in $S^{n+2}$ is an $(n + 1)$-fold $V$ with boundary equal to $K$. The following properties hold:

- Any knot $K$ in any dimension has a Seifert hypersurface.
- In $S^{n+2}$, we want the fiber of $S^{n+2} \setminus K \to S^1$ to be a Seifert hypersurface.
- In the classical case, there is a nice algorithm from the knot diagram. Follow along the knot, but every time you reach a branch, change strands. The loops you’ll get will correspond to pieces of a projected surface. Finally, whenever you have a crossing, add a “twisty strip”. This is called Seifert’s algorithm.

**Theorem 4.1.** Given $K$, there exists an $(n + 1)$-fold $V$ with boundary $K$.

**Sketch of proof.** Take a tubular neighborhood $N$, so that $N \approx K \times D^2$ and $\partial N \approx K \times S^1$. Let $X$ be the exterior of $K$. We have a map $\partial X \to S^1$, which we want to extend to all of $X$. Obstruction theory tells us that the only possible obstructions are nonzero elements of $H^2(X, \partial X; \mathbb{Z})$. But

$$H^2(X, \partial X; \mathbb{Z}) \cong H^2(S^{n+2}, N; \mathbb{Z}) \cong H^2(S^{n+2}, K; \mathbb{Z}) \cong H^1(K; \mathbb{Z})$$

which is zero provided $n \geq 2$.

Now we have a smooth map $\varphi : X \to S^1$ which on $\partial X$ is the projection map $K \times S^1 \to S^1$. Take $p \in S^1$ such that $\varphi^{-1}(p)$ is a manifold; call it $V_0$. Now

$$\partial V_0 = V_0 \cap \partial X = K \times \{p\}.$$  

Then fix this to get a manifold $V$ with $\partial V = K$. 

\[\square\]
We would like $V$ to not be especially complicated. We might ask when a knot $K$ has an $r$-connected Seifert hypersurface. Suppose this is the case, and choose such a hypersurface $V$. We will give a necessary condition involving the homotopy type of $\text{ext}(K)$.

Take a tubular neighborhood $N_V$ of $V$ which is topologically $V \times [-1, 1]$ with $V = V \times 0$. Now

$$S^{n+2} \setminus K = \text{int}(N_V) \cup S^{n+2} \setminus \underbrace{W}_{V}.$$  \hfill (4.5)

We know that $\text{int}(N_V)$ is $r$-connected. Also $V$ being $r$-connected and Alexander duality will give results for $W$.

**Lemma 4.2.** $W$ is simply connected.

**Proof.** We can write

$$S^{n+2} = W \cup (N_V \cup N).$$  \hfill (4.6)

Let $W' = N_V \cup N$. $W \cap W'$ is connected. ($W \cap N_V$ will have two components, $V \times \{>0\}$ and $V \times \{<0\}$. Then adding in $W \cap N$ will join those components on the outside.) So van Kampen gives us the amalgamated diagram

$$\begin{array}{ccc}
\pi_1(W \cap W') & \longrightarrow & \pi_1(W) \\
\downarrow & & \downarrow \\
\pi_1(W') & \longrightarrow & \pi_1(S^{n+2})
\end{array}$$  \hfill (4.7)

and we know that $\pi_1(S^{n+2}) = 1$ and that $\pi_1(W') = \pi_1(V) = 1$. Also $\pi_1(W \cap W') = 1$. Hence we must have $\pi_1(W) = 1$. \hfill $\square$

We’ll now show that $W$ is $r$-connected. It’s enough to show that $H_i(W; \mathbb{Z}) = 0$ for $2 \leq i \leq r$. But we have

$$H_i(W; \mathbb{Z}) \cong \tilde{H}^{n+1-i}(V; \mathbb{Z}) \cong H_i(V, \partial V; \mathbb{Z})$$  \hfill (4.8)

by Alexander and Poincaré duality. But now $V$ is $r$-connected and $\partial V$ is $(n-1)$-connected, so this relative homology is 0.

Also, $N_V \cap W \cong V \times \left([-1, 0) \cup (0, 1]\right)$ is a disjoint union of two $r$-connected components. Since $S^{n+2} \setminus K = N_V \cup W$, we can determine $\pi_i(S^{n+2} \setminus K)$ for $i \leq r$. We have an infinite cyclic cover of the union (alternating $N_V$ and $W$ pieces), which will then be $r$-connected. We end up with

$$\pi_i(S^{n+2} \setminus K) = \begin{cases} 1 & i = 0 \\ \mathbb{Z} & i = 1 \\ 0 & 2 \leq i \leq r. \end{cases}$$  \hfill (4.9)

**Theorem 4.3.** The following are equivalent:
(a) $K$ bounds an $r$-connected Seifert manifold $V$.

(b) $\pi_i(S^{n+2} \setminus K) \cong \pi_i(S^1)$ for $i \leq r$.

(c) The infinite cyclic cover $C_\infty$ of $\text{ext}(K)$ is $r$-connected.

**Sketch of proof.** We’ve just shown (a) implies (b) (via (c)), and (b) and (c) are equivalent.

For (b) implying (a), we follow Levine ’65. Choose some Seifert surface $V$ for $K$. Suppose $[\alpha]$ is a nontrivial class in $\pi_i$ for some $1 \leq i \leq r$. Then we can glue in a handle that makes $[\alpha]$ trivial.

**Corollary 4.4.** If such a $V$ exists with $r = \frac{n+1}{2}$, then $V$ is contractible. If $n \neq 2, 4$, then $V$ is homeomorphic to $D^{n+1}$. In this case, we can shrink $K$ to an unknot.

**Proof.** That $V$ contractible implies $V$ is homeomorphic to a disc follows from the h-cobordism theorem in high dimensions, and specific arguments in low dimensions.

An $n$-knot $K$ is said to be simple if it bounds an $\frac{n-1}{2}$-connected Seifert hypersurface.

- There will be different behavior depending on the parity of $n$.
- When $n = 1, 2$, every knot is simple.
- Knots arising from isolated singularities of complex hypersurfaces are simple.

## 5 The Seifert Endomorphism

Suppose $K \subseteq S^{n+2}$ and $V$ is such that $\partial V = K$. Let $W = S^{n+2} \setminus V$. For $\alpha \in H_i(V; \mathbb{Z})$, we can define $\alpha_+$ by pushing $\alpha$ along the normal bundle in the positive direction, and $\alpha_-$ by pushing $\alpha$ in the negative direction. We get two morphisms $H_i(V; \mathbb{Z}) \to H_i(S^{n+2} \setminus V; \mathbb{Z})$. These in fact arise from maps $i_+, i_- : V \to S^{n+2} \setminus V$ caused by pushing points.

(We only push along the interior. Here we take $V$ to be the interior of the Seifert hypersurface.)

**Fact.** The map $H_i(V; \mathbb{Z}) \to H_i(S^{n+2} \setminus V; \mathbb{Z})$ given by $(i_+)_* - (i_-)_*$ is an isomorphism. This formally follows from Mayer-Vietoris on a cover of $S^{n+2} \setminus K$ and $N_V \cup (S^{n+2} \setminus V)$.

The **Seifert endomorphism** of $H_i(V; \mathbb{Z})$ is the endomorphism defined by $a \mapsto za$, such that

$$(i_+)_* - (i_-)_*(za) = i_+_* (a). \quad (5.1)$$

Here is a geometric definition: given $a \in H_i(V; \mathbb{Z})$, take an $(i+1)$-chain $b$ in $S^{n+2} \setminus K$ with boundary equal to $i_+(a)$. Then $za$ is the class of $b \cap V$ in $H_i(V; \mathbb{Z})$.

Here is a classical example: take $K$ a trefoil knot and $V$ a Seifert surface.
We have $H_1(V; \mathbb{Z}) \cong \langle \alpha_1, \alpha_2 \rangle$. Here $\alpha_1$ loops between the first and second columns, and $\alpha_2$ loops between the second and third columns; they intersect. Meanwhile $H_1(S^3 \setminus V; \mathbb{Z}) \cong \langle \beta_1, \beta_2 \rangle$ where $\beta_1$ surrounds the first column and $\beta_2$ surrounds the second column.

We have $i_{+,*}(\alpha_1) = \beta_2$ (the knot gets pushed to the right near the twisty strips), and $i_{-,*}(\alpha_1) = -\beta_1$. Also $i_{+,*}(\alpha_2) = -\beta_1 - \beta_2$ (the class of the third column), and $i_{-,*}(\alpha_2) = -\beta_2$. 
Now observe that

\[(i_{+,\ast} - i_{-,\ast})(\alpha_1) = \beta_1 + \beta_2\]  \hspace{1cm} (5.2)
\[(i_{+,\ast} - i_{-,\ast})(\alpha_2) = -\beta_1\]  \hspace{1cm} (5.3)

Now \(z\alpha_1\) is the solution to \((i_{+,\ast} - i_{-,\ast})(z\alpha_1) = \beta_2\), which is \(\alpha_1 + \alpha_2\). Similarly, \(z\alpha_2 = -\beta_1 - \beta_2\).

The matrix of the Seifert endomorphism with respect to the \(\alpha_i\) is

\[
\begin{pmatrix}
1 & -1 \\
1 & 0
\end{pmatrix}
\]  \hspace{1cm} (5.4)
The Seifert endomorphism makes \( H_i(V; \mathbb{Z}) \) into a \( \mathbb{Z}[z] \)-module.

In the trefoil case, our module is isomorphic to \( \mathbb{Z}[z]/(z^2 - z + 1) \).

Now recall that we have a perfect pairing

\[
\langle \ , \ \rangle : H_i(V; \mathbb{Z}) \times H_{n+1-i}(V; \mathbb{Z}) \to \mathbb{Z} \tag{5.5}
\]

by Poincaré duality.

**Fact.** \( \langle za, b \rangle = \langle a, zb \rangle \), where \( z = 1 - \bar{z} \). The bar gives an involution on \( \mathbb{Z}[z] \).

**Proof.** We need to know that

\[
\langle a, b \rangle = \text{lk}(a, (i_{+,*} - i_{-,*})(b)). \tag{5.6}
\]

**Proof.** Consider a cylinder over \( b \) and intersect it with \( a \). Since \( b \) intersects the cylinder transversally, and the intersection is in \( V \), this intersection number is \( \langle a, b \rangle \). \( \square \)

Now we have, for \( j = n + 1 - i \),

\[
\text{lk}(a, i_{+,*}(b)) = (-1)^{ij+1} \text{lk}(i_{+,*}(b), a) \tag{5.7}
\]

\[
= (-1)^{ij+1} \text{lk}(b, i_{-,*}(a)) \tag{5.8}
\]

Suppose \( a \in H_i(V) \) and \( b \in H_j(V) \). Then

\[
\langle za, b \rangle = (-1)^{ij} \langle b, za \rangle \tag{5.9}
\]

\[
= (-1)^{ij} \text{lk}(b, (i_{+,*} - i_{-,*})(za)) \tag{5.10}
\]

\[
= (-1)^{ij} \text{lk}(b, i_{+,*}(a)) \tag{5.11}
\]

\[
= -\text{lk}(a, i_{-,*}(b)). \tag{5.12}
\]

Now \( i_{-,*}(b) = -(i_{+,*} - i_{-,*})(\bar{z}b) \). We end up with

\[
\text{lk}(a, (i_{+,*} - i_{-,*})(\bar{z}b)) = \langle a, \bar{z}b \rangle. \tag{5.13}
\]

So for \( K \) an \( n \)-knot and \( V \) of dimension \( n + 1 \) with \( \partial V = K \), we’ve given \( H_i(V; \mathbb{Z}) \) the structure of a module over \( P = \mathbb{Z}[z] \). For \( i = \frac{n+1}{2} \) we have additional structure.

An isometric structure is a \( P \)-module \( A \) with a \( \mathbb{Z} \)-bilinear form

\[
\langle \ , \ \rangle : A \times A \to \mathbb{Z} \tag{5.14}
\]

such that:

- \( A \) is a finitely generated abelian group. (Some definitions also require \( A \) to be torsion free.)
The pairing $\langle \cdot, \cdot \rangle$ is $\epsilon$-symmetric for $\epsilon = \pm 1$.

The two homomorphisms $A \to \text{Hom}_\mathbb{Z}(A, \mathbb{Z})$ are epimorphisms with kernels $T(A)$, the $\mathbb{Z}$-torsion subgroup of $A$.

We have $\langle za, b \rangle = \langle a, zb \rangle$ for $a, b \in A$.

For example, if $i = \frac{n+1}{2}$, $H_i(V; \mathbb{Z})$ is an isometric structure with $\epsilon = (-1)^i$.

We want equivalence relations on $P$-modules and isometric structures such that if $V, W$ are Seifert hypersurfaces for $K$, then $H_i(V; \mathbb{Z})$ is equivalent to $H_i(W; \mathbb{Z})$.

Two $P$-modules $A, B$ are $m$- adjoining if there exist $P$-module homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $\varphi \circ \psi = (z\overline{z})^m$ on $B$ and $\psi \circ \varphi = (z\overline{z})^m$ on $A$.

If $A$ and $B$ are $m$-adjoining and $B$ and $C$ are $n$-adjoining, then $A$ and $C$ are $(m+n)$-adjoining.

Two isometric structures are adjoining if we have $\varphi, \psi$ as above, along with the property that

$$\langle \varphi(a), b \rangle = \langle a, \psi(b) \rangle. \quad (5.15)$$

$R$-equivalence is the equivalence relation generated by $1$-adjoining.

Meanwhile, we say that two Seifert hypersurfaces $V, W$ with $\partial V = \partial W = K$ are $1$-adjoining if $\text{int}(V) \cap \text{int}(W) = \emptyset$. In this case, the union $V \cup W$ decomposes $S^{n+2}$ into 2 components, $M$ (the inside) and $N$ (the outside). A decomposition of $M$ into handles will give a sequence of surgeries from $V$ to $W$. We define $R$-equivalence on embedded hypersurfaces with boundary $K$ as the equivalence relation generated by being $1$-adjoining.

**Proposition 5.1.** If $V$ and $W$ are $1$-adjoining, then $H_i(V)$ and $H_i(W)$ are $1$-adjoining for every $i$.

**Proof.** Define $\psi : H_i(W) \to H_i(V)$ as follows:

$$
\begin{array}{ccc}
H_i(\text{int}(W)) & \xrightarrow{j_*} & H_i(S^{n+2} \setminus V) \\
\downarrow \sim & & \uparrow i_{+, -} - i_{-, +} \sim \\
H_i(W) & \xrightarrow{\psi} & H_i(V)
\end{array}
\quad (5.16)
$$

Here $j$ is the inclusion of $\text{int}(W)$ into $S^{n+2} \setminus V$. We may define $\varphi$ similarly, by switching the roles of $V$ and $W$.

We put a $P$-module structure on $H_i(V \cup W)$. If $U \subseteq S^{n+2}$ is a compact, connected $(n+1)$-manifold, then pick a disk $D^{n+1} \subseteq U$. For $i \leq n$, we have

$$H_i(U) \cong H_i(U \setminus \text{int}(D)). \quad (5.17)$$

Then $H_i(U \setminus \text{int}(D))$ is a $P$-module, so this induces a $P$-module structure on $H_i(U)$. This structure is independent of the choice of $D$.

**Lemma 5.2.** If $U^{n+1} \subseteq S^{n+2}$ is a closed connected oriented submanifold, then $z\overline{z}$ annihilates $H_i(U)$ for $1 \leq i \leq n$. 

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Proof. Write $S^{n+2} \setminus U$ as $M \amalg N$. Then we have a map
\[ r_* \oplus s_* : H_i(U) \rightarrow H_i(M) \oplus H_i(N) \] (5.18)
arising from Mayer-Vietoris on a cover of $S^{n+2}$ by thickened neighborhoods of $M$ and $N$. Hence any $a \in H_i(U)$ may be written as $a = a_1 + a_2$, where $s_*(a_1) = 0$ and $r_*(a_2) = 0$.

Claim. $za_1 = 0$ and $za_2 = 0$.

Proof. We have $i_+s_*(a_1) = 0$ in $H_i(N)$ and therefore in $H_i(S^{n+2} \setminus V)$.

But $i_+s_*(a_1) = (i_+s_* - i_-s_*)(za_1)$, so we must have $za_1 = 0$. Similarly $i_-s_*(a_2) = 0$, so $za_2 = 0$. \hfill \square

Now $z\varpi$ annihilates both $a_1$ and $a_2$, therefore $a$. \hfill \square

Without loss of generality, we can assume that $V \cup W$ is smooth. Write $U$ for the union. Because $V$ and $W$ are 1-adjoining, $V \cap W = K$. Now Mayer-Vietoris gives an exact sequence
\[ H_i(K) \rightarrow H_i(V) \oplus H_i(W) \rightarrow H_i(U) \rightarrow H_{i-1}(K) \] (5.19)

For $i < n$, the surrounding cohomologies vanish. Even if $i = n$, the class in $K$ maps to 0 as it is a boundary for both $V$ and $W$. Hence the map
\[ H_i(V) \oplus H_i(W) \rightarrow H_i(U) \] (5.20)
is an isomorphism for $1 \leq i \leq n$. Write $(a, b)$ for a class in $H_i(V) \oplus H_i(W)$. We may also orient $U$ compatibly with $V$. Then:

Claim. $z(a, 0) = (za, \varphi(a))$ and $z(0, b) = (\psi(b), \varpi b)$.

Proof. With the picture in place,
we have \( z(a,0) = \alpha \cap U \), \((za,0) = \alpha \cap V\), and \((0, \phi(a)) = \alpha \cap W\). Check that this follows from definitions.

Similar to the claim, we also have

\[
\bar{z}(a,0) = (\varphi, -\zeta(a)) \quad (5.21)
\]

\[
\bar{z}(0,b) = (-\psi(b), zb). \quad (5.22)
\]

From \((z\bar{z})(a,0) = 0\), we find

\[
0 = \bar{z}(za, \phi(a)) \quad (5.23)
\]

\[
= \left(\varphi za - \psi \varphi(a), -\varphi(za) + z\varphi(a)\right). \quad (5.24)
\]
We see that $\varphi$ is a $P$-module homomorphism and that $\psi \circ \varphi = z\varphi$. We get a similar result for $\varphi \circ \psi$.

Finally, for $i = \frac{n+1}{2}$, we need to check compatibility with the isometric structure. That is, we need to check that $\langle \varphi(a), b \rangle_W = \langle a, \psi(b) \rangle_V$. To do this, use the intersection pairing on $H_i(U)$; this is the direct sum of the pairings on $H_i(V)$ and $H_i(W)$. We then have

$$\langle \varphi(a), b \rangle_W = \left\langle \left( \varphi(a), (0, b) \right), z \right\rangle_U = \left\langle (a, 0), (0, b) \right\rangle_U = \left\langle (0, b), z \right\rangle_U = \left\langle (a, 0), (0, b) \right\rangle_U = \langle a, \psi(b) \rangle_V.$$ (5.29)

**Theorem 5.3.** Any two Seifert hypersurfaces for $K$ are $R$-equivalent.

**Proof.** Suppose $V$ and $W$ are such that $\partial V = \partial W = K$. We want to connect $V$ and $W$ by a sequence

$$V = V_0, V_1, V_2, \ldots, V_r = W$$ (5.30)

such that $V_j$ and $V_{j+1}$ are 1-adjoining for every $j$.

Fix a Riemannian metric on $S^{n+2}$.

**Lemma 5.4.** The unit normal vector fields on $K$ determined by $V$ and $W$ are isotopic.

**Proof.** In the case $n \geq 2$, any two unit normal fields on $K$ are isotopic (by obstruction theory). Indeed, dividing the vector fields gives a map $S^n \to S^1$, and this map must be nullhomotopic.

For $n = 1$, the homotopy classes of maps $S^1 \to S^1$ are determined by the winding number. Now for $K'$ given by pushing $K$ along the normal bundle for $V$,
the winding number of the map to $S^1$ is $lk(K, K')$. But this linking number is zero because $K'$ lies in $\text{int}(V)$. Similarly, we get a winding number of zero using the normal vector field from $W$. □

Lemma 5.5. Let $X$ be a compact manifold and $f, g : X \to S^1$ two smoothly homotopic continuous maps. Then there exists $(h_i)$ for $1 \leq i \leq N$ such that $h_0 = f$, $h_N = g$, and the graphs of $h_i$ and $h_{i+1}$ are disjoint.

Proof. Without loss of generality, let $g = 1$ (the identity), otherwise use the group structure to multiply by $g^{-1}$. Now $f \simeq 1$, so $f$ lifts to a map $\tilde{f} : X \to \mathbb{R}$. Take the image of $\tilde{f}$ to be of the form $[a, b]$ with $a > 0$. Now let $h_i = \frac{N-i}{N} \tilde{f}$, where $N > b$. Because $N > b$, $\frac{1}{N} \tilde{f}$ has image in $(0, 1)$, so $h_i$ and $h_{i+1}$ never agree modulo $\mathbb{Z}$. □

Lemma 5.6. Suppose $V \subseteq S^{n+2}$ with $\partial V = K$ and $v$ the normal vector field to $V$ along $K$. Suppose $w$ is another normal field along $K$ with $v(x) \neq w(x)$ for every $x$. Then in any neighborhood of $V$, there exists a Seifert hypersurface $W$ adjoining $V$ with normal field $w$.

Proof. Extend $w$ to a vector field on a neighborhood of $V$ that is transverse to $V$. Now pushing $K$ along $w$ gives the desired $W$. □
Corollary 5.7. If $V$ and $W$ are Seifert hypersurfaces for $K$, there exists $U$ a Seifert hypersurface such that $U$ is $R$-equivalent to $V$ and the normal vector fields $u$ and $w$ satisfy $u(x) \neq w(x)$. Hence

$$\text{int}(U) \cap \text{int}(W) \subseteq \text{int}(U).$$

(5.31)

Proof. Construct $v = v_0, v_1, \ldots, v_N = w$ such that $v_i(x) \neq v_{i+1}(x)$ for all $i$. Then construct manifolds $V_0, V_1, \ldots, V_{N-1}$ such that $V_i$ has normal vector field $v_i$ and $V_i$ adjoins $V_{i-1}$. Now let $U = V_{N-1}$. 

With $U$ and $W$ as in the corollary, $\text{int}(U) \cap W$ is compact. We may also assume the intersection is transverse. Write $V$ for this $U$ now.

Consider a cross-section:

We induct on the number of connected components of $V \cap W$.

Lemma 5.8 (technical). At least one component $A$ on $S^{n+2} \setminus (V \cup W)$ is such that $\overline{A}$ is a manifold with corners.

Now making $U$ equal to $V$ except on the boundary of $A$, and pushing up $V$ to get $U_1$, and pushing up $W$ to get $U_2$: 

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$U_2$

$U_1$

$U$

$V$ adjoins $U_1$, $W$ adjoins $U_2$, and each $U_i$ adjoin $U$. Now $U \cap U_2$ has fewer connected components than $V \cap W$, so we may induct.

We conclude that the $R$-equivalence class of a Seifert hypersurface for $K$ is an invariant of $K$. Hence the $R$-equivalence class of the $P$-module $H_i(V; \mathbb{Z})$ is an invariant of $K$.

6 Computing the Alexander Module

Recall that $C_\infty$ is the infinite cyclic cover of $S^{n+2} \setminus K$, the knot complement. The Alexander modules of $K$ are then the modules $H_i(C_\infty)$. We’ll adopt Farber’s convention that $H_i(X)$ means $\tilde{H}_i(X; \mathbb{Z})$.

The map $\text{int}(V) \hookrightarrow S^{n+2} \setminus K$ lifts to a map $f : \text{int}(V) \hookrightarrow C_\infty$. This is true because we have

$$\pi_1(\text{int}(V)) \rightarrow \ker \left( lk : \pi_1(S^{n+2} \setminus K) \rightarrow \mathbb{Z} \right)$$

(loops on $V$ do not link with $K$). We now obtain a map $f_* : H_i(V) \rightarrow H_i(C_\infty)$.

**Proposition 6.1.**
1. For $v \in H_i(V)$, $(1 - t)f_*(zv) = f_*(v)$.
2. $f_*(v) = 0$ if and only if $(z\mathbb{Z})^m v = 0$ for some nonnegative integer $m$.
3. For any $x \in H_i(C_\infty)$, there exists $\ell \geq 0$, $m \geq 0$, and $v \in H_i(V)$ such that

$$x = (1 - t)^{\ell}t^{-m}f_*(v).$$

**Proof.** For these calculations, $V$ simply means $\text{int}(V)$. 

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Let $Y$ be the region between $V$ and $tV$. Then $Y \to S^{n+2} \setminus K$ with image $S^{n+2} \setminus V$. So $Y \cong S^{n+2} \setminus V$.

1. We know that there exists a chain $b$ in dimension $i + 1$ such that

$$\partial b = i_{+,*}(za) - i_{-,*}(za) - i_{+,*}(a).$$  \hspace{1cm} (6.3)

The map $i_+: V \to S^{n+2} \setminus V$ lifts to a map $j_+: V \to Y$. We similarly get a map $j_-$. We can lift $b$ to get a chain $\tilde{b}$. Meanwhile, we can homotope $j_{+,*}(za)$ to $f_*(za)$ and similarly $j_{+,*}(a)$ to $f_*(a)$, and $j_{-,*}(za)$ to $tf_*(za)$.
We find that in $H_i(C_\infty)$,

$$f_*(za) - tf_*(za) = f_*(a). \quad (6.4)$$

Similarly, we have the formula

$$(1 - t^{-1})f_*(zv) = f_*(v). \quad (6.5)$$

2. For the direction $(\iff)$, if $z^m \bar{z}^m v = 0$, then

$$f_*(v) = (1 - t)m(1 - t^{-1})m f_*(z^m \bar{z}^m v) = 0. \quad (6.6)$$

For the other direction, let $\tilde{Y}$ be the preimage of $S^{n+2} \setminus V$. This is a disjoint union of the $t^i Y$ over $i \in \mathbb{Z}$. Calculating the relative homology exact sequence of the pair $(C_\infty, \tilde{Y})$,

$$H_{i+1}(C_\infty, \tilde{Y}) \to H_i(\tilde{Y}) \to H_i(C_\infty) \to H_i(C_\infty, \tilde{Y}) \to H_{i-1}(\tilde{Y}) \quad (6.7)$$

For $\Lambda = \mathbb{Z}[t, t^{-1}]$, all of the homology groups are $\Lambda$-modules. Then $H_i(\tilde{Y}) = \Lambda \otimes \mathbb{Z} H_i(Y)$. For $H_i(C_\infty, \tilde{Y})$, let $\tilde{V}$ be the preimage of $V$. Let $N_V$ be a thickened neighborhood of $V$ and $\tilde{N}_V$ the preimage. Then using excision,

$$H_i(C_\infty, \tilde{Y}) = H_i(\tilde{N}_V, \tilde{N}_V \setminus \tilde{V}) \quad (6.8)$$

$$= \Lambda \otimes \mathbb{Z} H_i(N_V, N_V \setminus V). \quad (6.9)$$

Now write $N_V = V \times [-1, 1]$. We claim that

$$H_i \left( V \times [-1, 1], V \times \left( [-1, 0) \cup (0, 1] \right) \right) \cong H_{i-1}(V). \quad (6.10)$$

The map here is the intersection of the relative chain with $V$. We can show this by using the exact sequence in relative homology.

Now our sequence with $C_\infty$ and $\tilde{Y}$ becomes

$$\Lambda \otimes \mathbb{Z} H_i(V) \xrightarrow{d} \Lambda \otimes \mathbb{Z} H_i(Y) \xrightarrow{e} H_i(C_\infty) \to \Lambda \otimes \mathbb{Z} H_{i-1}(V) \to \Lambda \otimes \mathbb{Z} H_{i-1}(Y). \quad (6.11)$$

We describe $d$ explicitly. Suppose $v \in H_i(V)$. We lift $v$ to a relative cycle $\tilde{v} \in H_{i+1}(C_\infty, \tilde{Y})$ with $\tilde{v} \cap V = v$. Taking the boundary of $\tilde{v}$, we get

$$d(v) = j_{+,*}(v) - t^{-1}j_{-, *}(v) \in \Lambda \otimes \mathbb{Z} H_i(Y). \quad (6.12)$$
We have $f_* = e \circ j_{+, \ast}$. Now if $w \in V$ is such that $f_*(w) = 0$, then $j_{+, \ast}(w)$ lies in the image of $d$. Hence we have

$$t^0 j_{+, \ast}(w) = d \left( \sum_k t^k v_k \right)$$

(6.13)

$$= \sum_k t^k j_{+, \ast}(v_k) - \sum_k t^{k-1} j_{-, \ast}(v_k)$$

(6.14)

$$= \sum_k t^k (j_{+, \ast}(v_k) - j_{-, \ast}(v_{k+1}))$$

(6.15)

Because we are concentrated in degree 0, we must have

$$j_{+, \ast}(v_k) - j_{-, \ast}(v_{k+1}) = \begin{cases} j_{+, \ast}(w) & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

(6.16)

On the other hand, we have

$$j_{+, \ast}(v_k) = (j_{+, \ast} - j_{-, \ast})(zv_k)$$

(6.17)

$$-j_{-, \ast}(v_k) = (j_{+, \ast} - j_{-, \ast})(\overline{z}v_k)$$

(6.18)

Applying these and inverting the isomorphism $j_{+, \ast} - j_{-, \ast}$, we find

$$zv_k + \overline{z}v_{k+1} = \begin{cases} zw & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

(6.19)

Now for $N \gg 0$, we have $v_N = v_{-N} = 0$. Now $z^k v_{N-k} = 0$ for $k \leq N - 1$. Similarly $z^k v_{-N+k} = 0$ for $k \leq N$. We find
\[(z\overline{z})^N w = (z\overline{z})^N v_0 + z^{N-1}\overline{z}^{N+1}v_1 = 0. \quad (6.20)\]

3. First, we'll use the long exact sequence.

**Claim.** \(d\) is injective.

**Proof.** If \(d\) annihilates \(\sum_k t^k v_k\), then \(zv_k + \overline{z}v_{k+1} = 0\) for all \(k\). Also \(v_N = v_{-N} = 0\) for \(N \gg 0\). So \(z^{2N}v_k = \overline{z}^{2N}v_k = 0\). As \(z\) and \(\overline{z}\) generate the unit ideal, so do \(z^{2N}\) and \(\overline{z}^{2N}\). So every \(v_k\) is 0.

Hence the maps \(H_i(C_\infty) \to \Lambda \otimes \mathbb{Z} H_{i-1}(V)\) are all zero, so \(e\) is surjective.

Now if \(x \in H_i(C_\infty)\) has the form

\[x = e \left( \sum_k t^k y_k \right) \quad (6.21)\]

for \(y_k \in H_i(Y)\). Write \(y_k = (j_{+,\ast} - j_{-,\ast})(v_k)\) for \(v_k \in H_i(V)\). Now

\[x = \sum_k t^k e(j_{+,\ast}(v_k)) - t^k e(j_{-,\ast}(v_k)) \quad (6.22)\]
\[= \sum_k t^k f_*(v_k) - t^{k+1}f_*(v_k) \quad (6.23)\]
\[= (1 - t) \sum_k t^k f_*(v_k) \quad (6.24)\]

For each \(m_1, m_2\), we have

\[f_*(v_k) = (1 - t)^{m_1}(1 - t^{-1})^{m_2}f_*(z^{m_1}\overline{z}^{m_2}v_k) \quad (6.25)\]
\[= (-1)^{m_2}t^{-m_2}(1 - t)^{m_1+m_2}f_*(z^{m_1}\overline{z}^{m_2}v_k) \quad (6.26)\]

For each \(k\), choose \(m_1 = M - k\) and \(m_2 = M + k\) for \(M \gg 0\) so that \(m\) is always positive. Then

\[\begin{align*}
(1 - t) \sum_k t^k f_*(v_k) &= (1 - t) \sum_k t^k(1 - t)^{M+k}t^{-M-k}(1 - t)^2M f_*(z^{M-k}\overline{z}^{M+k}v_k) \quad (6.27) \\
&= (1 - t)^{M+1}t^{-M} f_*( \sum_k (-1)^{M+k}z^{M-k}\overline{z}^{M+k}v_k ) \quad (6.28)
\end{align*}\]

**Corollary 6.2.** \(1 - t\) is an automorphism of \(H_i(C_\infty)\).
Proof. Surjectivity follows immediately from (3) and (1). For injectivity, $\Lambda = \mathbb{Z}[t, t^{-1}]$ is a Noetherian ring and $H_i(C_\infty)$ is a finitely generated $\Lambda$-module (e.g. by a cellular homology argument). But any surjective endomorphism of a finitely generated module over a Noetherian ring is injective: consider

$$\ker \varphi \subseteq \ker \varphi^2 \subseteq \ker \varphi^3 \subseteq \cdots \quad (6.29)$$

So there exists $i$ such that $\ker \varphi^i = \ker \varphi^{i+1}$, so $\varphi$ is injective on $\text{im} \varphi^i$, which is everything by surjectivity.

Corollary 6.3. $H_i(C_\infty)$ has a natural structure as a module over $L = \mathbb{Z}[t, t^{-1}, (1 - t)^{-1}]$.

Recall $P = \mathbb{Z}[z]$. We define a homomorphism $P \to L$ by $z \mapsto (1 - t)^{-1}$ (as suggested by (1)).

Corollary 6.4. $f_* : H_i(V) \to H_i(C_\infty)$ is a $P$-module homomorphism, with $H_i(C_\infty)$ viewed as a $P$-module via the map $P \to L$.

We can also write $L = P[z^{-1}, \bar{z}^{-1}]$, since $z$ maps to $(1 - t)^{-1}$ and $\bar{z}$ maps to $-\frac{t}{1-t}$.

Theorem 6.5. $f_*$ induces an isomorphism $L \otimes_P H_i(V) \xrightarrow{\sim} H_i(C_\infty)$ of $L$-modules.

Proof. Injectivity follows from (2) and surjectivity follows from (3).

7 Classification of Simple Knots

We’ll first classify the knots in terms of $R$-equivalence, and then go back and classify them in terms of the Alexander module.

Suppose $K$ is a $(2q-1)$-knot in $S^{2q+1}$, with $q \geq 3$. Recall that $K$ being simple means $\pi_i(S^{2q+1} \setminus K) \cong \pi_i(S^1)$ for $i \leq q - 1$. This is equivalent to $K$ having a $(q-1)$-connected Seifert hypersurface. Let $V$ be such a Seifert hypersurface. $V$ is called a simple (or special) Seifert hypersurface.

By Hurewicz, $H_i(V) \cong 0$ for $i \leq q - 1$, and $H_q(V) \cong \pi_q(V)$. For $q < i < 2q$, $H_i(V) \cong H^{2q-i}(V)$ by Poincaré duality, but $H^{2q-i}(V) = 0$. We have $H_{2q}(V) = 0$ since $V$ is a $2q$-manifold with boundary. Finally, the torsion in $H_q(V) \cong H^q(V)$ is dual to the torsion in $H_{q-1}(V)$, which is trivial. Hence $H_q(V)$ is torsion free over $\mathbb{Z}$.

A $\mathbb{Z}$-torsion-free (also called special) isometric structure is a $P$-module $A$ with a $(-1)^q$-symmetric $\mathbb{Z}$-valued bilinear pairing, with the usual isometric structure properties, and such that $A$ has no $\mathbb{Z}$-torsion.

Theorem 7.1. If $n = 2q - 1$ with $q \geq 3$, then there is a bijection between $n$-dimensional simple knots and $R$-equivalence classes of $\mathbb{Z}$-torsion-free isometric structures, where the map is $K \mapsto H_q(V)$ with $V$ a simple Seifert hypersurface for $K$.

Lemma 7.2. Suppose $V, W \subseteq S^{n+2}$ are simple Seifert hypersurfaces, not necessarily for the same knot. Suppose that the isometric structures $H_q(V)$ and $H_q(W)$ are 1-adjoining. Then there exists another simple Seifert hypersurface $V_1 \subseteq S^{n+2}$ such that $V_1$ is isotopic to $W$, $\partial V = \partial V_1$, and $V$ and $V_1$ are 1-adjoining.
This lemma will ultimately let us prove that two Seifert hypersurfaces are \( R \)-equivalent if and only if their isometric structures are \( R \)-equivalent.

**Lemma 7.5.** Every \( \mathbb{Z} \)-torsion-free isometric structure of parity \((-1)^q\) can be realized as the isometric structure of a \( 2q \)-dimensional simple Seifert hypersurface.

These two lemmas plus an extra lemma that we'll do later will imply Theorem 7.1.

The topological machinery we'll use is:

**Theorem 7.4.**

(a) Let \( U \) be an \( m \)-manifold and \( K \) a connected \( k \)-dimensional CW complex, with \( m \geq k + 3 \). Also suppose \( f : K \to U \) is \((2k - m + 1)\)-connected. Then there exists a compact \( m \)-submanifold \( V \subseteq U \) with \( \pi_1(\partial V) = \pi_1(V) \) and a homotopy equivalence \( g : K \to V \) such that \( g \) is homotopic to \( f \) inside \( U \). Such a \( V \) is called a thickening of \( K \) in \( U \).

(b) Suppose \( f \) is as above, but we now require \( f \) to be \((2k - m + 2)\)-connected and \( m \geq 6 \). Given thickenings \((V_1, g_1)\) and \((V_2, g_2)\) of \( K \) in \( U \), then there is an isotopy \( h_i \) of \( U \) with \( h_0 \) the identity and \( h_1(V_1) = V_2 \), so that \( h_1 \circ g_1 \) is homotopic to \( g_2 \) within \( V_2 \).

**Lemma 7.5.** If \( U^{2q} \subseteq S^{2q+1} \) is a closed \((q - 1)\)-connected manifold with \( q \geq 3 \), then there is a bijection between subgroups \( A \) of \( H_q(U) \) on which \( \langle \cdot, \cdot \rangle|_A \) is unimodular and ambient isotopy classes of almost closed \((q - 1)\)-connected submanifolds \( V \) in \( U \).

Here almost closed means \( \partial V \) is a homotopy \((2q - 1)\)-sphere, and hence homeomorphic to a \((2q - 1)\)-sphere.

**Proof.** Suppose \( V \subseteq U \) is an almost closed submanifold. We may write \( U = V \cup V' \) for \( V = \overline{U \setminus V} \), and \( V \cap V' = \partial V \), which is homotopy equivalent to \( S^q \).

**Claim.** \( \pi_1(V) \) is trivial.

**Proof.** Apply van Kampen to \( V \cup V' = U \).

**Claim.** \( V \) is \((q - 1)\)-connected.

**Proof.** \( H_i(V) \leftrightarrow H_i(U) \) for \( i \leq q - 1 \) from Mayer-Vietoris.

We also have \( i_* : H_q(V) \leftrightarrow H_q(U) \). Define \( A \) to be the image of \( i_* \). Note that \( H_q(V) = 0 \) for \( i > q \). Let \( r \) be the rank of \( H_q(V) = \pi_q(V) \). Then we get a map \( \bigvee S^q \to V \), which is bijective on \( H_q \).

**Claim.** \( V \) is a thickening of \( \bigvee S^q \).

**Proof.** For homotopy equivalence, use the homology version of Whitehead’s theorem.

Injectivity of the map \( V \mapsto A \) in our desired bijection follows from uniqueness of thickening. Surjectivity follows from existence of thickening. Unimodularity comes from Poincaré duality.

It remains to show that if \( \langle \cdot, \cdot \rangle \) is unimodular on \( H_q(V) \), then \( \partial V \) is almost closed. As before, let \( V' = U \setminus \text{int}(V) \). Then \( H_i(V) = 0 \) for \( i \neq q \). Then Poincaré duality implies \( H_{2q-i}(V, \partial V) = 0 \) for \( i \neq q \).

To show that \( H_q(\partial V) = H_{q-1}(\partial V) = 0 \), we need to show that the map \( H_q(V) \to H_q(V, \partial V) \cong H^q(V) \) is an isomorphism. This follows from unimodularity of the intersection pairing.
Finally, \( \partial V \) is simply connected because the thickening condition guarantees \( \pi_1(\partial V) = \pi_1(V) \), which is 0. We conclude that \( \partial V \) is a homotopy sphere.

**Lemma 7.6.** Suppose \( V, W \) are \( 2q \)-manifolds in \( S^{2q+1} \) which are simple Seifert hypersurfaces and \( \varphi : H_q(V) \to H_q(W) \) is an isomorphism of isometric structures. Then \( V \) is isotopic to \( W \) in \( S^{2q+1} \).

**Proof.** Given \( V \) and \( W \), we construct a one-sided neighborhood \( N_V \) of \( V \), with \( N_V \cong V \times [0, 1] \) with \( V \times 0 = V \). Construct \( N_W \) similarly. Take a map \( f : V \to W \) which is a homotopy equivalence and induces the map \( \varphi : H_q(V) \to H_q(W) \). This is possible since \( V \) and \( W \) are both homotopy equivalent to bouquets of spheres, and there are maps between bouquets which can induce any possible map on homology. Now \( f : V \to N_W \) is a thickening of \( V \), so \( N_V \) is isotopic to \( N_W \). So without loss of generality, we may assume \( N_V = N_W \); write \( N \) for both.

Let \( U = \partial N \), and \( M = S^{2q+1} \setminus \text{int}(N) \). Consider the maps \( i : V \hookrightarrow U, j : W \hookrightarrow U, r : U \hookrightarrow M, \) and \( s : U \hookrightarrow N \). The inclusions of \( V \) and \( W \) into \( N \) give the diagram on homology

\[
\begin{array}{ccc}
H_q(V) & \xrightarrow{\varphi} & H_q(W) \\
\downarrow{\sim} & & \downarrow{\sim} \\
H_q(N) & \xrightarrow{i_*} & H_q(N) \\
\end{array}
\tag{7.1}
\]

Now by Mayer-Vietoris, \( r_* \oplus s_* \) is an isomorphism. It suffices to show that the diagram

\[
\begin{array}{ccc}
H_q(V) & \xrightarrow{\varphi} & H_q(W) \\
\downarrow{i_*} & & \downarrow{j_*} \\
H_q(U) & & \\
\end{array}
\tag{7.2}
\]

commutes. Showing that \( j_* \circ \varphi = i_* \) is equivalent to \( s_* \circ j_* \circ \varphi = s_* \circ i_* \) (follows from the above square) and \( r_* \circ j_* \circ \varphi = r_* \circ i_* \). We examine the \( P \)-module behavior of \( i_* \):

**Claim.** \( i_*(zv) - zi_*(v) \) annihilates \( i_*(H_q(V)) \) inside \( H_q(U) \) with respect to the intersection pairing.

The linking pairing \( H_q(N) \times H_q(M) \to \mathbb{Z} \) is nondegenerate by Alexander duality. We compute

\[
\text{lk}(s_*i_*v_1, r_*(i_*v - j_*f_*v)) = \text{lk}(s_*i_*v_1, r_*i_*v) - \text{lk}(s_*i_*v_1, r_*j_*f_*v) = \text{lk}(s_*j_*f_*v_1, r_*j_*f_*v). \tag{7.3}
\]

**Claim.** \( \text{lk}(s_*i_*v_1, r_*i_*v) = \langle v_1, zv \rangle_V \).

So now our difference of linking numbers equals

\[
\langle v_1, zv \rangle_V - \langle f_*v_1, zf_*v \rangle_W = \langle v_1, zv \rangle_V - \langle \varphi v_1, \varphi zv \rangle_W \tag{7.5}
\]

and this is 0 since \( \varphi \) is an isomorphism of isometric structures.

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Now that our triangle commutes, by Lemma 7.5 we have that \( V \) is isotopic to \( W \) inside \( U \). We extend this isotopy to \( S^{2q+1} \) via a tubular neighborhood of \( U \).

**Lemma 7.7.** Let \( A \) be a \( \mathbb{Z} \)-torsion-free isometric structure of parity \((-1)^q\) such that \( za = 0 \) for every \( a \in A \). Then there exists a \((q - 1)\)-connected \( V \subset S^{2q+1} \) a closed \( 2q \)-manifold with isometric structure \( A \).

*Proof.* Let \( A_+ = \{ a | za = 0 \} \) and \( A_-= \{ a | za = 0 \} \). Then \( A_+ = \text{im} \, z \) and \( A_- = \text{im} \, z \), with \( A = A_+ \oplus A_- \). In addition, \( ( , , ) \) becomes restricted to either \( A_+ \) or \( A_- \). So \( A_+ \cong \mathbb{Z}^r \), \( A_- = \text{Hom}(A_+, \mathbb{Z}) \), and \( A = A_+ \oplus A_- \).

To construct \( V \), take \( r \) copies of \( S^q \), expand them to dimension \( 2q + 1 \) by taking tubular neighborhoods, and considering their boundaries. Each boundary is of the form \( S^q \times S^q \), and then we take the connected sum of the \( r \) copies. This gives us the desired \( V \).

*Proof of Lemma 7.2.* Let \( A = H_q(V) \) and \( B = H_q(W) \). Then there exist \( P \)-module maps \( \varphi : A \to B \) and \( \psi : B \to A \) such that \( \varphi \circ \psi = za \), \( \psi \circ \varphi = za \), and \( \langle a, \psi(b) \rangle_A = \langle \varphi(a), b \rangle_B \).

We’ll construct a \((q - 1)\)-connected closed hypersurface \( U \) such that \( H_q(U) \cong H_q(A) \oplus H_q(B) \) as \( \mathbb{Z} \)-modules, and such that \( H_q(U) \) has an isometric structure defined by

\[
za = (za + \psi(b), \varphi(a) + \bar{z}b) \quad (7.6)
\]

\[
za = (za - \psi(b), -\varphi(a) + zb) \quad (7.7)
\]

\[
\langle (a, b), (a', b') \rangle_U = \langle a, a' \rangle_A - \langle b, b' \rangle_B. \quad (7.8)
\]

Check that this is an isometric structure annihilated by \( za \). By Lemma 7.7, there exists \( U \subset S^{2q+1} \) \((q - 1)\)-connected of dimension \( 2q \), such that \( H_q(U) \cong A \oplus B \) with the above isometric structure. Now applying Lemma 7.5 to the subgroup \( A \subset H_q(U) \), we can find a submanifold \( V_1 \subset U \) such that \( H_q(V_1) \cong A \) (as abelian groups, respecting the bilinear pairing) and such that \( \partial V_1 \) is homeomorphic to \( S^{2q-1} \).

Now let \( W_1 = U_1 \setminus \text{int}(V_1) \). We have \( U = V_1 \cup W_1 \) with \( V_1 \cap W_1 \) equal to their common boundary. We have \( H_q(W_1) \subset B \), and by comparing ranks, we must have equality. Once we show that the \( P \)-module structures on \( H_q(V_1) \) and \( A \) agree, we find that \( H_q(V_1) \cong A \cong H_q(U) \) as isometric structures. Similarly, \( H_q(W_1) \cong H_q(W) \) as isometric structures. By Lemma 7.6, \( V \) is isotopic to \( V_1 \) and \( W \) is isotopic to \( W_1 \).

*Proof of Lemma 7.3.* First note that \( A \) 1-adjoins \( A \), taking \( \varphi = z \) and \( \psi = \bar{z} \). By the proof of Lemma 7.2, there exists a closed \((q - 1)\)-connected hypersurface \( U \) such that \( H_q(U) \cong A \oplus A \) with the structure as before. The map \( A \mapsto \partial A \) mapping onto the first summand comes from an inclusion \( V \hookrightarrow U \) where \( V \) is an almost closed \( 2q \)-submanifold, by Lemma 7.5. It follows that \( H_q(V) \cong A \) as isometric structures.

Finally, we need one more lemma, whose proof we’ll defer:

**Lemma 7.8.** If \( A \) and \( B \) are \( \mathbb{Z} \)-torsion-free isometric structures and \( A \) is \( R \)-equivalent to \( B \), then \( A \) and \( B \) can be joined by a chain.
\[ A = A_1, A_2, \ldots, A_m = B \]  

such that \( A_i \) 1-adjoins \( A_{i+1} \) and all \( A_i \) are \( \mathbb{Z} \)-torsion-free.

Given this lemma, we can prove the classification theorem.

**Proof of Theorem 7.1.** We need to show that if \( K_1 \) and \( K_2 \) are simple knots having simple Seifert hypersurfaces \( V_1 \) and \( V_2 \), then \( K_1 \sim K_2 \) if and only if \( H_q(V_1) \) is \( R \)-equivalent to \( H_q(V_2) \).

For ( \( \Rightarrow \) ), any two Seifert hypersurfaces of the same knot have isomorphic isometric structures. For ( \( \Leftarrow \) ), by Lemma 7.8, there exists a sequence \( A_1, \ldots, A_m \) of consecutively 1-adjoining \( \mathbb{Z} \)-torsion-free isometric structures with \( H_q(V_1) = A_1, H_q(V_2) = A_m \). By Lemma 7.5, there exist simple Seifert hypersurfaces \( W_i \) such that \( A_i \sim H_q(W_i) \) as isometric structures. Then Lemma 7.2 implies that \( \partial W_i \sim \partial W_{i+1} \). In particular, \( K_1 \sim K_2 \).

\[ \square \]

### 8 Minimal Isometric Structures

An isometric structure \( A \) is minimal if the map \( z\bar{z} : A \to A \) is injective. If \( A \) is minimal, then \( A \) has no \((z\bar{z})^m\)-torsion for any \( m \), and so the natural map \( A \to A \otimes PR \) is injective. More generally, a \( P \)-module is minimal if multiplication by \( z\bar{z} \) is injective.

**Lemma 8.1.** If \( A \) is an isometric structure, then \( A \) is \( R \)-equivalent to some \( A' \) such that the \( \mathbb{Z} \)-torsion subgroup of \( A' \) is a minimal \( P \)-module.

**Proof.** For \( m \gg 0 \), \( \ker((z\bar{z})^m) \) agrees with \( \ker((z\bar{z})^{m+1}) \) on \( \mathbb{Z} \)-torsion. Let \( K \) be this kernel. Consider the projection map \( \pi : A \to A/K = A' \).

We have \( T_{\mathbb{Z}}(A') = T_{\mathbb{Z}}(A)/K \). The kernel of \( z\bar{z} \) on \( T_{\mathbb{Z}}(A') \) is then contained in \( \ker(z\bar{z})^{m+1}/K \), which is zero.

To show that \( A \) and \( A' \) are \( R \)-equivalent, let \( A_{z} = A/(\ker z \cap T_{\mathbb{Z}}(A)) \). We claim this 1-adjoins \( A \), for we have maps \( \bar{z} : A \to A_z \) and \( z : A_z \to A \). Composing both in either order gives \( z\bar{z} \), and they give morphisms of isometric structures: \( A_{z} \) has an inherited structure from \( A \) because the pairing vanishes on torsion. It’s easy to check that the morphisms are compatible with this pairing.

Similarly \( A \) is \( R \)-equivalent to \( A_{z^{-1}} \) by iterating this. Also \( A \) adjoins \( A_{\bar{z}} \), and so \( A \) is \( R \)-equivalent to \( A_{(z\bar{z})^m} = A' \). \( \square \)

Now let \( A \) be an isometric structure such that \( T_{\mathbb{Z}}(A) \) is minimal. Write \( Z = z\bar{z} \). So \( Z \) acts injectively on the finite set \( T_{\mathbb{Z}}(A) \), therefore is an isomorphism on this set.

We might guess to take \( A' = Z^mA \) to produce an \( R \)-equivalent minimal structure. Put a bilinear pairing on \( Z^mA \) by

\[ \langle Z^ma, Z^mb \rangle_{Z^mA} = \langle a, Z^mb \rangle_A = \langle Z^ma, b \rangle_A \]  

(8.1)

which is well-defined.
Lemma 8.2. There is a short exact sequence

\[ 0 \to \mathbb{Z}^m A/T_\mathbb{Z}(\mathbb{Z}^m A) \to \text{Hom}(\mathbb{Z}^m A, \mathbb{Z}) \to \text{Ext}_\mathbb{Z}^1(A/\mathbb{Z}^m A, \mathbb{Z}) \to 0. \tag{8.2} \]

Note that for any finitely generated abelian group \( B \),

\[ \text{Ext}_\mathbb{Z}^1(B, \mathbb{Z}) = \text{Hom}_\mathbb{Z}(T_\mathbb{Z}(B), \mathbb{Q}/\mathbb{Z}). \tag{8.3} \]

Proof. We have a short exact sequence

\[ 0 \to \mathbb{Z}^m A \to A \to A/\mathbb{Z}^m A \to 0. \tag{8.4} \]

Now take the long exact sequence with respect to \( \text{Hom}(\square, \mathbb{Z}) \). We obtain

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Hom}(A/\mathbb{Z}^m A, \mathbb{Z}) & \longrightarrow & \text{Hom}(A, \mathbb{Z}) & \xrightarrow{f} \text{Hom}(\mathbb{Z}^m A, \mathbb{Z}) \\
\uparrow & & \uparrow & & \uparrow & \\
\text{Ext}_\mathbb{Z}^1(A/\mathbb{Z}^m A, \mathbb{Z}) & \longrightarrow & \text{Ext}_\mathbb{Z}^1(A, \mathbb{Z}) & \longrightarrow & \text{Ext}_\mathbb{Z}^1(\mathbb{Z}^m A, \mathbb{Z}) & \\
\end{array}
\tag{8.5}
\]

We first show that \( \text{Hom}(\mathbb{Z}^m A, \mathbb{Z}) \to \text{Ext}_\mathbb{Z}^1(A/\mathbb{Z}^m, \mathbb{Z}) \) is surjective. Equivalently, we need to show that the map

\[
\begin{array}{cccccc}
\text{Ext}_\mathbb{Z}^1(A, \mathbb{Z}) & \longrightarrow & \text{Ext}_\mathbb{Z}^1(\mathbb{Z}^m A, \mathbb{Z}) \\
\downarrow & & \downarrow \\
\text{Hom}_\mathbb{Z}(T_\mathbb{Z}(A), \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \text{Hom}_\mathbb{Z}(T_\mathbb{Z}(\mathbb{Z}^m A), \mathbb{Q}/\mathbb{Z}) & \\
\end{array}
\tag{8.6}
\]

is injective. But \( \mathbb{Z} \) acts surjectively on \( T_\mathbb{Z}(A) \), so \( T_\mathbb{Z}(A) = T_\mathbb{Z}(\mathbb{Z}^m A) \) implying the above map is an isomorphism.

We also need to show that \( \text{im } f \cong \mathbb{Z}^m A/T_\mathbb{Z}(\mathbb{Z}^m A) \). For injectivity, suppose \( a \in \mathbb{Z}^m A/T_\mathbb{Z}(\mathbb{Z}^m A) \), and write \( a = [\mathbb{Z}^m a'] \). Then the image of \( a \) in \( \text{Hom}_\mathbb{Z}(\mathbb{Z}^m A, \mathbb{Z}) \) is

\[
b \mapsto \langle \mathbb{Z}^m a', b \rangle_{\mathbb{Z}^m A} = \langle a', b \rangle_A. \tag{8.7}
\]

If this pairing is identically zero, then by unimodularity of the pairing, \( a' \) must be torsion. Hence \( a = 0 \).

For surjectivity, we have maps

\[
\begin{array}{cccc}
A/T_\mathbb{Z}(A) & \xrightarrow{\sim} & \text{Hom}(A, \mathbb{Z}) & \\
\downarrow^{\mathbb{Z}^m} & & \downarrow^{f} & \\
\mathbb{Z}^m A/T_\mathbb{Z} & \xrightarrow{Z^m a' \mapsto (a', \square)} & \text{Hom}_\mathbb{Z}(\mathbb{Z}^m A, \mathbb{Z}) & \\
\end{array}
\tag{8.8}
\]

where the top map is an isomorphism by unimodularity of the pairing on \( A \). \qed
However, in order to have an isometric structure, \(Z^m A/T_z(Z^m A) \to \text{Hom}_\mathbb{Z}(Z^m A, \mathbb{Z})\) would have to be an isomorphism, and this isn’t quite true.

Let \(L = Z^m A/T_z(Z^m A)\) and \(L^* = \text{Hom}(Z^m A, \mathbb{Z}) = \text{Hom}(L, \mathbb{Z})\). Then we will show that \(L^*/L\) carries a perfect \(\mathbb{Q}/\mathbb{Z}\)-valued pairing. Let \(L_Q = L \otimes \mathbb{Q}\) and \(L_Q^* = L^* \otimes \mathbb{Q}\), so that \(L_Q^* \cong L_Q\). Then the pairing on \(L\) given by

\[
\langle a_1, a_2 \rangle_L = \langle a_1', a_2 \rangle_A
\]

(8.9)

(where \(a_1 = Z^m a'_1\)) extends linearly to a \(\mathbb{Q}\)-valued pairing on \(L_Q\), and then induces one on \(L_Q^*\). We can restrict this to a \(\mathbb{Q}\)-valued pairing on \(L^*\).

**Claim.** This descends to a \(\mathbb{Q}/\mathbb{Z}\)-valued pairing on \(L^*/L\).

To see this, suppose \(a' \in L\) and \(b \in L^*\). Then

\[
\langle a', b \rangle_{L_Q^*} = b(a') \in \mathbb{Z}.
\]

(8.10)

The pairing on \(L^*/L\) is a perfect pairing. This follows from what we’ve done earlier.

To see this, let \(B = L^*/L\) and let \(\varphi \in \text{Hom}_\mathbb{Z}(B, \mathbb{Q}/\mathbb{Z})\). We can lift \(\varphi\) to \(\bar{\varphi} \in \text{Hom}(L^*, \mathbb{Q}/\mathbb{Z})\), which we can then lift to some \(\psi \in \text{Hom}(L^*, \mathbb{Q})\). Now we must have \(\psi(L) \subseteq \mathbb{Z}\), so \(\psi|_L \in L^*\). Letting \(b\) be the image of \(\psi|_L\) in \(L^*/L\), we can check that the pairing takes \(b\) to \(\varphi\). Injectivity is similar.

We now have an isomorphism

\[
\begin{array}{ccc}
B & \sim & \text{Hom}_\mathbb{Z}(B, \mathbb{Q}/\mathbb{Z}) \\
\| & & \| \\
\text{Hom}_\mathbb{Z}(T_z(A/Z^m A), \mathbb{Q}/\mathbb{Z}) & \longrightarrow & T_z(A/Z^m A)
\end{array}
\]

(8.11)

So \(T_z(A/Z^m A)\) also carries a dual perfect pairing. The \(\mathbb{Q}/\mathbb{Z}\)-valued pairing on \(T_z(A/Z^m A)\) is given as follows: suppose \([c_1], [c_2] \in T_z(A/Z^m A)\) are represented by \(c_1, c_2 \in A\). Then for some \(N\), we have \(Nc_1 = Z^m c'_1\) for some \(c'_1 \in A\). Then we have

\[
\{c_1, c_2\}_{T_z(A/Z^m A)} = \frac{1}{N} \langle c'_1, c_2 \rangle_A
\]

(8.12)

(From now on, curly braces will be used for \(\mathbb{Q}/\mathbb{Z}\)-valued pairings.) If \(Mc_2 = Z^m c'_2\), then this pairing also equals \(\frac{1}{N} \langle c_1, c'_2 \rangle_A\), and \(\frac{1}{MN} \langle c'_1, c'_2 \rangle Z^m A\). It is an exercise to check that this is the pairing defined previously.

A \(\mathbb{Q}/\mathbb{Z}\)-valued isometric structure (or “periodic” isometric structure) of parity \(\epsilon = \pm 1\) is a finite \(P\)-module \(B\) with a \(\mathbb{Z}\)-bilinear pairing

\[
\{ \ , \ \} : B \times B \to \mathbb{Q}/\mathbb{Z}
\]

(8.13)

such that \(\{ \ , \ \}\) is a perfect \(\epsilon\)-symmetric pairing and such that \(\{ zb_1, b_2 \} = \{ b_1, zb_2 \}\).

For \(B = \text{Hom}_\mathbb{Z}(T_z(A/Z^m A), \mathbb{Q}/\mathbb{Z})\), we have constructed a \(\mathbb{Q}/\mathbb{Z}\)-valued isometric structure \(\{ \ , \ \}_B\) on \(B\). Observe that \(B\) is a \(P\)-module. More generally, if \(C\) is a \(P\)-module, then \(C^* = \text{Hom}_\mathbb{Z}(C, \mathbb{Q}/\mathbb{Z})\)
is a $P$-module by $z\varphi(c) = \varphi(zc)$. This module structure makes the map $B \to \text{Hom}_\mathbb{Z}(B, \mathbb{Q}/\mathbb{Z})$ (induced by the pairing) $P$-linear. (The target doesn’t have to be $\mathbb{Q}/\mathbb{Z}$ for this to work.)

We also have a natural $\mathbb{Q}/\mathbb{Z}$-valued isometric structure on $C = T_\mathbb{Z}(A/\mathbb{Z}^m A)$ given by identifying $C$ with $\text{Hom}_\mathbb{Z}(B, \mathbb{Q}/\mathbb{Z})$ which is isomorphic to $B$.

With $L$ and $L^*$ as before, $L$ is a $P$-module, so $L^* = \text{Hom}(L, \mathbb{Z})$ is also a $P$-module and $L \to L^*$ is a $P$-module homomorphism. Suppose we had $M \subseteq L^*/L$ a $P$-module such that $M = M^\perp$, where $M^\perp$ is the set of $b \in B$ such that $\{b, m\}_B = 0$ for every $m \in M$. Such an $M$ will be called *metabolic*.

Let $\pi : L^* \to L^*/L$ be the projection and $\tilde{M} = \pi^{-1}(M)$, a $P$-module. Restricting the pairing on $L^*_Q$ to $\tilde{M}$ gives a $\mathbb{Q}$-valued pairing on $\tilde{M}$. But $M = M^\perp$, so this pairing is actually $\mathbb{Z}$-valued. Denote this pairing by $\langle \ , \ \rangle_{\tilde{M}}$.

**Claim.** $\tilde{M}$ with its pairing $\langle \ , \ \rangle_{\tilde{M}}$ is an isometric structure.

The only thing to check is that $\langle \ , \ \rangle_{\tilde{M}}$ is a perfect pairing. To show that $\tilde{M} \to \text{Hom}(\tilde{M}, \mathbb{Z})$ is an isomorphism, since both are free abelian of the same rank, it’s enough to show that the map is surjective.

If $\varphi \in \text{Hom}(\tilde{M}, \mathbb{Z})$, then $\varphi|_L \in L^*$. We need to show that $\pi(\varphi|_L) \in M$, or equivalently $M^\perp$. For $m \in M$, lift to $\tilde{m} \in \tilde{M}$, which is then inside $L^*_Q = L^*_Q$. We can also extend $\varphi$ to a map $\varphi_Q : L_Q \to \mathbb{Q}$. But then

$$\varphi_Q(\tilde{m}) = \varphi(\tilde{m}) \in \mathbb{Z}$$

so $\pi(\varphi|_L)$ is zero when paired with $m$.

Now if $A$ is an isometric structure such that $T_\mathbb{Z} A$ is minimal, and $B$ $m$-adjoins $A$, then $T_\mathbb{Z} A \hookrightarrow T_\mathbb{Z} B$ and $T_\mathbb{Z} B \to T_\mathbb{Z} A$. These would be isomorphisms if $T_\mathbb{Z} B$ were also minimal. However, $\tilde{M}$ never has $\mathbb{Z}$-torsion, being contained in $L^*$.

To fix this, we will make a similar construction to $L^*$ which includes $T_\mathbb{Z} A$. Define

$$[A]_m = \{ a \in A \big| Na \in Z^m a \text{ for some } N \}.$$  

$[A]_m$ maps to $L_Q$ with kernel $T_\mathbb{Z} A$.

**Claim.** The image of the map $[A]_m \to L_Q$ lies inside $L^*$.

The map $[A]_m \to L_Q$ is given as follows: if $b \in [A]_m$ is such that $Nb = Z^m a$, then $b \mapsto \frac{1}{N} \langle a, \square \rangle_A$. Now if $c \in L$, $c$ has the form $Z^m d$, and then

$$\frac{1}{N} \langle a, Z^m d \rangle = \frac{1}{N} \langle Z^m a, d \rangle = \langle b, d \rangle \in \mathbb{Z}.$$  

(8.16)

This shows that $b$ maps to an element of $L^*$. We have that $[A]_m \to L^*$ with kernel $T_\mathbb{Z} A$.

**Claim.** This map is surjective.

Suppose $\varphi \in \text{Hom}(L, \mathbb{Z}) = \text{Hom}(Z^m A, \mathbb{Z})$. Then $\varphi(Z^m \cdot \square) \in \text{Hom}(A, \mathbb{Z})$, which is $A/T_\mathbb{Z} A$. So there exists $b \in A$ such that $\varphi(Z^m a) = \langle b, a \rangle_A$.

To see that $b \in [A]_m$, we have $N \varphi \in L$ for some $N$. Then $N \varphi$ is a map $L^* \to \mathbb{Z}$, so should come from an element of $Z^m A$. **Fix Me (2)**
We conclude that \( L^* \cong [A]_m/T_Z A \). Now let \( A_M \) be the preimage of \( \bar{M} \) in \([A]_m\). The isometric structure on \( \bar{M} \) lifts to an isometric structure on \( A_M \). We can describe this pairing explicitly: if \( a, b \in A_M \), then we may write \( N_a a = Z^m a' \) and \( N_b b = Z^m b' \). Then

\[
\langle a, b \rangle_{A_M} = \frac{1}{N_a N_b} \langle Z^m a', Z^m b' \rangle_L = \frac{1}{N_a N_b} \langle Z^m a, b \rangle_A.
\]  

(8.17)

**Lemma 8.3.** \( A \) adjoins \( A_M \).

**Proof.** Let \( \varphi : A_M \hookrightarrow A \) be the inclusion. To define \( \psi \), compose multiplication by \( Z^m \) with the inclusion \( Z^m A \hookrightarrow A_M \). Either composition of these maps gives multiplication by \( Z^m \). We now just have to show that for \( a \in A \) and \( b \in A_M \),

\[
\langle \psi(a), b \rangle = \langle a, \varphi(b) \rangle.
\]  

(8.18)

This means \( \langle Z^m a, b \rangle_{A_M} = \langle a, b \rangle_A \). Now if \( N b = Z^m c \), then

\[
\langle Z^m a, b \rangle_{A_M} = \frac{1}{N} \langle Z^m a, c \rangle_A = \frac{1}{N} \langle a, Z^m c \rangle_A = \langle a, b \rangle_A.
\]  

(8.19)

To finish up, we need to be able to produce a metabolic \( M \).

**Lemma 8.4.** If \( B \) is a \( \mathbb{Q}/\mathbb{Z} \)-valued isometric structure such that \( Z \) annihilates \( B \), then

\[
B = zB \oplus \overline{z}B
\]  

(8.20)

with \( zB \) and \( \overline{z}B \) both metabolic submodules of \( B \).

**Proof.** We know that \( (zB)^\perp \supseteq zB \) since \( \langle z b, z b' \rangle = \langle Z b, Z b' \rangle = 0 \). Similarly, \( (\overline{z}B)^\perp \supseteq \overline{z}B \). But \( B = zB \oplus \overline{z}B \) as \( P \)-modules. Now \( (zB)^\perp \cap \overline{z}B = 0 \) (otherwise some element of \( \overline{z}B \) would be orthogonal to both \( zB \) and \( \overline{z}B \)), so \( (zB)^\perp = zB \).

Here is a generalization, which can be proved similarly:

**Lemma 8.5.** If \( Z^m \) annihilates \( B \), then

\[
B = z^m B \oplus \overline{z}^m B
\]  

(8.21)

and \( z^m B \) and \( \overline{z}^m B \) are both metabolic.

**Theorem 8.6.** If \( A \) is an isometric structure, then \( A \) is \( R \)-equivalent to a minimal isometric structure.

**Proof.** Without loss of generality, we may assume that \( T_Z A \) is minimal. We induct on the rank of \( A \) as a \( \mathbb{Z} \)-module.

We now use our previous analysis for \( m = 1 \). Here \( L = Z A / T_Z(ZA) \), and we have \( L^* / L \cong T_Z(A/ZA) \). In particular, \( L^* / L \) is annihilated by \( Z \), so there exists a metabolic \( M \). So \( A \) adjoins \( A_M \).
If \( A \) is not minimal, then \( ZA \) has lower rank than \( A \). But \( ZA \) and \( L \) have the same rank, so \( ZA \) and \( A_L \) have the same rank. So \( A_L \) has lower rank than \( A \), so we can induct.

Our next goal is to show that \( A \) is \( R \)-equivalent to \( B \) if and only if \( A \) is \( m \)-adjoining to \( B \) for some \( m \). We know the “only if” case since if \( A \) \( m \)-adjoins \( B \) and \( B \) \( n \)-adjoins \( C \), then \( A \) \((m + n)\)-adjoins \( C \).

To prove the converse, we’ll first prove it in the case where \( A \) and \( B \) are minimal.

**Proposition 8.7.** If \( A \) and \( B \) are minimal isometric structures and \( A \) \( m \)-adjoins \( B \), then \( B \overset{\sim}{=} A_M \) for some \( M \subseteq L^*/L \cong T_Z(A/Z^mA) \).

**Proof.** Let \( \varphi : A \rightarrow B \) and \( \psi : B \rightarrow A \) be appropriate maps. Then both compositions of the two maps are injective, so \( \varphi \) and \( \psi \) are injective. We can then identify \( B \) with a \( P \)-submodule of \( A \), with \( \psi \) the inclusion. Then \( Z^mA \subseteq B \). We consider the diagram

\[
\begin{array}{c}
0 \longrightarrow Z^mA/T_Z(Z^mA) \longrightarrow \text{Hom}_Z(Z^mA, Z) \longrightarrow L^*/L \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
B/T_ZB \overset{\sim}{\longrightarrow} \text{Hom}_Z(B, Z)
\end{array}
\]

(8.22)

We claim this commutes. We have

\[
\langle a, b \rangle_A = \langle a, \psi(b) \rangle_A = \langle \varphi(a), b \rangle_B = \langle Z^mA, b \rangle_B.
\]

(8.23)

Let \( M \) be the image of \( B \) in \( L^*/L \).

**Claim.** \( M \) is metabolic.

**Proof.** \( M \subseteq M^\perp \) is true because the pairing on \( L_\mathbb{Q} \) takes \( \mathbb{Z} \)-values on \( B \). Conversely, to show \( M^\perp \subseteq M \), suppose \( \ell \in L^* \) is such that \( \langle \ell, b \rangle_{L_\mathbb{Q}} \in \mathbb{Z} \) for every \( b \in B \). Then \( \ell \) may be represented by some \( b' \in B \) because \( B/T_ZB \overset{\sim}{\longrightarrow} \text{Hom}_Z(B, \mathbb{Z}) \). \( \square \)

Now to show that \( B = A_M \), \( A \) being minimal implies \( Z : A \rightarrow A \) is injective, in particular becomes an isomorphism after tensoring with \( \mathbb{Q} \). The same is true of \( Z^m \), and so \([A]_m = A \). Now \( L^*/L = A/Z^mA \), and so \( M = B/Z^mA \). Thus \( B = A_M \). \( \square \)

Given \( C \) a \( \mathbb{Q} \)-valued isometric structure, suppose there exists \( m > 0 \) such that \( Z^mC = 0 \). We have

\[
C = \mathbb{Z}^mC \oplus z^mC = \ker(z^m) \oplus \ker(\mathbb{Z}^m) = C^+ \oplus C^-.
\]

(8.24)

We know that \((C^\pm)^\perp = C^\pm \). Now if \( M \subseteq C \) is a \( P \)-submodule, then

\[
M = M^+ \oplus M^-
\]

for \( M^\pm = M \cap C^\pm \).

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**Claim.** $M$ is metabolic if and only if $M^\pm = C^\pm \cap (M^\mp)^\perp$. (Because we have a perfect pairing, we only need to know that one of these will be satisfied.)

**Proof.** First suppose $M$ is metabolic, so $M^\perp = M$. We have $M^- \subseteq C^- \cap (M^+)\perp$. On the other hand, if $c^- \in C^-$ is such that $c^- \in (M^+)\perp$, then $C^- = (C^-)\perp \subseteq (M^-)^\perp$ shows that $c^- \in M^\perp = M$. Hence $c \in M^-$. Conversely, if $M^\pm = C^\pm \cap (M^\pm)^\perp$, then $M^\perp = (M^+)\perp \oplus (M^-)^\perp$. We wish to show $(M^+)\perp = M^+$. For we have

\[
(M^+)\perp = M^- \cap C^+ = (M^+)\perp \cap (M^-)\perp \cap C^+.
\]

thus $(M^+)\perp$ contains $(C^+)\perp = C^+$ and therefore $M^+$. \hfill \Box

**Proposition 8.8.** Suppose $A$ is a minimal isometric structure. Let $L^*/L$ be as before and $M \subseteq L^*/L$ metabolic. Write $M = M^+ \oplus M^-$. Take

\[
M_1 = zM^+ \oplus ((zM^+)\perp \cap (L^*/L)^-).
\]

Then $M_1$ is metabolic and $A_{M_1}$ 1-adopts $A_M$.

**Proof.** The fact that $M_1$ is metabolic follows from the previous claim. We can define $\varphi : A_M \to A_{M_1}$ and $\psi : A_{M_1} \to A_M$ by letting $\varphi$ be multiplication by $z$ and $\psi$ be multiplication by $z^{-1}$. So we just need to know that $zM \subseteq M_1$ and $zM_1 \subseteq M$. To see that $zM \subseteq M_1$, observe that $zM^- \subseteq M_1^- = (zM^+)\perp \cap (L^*/L)^-$. \hfill \Box

Now for each $i$, define

\[
M_i = z^iM^+ \oplus (z^iM^+)\perp \cap (L^*/L)^-.
\]

Then each $A_{M_i}$ 1-adopts $A_{M_{i+1}}$. Also $M_m$ is simply $(L^*/L)^-$. **Proposition 8.9.** In the setting as before, with $A$ minimal, suppose $M = (L^*/L)^-$. Then $A_M \cong A$.

**Proof.** $M = (A/Z^mA)^-$, which is the image of $z^mA$ in $A/Z^mA$. So now $A_M = z^mA$. Now multiplication by $z^m : A \to A_M$ is surjective by definition and injective since $A$ is minimal, and respects the pairings. \hfill \Box

**Theorem 8.10.** Let $A$ and $B$ be isometric structures. Then $A$ is $R$-equivariant to $B$ if and only if $A$ $m$-adopts $B$ for some $m$.

**Proof.** We’ve already seen the “only if” direction. For the “if” direction, choose minimal $A'$ and $B'$ with $A$ $R$-equivariant to $A'$ and $B$ $R$-equivariant to $B'$. Then $A'$ will $m'$-adopts $B'$ for some $m'$, and the above work then shows that $A'$ and $B'$ are $R$-equivariant. \hfill \Box
Finally, we can prove Lemma 7.8. Indeed, the 1-adjoining relations found above will all go through $\mathbb{Z}$-torsion-free isometric structures if $A$ and $B$ are $\mathbb{Z}$-torsion-free.

9 Bilinear Pairings on the Alexander Module

Recall that we have classified simple knots by $\mathbb{Z}$-torsion-free isometric structures. Given a knot $K$, we take $H_q(V, \mathbb{Z})$ for $V$ a Seifert hypersurface for $V$. An invariant of $K$ is the Alexander module $\text{Alex}_q(K) = H_q(C_\infty, \mathbb{Z})$ for $C_\infty$ the infinite cyclic cover of $\text{ext}(K)$. $C_\infty$ is a manifold with boundary homeomorphic to $K \times \mathbb{R}$. Recall that

$$H_q(C_\infty, \mathbb{Z}) = H_q(V, \mathbb{Z}) \otimes_{\mathbb{Z}[z]} L$$

for $L = \mathbb{Z}[z]/((z\bar{z})^{-1}] = \mathbb{Z}[t, t^{-1}, (1 - t)^{-1}]$ for $t = 1 - z^{-1}$. The induced $L$-module structure on $H_q(C_\infty, \mathbb{Z})$ agrees with the natural structure of $H_q(C_\infty, \mathbb{Z})$ as a $\mathbb{Z}[t, t^{-1}]$-module.

Observe that if $H_q(V, \mathbb{Z})$ $m$-adjoins $H_q(V', \mathbb{Z})$ for some $m$, then the two become isomorphic after tensoring with $L$. However, the tensor product forgets the isometric structure, so the converse is not quite true. So we wish to put additional structure on our $L$-modules to fix this.

$H_q(C_\infty, \mathbb{Z})$ is not a finitely generated $\mathbb{Z}$-module, but $H_q(C_\infty, \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{Q}$ will still be a finite dimensional vector space over $\mathbb{Q}$. $H_q(C_\infty, \mathbb{Z})$ will have two natural pairings: the $\mathbb{Z}$-bilinear Milnor pairing

$$H_q(C_\infty, \mathbb{Z}) \times H_q(C_\infty, \mathbb{Z}) \to \mathbb{Q}$$

which becomes a perfect pairing after tensoring with $\mathbb{Q}$, and the perfect $\Lambda$-bilinear $\epsilon$-hermitian Blanchfield pairing which takes values in $\text{Frac}(\Lambda)/\Lambda$. Recall that $\Lambda = \mathbb{Z}[t, t^{-1}]$.

We’ll consider the Milnor pairing first. We have a cup product

$$H^q(C_\infty, \partial C_\infty; \mathbb{Q}) \times H^q(C_\infty, \mathbb{Q}) \to H^{2q}(C_\infty, \partial C_\infty; \mathbb{Q})$$

Consider the picture of $C_\infty$: 

![Diagram of C_infinity]

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We have $\partial U \cong \pi V$, and $D$ is homeomorphic to $K \times (0, 1)$ since $\partial C_\infty = K \times \mathbb{R}$. Also

$$H^{2q-1}(D, \mathbb{Q}) \to H^{2q}(C_\infty, \partial C_\infty; \mathbb{Q}) \to H^{2q}(U, D; \mathbb{Q}) \to H^{2q}(U, \mathbb{Q}) \cong \tilde{H}_0(V, \mathbb{Q}) = 0 \quad (9.4)$$

For the last nonzero term, $H^{2q}(U, K \times (0, 1)) = \mathbb{Q}$. We also have $H^{2q}(V, K) = \mathbb{Q}$. To relate these, we have a map $i_+: (V, K) \to (U, K \times (0, 1))$. The map

$$i^+_*: H^{2q}(U, K \times (0, 1)) \to H^{2q}(V, K) \quad (9.5)$$

is an isomorphism, as it is dual to $i_+^\ast$.

Let $U_\infty = \prod t^i U$ and $V_\infty = \prod t^i V$. Then

$$0 \longrightarrow H^{2q}(C_\infty, \partial C_\infty) \quad (9.6)$$

$$H^{2q}(U_\infty, K \times (\mathbb{R} \setminus \mathbb{Z})) \oplus H^{2q}(V_\infty, K \times \mathbb{Z}) \longrightarrow H^{2q}(V_\infty, K \times \mathbb{Z}) \oplus H^{2q}(V_\infty, K \times \mathbb{Z})$$

$$\prod_{i=-\infty}^{\infty} H^{2q}(t^i U, t^i (K \times [0, 1])) \oplus \prod_{i=-\infty}^{\infty} \mathbb{Q}$$

$$\prod_{i=-\infty}^{\infty} \mathbb{Q} \oplus \prod_{i=-\infty}^{\infty} \mathbb{Q}$$

Our second map ends up being

$$\prod_{i=-\infty}^{\infty} u_i \oplus \prod_{i=-\infty}^{\infty} v_i \mapsto \left( \prod_{i=-\infty}^{\infty} u_i, \prod_{i=-\infty}^{\infty} -u_{i+1} \right) + \left( \prod_{i=-\infty}^{\infty} v_i, \prod_{i=-\infty}^{\infty} -v_{i+1} \right) \quad (9.7)$$

The kernel of this map consists of those sequences with $u_i = -v_i$ and $u_{i+1} = -v_i$. This is a 1-dimensional $\mathbb{Q}$-vector space, so $H^{2q}(C_\infty, \partial C_\infty; \mathbb{Q}) \cong \mathbb{Q}$.

Claim. The pairing

$$H^q(C_\infty, \partial C_\infty; \mathbb{Q}) \times H^q(C_\infty, \mathbb{Q}) \to H^{2q}(C_\infty, \partial C_\infty; \mathbb{Q}) \cong \mathbb{Q} \quad (9.8)$$

is a perfect pairing.

Assuming the claim, $H^q(C_\infty, \partial C_\infty; \mathbb{Q}) \cong H_q(C_\infty, \mathbb{Q})$ and $H^q(C_\infty, \mathbb{Q}) \cong H_q(C_\infty, \partial C_\infty; \mathbb{Q})$. But using the relative long exact sequence, $H_q(C_\infty, \partial C_\infty; \mathbb{Q}) \cong H_q(C_\infty, \mathbb{Q})$. Combining these, we obtain a perfect pairing on $H_q(C_\infty, \mathbb{Q})$. This pairing is called the Milnor pairing.

Here is an algebraic approach. We already have a bilinear $\mathbb{Z}$-valued pairing on $A = H_q(V, \mathbb{Z})$ and a $\mathbb{Z}$-linear map $\varphi : A \to H_q(C_\infty, \mathbb{Z}) = \bar{A}$, since the latter can be identified with $A \otimes_p L$. We might try to transfer $\langle \cdot, \cdot \rangle_A$ to a pairing on $\bar{A}$. The naive approach, trying to define $\langle \varphi(a_1), \varphi(a_2) \rangle_{\bar{A}} = \langle a_1, a_2 \rangle_A$, is not well defined if $\varphi$ has any non-torsion elements in the kernel. We will fix this deficiency.

Let $A$ be an isometric structure. $B \subseteq A$ is basic if:
(a) \( B \supseteq (z\overline{z})^k A \) for some \( k \geq 0 \).

(b) If \( a \in A \) is such that \( Na \in B \), then \( a \in B \).

(c) The kernel of \( z\overline{z} : B \to B \) is contained in \( T_{\mathbb{Z}}B \).

**Theorem 9.1 (Farber).** *Any isometric structure has a unique basic submodule.*

To do: Fill in this gap. (3)

With \( B \) basic, the map \( B \otimes_{\mathbb{Z}} \mathbb{Q} \to \tilde{A} \otimes_{\mathbb{Z}} \mathbb{Q} \) is an isomorphism. The intersection pairing on \( B \) extends to \( \tilde{A} \otimes_{\mathbb{Z}} \mathbb{Q} \). This can be shown to be a perfect pairing.

We would like to know why our two defined pairings are compatible. We have \( A \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_q(V, \mathbb{Q}) \).

The topological definition worked more naturally with cohomology; we have

\[
\begin{align*}
H^q(C_{\infty}, \mathbb{Q}) & \to H^q(V, \mathbb{Q}) \to (A \otimes_{\mathbb{Z}} \mathbb{Q})^* \to A \otimes_{\mathbb{Z}} \mathbb{Q} \\
\sim & \sim \sim \sim \\
(\tilde{A} \otimes_{\mathbb{Z}} \mathbb{Q})^* & \to (A \otimes_{\mathbb{Z}} \mathbb{Q})^*
\end{align*}
\]

(9.9)

The kernel of \( i^* \) turns out to be 0. We would like to know the image of this map; it is dual to the kernel of \( i_* \), which is the kernel of \( Z^N : A \otimes_{\mathbb{Z}} \mathbb{Q} \to A \otimes_{\mathbb{Z}} \mathbb{Q} \) for some \( N \gg 0 \). Dualizing again, the image of \( i^* \) is the image of \( (Z^N)^* \). By self-duality of \( A \otimes_{\mathbb{Z}} \mathbb{Q} \), we get the image of \( Z^N \) from \( A \otimes_{\mathbb{Z}} \mathbb{Q} \) to itself.

We find that the pairing on \( (\tilde{A} \otimes_{\mathbb{Z}} \mathbb{Q})^* \) is identified with the pairing on \( Z^N(A \otimes_{\mathbb{Z}} \mathbb{Q}) \), which is exactly \( B \otimes_{\mathbb{Z}} \mathbb{Q} \).

**Theorem 9.2.** Let \( A_1, A_2 \) be isometric structures. \( A_1 \) is \( R \)-equivalent to \( A_2 \) if and only if there exists an isomorphism \( \tilde{A}_1 \to \tilde{A}_2 \) of \( L \)-modules which preserves the Milnor pairing.

**Proof.** For the “only if” case, suppose \( A_1 \) and \( A_2 \) are \( m \)-adjoining. Choose \( \varphi : A_1 \to A_2 \) and \( \psi : A_2 \to A_1 \) as appropriate. We define \( \Phi : \tilde{A}_1 \to \tilde{A}_2 \) by \( \Phi = \varphi \otimes z^{-m} \), and \( \Psi : \tilde{A}_2 \to \tilde{A}_1 \) by \( \Psi = \psi \otimes \overline{z}^{-m} \). These are inverses, so it remains to show they preserve the Milnor form. We have

\[
\begin{align*}
\tilde{A}_1 \otimes_{\mathbb{Z}} \mathbb{Q} & \to \tilde{A}_2 \otimes_{\mathbb{Z}} \mathbb{Q} \\
\sim & \sim \\
Z^N A_1 \otimes_{\mathbb{Z}} \mathbb{Q} & \to Z^N A_2 \otimes_{\mathbb{Z}} \mathbb{Q}
\end{align*}
\]

(9.10)

Now \( Z^N A_1 \otimes_{\mathbb{Z}} \mathbb{Q} = B_1 \otimes_{\mathbb{Z}} \mathbb{Q} \). Now we have

\[
\langle (z^{-m}\varphi(a_1)), z^{-m}\varphi(a_2) \rangle_{B_2 \otimes_{\mathbb{Z}} \mathbb{Q}} = \langle ((z\overline{z})^{-m}\varphi(a_1)), \varphi(a_2) \rangle \\
= \langle (z\overline{z})^{-m}\psi(\varphi(a_1)), a_2 \rangle \\
= \langle a_1, a_2 \rangle.
\]

(9.11)
For the other direction, suppose $\Phi : \tilde{A}_1 \to \tilde{A}_2$ is an isomorphism. Let $\Psi$ be its inverse. Without loss of generality, we can assume that $A_1$ and $A_2$ are minimal, replacing them by $R$-equivalent structures if necessary. In this situation, $A_1$ and $A_2$ are already basic, since $Z^m A_1 \otimes_\mathbb{Z} \mathbb{Q} \cong A_1 \otimes_\mathbb{Z} \mathbb{Q}$. Also the map $A_1 \to \tilde{A}_1$ is injective.

Now consider $\Phi|_{A_1} : A_1 \to \tilde{A}_2$. The image of this map is contained in $Z^{-m} A_2$ for some $m$. Choose $m$ such that $\Psi|_{A_2}$ has image contained in $Z^{-m} A_1$ as well. Now we may define $\varphi = Z^m \Phi|_{A_1}$ and $\psi = Z^m \Psi|_{A_2}$. It remains to show that these maps are $m$-adjoining; these is by the reverse reasoning of the calculation for the other direction.

We’ve shown that simple knots $K_1$ and $K_2$ are equivalent if and only if their isometric structures associated to Seifert hypersurfaces are $R$-equivalent. In turn, this happens if and only if the corresponding $H_q(C_\infty, \mathbb{Z})$ are isomorphic as $L$-modules with the isomorphism preserving the Milnor pairing.
To do...

- 1 (p. 5): Examples of positive and negative crossings.
- 2 (p. 30): Fix Me
- 3 (p. 36): Fill in this gap.