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1 Basic Definitions

A $C^\infty$ manifold is a set $X$ with the structure of a topological space and an atlas: an open cover $\{U_\alpha\}$ and homeomorphisms $\varphi_\alpha : U_\alpha \xrightarrow{\sim} \Delta \cong \mathbb{R}^n$ such that $\varphi_\alpha \circ \varphi^{-1}_\beta$ is $C^\infty$ where defined.

We can talk about $C^\infty$ functions on $X$ and $C^\infty$ maps $X \to Y$ via working with atlases.

Recall that a group is a set $X$ with the structure of $e \in X$, $m : X \times X \to X$, and $i : X \to X$ satisfying appropriate axioms.

A Lie group is a $C^\infty$ manifold $G$ with a group structure such that $m$ and $i$ are $C^\infty$. We say that a map $G \to H$ is a Lie group homomorphism if it is a group homomorphism which is also a $C^\infty$ map.

A Lie subgroup $H$ of $G$ is an $H \subseteq G$ which is a subgroup and a closed submanifold.

As a nonexample, consider a line through the origin in $\mathbb{R}^2$ having an irrational slope. The map $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ takes this line to a dense subset of the torus, called an immersed submanifold of the torus.

A representation of $G$ on a vector space $V$ is a map $G \to GL(V)$, the set of invertible linear maps on $V$.

2 Examples of Lie Groups

- $GL_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ is a Lie group because of the explicit descriptions of matrix multiplication and inversion. This makes $GL(V)$ a Lie group for any finite dimensional vector space $V$.

- $SL_n(\mathbb{R})$ is a submanifold of $GL_n(\mathbb{R})$ because $\det : M_n(\mathbb{R}) \to \mathbb{R}$ has nonvanishing differential. (It’s enough to check at the identity.) We can also look at $SL(V)$, the set of $\varphi$ preserving a volume form. That is, for some nonzero $\alpha \in \Lambda^n V^*$, we have $\alpha(\varphi(v_1), \ldots, \varphi(v_n)) = \alpha(v_1, \ldots, v_n)$.

- A flag $\mathcal{V}$ in $V$ is a nested sequence of subspaces

$$0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k \subsetneq V \tag{2.1}$$
Define $B(V)$ to be the set of automorphisms of $V$ preserving $V$; that is, $\varphi$ such that $\varphi(V_i) \subseteq V_i$ for every $i$ (in this case, $\varphi(V_i)$ necessarily equals $V_i$.)

A full flag is a $V$ where $V_i$ has dimension $i$. In an appropriate basis for $V$, $B(V)$ consists of upper triangular matrices. More generally, $B(V)$ consists of block upper triangular matrices in an appropriate basis.

- Let $N(V)$ be the subgroup of $B(V)$ consisting of those $\varphi$ such that $(\varphi - I)(V_i) \subseteq V_{i-1}$. That is, $\varphi$ is the identity on successive quotients.

- Let $Q$ be a symmetric bilinear form on $V$. $Q : V \times V \to \mathbb{R}$ is an element of $\text{Sym}^2 V^*$. Assume that $Q$ is nondegenerate. Define $O(V,Q)$ to be the set of $\varphi$ preserving $Q$; that is, $Q(\varphi(v),\varphi(w)) = Q(v,w)$. In matrix form, express $Q$ as a symmetric $n \times n$ matrix $M$, so $Q(v,w) = v^T M w$. Then

\[ O(V,Q) = \{ A \in \text{GL}_n(\mathbb{R}) : A^T MA = M \}. \] (2.2)

If $M = I$, then $O(V,Q)$ is called $O(n)$. More generally, given $Q$, we can find a basis such that $M$ is diagonal with diagonal entries 1 and $-1$. We say that $M$ has signature $(k, n-k)$, and $O(V,Q)$ is called $O_{k,n-k}$.

$SO(V,Q)$ is the further subgroup of determinant-1 transformations.

- Take $Q$ to be a skew-symmetric nondegenerate bilinear form, so $Q \in \Lambda^2 V^*$. $Q$ can be specified by a skew-symmetric $M$, and under an appropriate basis, we can write

\[ M = \begin{pmatrix} O & I \\ -I & 0 \end{pmatrix} \] (2.3)

The group of matrices preserving $Q$ is called $Sp_{2n}$.

- If $V$ is 7-dimensional and $\alpha$ is a general skew-symmetric trilinear form ($\alpha \in \Lambda^3 V^*$), then the group of automorphisms of $V$ preserving $\alpha$ is nontrivial and called $G_2$.

3 Isogenies of Lie Groups

**Proposition 3.1.** If $G$ is a connected Lie group and $U$ is an open subset containing $e$, then $U$ generates $G$.

**Proof.** By replacing $U$ by $U \cap U^{-1}$, we can without loss of generality assume $U = U^{-1}$. Let $H$ be the subgroup generated by $U$. Suppose $H \neq G$. Then there exists $g \in \partial H = \overline{H} \setminus H$. Then $gU \cap H \neq \emptyset$; choose $h$ in the intersection. If $h = gu$ for $u \in U \subseteq H$, then $g = hu^{-1} \in H$, a contradiction. \qed

Let $H$ be a Lie group and $\Gamma \subseteq Z(H)$ discrete. Set $G = H/\Gamma$.

**Claim.** $G$ has the unique structure of a Lie group such that the quotient map $H \to G$ is a map of Lie groups. (In fact, the map is a covering map.)

On the other hand, now start with $G$ a Lie group and let $H$ be a connected topological space such that $H \to G$ is a covering map. Clearly $H$ inherits the structure of a manifold. Choose an element $e' \in H$ mapping to $e$. Then:
Theorem 3.2. There exists a unique group structure on $H$ such that $H$ is a Lie group and $H \to G$ is a map of Lie groups.

Proof. First consider the case where $H$ is the universal cover of $G$. Consider the group law on $G$ as a map $G \times G \to G$. We can uniquely lift

$$
\begin{array}{c}
H \times H \xrightarrow{m_H} H \\
\downarrow \quad \downarrow \\
G \times G \xrightarrow{m_G} G
\end{array}
$$

We get a unique map $H \times H \to H$ mapping $(e', e')$ to $e'$. Inversion is similar.

To see that $m_H$ is associative, consider $a(bc)((ab)c)^{-1}$. This is a continuous map $H \times H \times H \to H$ lying in the fiber of $e$. So this map is a constant, identically $e'$.

Proposition 3.3. If $G$ is a connected Lie group and $\Gamma \subseteq G$ is discrete and normal, then $\Gamma \subseteq Z(G)$.

Proof. For $h \in \Gamma$, the map $G \to \Gamma$ by $g \mapsto ghg^{-1}$ is continuous, so must be constant. The value at $e$ is $h$, so $ghg^{-1} = g$ for every $g \in G$.

We say that connected groups $G, H$ are isogenous if there exists a Lie group map $G \to H$ with discrete kernel (that is, a covering space map). The isogeny classes are the equivalence classes of the equivalence relation generated by isogenies. In each isogeny class, there exists a unique initial member, which we’ll call $H$, the simply connected form: the universal covering space of any member. If $Z(H)$ is discrete, then there also exists a final object $H/Z(H)$, called the adjoint form.

Our basic approach is to describe all representations of the simply connected form, and then say which ones descend to a given member of the isogeny class.

4 Maps Between Lie Groups

Here is a problem: given $G, H$, describe all Lie group maps $G \to H$. (The case $H = \text{GL}(V)$ corresponds to representation theory.) We’ll assume that $G$ is simply connected. In fact, $\rho : G \to H$ is determined not just by its values on any open neighborhood of $e$, but even $d\rho_e : T_eG \to T_eH$. In other words we have an inclusion

$$
\text{Hom}_{\text{Lie}}(G, H) \hookrightarrow \text{Hom}(T_eG, T_eH).
$$

We would like to know which linear maps $T_eG \to T_eH$ actually arise as differentials of Lie group maps.

If $\rho$ is a Lie group map, then $\rho(gh) = \rho(g)\rho(h)$. In other words, the following diagram commutes for every $g$:
where \( m_g \) is multiplication by \( g \). To focus our attention on \( G, H \) near the identity, consider instead \( \psi_g \) which is conjugation by \( g \). We also have the commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\rho} & H \\
\downarrow m_g & & \downarrow m_{\rho(g)} \\
G & \xrightarrow{\rho} & H
\end{array}
\]  \hspace{1cm} (4.2)

But now \( \psi_g(e) = e \) so we can consider \( (d\psi_g)_e : T_eG \to T_eG \). A get a map \( G \to GL(T_eG) \) by \( g \mapsto (d\psi_g)_e \), called the adjoint representation \( \text{Ad} \), a distinguished representation of \( G \). Taking the differential of \( \text{Ad} \), we get a map \( \text{ad} : T_eG \to \text{End}(T_eG) \). By taking transpose, we can also view \( \text{ad} \) as a map \( T_eG \times T_eG \to T_eG \) by \( (X,Y) \mapsto \text{ad}(X)(Y) = [X,Y] \).

The point is that if \( \rho \) is any Lie group map, then \( (d\rho)_e \) respects the binary operation \( \text{ad} \) on \( T_eG \). That is,

\[
\begin{array}{ccc}
T_eG & \times & T_eG \\
\downarrow \text{[ , ]} & & \downarrow \text{[ , ]} \\
T_eG & \xrightarrow{d\rho} & T_eH
\end{array}
\]  \hspace{1cm} (4.4)

In fact, if \( G \) is simply connected, then any linear map \( \varphi : T_eG \to T_eH \) is the differential of a Lie group map \( \rho : G \to H \) if and only if \( [\varphi(X),\varphi(Y)] = \varphi([X,Y]) \).

**Fact.** We can describe the map \( [ , ] \) explicitly.

Start with \( G = GL(V) \). Then \( T_eG = \text{End}(V) \). Observe that \( \psi_G : G \to G \) by \( h \mapsto ghg^{-1} \) extends naturally to \( \text{End}(V) \), and this extension is the differential of \( \psi_g \), since \( \psi_g \) is linear. We have \( d\psi_g(X) = gXg^{-1} \).

To describe \( \text{ad} \), we start with a given \( X \in \text{End}(V) \) and choose an arc \( \gamma : (-\varepsilon,\varepsilon) \to G \) with \( \gamma(0) = e \) and \( \gamma'(0) = X \). Then

\[
[X,Y] = \text{ad}(X)(Y) 
\]

\[
= \frac{d}{dt} \bigg|_{t=0} \gamma(t)Y\gamma_t^{-1} 
\]

\[
= \gamma_0'Y\gamma_0^{-1} + \gamma_0Y(\gamma^{-1})_0' 
\]

\[
= \gamma_0'Y\gamma_0^{-1} - \gamma_0Y(\gamma_0^{-1}\gamma_0')_0 
\]

\[
= XY - YX. 
\]

This formula also applies to any subgroup of \( GL(V) \). This has two consequences:
• \([X,Y] = -[Y,X]\); that is, the bracket is skew-symmetric.

• The Jacobi identity holds: for \(X,Y,Z \in T_eG\),

\[
[[X,Y], Z] + [[Y,Z], X] + [[Z,X], Y] = 0. \tag{4.10}
\]

5 Lie Algebras

We define a Lie algebra \(\mathfrak{g}\) to be a vector space with a skew-symmetric bilinear map \(\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\) satisfying the Jacobi identity.

So one of our above claims is that for \(G\) simply connected and \(H\) any Lie group, if \(\mathfrak{g} = T_eG\) and \(\mathfrak{h} = T_eH\), then

\[
\text{Hom}_{\text{Lie grp}}(G,H) = \text{Hom}_{\text{Lie alg}}(\mathfrak{g}, \mathfrak{h}). \tag{5.1}
\]

We will split this assertion into the following two theorems:

**Theorem 5.1.** If \(G\) is connected, any Lie group map \(\rho : G \to H\) is determined by its differential \(d\rho : \mathfrak{g} \to \mathfrak{h}\).

**Theorem 5.2.** If \(G\) is simply connected, a linear map \(\varphi : \mathfrak{g} \to \mathfrak{h}\) is the differential of a Lie group map \(\rho : G \to H\) if and only if \(\varphi\) is a Lie algebra map.

Why couldn’t we just define \([X,Y] = XY - YX\)? The problem is that \(XY\) is not always defined: given \(G \hookrightarrow GL(V)\) and \(X,Y \in T_eG\) viewed as endomorphisms of \(V\), \(XY\) depends on the embedding of \(G\) into \(GL(V)\), and is not canonical. In fact, it may not even lie in the image of \(T_eG\).

We may sometimes write \(\text{End}(V)\) as \(\mathfrak{gl}(V)\) if we are interested in the Lie algebra structure.

A Lie algebra morphism from \(\mathfrak{g}\) to \(\mathfrak{h}\) is a linear map \(\varphi : \mathfrak{g} \to \mathfrak{h}\) such that \(\varphi([X,Y]) = [\varphi(X), \varphi(Y)]\). A representation of \(\mathfrak{g}\) on \(V\) is a Lie algebra morphism \(\mathfrak{g} \to \mathfrak{gl}(V)\). This can be viewed as an action of \(\mathfrak{g}\) on \(V\) such that for every \(X,Y \in \mathfrak{g}\), \(X(Y(v)) - Y(X(v)) = [X,Y](v)\).

Observe that given a group \(G\) acting on vector spaces \(V\) and \(W\), we get an action of \(G\) on \(V \otimes W\) by \(g(v \otimes w) = g(v) \otimes g(w)\). In other words, given \(\rho_V : G \to GL(V)\) and \(\rho_W : G \to GL(W)\), we obtain a map \(\rho_{V \otimes W} = \rho_V \otimes \rho_W\).

How do the corresponding Lie algebra maps relate to this? Suppose \(\varphi_V : \mathfrak{g} \to \mathfrak{gl}(V)\) and \(\varphi_W : \mathfrak{g} \to \mathfrak{gl}(W)\) are given. Let \(\gamma\) be an arc in \(G\) with \(\gamma_0 = e\) and \(\gamma'_0 = X\). We have \(\gamma_t(v \otimes w) = \gamma_t(v) \otimes \gamma_t(w)\). Differentiating at \(t = 0\),

\[
X(v \otimes w) = X(v) \otimes w + v \otimes X(w). \tag{5.2}
\]

In general, given two Lie algebra representations, the tensor product representation will be defined by the above.

Here are some other examples:

• If \(G\) acts on \(V\), then \(G\) correspondingly acts on \(\text{Sym}^2 V\) by \(g(v^2) = g(v)^2\). The corresponding representation on \(\mathfrak{g}\) is given by \(X(v^2) = 2vX(v)\).
• Given \( \varphi : V \to W \), recall that the transpose map \( \varphi^T : W^* \to V^* \) is the map described by composition: \( [\varphi^T(\lambda)](v) = \lambda(\varphi(v)) \).

Now if \( G \) acts on \( V \), then \( G \) acts on \( V^* \) by \( \rho_V^*(g) = \rho_V(g^{-1})^T \). The corresponding representation of \( \mathfrak{g} \) is given by \( \varphi_{V^*}(X) = -\varphi_V(X)^T \).

**Remark.**

• The definition of a Lie algebra doesn’t require \( \mathfrak{g} \) to be finite dimensional, but we’ll only deal with finite dimensional ones.

• The definition also doesn’t specify the base field. We’ll assume that the base field is either \( \mathbb{R} \) or \( \mathbb{C} \), and will use \( \mathbb{R} \) by default.

• In fact, every Lie algebra can be embedded in \( \mathfrak{gl}_n(\mathbb{R}) \) for some \( n \). This implies that every Lie algebra arises from a Lie group.

• But not every Lie group is a subgroup of \( GL_n(\mathbb{R}) \). An example is the universal cover \( \widetilde{SL_2(\mathbb{R})} \) of \( SL_2(\mathbb{R}) \). Another example is given by

\[
\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \right\} / \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : b \in \mathbb{Z} \right\}.
\] (5.3)

### 6 The Exponential Map

Suppose \( G \) is any Lie group, and \( X \in \mathfrak{g} = T_eG \). We can define a vector field \( V = V_X \) on \( G \) by \( V(g) = (dm_g)_e(X) \). (We may write \( \alpha_s \) instead of \( d\alpha \) for a map \( \alpha \) on manifolds. If the domain is a subset of \( \mathbb{R} \), then \( \alpha' \) may be used.)

Now recall from differential geometry the following

**Theorem 6.1.** If \( M \) is any manifold, \( p \in M \), and \( V \) is a vector field on \( M \), then there exists a unique germ of an arc \( \gamma : (-\varepsilon, \varepsilon) \to M \) such that \( \gamma(0) = p \) and \( \gamma'(t) = v(\gamma(t)) \).

Apply this theorem to the above situation. Given \( X \in \mathfrak{g} = T_eG \) and \( V_X \) the corresponding vector field, let \( \gamma : (-\varepsilon, \varepsilon) \to G \) be the corresponding integral curve with \( \gamma(0) = e \).

**Claim.** \( \gamma \) is a group homomorphism where defined. That is, \( \gamma(s + t) = \gamma(s)\gamma(t) \).

**Proof.** Fix \( s \), and set \( \alpha(t) = \gamma(s + t) \) and \( \beta(t) = \gamma(s)\gamma(t) \). Then \( \alpha'(t) = V(\gamma(s + t)) \) while \( \beta'(t) = (dm_{\gamma(s)}\gamma(t))V(\gamma(t)) \). Since \( V \) is a left invariant vector field, \( \alpha(t) \) and \( \beta(t) \) are both integral curves for \( V \) starting at \( \gamma(s) \), so they must be the same arc. \( \square \)

Now using the claim, we can extend \( \gamma \) to \( \mathbb{R} \).

In summary, for each \( X \in \mathfrak{g} \), we get a Lie group morphism \( \varphi_X : \mathbb{R} \to G \) such that \( \varphi_X(0) = X \). These properties uniquely determine \( \varphi_X \). Such a map is called a 1-parameter subgroup.

By uniqueness, we get the following naturality: if \( \rho : G \to H \) is any Lie group map and \( X \in \mathfrak{g} \), then

\[
\rho \circ \varphi_X = \varphi_{\rho(X)}.
\] (6.1)
Remark. • \( \varphi_X \) does not need to be injective. However, if \( X \neq 0 \), the kernel will be a nontrivial closed subgroup of \( \mathbb{R} \), therefore either \( \{0\} \) or \( r\mathbb{Z} \) for some \( r > 0 \). In these cases, the image of \( \varphi \) will be isomorphic to \( \mathbb{R} \) or \( S^1 \).

• If \( \varphi_X \) is injective, the image may be a closed subgroup or simply an immersed subgroup.

• We have \( \varphi_{\lambda X}(t) = \varphi_X(\lambda t) \).

We now define the exponential map \( \exp : \mathfrak{g} \to G \) by \( \exp(X) = \varphi_X(1) \). So \( \exp \) restricted to each line through \( 0 \in \mathfrak{g} \) is a 1-parameter subgroup.

Fact. The map \( \exp \) is the unique \( C^\infty \) map \( \mathfrak{g} \to G \) such that the differential at 0 is the identity and the restriction to lines is a 1-parameter subgroup.

By this characterization of \( \exp \), given a Lie group map \( \rho : G \to H \), we have

\[
\begin{array}{ccl}
G & \xrightarrow{\rho} & H \\
\exp & \downarrow & \exp \\
\mathfrak{g} & \xrightarrow{d\rho} & \mathfrak{h}
\end{array}
\] (6.2)

We’ll now show that \( \exp \) is \( C^\infty \). Explicitly, if \( G = GL(V) \) and \( \mathfrak{g} = gl(V) \), then

\[ \exp(X) = 1 + X + \frac{X^2}{2} + \frac{X^3}{6} + \cdots \] (6.3)

This is also true for any subgroup of \( GL(V) \), and therefore any Lie group whose Lie algebra embeds into \( gl(V) \) for some \( V \).

On a small neighborhood of \( e \in G \), we can define an inverse to \( \exp \), called \( \log \). In the case of \( GL(V) \),

\[ \log(g) = (g - I) - \frac{(g - I)^2}{2} + \frac{(g - I)^3}{3} - \cdots \] (6.4)

Given \( X, Y \in \mathfrak{g} \) sufficiently close to 0, we will define \( X \ast Y = \log(\exp(X) \exp(Y)) \). We then have

\[ X \ast Y = \log \left( \left( 1 + X + \frac{X^2}{2} + \cdots \right) \left( 1 + Y + \frac{Y^2}{2} + \cdots \right) \right) \] (6.5)

\[ = \log \left( 1 + (X + Y) + \left( \frac{X^2}{2} + XY + \frac{Y^2}{2} \right) + \cdots \right) \] (6.6)

\[ = \left( X + Y + \left( \frac{X^2}{2} + XY + \frac{Y^2}{2} \right) + \cdots \right) - \frac{1}{2} \left( X + Y + \left( \frac{X^2}{2} + XY + \frac{Y^2}{2} \right) \right) + \cdots \] (6.7)

\[ = X + Y + \left( \frac{X^2}{2} + XY + \frac{Y^2}{2} - \frac{(X + Y)^2}{2} \right) + \cdots \] (6.8)

\[ = X + Y + \frac{[X,Y]}{2} + \cdots \] (6.9)

The final formula of Campbell-Hausdorff-Dynkin-Baker-etc. is
\[ X \ast Y = X + Y + \frac{[X,Y]}{2} + \frac{1}{12} ([X,[X,Y]] - [Y,[X,Y]]) + \cdots \quad (6.10) \]

The important fact is that this expression involves only addition and Lie brackets.

**Corollary 6.2.** Given \( \mathfrak{h} \) a vector subspace of \( \mathfrak{g} \), the subgroup \( H \) of \( G \) generated by \( \exp(\mathfrak{h}) \) is an immersed subgroup with tangent space \( T_eH = \mathfrak{h} \) if and only if \( \mathfrak{h} \) is a Lie subalgebra of \( G \).

We get a bijection between connected immersed subgroups of \( G \) and Lie subalgebras of \( \mathfrak{g} \).

We conclude by proving Theorems 5.1 and 5.2.

**Theorem 5.1** follows from the naturality of \( \exp \): as \( \exp \) is locally a diffeomorphism, \( d\rho \) determines \( \rho \) on a neighborhood of the identity in \( G \).

**Proof of Theorem 5.2.** Given \( \varphi : \mathfrak{g} \rightarrow \mathfrak{h} \), let \( j \subseteq \mathfrak{g} \times \mathfrak{h} \) be the graph. This is a Lie subalgebra if and only if \( \varphi \) is a Lie algebra map. In this case, there exists \( J \subseteq G \times H \) an immersed subgroup. Composing with the projection \( G \times H \rightarrow G \) gives a covering map \( J \rightarrow G \). \( G \) simply connected implies \( J = G \), so \( J \) is the graph of a Lie group map \( G \rightarrow H \).

\[ \square \]

7 Classes of Lie Algebras

Assume \( G \) is connected, and let \( \mathfrak{g} \) be its Lie algebra. \( \mathfrak{g} \) is said to be abelian if \([X,Y] = 0\) for every \( X,Y \in \mathfrak{g} \). More generally, for any \( \mathfrak{g} \), we define the center

\[ Z(\mathfrak{g}) = \{ X \in \mathfrak{g} : [X,Y] = 0 \forall Y \in \mathfrak{g} \}. \quad (7.1) \]

Connected subgroups \( H \subseteq G \) correspond to subalgebras \( \mathfrak{h} \) of \( \mathfrak{g} \). Now suppose \( H \) is normal. Then \( gHg^{-1} = H \) for every \( G \in G \). Choose \( \gamma : I \rightarrow G \) with \( \gamma(0) = e \) and \( \gamma'(0) = X \in \mathfrak{g} \). Then \( \gamma_tH\gamma_t^{-1} = H \) for every \( t \), so \( \text{ad}(X) = \text{Ad}(\gamma_t) \) takes \( \mathfrak{h} \) to \( \mathfrak{h} \).

We say that \( \mathfrak{h} \) is an ideal if \([\mathfrak{g},\mathfrak{h}] \subseteq \mathfrak{h} \). Then \( H \) is normal if and only if \( \mathfrak{h} \) is an ideal.

**Remark.** If \( \mathfrak{h} \subseteq \mathfrak{g} \) is a subalgebra, then \([ , ]\) descends to \( \mathfrak{g}/\mathfrak{h} \) (meaning there exists a Lie algebra structure on the vector space \( \mathfrak{g}/\mathfrak{h} \) such that \( \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h} \) is a Lie algebra map) if and only if \( \mathfrak{h} \) is an ideal.

We say that \( \mathfrak{g} \) is simple if \( \mathfrak{g} \) has no proper nontrivial ideals.

Given a Lie algebra \( \mathfrak{g} \), we define two sequences of ideals in \( \mathfrak{g} \):

1. The lower central series: start with \( D_0\mathfrak{g} = \mathfrak{g} \), and inductively define \( D_i\mathfrak{g} = [\mathfrak{g}, D_{i-1}\mathfrak{g}] \). (In particular, \( D_1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \). We get a sequence of subalgebras

\[ D_0\mathfrak{g} \supseteq D_1\mathfrak{g} \supseteq D_2\mathfrak{g} \supseteq \cdots \quad (7.2) \]

2. The derived series: start with \( D^0\mathfrak{g} \), and this time \( D^i\mathfrak{g} = [D^{i-1}\mathfrak{g}, D^{i-1}\mathfrak{g}] \). The Jacobi relation can be used to show that the \( D^i\mathfrak{g} \) are in fact ideals.
We say that $g$ is nilpotent if $D_k g = 0$ for some $k$, and that $g$ is solvable if $D_k g = 0$ for some $k$.

We say that $g$ is semisimple if $g$ has no nonzero solvable ideals. (We’ll see that this is equivalent to being a direct sum of simple algebras.)

A key example of a nilpotent Lie algebra is the Lie algebra $n$ of strictly upper triangular matrices in $\mathfrak{gl}_n$. This is the Lie algebra of the subgroup of unipotent upper triangular matrices. We will prove that any nilpotent Lie algebra can be realized as a subalgebra of $n$ (for some $n$).

Another example of a solvable Lie algebra is $b$, the space of all upper triangular matrices in $\mathfrak{gl}_n$. In fact, every solvable Lie algebra is a subalgebra of $b$ (for some $n$).

Observe that $g$ is solvable if and only if there exists a sequence of subalgebras

$$0 \subseteq g_1 \subseteq g_2 \subseteq \cdots \subseteq g_k \subseteq g$$

such that $g_i$ is an ideal in $g_{i+1}$ and $g_{i+1}/g_i$ is abelian. As a consequence, if $\mathfrak{h} \subseteq g$ is an ideal, then $g$ is solvable if and only if $\mathfrak{h}$ and $g/\mathfrak{h}$ are solvable.

Remark. If $\mathfrak{a}, \mathfrak{b} \subseteq g$ are solvable ideals, then $\mathfrak{a} + \mathfrak{b}$ is also solvable. This is because $(\mathfrak{a} + \mathfrak{b})/\mathfrak{b} = \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b})$, which is solvable.

**Corollary 7.1.** There exists a maximal solvable ideal in $g$. This is called the radical of $g$ and denoted $\text{Rad}(g)$.

We now have an exact sequence

$$0 \to \text{Rad}(g) \to g \to g/\text{Rad}(g) \to 0$$

with $g/\text{Rad}(g)$ semisimple. We will denote $g/\text{Rad}(g)$ by $g_{ss}$.

**Theorem 7.2 (Engel).** If $\varphi : g \to \mathfrak{gl}(V)$ is a representation of $g$ such that $\forall X \in g$, $\varphi(X)$ is nilpotent, then there exists $0 \neq v \in V$ such that $X(v) = 0$ for every $X \in g$.

Observe that in this case, there exists a basis $v_1, \ldots, v_n$ of $V$ such that $\varphi(g) \subseteq n$.

Remark. If $X \in \mathfrak{gl}(V)$ is nilpotent, then there is a sequence of spaces

$$V \supseteq V_1 \supseteq V_2 \supseteq \cdots \supseteq V_k \supseteq 0$$

such that $X$ maps $V_i$ into $V_{i+1}$. Then $\text{ad}(X)$ viewed as an endomorphism of $\mathfrak{gl}(V)$ is also nilpotent.

**Proof of Engel’s theorem.** Induct on $\dim g$.

**Lemma 7.3.** $g$ contains an ideal $\mathfrak{h}$ of codimension 1.

**Proof.** Let $\mathfrak{h}$ be any maximal proper subalgebra of $g$. We claim that $\mathfrak{h}$ is an ideal and that $\text{codim}(\mathfrak{h} \subseteq g) = 1$.

Look at the adjoint action $\text{ad}(\mathfrak{h}) \subseteq \mathfrak{gl}(g)$. Since $\text{ad}(\mathfrak{h})$ carries $\mathfrak{h}$ into itself, $\text{ad}(\mathfrak{h})$ acts on $g/\mathfrak{h}$. Every element acts nilpotently. For if $X^k = 0$ on $V$, then $\text{ad}(X)^{2k-1}Y = 0$ for every $Y \in g$.

By the inductive hypothesis, there exists $Y \in g/\mathfrak{h}$ nonzero such that $[\mathfrak{h}, Y] \subseteq \mathfrak{h}$. By maximality, we must have $\mathfrak{h} + Y = g$, so $\mathfrak{h}$ has codimension 1.
Given this, by the inductive hypothesis, there exists a nonzero $v \in V$ such that $X(v) = 0$ for every $X \in \mathfrak{h}$. Set

$$W = \{ v \in V : X(v) = 0 \ \forall \ x \in \mathfrak{h} \}. \quad (7.6)$$

Claim. $Y(W) \subseteq W$.

Given the claim, because $Y$ is nilpotent, it must be nilpotent on $W$. So there exists nonzero $w \in W$ for which $Y(w) = 0$. Hence $X(w) = 0$ for every $X \in \mathfrak{g}$.

Proof of claim. Say $w \in W$, and consider $Y(w)$. For $X \in \mathfrak{h}$,

$$X(Y(w)) = Y(X(w)) + [X,Y](w) = 0$$

so that $Y(w) \in W$. \hfill $\Box$ \hfill $\Box$

**Theorem 7.4** (Lie). If $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ is any solvable Lie algebra, then there exists $0 \neq v \in V$ such that $v$ is an eigenvector for all $X \in \mathfrak{g}$.

A consequence is that there exists a basis $v_1, \ldots, v_n \in V$ for which $\mathfrak{g} \subseteq \mathfrak{b}$.

**Proof of Lie’s Theorem.**

Claim. There exists an ideal $\mathfrak{h} \subseteq \mathfrak{g}$ of codimension 1.

Proof of claim. $D_\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ is a proper ideal of $\mathfrak{g}$, with the quotient $\mathfrak{g}/D_\mathfrak{g}$ abelian. Take $\overline{\mathfrak{h}} \subseteq \mathfrak{g}/D_\mathfrak{g}$ any vector subspace of codimension 1 and $\mathfrak{h}$ the preimage. \hfill $\Box$

Now apply the inductive hypothesis to the action of $\mathfrak{h}$ on $V$. So there exists a nonzero $v \in V$ such that $X(v) = \lambda(X)v$ for every $X \in \mathfrak{h}$. $\lambda$ is necessarily linear, so $\lambda \in \mathfrak{h}^*$. With $\lambda$ determined, set

$$W = \{ v \in V : X(v) = \lambda(X)v \ \forall \ x \in \mathfrak{h} \}. \quad (7.8)$$

Choose any $Y \in \mathfrak{g} \setminus \mathfrak{h}$.

Claim. $Y(W) \subseteq W$.

Given the claim, $Y$ acting on $W$ has an eigenvector. (Remember, we’re working over $\mathbb{C}$ here.)

Proof of claim. Let $w \in W$; we have to show that $Y(w) \in W$. Let $X \in \mathfrak{h}$. Then

$$X(Y(w)) = Y(X(w)) + [X,Y](w) = \lambda(X)Y(w) + \lambda([X,Y])(w). \quad (7.9)$$

It remains to show that $\lambda([X,Y]) = 0$. Consider the subspace

$$U = \langle w, Yw, Y^2w, \ldots \rangle. \quad (7.10)$$
Choose $n$ such that \{w, Yw, Y^2w, \ldots, Y^nw\} is a basis for $U$. For $X \in \mathfrak{h}$, we then have

\begin{align*}
X(w) &= \lambda(X)w \\
X(Y(w)) &= \lambda(X)Y(w) + \lambda([X,Y])w \\
X(Y^2(w)) &= \lambda(X)Y^2(w) + \left(\text{linear combination of } w, Yw\right)
\end{align*}

(7.11) (7.12) (7.13)

Repeating, we see that $X$ maps $U$ into $U$ as an upper triangular matrix with diagonal entries all equal to $\lambda(X)$. In particular, $\text{tr}(X : U \to U) = \lambda(X) \dim U$ for every $X \in \mathfrak{h}$. For the case of $[X,Y]$, $\text{tr}([X,Y]) = 0$ implying $\lambda([X,Y]) = 0$.

\begin{proof}
Set $\mathfrak{h} = \text{Rad}(\mathfrak{g})$. Choose $\lambda \in \mathfrak{h}^*$ such that $W_\lambda \neq 0$ (possible by Lie’s theorem). Since $V$ is irreducible, $V = W_\lambda$. In particular, $\lambda = 0$ on $\mathfrak{h} \cap D\mathfrak{g}$. So extend $\lambda : \mathfrak{h} \to \mathbb{C}$ to a linear function $\tilde{\lambda} : \mathfrak{g} \to \mathbb{C}$ such that $\tilde{\lambda} = 0$ on $D\mathfrak{g}$, and let $L$ be the corresponding 1-dimensional representation of $\mathfrak{g}$. Now $\mathfrak{h}$ acts trivially on $V_0 = V \otimes L^*$.
\end{proof}

Proposition 7.5. If $V$ is an irreducible representation of $\mathfrak{g}$, then $V \cong V_0 \otimes L$ where $V_0$ descends to $\mathfrak{g}_{ss}$ and $L$ is 1-dimensional.

Remark. 1-dimensional representations of a given Lie algebra $\mathfrak{g}$ correspond exactly to linear functions on $\mathfrak{g}/D\mathfrak{g}$.

For any ideal $\mathfrak{h} \subseteq \mathfrak{g}$, if $V$ is a representation of $\mathfrak{g}$, then for $\lambda \in \mathfrak{h}^*$, set

$$W_\lambda = \{v \in V : X(v) = \lambda(X)v \forall X \in \mathfrak{h}\}.$$  

(7.14)

Recall then that $Y(W_\lambda) \subseteq W_\lambda$ for every $Y \in \mathfrak{g}$.

Proof. Set $\mathfrak{h} = \text{Rad}(\mathfrak{g})$. Choose $\lambda \in \mathfrak{h}^*$ such that $W_\lambda \neq 0$ (possible by Lie’s theorem). Since $V$ is irreducible, $V = W_\lambda$. In particular, $\lambda = 0$ on $\mathfrak{h} \cap D\mathfrak{g}$. So extend $\lambda : \mathfrak{h} \to \mathbb{C}$ to a linear function $\tilde{\lambda} : \mathfrak{g} \to \mathbb{C}$ such that $\tilde{\lambda} = 0$ on $D\mathfrak{g}$, and let $L$ be the corresponding 1-dimensional representation of $\mathfrak{g}$. Now $\mathfrak{h}$ acts trivially on $V_0 = V \otimes L^*$.

Recall that if $G$ is a finite group and $\rho : G \to GL(V)$ is a representation over $\mathbb{C}$, then:

1. We have complete reducibility: if $W \subseteq V$ such that $GW = W$, then there exists a subspace $W'$ of $V$ such that $GW' = W'$ and $V = W \oplus W'$.

2. For every $g \in G$, $\rho(g)$ is diagonalizable.

Here’s what’s true for representations of Lie algebras:

1. Reducibility fails, as illustrated by $\{(0 \ 1 \ 0 \ 0)\}$ fixing $\langle e_1 \rangle$. The corresponding Lie group representation (given by exponentiating) is $\{(0 \ 1 \ 0 \ 0)\}$. But reducibility holds for $\mathfrak{g}$ semisimple.

2. Diagonalizability fails in general, shown by the same example as above. Furthermore, $\{(0 \ 1 \ 0 \ 0)\}$ is neither diagonalizable nor nilpotent.

Recall that if $A \in \text{End}(V)$, then we may write $A = A_{ss} + A_n$ where $A_{ss}$ is diagonalizable and $A_n$ is nilpotent.

Theorem 7.6. If $\mathfrak{g}$ is semisimple, then every $X \in \mathfrak{g}$ has a decomposition $X = X_{ss} + X_n$ such that under any representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, $\rho(X)_{ss} = \rho(X_{ss})$ and $\rho(X)_n = \rho(X_n)$. 

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8 Low-Dimensional Simple Lie Algebras

A Lie algebra consists of \( g \) along with \([ , ] : \Lambda^2 g \to g\) satisfying the Jacobi identity. We would like to know the smallest \( n \) such that there exists a simple Lie algebra with dimension \( n \). We necessarily require \([ , ]\) to be injective. In particular, \( \binom{n}{2} \geq n \), so \( n \geq 3 \).

If \( g \) is a simple Lie algebra of dimension 3, then \([ , ] : \Lambda^2 g \to g\) is an isomorphism. In particular, for every nonzero \( H \in g \), \( \ker(\text{ad} H) = \langle H \rangle \).

**Claim.** There exists \( H \in g \) such that \( \text{ad} H \) has an eigenvector with nonzero eigenvalue.

**Proof.** Start with any \( X \in g \) and consider the map \( \text{ad} X : g \to g \). If \( \text{ad} X \) is not nilpotent, then we’re done. If \( \text{ad} X \) is nilpotent, then there exists \( Y \in \ker((\text{ad} X)^2) \setminus \ker(\text{ad} X) \). Now \( (\text{ad} X)(Y) = \alpha X \) for \( \alpha \neq 0 \), so we can take \( H = Y \).  

Now start with such an \( H \), and let \( X \) be an eigenvector for \( \text{ad} H \). We have \([H, X] = \alpha X\) for some nonzero \( \alpha \). Now as \( \text{ad} H \) has trace 0 and \( H \) as an eigenvector with eigenvalue 0, there must exist \( Y \in g \) such that \([H, Y] = -\alpha Y\). Now \{\( H, X, Y \)\} is necessarily a basis for \( g \). Jacobi implies \([H, [X, Y]] = 0\) so \([X, Y] = \beta H\) for some \( \beta \). By scaling \( X \) and \( Y \), we may take \( \beta = 1 \).

We conclude that \( g = \langle X, Y, H \rangle \) with \([H, X] = \alpha X\), \([H, Y] = -\alpha Y\), and \([X, Y] = H\). In fact, \( g = \mathfrak{sl}_2(\mathbb{C}) \) with

\[
H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
Y = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \tag{8.1, 8.2, 8.3}
\]

Here \( \alpha = 2 \).

### 8.1 \( \mathfrak{sl}_2 \)

Observe that \( \mathfrak{sl}_2(\mathbb{C}) \) is the complexification of both \( \mathfrak{sl}_2(\mathbb{R}) \) and \( \mathfrak{su}_2 \), so understanding of \( \mathfrak{sl}_2(\mathbb{C}) \) will give us information about both of these real Lie algebras (hence Lie groups).

Recall that \( \mathfrak{sl}_2(\mathbb{C}) \) consists of those \( 2 \times 2 \) matrices of trace 0, with \( H, X, Y \) as described above as a basis.

Let \( V \) be an irreducible representation of \( \mathfrak{sl}_2(\mathbb{C}) \). The semisimple-nilpotent decomposition shows that the action of \( H \) acts diagonalizably on \( V \). We get a decomposition of \( V \) into eigenspaces

\[ V = \bigoplus V_\alpha, \quad V_\alpha = \{ v \in V : Hv = \alpha v \}. \tag{8.4} \]

We want to determine what \( X \) does to these spaces. Start with \( v \in V_\alpha \). Then we have

\[ H(Xv) = X Hv + [H, X]v = \alpha Xv + 2Xv = (\alpha + 2)Xv \tag{8.5} \]
so $Xv \in V_{\alpha+2}$. Similarly, $Yv \in V_{\alpha-2}$. In particular, for any $\alpha$, set

$$W = \bigoplus_{n \in \mathbb{Z}} V_{\alpha+2n} \subseteq V. \quad (8.6)$$

$W$ is a subrepresentation of $V$, hence all of $V$ by irreducibility. Similarly, the eigenvalues must be an unbroken string of eigenvalues differing by 2; if $V_{\alpha+2N} = 0$ for positive $N$, then we can consider the sum over $n < N$, and similarly if $V_{\alpha-2N} = 0$.

Now start with an eigenspace $V_\alpha$ with $\alpha$ as large as possible. In particular, for $v \in V_\alpha$, $Xv = 0$.

**Claim.** $V$ has a basis $\{v, Yv, Y^2v, \ldots \}$.

**Proof.** Let $W = \langle v, Yv, Y^2v, \ldots \rangle$ be the span. We’ll show that $W$ is a subrepresentation of $V$, and therefore equal to $V$. Since the eigenvalues of $v, Yv, Y^2v, \ldots$ are all distinct, they will be linearly independent as long as they are nonzero, so $W$ has a basis consisting of $Y^i v$.

We know that $Y$ and $H$ carry $W$ into itself, so we look at the action of $X$ on $W$: first, $Xv = 0$.

Second,

$$X(Yv) = Y(Xv) + [X,Y]v = Hv = \alpha v. \quad (8.7)$$

Continuing,

$$X(Y^2v) = Y(XY(v)) + [X,Y](Yv) = \alpha Yv + (\alpha - 2)Yv = (2\alpha - 2)Yv. \quad (8.8)$$

It is easy to show by induction that $X(Y^kv)$ is a multiple of $Y^{k-1}v$. Specifically,

$$X(Y^kv) + (\alpha + (\alpha - 2) + \cdots + (\alpha - 2k + 2))Y^{k-1}v. \quad (8.9)$$

We can say more: as $W$ is finite dimensional, $Y^kv$ must be zero for some $v$. Looking at the smallest such $k$ gives a constraint on $\alpha$.

A simpler way to constrain $\alpha$ is to notice that $H = [X,Y]$, as a commutator, has trace zero, so the sum of the eigenvalues is zero. We observe that $\alpha$ must be a nonnegative integer. Writing $\alpha = n$, we have

$$V = V_n \oplus V_{n-2} \oplus \cdots \oplus V_{-n}. \quad (8.10)$$

We conclude that for every nonnegative integer $n$, there exists a unique irreducible representation $W_n$ of $\mathfrak{sl}_2(\mathbb{C})$ of dimension $n + 1$, with eigenvalues $n, n-2, \ldots, -n$.

Here are some examples:

- For $n = 0$, this is the trivial representation on $\mathbb{C}$.
- For $n = 1$, this is the standard representation on $\mathbb{C}^2$, which is clear by the matrix representation of $H$.
- For $n = 2$, this is the adjoint representation, for $\text{ad } H$ takes $X$ to $2X$, $H$ to 0, and $Y$ to $-2Y$. 

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If $V \cong \mathbb{C}^2$ is the standard representation, then $W_n = \text{Sym}^n V$. For example, if $V = \langle a, b \rangle$, then $\text{Sym}^2 V = \langle a^2, ab, b^2 \rangle$. If $H$ maps $a$ to $a$ and $b$ to $-b$, then $X$ maps $a$ to 0 and $b$ to $a$. $Y$ maps $a$ to $b$ and $b$ to 0. Recall the product rule on the symmetric square,

$$H(ab) = H(a)b + H(b)a = 0.$$  (8.11)

Similarly, $H(a^2) = 2a^2$ and $H(b^2) = -2b^2$.

Here are some consequences:

- If $V$ is any representation of $\mathfrak{sl}_2(\mathbb{C})$, then $V \cong V^*$.
- $V$ is irreducible if and only if $V_0 \oplus V_1$ is 1-dimensional. More generally, the number of irreducible factors is $\dim(V_0 + V_1)$.
- The dimensions of the eigenspaces determines $V$.

Remark. There are exactly two complex Lie groups with Lie algebra $\mathfrak{sl}_2(\mathbb{C})$: $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\pm I \cong SO_3(\mathbb{C})$.

Since $SL_2(C)$ is simply connected, its representations are exactly the $W_n = \text{Sym}^n V$. These representations descend to $PSL_2(\mathbb{C})$ if and only if $\pm I$ acts trivially (that is, for $n$ even).

### 8.2 Plethysm

We aim to describe representations obtained from some (irreducible) representation via tensor product and symmetric and exterior powers.

We will illustrate an example: let $V$ be the standard representation and consider $V \otimes V$. This has eigenvalues $\{-2, 0, 0, 2\}$, implying $V \otimes V \cong \text{Sym}^2 V \oplus \mathbb{C}$. (Of course, in general we have $W \otimes W = \text{Sym}^2 \oplus \Lambda^2 W$, so we already knew this.)

Another example might be to look at $\text{Sym}^2 V \otimes \text{Sym}^2 V$. The eigenvalues (pairwise sums of those of $\text{Sym}^2 V$) are $\{-4, -2, -2, 0, 0, 2, 2, 4\}$, so this representation decomposes as $\text{Sym}^4 V \oplus \text{Sym}^2 V \oplus \mathbb{C}$.

As an immediate consequence, we see that $\text{Sym}^2(\text{Sym}^2 V) \cong \text{Sym}^4 V \oplus \mathbb{C}$ and that $\Lambda^2(\text{Sym}^2 V) \cong \text{Sym}^2 V$, as it’s the only possible further decomposition that matches up dimensions. Let’s do these directly. The eigenvalues of $\text{Sym}^2(\text{Sym}^2 V)$ are the unordered pairwise sums of two of $\{-2, 0, 2\}$, which is $\{-4, -2, 0, 0, 2, 4\}$, so we get $\text{Sym}^4 V \oplus \mathbb{C}$. For $\Lambda^2\text{Sym}^2 V$, the eigenvalues are the unordered sums of distinct elements of $\{-2, 0, 2\}$, which is again $\{-2, 0, 2\}$. So $\Lambda^2(\text{Sym}^2 V) \cong \text{Sym}^2 V$.

The $\mathbb{C}$ summand of $\text{Sym}^2(\text{Sym}^2 V)$ means that the action of $\mathfrak{sl}_2(\mathbb{C})$ must preserve some symmetric bilinear form on $W_2$. Hence $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{so}_3(\mathbb{C})$.

### 8.3 $\mathfrak{sl}_3$

Recall what made our analysis of $\mathfrak{sl}_2$ successful was that $H$ was a diagonal matrix and that $X$ and $Y$ were eigenvalues of $\text{ad} H$. In $\mathfrak{sl}_3(\mathbb{C})$, the role of $H$ will be played by the 2-dimensional vector space
A crucial observation: commuting diagonalizable endomorphisms of a vector space are simultaneously diagonalizable.

For every $\alpha \in \mathfrak{h}^*$, and $V$ any representation of $\mathfrak{sl}_3(\mathbb{C})$, we set

$$V_\alpha = \{ v \in V : Hv = \alpha(H)v \forall H \in \mathfrak{h} \}$$

the eigenspace (for action of $\mathfrak{h}$) with eigenvalue $\alpha$. We can write, dual to $\mathfrak{h}$,

$$\mathfrak{h}^* = \mathbb{C}<L_1,L_2,L_3>/ (L_1 + L_2 + L_3).$$

Then any representation $V$ can be expressed as $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$.

Consider the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}_3(\mathbb{C})$. We get a decomposition

$$\mathfrak{sl}_3(\mathbb{C}) = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha.$$  

Here we have

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} : (\text{ad} H)(X) = \alpha(H)X \forall H \in \mathfrak{h} \}.$$  

The eigenvectors for $\text{ad}(\mathfrak{h})$ are given as follows: letting $E_{ij}$ be the matrix with $(i,j)$ entry one and the rest zero for $i \neq j$, it turns out that the $E_{ij}$ are eigenvectors. The eigenvalue of $E_{ij}$ is $L_i - L_j$.

We will see that for $X \in \mathfrak{g}_\alpha$, $\text{ad} X$ maps $\mathfrak{g}_\beta$ into $\mathfrak{g}_{\alpha + \beta}$. Indeed, if $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$, then

\[
\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} : a_1 + a_2 + a_3 = 0.
\]  

(8.12)
ad \, H((ad \, X)Y) = ad \, (X)((ad \, H)Y) + (ad \, [H, X])Y \tag{8.18}
= \beta(H)(ad \, X)Y + \alpha(H)(ad \, X)Y. \tag{8.19}

Observe that for \( \alpha \) one of the eigenvalues, \([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \in \mathfrak{h} \), and so

\[ s_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \cong \mathfrak{sl}_2. \tag{8.20} \]

Write \( s_{L_i - L_j} = CE_{ij} \oplus CE_{ji} \oplus CH_{ij} \).

Now let \( V \) be an irreducible representation of \( \mathfrak{sl}_3 \). Start by writing

\[ V = \bigoplus V_\alpha \tag{8.21} \]
\[ V_\alpha = \{ v \in V : Hv = \alpha(H)v \forall H \in \mathfrak{h} \}. \tag{8.22} \]

If \( v \in V_\beta \) and \( X \in \mathfrak{g}_\alpha \), then \( Xv \in V_{\alpha + \beta} \) by the usual argument. So \( \mathfrak{g}_\alpha \) carries \( V_\beta \) into \( V_{\alpha + \beta} \). Hence we can say that \( \{ \alpha : V_\alpha \neq 0 \} \) is contained in a translate of the lattice given by \( L_i - L_j \).

In general, the \( \alpha \) such that \( V_\alpha \neq 0 \) are called the weights of \( V \), \( V_\alpha \) is called a weight space, and \( v \in V_\alpha \) is called a weight vector. The weights of the adjoint representation are called roots; denote this set by \( R \). The lattice spanned by the roots is called the root lattice, and denoted by \( \Lambda_R \).

Choose a linear function \( \ell \) on \( \mathfrak{h}^* \) which is irrational with respect to \( \Lambda_R \). Say \( \ell(aL_1 + bL_2 + cL_3) = a_1a + a_2b + a_3c \), with \( a_1 > a_2 > a_3 \) (and \( a_1 + a_2 + a_3 = 0 \)). Let \( \alpha_0 \) be the weight of \( V \) with \( \ell(\alpha) \) maximal (or rather, \( \Re(\ell(\alpha)) \) maximal). Choose \( v \in V_{\alpha_0} \). Such a \( v \) is called a highest weight vector. Observe that \( Hv = \alpha_0(H)v \) for every \( H \in \mathfrak{h} \), and that

\[ \mathfrak{g}_{L_1 - L_2}(v) = \mathfrak{g}_{L_1 - L_3}(v) = \mathfrak{g}_{L_2 - L_3}(v) = 0. \tag{8.23} \]

Write \( R^+ = \{ L_1 - L_2, L_2 - L_3, L_1 - L_3 \} \) and \( R^- = \{ L_2 - L_1, L_3 - L_2, L_3 - L_1 \} \).

Claim. \( V \) is spanned by the images of \( v \) under successive applications of \( E_{21} \), \( E_{32} \), and \( E_{31} \). (Actually, the third is unnecessary since \( E_{31} \) is a commutator of \( E_{21} \) and \( E_{32} \).)

Consequences:

- All of the weights of \( V \) lie in a \( \frac{1}{3} \)-plane.
- \( V_{\alpha_0} \) is 1-dimensional, so \( V \) has a unique highest weight vector up to scalars. The same is true for \( V_{\alpha_0 + n(L_2 - L_1)} \) and \( V_{\alpha_0 + n(L_3 - L_2)} \).

Proof of claim. Let \( w_n \) denote any word of length \( n \) or less in \( E_{21} \) and \( E_{32} \). Let \( W_n \) be the subspace of \( v \) spanned by all \( w_n(v) \) (here \( W_0 = C(v) \) and \( W_{-1} = 0 \)), and let \( W = \bigcup_n W_n \) be the subspace spanned by successive applications of \( E_{21} \) and \( E_{32} \) to \( v \).

Claim. \( E_{12}(W_n) \subseteq W_{n-1} \) and \( E_{23}(W_n) \subseteq W_{n-1} \).
Proof. Induct on $n$. Say $u \in W_n$. We can write $u = E_{21} w_{n-1}(v)$ or $u = E_{32} w_{n-1}(v)$. In the first situation,

$$E_{12} u = E_{12} E_{21} w_{n-1}(v) = E_{21} E_{12} w_{n-1}(v) + [E_{12}, E_{21}] w_{n-1}(v).$$

(8.24)

The bracket is in $\mathfrak{h}$, so the second term is in $W_{n-1}$. So is the first, by induction. If $u = E_{32} w_{n-1}(v)$, then

$$E_{12} u = E_{12} E_{32} w_{n-1}(v) = E_{32} E_{12} w_{n-1}(v) + [E_{12}, E_{32}] w_{n-1}(v).$$

(8.25)

(8.26)

(8.27)

Now $[E_{12}, E_{32}] = 0$, so the result is in $W_{n-1}$. The $E_{23}$ cases are similar.

Now $\bigoplus V_{\alpha_0 + n(L_2 - L_1)}$ is a representation of $\mathfrak{s}_{L_1 - L_2}$. So the eigenvalues under $H_{12}$ are an unbroken string of integers of the same parity which are symmetric about 0. The same goes for considering the other boundary as a representation of $\mathfrak{s}_{L_2 - L_3}$. In particular, the eigenvalues of $V$ actually lie in the hexagonal lattice.

Now go to the last vector on one of the boundaries, $v'$, obtained by applying $E_{21}$ enough times to $v$. Now $E_{21} v' = 0$, $E_{23} v' = 0$, and $E_{13} v' = 0$. So $v'$ is a highest weight vector with respect to a different order. We can proceed by a similar argument.

We conclude that the weights form a hexagonal configuration, with 120 degree interior angles and alternating side lengths. The interior lattice points are also weights, by looking at a line parallel to one of the edges as a representation of some $\mathfrak{s}_a$.

If $V$ has a highest weight vector with weight $\alpha_0$, then the weights of $V$ is given by those $\alpha \in \mathfrak{h}^*$ with $\alpha \equiv \alpha_0 \ (\text{mod } \Lambda_R)$ and lying inside the hexagon with vertices obtained by reflecting $\alpha_0$ in lines $L(H_{ij}) = 0$.

In general, if $V$ is not necessarily irreducible, we get a bijection between irreducible subrepresentations of $V$ and highest weight vectors (modulo scalars).

**Theorem 8.1.** For all pairs $a, b \in \mathbb{Z}_{\geq 0}^2$, there exists a unique irreducible representation $\Gamma_{a,b}$ of $\mathfrak{sl}_3$ with highest weight $aL_1 - bL_3$.

Here are some examples:

- The standard representation on $V \cong \mathbb{C}(e_1, e_2, e_3)$ has $E_{ij} : e_j \mapsto e_i$ and $e_k \mapsto 0$ for $k \neq j$.
  
  For matrices in $\mathfrak{h}$, $\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$ maps $e_i$ to $a_i e_i$, so $e_i$ is an eigenvector for $\mathfrak{h}$ with eigenvalue $L_i$.
  
  This representation has highest weight $L_1$, and the weights form a triangle.

- With $V$ as above, $V^* = \mathbb{C}(e_1^*, e_2^*, e_3^*)$ has $E_{ij} : e_i^* \mapsto -e_j^*$ and $e_k^* \mapsto 0$ for $k \neq i$. Also $\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$ has $e_i \mapsto -a_i e_i^*$. The weights form an opposite triangle, and the highest weight is $-L_3$. In other words, $V = \Gamma_{1,0}$ and $V^* = \Gamma_{0,1}$.
• If we were to look at $\Lambda^2 V$, the weights would be $L_i + L_j$ for $i < j$, but these are just $-L_k$. Indeed, $\Lambda^2 V \cong V^*$, because $\Lambda^3 V \cong \mathbb{C}$.

• The weights of $\text{Sym}^2 V$ have weights $L_i + L_j$ for $i \leq j$. We get a triangle twice as large as the ones in the standard representations, so $\text{Sym}^2 V$ contains a copy of $\Gamma_{2,0}$. But all weights of $\text{Sym}^2 V$ have multiplicity one and appear as weights of $\Gamma_{2,0}$. So $\text{Sym}^2 V \cong \Gamma_{2,0}$.

• In general, $\text{Sym}^n V \cong \Gamma_{n,0}$ and $\text{Sym}^n V^* \cong \Gamma_{0,n}$.

  **Remark.** If $V, W$ are any representations of $\mathfrak{sl}_3$ with highest weight vectors $v \in V_\alpha$ and $w \in W_\beta$, then $v \otimes w \in V \otimes W$ is a highest weight vector with weight $\alpha + \beta$. So $\Gamma_{a,b} \otimes \Gamma_{c,d}$ necessarily contains $\Gamma_{a+c,b+d}$.

  We have just shown existence. Specifically, $\Gamma_{a,b} \subseteq \text{Sym}^a V \otimes \text{Sym}^b V^*$.

• $V \otimes V^*$ has weights $L_i - L_j$ for every $i,j$. The weight 0 appears three times. We see that $V \otimes V^* \cong \Gamma_{1,1} \otimes \mathbb{C}$. (We know that $\Gamma_{1,1}$ is the adjoint representation.) We can consider $\Gamma_{1,1}$ as the kernel of the trace $V \otimes V^* \rightarrow \mathbb{C}$.

• Now look at $\text{Sym}^2 V \otimes V^*$, which contains $\Gamma_{2,1}$. All of the weights on the boundary of the $(2,1)$ hexagon appear with multiplicity one. (The $2L_i - L_j$ and the $-2L_i$ occur with multiplicity one.) The inner triangle appears with multiplicity three. So some multiple of the standard remains.

**Lemma 8.2.** If $V$ is any representation and $v \in V$ is a highest weight vector (so $v \in V_\alpha$ and $g_\alpha v = 0$ for every $\alpha \in \mathbb{R}^+$), introduce $b = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathbb{R}^+} \mathfrak{g}_\alpha$. Then $bv$ is an irreducible subrepresentation.

**Proof.** If $W = bv$ and we had $W = W_1 \oplus W_2$, then

$$\mathbb{C}v = W_\alpha = (W_1)_{\alpha_0} \oplus (W_2)_{\alpha_0} \quad (8.28)$$

implying $W_1 = W$ or $W_2 = W$. 

We conclude that $\text{Sym}^a V \otimes \text{Sym}^b V^*$ contains an irreducible subrepresentation with highest weight $aL_1 - bL_3$. The highest weight vector is $e_1^a \otimes (e_3^*)^b$.

For uniqueness, if $V$ and $W$ are both irreducible representations with highest weight vectors $v, w$, both of weight $\alpha$, then $(v, w) \in V \oplus W$ is a highest weight vector of weight $\alpha$. Letting $U = b(v, w)$, $U$ is also irreducible. The projection maps from $U$ onto $V$ and $W$ are nonzero, so must be isomorphisms. Hence $V \cong W$.

Now $\Gamma_{a,b}$ is well-defined.

We return to looking for $\Gamma_{2,1}$ in $\text{Sym}^2 V \otimes V^*$. Consider the vector $e_1^2 \otimes e_3^*$ and apply $b$. There are only two ways of reaching the first vertex in the inner triangle, so the multiplicity of the weight space $L_1$ in $\Gamma_{2,1}$ is at most two.

Now look at $E_{21}(E_{32}(e_1^2 \otimes e_3^*))$ and $E_{32}(E_{21}(e_1^2 \otimes e_3^*))$. We have

$$E_{32}(E_{21}(e_1^2 \otimes e_3^*)) = E_{21}(-e_1^2 \otimes e_2^*) \quad (8.29)$$

$$= -e_1 e_2 \otimes e_3^* + e_1^2 \otimes e_1^* \quad (8.30)$$

$$E_{32}(E_{21}(e_1^2 \otimes e_3^*)) = E_{32}(2e_1 e_2 \otimes e_3^*) \quad (8.31)$$

$$= 2e_1 e_3 \otimes e_3^* - 2e_1 e_2 \otimes e_2^* \quad (8.32)$$
These are linearly independent, so the multiplicity is two. Hence $\text{Sym}^2 V \otimes V^* \cong \Gamma_{2,1} \oplus V$.

There is a projection map $\iota : \text{Sym}^2 V \otimes V^* \to V$, called the contraction map, by $uv \otimes w^* \mapsto w^*(u)v + w^*(v)u$. Then $\Gamma_{2,1}$ can be realized as the kernel.

In general, we have contraction maps $\iota_{a,b} : \text{Sym}^a V \otimes \text{Sym}^b V^* \to \text{Sym}^{a-1} V \otimes \text{Sym}^{b-1} V^*$. 

Claim. $\Gamma_{a,b} = \ker \iota_{a,b}$. Equivalently, we have a decomposition

$$\text{Sym}^a V \otimes \text{Sym}^b V^* = \bigoplus_{k=0}^{\min(a,b)} \Gamma_{a-k,b-k}.$$  

(8.33)

9 Simple Lie Algebras

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Our steps will be:

1. Find a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$; that is, a maximal abelian subalgebra acting diagonalizably.

2. We get a Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R \subseteq \mathfrak{h}^*} \mathfrak{g}_\alpha$$

(9.1)

where $\mathfrak{g}_\alpha$ is the eigenspace of $\mathfrak{g}$ with eigenvalue $\alpha$. $R$ is called the set of roots, and the $\mathfrak{g}_\alpha$ are called root spaces. We may write $\mathfrak{h} = \mathfrak{g}_0$.

(Had we chosen $\mathfrak{h}$ not maximal, an additional zero eigenspace would show up.)

Observe that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. Also observe that for $V$ any representation of $\mathfrak{g}$, we can write $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$ as eigenspaces for $\mathfrak{h}$.

Fact. • Each $\mathfrak{g}_\alpha$ is 1-dimensional.

• $R = -R$. ($\alpha \in R \iff -\alpha \in R$.)

• $R$ generates a sublattice $\Lambda_R$ of $\mathfrak{h}^*$ of rank equal to $\dim \mathfrak{h}$. In particular, $\mathfrak{h}_R^* = \Lambda_R \otimes \mathbb{R}$ is a real Lie algebra with complexification $\mathfrak{h}^*$.

3. Introduce the distinguished subalgebras: for every $\alpha$, we have

$$\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$$

(9.2)

Fact. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$, and the adjoint action of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ on $\mathfrak{g}_\alpha$ is nontrivial. Hence $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$.

We also have distinguished elements $H_\alpha \in \mathfrak{h}$. Specifically, $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ acts on $\mathfrak{g}_\alpha$ with eigenvalue 2. For every representation $V$ of $\mathfrak{g}$, the eigenvalues of $H_\alpha$ are integers, and symmetric about the origin. These properties encompass the next two steps:

4. Introduce the weight lattice $\Lambda_W \subseteq \mathfrak{h}^*$ defined as

$$\Lambda_W = \{ \alpha \in \mathfrak{h}^* : \alpha(H_\beta) \in \mathbb{Z} \forall \beta \in R \} \supseteq \Lambda_R.$$  

(9.3)
Fact. Let $G_0$ be the adjoint form of $g$ (or the simply connected form modded out by the center). Then $\Lambda_W/\Lambda_R = \pi_1(G_0)$. So forms $G$ of $g$ correspond to subgroups $S_G$ of $\pi_1(G_0)$, and $V$ lifts to $G$ if and only if the weights of $V$ lie in $\Lambda_R + S_G$.

5. Introduce the hyperplanes

$$\Omega_\alpha = \{ \beta \in h^* : \beta(H_\alpha) = 0 \}$$

(9.4)

Let $W_\alpha : h^* \to h^*$ be the involution with eigenvalue 1 on $\Omega_\alpha$ and $-1$ on $\mathbb{C}\alpha$. Now write $\mathfrak{W}$ to be the group generated by the $W_\alpha$ for $\alpha \in R$, called the Weyl group. For $V$ any representation of $g$, its weights of $V$ are invariant under $W$.

6. We have the Killing form, a positive definite inner product on $h^*_R$ invariant under $\mathfrak{W}$. This is given by a symmetric bilinear map $B : g \times g \to \mathbb{C}$. Also we denote by $B$ its restriction to $h \times h$ and the induced map on $h^* \times h^*$.

The definition is

$$B(X,Y) = \text{tr} \left( \text{ad}(X) \circ \text{ad}(Y) : g \to g \right).$$

(9.5)

Observe that $B(g_\alpha, g_\beta) = 0$ unless $\beta = -\alpha$. So we have an orthogonal decomposition

$$g = h \oplus \bigoplus_\alpha (g_\alpha \oplus g_{-\alpha})$$

(9.6)

On $h$, if $X, Y \in h$, then

$$B(X,Y) = \sum_{\alpha \in R} \alpha(X)\alpha(Y).$$

(9.7)

$B$ is positive definite on the subspace $h_R$ spanned by the $H_\alpha$; $B$ is also positive definite on $h^*_R$.

Claim. The isomorphism $h \to h^*$ given by the Killing form carries $\Omega_\alpha$ to $\ker \alpha$ and $\alpha$ to $\frac{2H_\alpha}{B(H_\alpha, H_\alpha)}$.

Proof. For $A, B, C \in \text{End}(V)$, we have

$$\text{tr}([A, B]C) = \text{tr}(A[B, C]).$$

(9.8)

Indeed, $[A, B]C - A[B, C] = [AC, B]$ has trace zero. Now since $H_\alpha = [X_\alpha, Y_\alpha]$, we can write, for $H \in h$,

$$B(H_\alpha, H) = B([X_\alpha, Y_\alpha], H) = B(X_\alpha, [Y_\alpha, H]) = \alpha(H)B(X_\alpha, Y_\alpha).$$

(9.9)

In particular, $\ker \alpha$ is carried into $\Omega_\alpha$ (so these hyperplanes must be identified under our isomorphism). Now suppose $T_\alpha$ is the dual element of $\alpha$. Then $B(T_\alpha, H) = \alpha(H)$. Hence

$$T_\alpha = \frac{H_\alpha}{B(X_\alpha, Y_\alpha)} = \frac{2H_\alpha}{B(H_\alpha, H_\alpha)}.$$ 

(9.10)
7. Choose an ordering of $R$, namely a decomposition $R = R^+ \sqcup R^-$ where $R^+$ and $R^-$ lie in halfspaces.

An ordering of the roots provides us with a Borel subalgebra

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^-} \mathfrak{g}_\alpha.$$  \hspace{1cm} (9.11)

This also provides us with a highest weight vector: for $V$ a representation, $v \in V$ is a highest weight vector if $v \in V_{\alpha_0}$ and $g_\alpha v = 0 \forall \alpha \in R^+$. Every representation has a highest weight vector, and every irreducible representation has a unique one modulo scaling. In this case, the weight of $v$ is called the highest weight of $V$.

If $V$ is any representation and $v \in V$ is any highest weight vector, then $\mathfrak{b}v$ is an irreducible subrepresentation of $V$.

If $V$ is any irreducible representation with highest weight $\alpha_0$, then

$$\{\text{weights of } V\} = (\alpha_0 + \Lambda_R) \cap (\text{convex hull of } \mathfrak{W}(\alpha_0)).$$  \hspace{1cm} (9.12)

(This does not tell us multiplicities.)

8. Introduce the Weyl chamber

$$\mathcal{W} = \{\alpha \in \mathfrak{h}^* : B(\alpha, \beta) = \alpha(H_\beta) \geq 0 \forall \beta \in R^+\}. \hspace{1cm} (9.13)$$

$\mathcal{W}$ is the closure of a connected component of $\mathfrak{h}^*_R \setminus \bigcup \Omega_\alpha$.

9. \textbf{Theorem 9.1 (Main theorem).} $V$ an irreducible representation $\leadsto$ its highest weight $\alpha \in \mathfrak{h}^*$ induces a bijection

$$\{\text{irreducible representations of } \mathfrak{g}\} \leftrightarrow \Lambda_\mathcal{W} \cap \mathcal{W}. \hspace{1cm} (9.14)$$

10. A simple (or primitive) root is an $\alpha \in R^+$ which cannot be expressed as the sum of two other positive roots. Observe that every positive root is a sum of simple roots.

\textbf{Fact.} The number of simple roots equals $\dim \mathfrak{h} = m$.

This says that $\mathcal{W}$ is a simplicial cone (an intersection of $m$ half-spaces).

11. In fact, if $\omega_1, \ldots, \omega_m$ are the smallest weights of $\Lambda_\mathcal{W}$ along edges of $\mathcal{W}$, then the $\omega_i$ generate $\Lambda_\mathcal{W} \cap \mathcal{W}$ as a semigroup. The $\omega_i$ are called fundamental weights.

\textbf{The main theorem} implies we have a bijection between the set of irreducible representations of $\mathfrak{g}$ and $\mathbb{Z}_{\geq 0}^m$. Write $\Gamma_{a_1 \ldots a_m}$ for the irreducible representation with highest weight $a_1\omega_1 + \cdots + a_m\omega_m$.

To prove the existence half of the main theorem, it’s sufficient to show existence of representations with highest weights the $\omega_i$. 

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10 Analysis of $\mathfrak{sl}_n$

$V \cong \mathbb{C}^n$ is the standard representation of $\mathfrak{sl}_n$.

Recall that $\mathfrak{sl}_n$ consists of the traceless $n \times n$ matrices. Let $\mathfrak{h}$ be the subspace of traceless diagonal matrices, associated with those $n$-tuples $(a_1, \ldots, a_n)$ with $\sum a_i = 0$. The dual algebra is

$$\mathfrak{h}^* = \mathbb{C}\langle L_1, \ldots, L_n \rangle/\langle L_1 + \cdots + L_n \rangle.$$  \hspace{1cm} (10.1)

Introduce, as before, the maps (for $i \neq j$)

$$E_{ij} : e_k \mapsto \begin{cases} e_i & k = j \\ 0 & k \neq j \end{cases}$$  \hspace{1cm} (10.2)

and the diagonal (not traceless!) matrices

$$H_i : e_k \mapsto \begin{cases} e_i & k = i \\ 0 & k \neq i \end{cases}$$  \hspace{1cm} (10.3)

$E_{ij}$ is an eigenvector for $\mathfrak{h}$ with eigenvalue $L_i - L_j$. So $g_{L_i - L_j} = CE_{ij}$. Thus we have found the entire Cartan decomposition

$$\mathfrak{sl}_n = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C}E_{ij}.$$  \hspace{1cm} (10.4)

In particular, no zero eigenspace besides $\mathfrak{h}$ appears, so $\mathfrak{h}$ is in fact maximal.

For $n = 4$, the root diagram appears as follows:

The roots lie on the edge midpoints of a larger cube.

The distinguished subalgebras are

$$g_{L_i - L_j} = \mathbb{C}\langle E_{ij}, E_{ji}, H_i - H_j \rangle \cong \mathfrak{sl}_2$$  \hspace{1cm} (10.5)

$$g_{L_i - L_j} = \mathbb{C}\langle E_{ij}, E_{ji}, H_i - H_j \rangle \cong \mathfrak{sl}_2$$  \hspace{1cm} (10.6)
with distinguished elements $H_i - H_j$. The weight lattice is $\mathbb{Z}\langle L_1, \ldots, L_n \rangle$ and the root lattice is the sublattice spanned by $L_i - L_j$.

The hyperplane $\Omega_{L_i - L_j}$ orthogonal to $L_i - L_j$ is given by

$$\{ \sum a_i L_i : a_i = a_j \}.$$  \hfill (10.7)

The reflection on $\Omega_{L_i - L_j}$ exchanges $a_i$ and $a_j$, so $\mathfrak{W} \cong S_n$.

To order the roots, set $R^+ = \{ L_i - L_j : i < j \}$. The simple roots are $L_i - L_{i+1}$ for $i = 1, \ldots, n-1$. The Weyl chamber is

$$\mathcal{W} = \{ \sum a_i L_i : a_1 \geq a_2 \geq \cdots \geq a_n \}.$$  \hfill (10.8)

The edges of $\mathcal{W}$ are spanned by the vectors

$$L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{n-1} = -L_n.$$  \hfill (10.10)

These are the fundamental weights.

**Remark.** $\Lambda \cap \mathcal{W} = \mathbb{Z}_{\geq 0}\langle L_1, L_1 + L_2, \ldots, -L_n \rangle$.

Consider the unique representation of $\mathfrak{sl}_n$ with highest weight $a_1 L_1 + a_2 L_2 + \cdots + a_n L_n$, with $a_i \geq a_{i+1}$, then we can rewrite as

$$b_1 L_1 + b_2 (L_1 + L_2) + \cdots + b_{n-1} (L_1 + \cdots + L_{n-1}).$$  \hfill (10.11)

The corresponding representation is called $\Gamma_{b_1, \ldots, b_{n-1}}$.

Here are some examples for $n = 4$:

- The standard representation $V \cong \mathbb{C}^4$ has weights $L_i$, highest weight $L_1$, and so $V = \Gamma_{1,0,0}$.
- The dual $V^*$ has weights $-L_i$, so $V^* = \Gamma_{0,0,1}$. We can also write $V^*$ as $\Lambda^3 V$.
- Now look at $\Lambda^2 V$. The weights are $L_i + L_j$ for $i < j$. (These correspond to the face midpoints of the cube.) This is irreducible, and has highest weight $L_1 + L_2$, so $\Lambda^2 V = \Gamma_{0,1,0}$. We have just produced existence for $\mathfrak{sl}_4$.

More generally, for $\mathfrak{sl}_n$, $\Lambda^k V$ has weights $L_{i_1} + \cdots + L_{i_k}$ for $i_1 < \cdots < i_k$, so the highest weight is $L_1 + \cdots + L_k$. We just produced representations with highest weight a fundamental weight, and thus have proven existence of $\Gamma_{a_1, \ldots, a_{n-1}}$. Specifically
as the subrepresentation generated by the element
\[ e_1^{a_1} \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{n-1})^{a_{n-1}}. \]

We now return to the case of \( \mathfrak{sl}_4 \). Consider:

- \( V \otimes \Lambda^2 V \) has weights \( 2L_i + L_j \) (appearing once) and \( L_i + L_j + L_k \) (appearing three times).

We have a surjective map \( V \otimes \Lambda^2 V \xrightarrow{\varphi} \Lambda^3 V \).

Claim. \( \ker \varphi = \Gamma_{1,1,0} \). Hence \( V \otimes \Lambda^2 V \cong \Gamma_{1,1,0} \oplus V^* \).

\( \Gamma_{1,1,0} \) is the subrepresentation obtained by applying \( E_{21}, E_{32}, \) and \( E_{43} \) to \( e_1 \otimes (e_1 \wedge e_2) \). \( E_{43} \) will annihilate everything, so we need to see that \( E_{21} E_{32} (e_1 \otimes (e_1 \wedge e_2)) \) and \( E_{32} E_{21} (e_1 \otimes (e_1 \wedge e_2)) \) are linearly independent:

\[
E_{21} E_{32} (e_1 \otimes (e_1 \wedge e_2)) = E_{21} (e_1 \otimes (e_1 \wedge e_3)) = e_2 \otimes (e_1 \wedge e_3) + e_1 \otimes (e_2 \wedge e_3) \] (10.14)

\[
E_{32} E_{21} (e_1 \otimes (e_1 \wedge e_2)) = E_{32} (e_2 \otimes (e_1 \wedge e_2)) = e_3 \otimes (e_1 \wedge e_2) + e_2 \otimes (e_1 \wedge e_3). \] (10.15)

- \( V \otimes V^* \): The weights are \( L_i - L_j \) for \( i \neq j \) (each appearing once) and \( 0 \) (with multiplicity four). We have a map \( V \otimes V^* \xrightarrow{\varphi} \mathbb{C} \).

Claim. \( \ker \varphi = \Gamma_{1,0,1} \), the adjoint representation.

The map \( V \otimes V^* \to \text{End}(V) \to \mathbb{C} \) is the trace map, and the kernel is \( \mathfrak{sl}_4 \) itself.

Schur Functors: (§6.1 (builds on 4.1, 4.2) and §15.3)

Recall that \( V \otimes V = \text{Sym}^2 V \oplus \Lambda^2 V \). Is there an analogous decomposition of \( V \otimes V \otimes V \)? Think of \( V \otimes V \otimes V \) as a representation both of \( G \) and of \( S_3 \) (and these actions commute). Breaking this up into canonical components under the \( S_3 \) representation, we get

\[ V \otimes V \otimes V = \text{Sym}^3 V \oplus \Lambda^3 V \oplus S_{2,1}(V)^{\otimes 2}. \] (10.18)

Applying the Schur functors to \( V \) gives the \( \Gamma_\alpha \).

11 Geometric Plethysm

The basic idea is to look at actions of groups and algebras on projective spaces.

For \( V \) any vector space, we will set \( \mathbb{P}V \) to be the set of 1-dimensional subspaces of \( V \), equal to \( (V \setminus \{0\})/\mathbb{C}^\times \). For nonzero \( v \in V \), write \([v]\) for the point of \( \mathbb{P}V \) corresponding to \( Cv \).
We’ll look at $SL_2$ for now. Let $V$ be the standard representation, and $\Gamma_n = \text{Sym}^n V$. $SL_2$ acts on $\mathbb{P} V = \mathbb{P}^1$ and on $\mathbb{P}(\text{Sym}^n V) = \mathbb{P}^n$. This action is transitive for $n = 1$, but not in general for $n \geq 2$.

For the case $n = 2$, there are two orbits (squares of linear forms, and everything else). The squares of linear forms sweep out a conic in $\mathbb{P}^2$.

In general, the action of $SL_2$ on $\mathbb{P}(\text{Sym}^n V) \cong \mathbb{P}^n$ preserves the locus of $n$th powers, given by the map $\mathbb{P}^1 = \mathbb{P} V \to \mathbb{P}(\text{Sym}^n V) = \mathbb{P}^n$ with $[v] \mapsto [v^n]$. This image $C$ is the rational normal curve of degree $n$. In coordinates, $C$ is the image of $[x, y] \mapsto [x^n, x^{n-1}y, \ldots, y^n]$. $I(C)$ is generated by quadratic polynomials $Z_i Z_j - Z_k Z_\ell$ for $i + j = k + \ell$.

Now look at $\text{Sym}^2(\text{Sym}^2 V) = \text{Sym}^2 \Gamma_2$. We have an exact sequence

$$0 \to C \to \text{Sym}^2 \Gamma_2 \to \text{Sym}^4 V \to 0 \quad (11.1)$$

with the $\mathbb{C}$ subspace being the span of the unique (up to scaling) quadratic polynomial on $\mathbb{P}^2$ vanishing on $C$. Now $\text{Sym}^2 \Gamma_2$ splits naturally into $\mathbb{C} \oplus \text{Sym}^4 V$. How do we identify $\text{Sym}^4 V$ as a natural subspace of $\text{Sym}^2 \Gamma_2$? This turns out to be the subspace spanned by squares of tangent lines to the conic $C$. This is a degree four map, so cannot be surjective. Because this subspace is invariant under $SL_2$, it must be $\text{Sym}^4 V$.

Now we have a linear map $\text{Sym}^4 V \to \text{Sym}^2 \Gamma_2$; that is, given $f$ of degree 4 on $\mathbb{P}^1$, there is a “natural” way to write it as a quadratic polynomial in quadratic polynomials on $\mathbb{P}^1$. In other words, given four points $p_1, \ldots, p_4 \in C$, there exists a natural choice of conic $C' \subseteq \mathbb{P}^2$ such that $C \cap C' = \{p_1, \ldots, p_4\}$.

Now we’ll look at $n = 3$ and $d = 2$. We have $\mathbb{P}^1 \to C \subseteq \mathbb{P}^3$ as a twisted cubic. We have an exact sequence (using an evaluation map)

$$0 \to \text{Sym}^2 V \to \text{Sym}^2 \Gamma_3 \to \text{Sym}^6 V \to 0 \quad (11.2)$$

Given six points on a twisted cubic, there is a natural choice of quadric cutting out those six points.’

For $\text{Sym}^2 V$ embedded as above, the distinguished conic consists of the singular quadrics containing $C$.

Next look at $n = 3$ and $d = 4$:

$$\text{Sym}^4 \Gamma_3 = \Gamma_{12} \oplus \Gamma_8 \oplus \Gamma_6 \oplus \Gamma_4 \oplus \mathbb{C}. \quad (11.3)$$

$\Gamma_{12}$ appears as the image of the evaluation map, so the kernel $I$, the ideal of $C$ of degree 4, consists of the other summands. The trivial subspace corresponds to the tangential surface of the twisted cubic.

Look at $n = 4$ now, so $C$ is a rational normal quartic curve. For $d = 2$,

$$0 \to \mathbb{C} \oplus \Gamma_4 \to \text{Sym}^2 \Gamma_4 \to \text{Sym}^8 V \to 0 \quad (11.4)$$

The $\mathbb{C}$ part means that given $C$, there is a distinguished quadric containing it. One description is given by considering the $j$-function as a rational function on $\mathbb{P}^4$, and the distinguished quadric is the zero locus of $j$. In general, the fibers of $j$ is a sextic.
Now suppose $2|n$ and $d = 2$. Then

$$\text{Sym}^2 \Gamma_n = \Gamma_{2n} \oplus \Gamma_{2n-4} \oplus \cdots \oplus \mathbb{C}. \quad (11.5)$$

In particular, we get $\mathbb{C}$ as a subrepresentation. Where does it appear?

12 The Symplectic Groups $Sp_{2n}$

Let $V$ be a vector space and $Q : V \times V \to \mathbb{C}$ be a nondegenerate skew-symmetric bilinear form. Then $\dim V$ is necessarily even; write $\dim V = 2n$.

We can think of $Q$ as being an element of $\Lambda^2 V^*$, and nondegeneracy means that $Q \wedge \cdots \wedge Q \in \Lambda^{2n} V^*$.

We define

$$Sp(V, Q) = \{ A : V \to V : Q(Av, Aw) = Q(v, w) \forall v, w \in V \}. \quad (12.1)$$

(Observe that $Sp(V, Q) \subseteq SL(V)$.) Its Lie algebra is

$$\mathfrak{sp}(V, Q) = \{ X : V \to V : Q(Xv, w) + Q(v, Xw) = 0 \forall v, w \in V \}. \quad (12.2)$$

We can choose a basis $V \cong \mathbb{C}^{2n}$ under which $Q(v, w) = v^T M w$, where

$$M = \begin{pmatrix} O & I \\ -I & 0 \end{pmatrix} \quad (12.3)$$

If $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then $X \in \mathfrak{sp}(V, Q)$ if and only if $X^T M + M X = 0$; this is equivalent to having $B, C$ symmetric and $D = -A^T$. This defines $\mathfrak{sp}_{2n}$.

We will take $\mathfrak{h}$ to be the subalgebra of $\mathfrak{sp}_{2n}$ consisting of the diagonal matrices (here the second block is the negative of the first). Writing $H_i = E_{ii} - E_{n+i,n+i}$, we have $\mathfrak{h} = \mathbb{C}\langle H_1, \ldots, H_n \rangle$. Let $L_i^*$ be the dual basis to the $H_i$ in $\mathfrak{h}^*$.

To work out the adjoint representation, recall that for any $i, j$, we have

$$\left[ \sum a_k E_{kk}, E_{ij} \right] = (a_i - a_j) E_{ij}. \quad (12.4)$$

Now for $i \neq j$, consider the matrix $E_{ij} - E_{n+j,n+i}$. This is a root with eigenvalue $L_i - L_j$. Next, $E_{i,n+j} + E_{j,n+i}$ is a root with eigenvalue $L_i + L_j$, and similarly, $E_{n+i,j} + E_{n+j,i}$ is a root with eigenvalue $-L_i - L_j$. Finally, $E_{i,n+i}$ and $E_{n+i,i}$ have eigenvalues $2L_i$ and $-2L_i$, respectively. These are all of the roots (and in particular, $\mathfrak{h}$ is maximal).

The roots $\pm L_i \pm L_j$ are called the short roots, while $\pm 2L_i$ are called the long roots.

For the case $n = 2$, the configuration of roots look like:
For $n = 3$, it is given by:

$$s_{L_i - L_j} = \mathbb{C}\langle E_{ij} - E_{n+j,n+i}, E_{ji} - E_{n+i,n+j}, H_i - H_j \rangle$$  \hspace{1cm} (12.7)

$$H_{L_i - L_j} = H_i - H_j$$  \hspace{1cm} (12.8)

$$H_{L_i + L_j} = H_i + H_j$$  \hspace{1cm} (12.9)

$$H_{-L_i - L_j} = -H_i - H_j$$  \hspace{1cm} (12.10)

$$H_{2L_i} = H_i$$  \hspace{1cm} (12.11)

$$H_{-2L_i} = -H_i$$  \hspace{1cm} (12.12)

The weight lattice is simply

$$\Lambda_W = \mathbb{Z}\langle L_1, \ldots, L_n \rangle.$$  \hspace{1cm} (12.13)

The distinguished subalgebras and elements are given by

$$\mathfrak{h}_R^* \to \mathbb{R}$$ given by $\sum a_iL_i \mapsto \sum c_i a_i$ with $c_1 > c_2 > \cdots > c_n > 0$. Here the positive roots are $2L_i, L_i + L_j, \text{ and } L_i - L_j$ for $i < j$. The primitive positive roots are then $L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, 2L_n$. The Weyl chamber is

$$W = \left\{ \sum a_iL_i : a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \right\}.$$  \hspace{1cm} (12.15)
with edges \( \{a_1 = a_2 = \cdots = a_k \geq a_{k+1} = \cdots = a_n = 0\} \). The fundamental weights are given by

\[
L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_n.
\]

(12.16)

Observe that \( \mathcal{W} \cap \Lambda W \) is the semigroup generated by these. Let \( \Gamma_{a_1, \ldots, a_n} \) be the irreducible representation with highest weight \( a_1L_1 + \cdots + a_n(L_1 + \cdots + L_n) \).

For \( \mathfrak{sp}_{2n} \), every representation is isomorphic to its dual, because \( \mathcal{W} \) contains an element which changes the sign of a coordinate, and the weights are invariant under \( \mathcal{W} \).

Examples of representations of \( \mathfrak{sp}_4 \):

- \( V = \mathbb{C}^4 \) the standard representation; then \( e_1, \ldots, e_4 \) are eigenvectors for \( \mathfrak{h} \) with eigenvalues \( L_1, L_2, -L_1, -L_2 \). This representation is \( \Gamma_{1,0} \).

- \( \Lambda^2 V \) has weights \( \pm L_i \pm L_i \), and 0 with multiplicity 2. There is only one way to get from \( L_1 + L_2 \) to 0 by applying primitive negative roots, so \( \Gamma_{0,1} \) has 0 as a weight with multiplicity only one. So \( \Lambda^2 V \cong \Gamma_{0,1} \oplus \mathbb{C} \).

Observe that we also have a natural contraction map \( \Lambda^2 \xrightarrow{Q} \mathbb{C} \) (actually \( \Lambda^2 V^* \), but we don’t need to worry about dualizing here). The kernel then has all of its weights of multiplicity one, so is necessarily irreducible. We again see that \( \Lambda^2 V \cong \Gamma_{0,1} \oplus \mathbb{C} \). We conclude existence of \( \Gamma_{a,b} \) for \( \mathfrak{sp}_4 \). Let \( W = \Gamma_{0,1} \).

- \( \text{Sym}^2 V \) has weights \( \pm L_i \pm L_j, \pm 2L_i, \) and zero (twice), which is exactly the weights of the adjoint representation. So the adjoint representation is \( \Gamma_{2,0} \) (check that it’s irreducible).

- \( V \otimes W \) has weights \( \pm 2L_i \pm L_j, \pm 2L_i \) (each with multiplicity three). Now \( \Gamma_{1,1} \) has \( \pm L_i \) as weights with multiplicity at most two, so \( V \otimes W \) contains at least one copy of \( V \). We can see this from

\[
V \otimes W \subseteq V \otimes \Lambda^2 V \rightarrow \Lambda^3 V \cong V^* \cong V
\]

(12.17)

and the composite map is not zero. Now

\[
\text{Hom}(V \otimes W, V) = \text{Hom}(W, V^* \otimes V)
\]

(12.18)

has multiplicity one, so the remaining factor is irreducible. (This can also be checked by computation.)

- \( \text{Sym}^2 W \) has weights \( \pm 2L_i \pm 2L_j, \pm L_i \pm L_j, 2L_i, \) and zero (with multiplicity three). This contains \( \Gamma_{0,2} \). It also contains a copy of \( \mathbb{C} \), since

\[
\Lambda^2 V \times \Lambda^2 V \rightarrow \Lambda^4 V = \mathbb{C}
\]

(12.19)

is a symmetric pairing, so restricts to \( \text{Sym}^2 W \). It turns out that \( \mathbb{C} \) appears only once.

- \( \Lambda^2 W \) has weights \( \pm 2L_i, \pm L_i \pm L_j, \) and zero (twice). This is again the adjoint representation \( \Gamma_{2,0} \).

Observe that \( \text{Sym}^2 V \cong \Lambda^2 W \). This reflects the fact that \( \mathfrak{sp}_4 \cong \mathfrak{so}_5 \).
In general, for $\mathfrak{sp}_{2n}$, let $V$ be the standard representation, equal to $\Gamma_{1,0,\ldots,0}$. Then $\Lambda^k V$ contains a copy of $\Gamma_{0,1,\ldots,0}$ (having highest weight $L_1 + \cdots + L_k$), so we establish existence.

Explicitly, we have a contraction map $\Lambda^k V \xrightarrow{Q} \Lambda^{k-2} V$, by

$$v_1 \wedge \cdots \wedge v_k \mapsto \sum Q(v_i,v_j) v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots v_k.$$  \hfill (12.20)

The kernel turns out to be irreducible.

## 13 Orthogonal Lie Algebras

Start with the case $m = 2n$. Given a quadratic form $Q$, choose a basis such that $Q(e_i,e_{n+i}) = Q(e_{n+i},e_i) = 1$, and all others are 0. So $Q$ as given is

$$Q(x,y) = x^T \begin{pmatrix} O & I \\ I & O \end{pmatrix} y.$$  \hfill (13.1)

The orthogonal Lie algebra can then be expressed as

$$\mathfrak{so}_{2n} = \left\{ X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} : B,C \text{ skew-symmetric} \right\}.$$  \hfill (13.2)

Take $\mathfrak{h}$ to be the subalgebra of diagonal matrices, with

$$H_i = E_{ii} - E_{n+i,n+i}$$  \hfill (13.3)

$$\mathfrak{h}^* = \mathbb{C} \langle L_1, \ldots, L_n \rangle.$$  \hfill (13.4)

The roots are given by:

- $E_{ij} - E_{n+j,n+i}$, with eigenvalue $L_i - L_j$.
- $E_{i,n+j} - E_{j,n+i}$, with eigenvalue $L_i + L_j$.
- $E_{n+i,j} - E_{n+j,i}$ with eigenvalue $-L_i - L_j$.

These are all of the roots of $\mathfrak{so}_{2n}$. The distinguished elements are

$$H_{\pm L_i \pm L_j} = \pm H_i \pm H_j.$$  \hfill (13.5)

In the case $n = 2$, the roots lie in a union of two orthogonal lines. We see that the Lie algebra splits, and $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$. For $n = 3$, the roots are given by the midpoints of the edges of a cube. This is identical to the root diagram of $\mathfrak{sl}_4$. In fact, $\mathfrak{so}_6 \cong \mathfrak{sl}_4$. (We’ll later prove that a semisimple Lie algebra is determined by its root diagram.)
Now we’ll look at \( \mathfrak{so}_{2n+1} \). Choose a basis such that

\[
Q(e_i, e_{n+i}) = Q(e_{n+i}, Q_i) = 1, \quad Q(e_{2n+1}, e_{2n+1}) = 1,
\]

and all others are 0. Then \( Q \) is represented by the block matrix

\[
\begin{pmatrix}
O & I & O \\
I & O & O \\
O & O & 1
\end{pmatrix}
\]

Now we have

\[
\mathfrak{so}_{2n+1} = \left\{ X = \begin{pmatrix}
A & B & E \\
C & -A^T & F \\
-F^T & -E^T & 0
\end{pmatrix} : B, C \text{ skew-symmetric.} \right\}
\]

Take \( \mathfrak{h} = \mathbb{C}\langle H_i \rangle \) and \( \mathfrak{h}^* = \mathbb{C}\langle L_i \rangle \). The roots (in addition to the ones appearing for \( \mathfrak{so}_{2n} \)) are:

- \( E_{i,2n+1} - E_{2n+1,n+i} \) with eigenvalue \( L_i \).
- \( E_{n+i,2n+1} - E_{2n+1,i} \) with eigenvalue \( -L_i \).

We have \( H_{\pm L_i} = \pm 2H_i \).

For \( n = 2 \), the root diagram is the same as that of \( \mathfrak{sp}_4 \), rotated by 45 degrees. In fact \( \mathfrak{so}_5 \cong \mathfrak{sp}_4 \).

For \( n = 3 \), the roots are the edge midpoints along with the face midpoints of the cube.

Observe that for \( \mathfrak{sl}_n \) and \( \mathfrak{so}_{2n} \), all roots have the same length. Meanwhile for \( \mathfrak{sp}_{2n} \) and \( \mathfrak{so}_{2n+1} \), there are two different lengths. The Lie algebras in the first case are called simply laced.
The Weyl group, in the case of $\mathfrak{so}_{2n+1}$, is the same as that of $\mathfrak{sp}_{2n}$. In the case of $\mathfrak{so}_{2n}$, though, we get fewer reflections. $\mathfrak{W}$ fits into an exact sequence

$$0 \to (\mathbb{Z}/2)^{n-1} \to \mathfrak{W} \to S_n \to 0$$

(13.11)

where the kernel consists of those matrices with an even number of $-1$ entries.

Now we examine the weight lattice. For both $\mathfrak{so}_{2n}$ and $\mathfrak{so}_{2n+1}$, the weight lattice is

$$\mathbb{Z}\left\langle L_1, \ldots, L_n, \alpha = \frac{L_1 + \cdots + L_n}{2} \right\rangle.$$  

(13.12)

For now, say $V$ is a real vector space with quadratic form $Q : V \times V \to \mathbb{R}$. As usual, write $SO(V, Q)$ for the group of automorphisms of $V$ preserving $Q$, and $\mathfrak{so}(V, Q)$ its Lie algebra. Take $\mathfrak{h}_\mathbb{R} = \mathfrak{h} \cap \mathfrak{so}(V, Q)$ a real Cartan subalgebra. Then consider $H = \exp(\mathfrak{h}_\mathbb{R})$. In the case $Q = I$, $H \cong (S^1)^n$. But for $Q$ as we have considered earlier, $H \cong \mathbb{R}^n$. $I$ is called the compact form of the Lie algebra, while the second $Q$ is called the split form. (Similarly, $\mathfrak{sl}_n$ has split form the real $\mathfrak{sl}_n$ and compact form the real $\mathfrak{su}_n$.)

The Weyl chamber for $\mathfrak{so}_{2n+1}$ is

$$W = \left\{ \sum a_i L_i : a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \right\}.$$ 

(13.13)

But for $\mathfrak{so}_{2n}$, we get

$$W = \left\{ \sum a_i L_i : a_1 \geq a_2 \geq \cdots \geq a_n-1 \geq |a_n| \right\}.$$ 

(13.14)

In the case $\mathfrak{so}_{2n+1}$, the fundamental weights are

$$L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{n-1}, \alpha.$$  

(13.15)

Meanwhile, for $\mathfrak{so}_{2n}$, they are

$$L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{n-2}, \alpha, \beta = \frac{L_1 + \cdots + L_{n-1} - L_n}{2}.$$  

(13.16)

Remark. • The weights of the standard representation for $\mathfrak{so}_{2n}$ are $\pm L_i$. For $\mathfrak{so}_{2n+1}$, they are $\pm L_i$ and 0.

• Observe that the adjoint representation is given by $\Lambda^2$ of the standard representation. For $Q$ a bilinear form gives $V \cong V^*$, and then

$$\mathfrak{gl}(V) = \text{Hom}(V, V) = \text{Hom}(V, V^*) = V \otimes V \cong \Lambda^2 V \oplus \text{Sym}^2 V.$$  

(13.17)

Under this identification, $\mathfrak{so}(V)$ is sent to $\Lambda^2 V$. (We could have determined that the adjoint representation for $\mathfrak{sp}_{2n}$ was $\text{Sym}^2 V$ in a similar way.)

We now exhibit some exceptional isomorphisms (expressing $\mathfrak{so}_{3,4,5,6}$ as isomorphic to previously encountered Lie algebras) using algebraic geometry.
• $\mathfrak{so}_3 \cong \mathfrak{sl}_2$: $\mathfrak{sl}_2$ has adjoint form $PGL_2 \cong Aut(\mathbb{P}^1)$. $\mathbb{P}^1$ embeds in $\mathbb{P}^2$ as a conic $C$ under a Veronese map. So

$$Aut(\mathbb{P}^1) \cong Aut(\mathbb{P}^2, C) = PSO_3.$$  \hspace{1cm} (13.18)

The standard representation of $SO_3$ is taken to $\text{Sym}^2(\text{std})$ of $\mathfrak{sl}_2$. The “spin” representation (we haven’t described them in general yet) of $\mathfrak{so}_3$ is given by the standard representation of $\mathfrak{sl}_2$.

• $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$: The adjoint representation is irreducible, and splits into representations with highest weights $L_1 + L_2$ and $L_1 - L_2$. This implies $\mathfrak{so}_4$ is a direct sum of Lie algebras; these can be seen to be $\mathfrak{sl}_2$. The spin representations are given by the two standards on each of the $\mathfrak{sl}_2$ factors.

We can obtain this isomorphism as follows: smooth quadric surfaces in $\mathbb{P}^3$ are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, so

$$PSO_4 \cong Aut(\mathbb{P}^3, \mathbb{P}^1 \times \mathbb{P}^1)^0 = Aut(\mathbb{P}^1 \times \mathbb{P}^1)^0 = PSL_2 \times PSL_2.$$  \hspace{1cm} (13.19)

(The connected component consists of those automorphisms preserving each ruling.)

• $\mathfrak{so}_5$: The roots consist of the vertices and edge midpoints of a square. The Weyl chamber $\mathcal{W}$ is the cone generated by $L_1$ and $\alpha = \frac{L_1 + L_2}{2}$; these are the two fundamental weights. The standard representation $V = V_{\mathfrak{so}_5}$ has weights $\pm L_i$ and 0, so highest weight $L_1$. Meanwhile, $\Lambda^2 V$ has weights $\pm L_i \pm L_j$, $\pm L_i$, and 0 twice. The highest weight is $L_1 + L_2$ (twice that of the spin representation).

In terms of the isomorphism $\mathfrak{so}_5 \cong \mathfrak{sp}_4$, $V_{\mathfrak{so}_5} = W = \Lambda^2 V_{\mathfrak{sp}_4} / \mathbb{C}$. Now $\Lambda^2 V_{\mathfrak{so}_5} = \text{Sym}^2 V_{\mathfrak{sp}_4}$, so $V_{\mathfrak{sp}_4}$ corresponds to the spin representation.

• $\mathfrak{so}_6$: The standard representation $V_{\mathfrak{so}_6}$ has weights $\pm L_i$, so highest weight $L_1$. $\Lambda^2 V$ (the adjoint representation) is the irreducible representation with highest weight $L_1 + L_2$.

$\Lambda^3 V$ has weights $\pm L_i \pm L_j \pm L_k$, and $\pm L_i$ each with multiplicity 2. This representation can’t be irreducible, since the weights of the irreducible representation with highest weight $L_1 + L_2 + L_3$ lie in a tetrahedron (which only includes four of the eight vertices). This representation also includes $\pm L_i$ (the edge midpoints of the tetrahedron). We conclude that $\Lambda^3 V = \Gamma_2 \oplus \Gamma_3$.

We now describe the isomorphism $\mathfrak{so}_6 \cong \mathfrak{sl}_4$. We have $PSL_4 \cong Aut(G(2, 4))^0$. ($G(2, 4)$ has additional automorphisms: by identifying $\mathbb{C}^4$ with its dual, we get an involution taking each 2-plane to its annihilator in the dual space.)

Now given a 2-plane $\Lambda = \langle v, w \rangle$, associate $[v \wedge w] \in \mathbb{P}(\Lambda^2 \mathbb{C}^4) \cong \mathbb{P}^5$. This element is independent of choice of basis of $\Lambda$. We get a map $G(2, 4) \to \mathbb{P}^5$. The image is $\{ \eta \in \Lambda^2 \mathbb{C}^4 : \eta \wedge \eta = 0 \}$, the kernel of a nondegenerate symmetric bilinear form $Q$. Hence

$$Aut(G(2, 4))^0 \cong Aut(\mathbb{P}^5, Q) \cong PSO_6.$$  \hspace{1cm} (13.20)

Using this isomorphism, we can also describe $\mathfrak{so}_5 \cong \mathfrak{sp}_4$. Fixing a hyperplane (or equivalently point by self-duality) in $\mathbb{P}^5$ corresponds to preserving a skew-symmetric bilinear form in $\mathbb{C}^4$.

Using the isomorphism $\mathfrak{so}_6 \cong \mathfrak{sl}_4$, we can describe the spin representations. $\Gamma_\alpha$ is the standard representation of $\mathfrak{sl}_4$, and $\Gamma_\beta$ is its dual. We have $V_{\mathfrak{so}_6} \cong \Lambda^2 V_{\mathfrak{sl}_4}$. 

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We now look at $\mathfrak{so}_{2n+1}$ in general.

**Theorem 13.1.** $\Lambda^k V_{\mathfrak{so}_{2n+1}}$ is the irreducible representation with highest weight $L_1 + \cdots + L_k$ for $1 \leq k \leq n$.

This gives us all but one of the fundamental representations (and one with weight $2\alpha$).

For $\mathfrak{so}_{2n}$ in general, if $V$ is the standard representation, then $\Lambda^k V$ is irreducible for $k = 1, \ldots, n-1$, and $\Lambda^n V = \Gamma_{2\alpha} \oplus \Gamma_{2\beta}$.

For $V$ a complex vector space of dimension $m$ and $Q$ a nondegenerate symmetric bilinear form, equivalently given as $q : V \to V^*$ with $v \mapsto Q(\bullet, v)$, we say $\Lambda \subseteq V$ is isotropic for $Q$ if $Q|_{\Lambda \times \Lambda} = 0$, or equivalently, $q(\Lambda) \subseteq \text{Ann}(\lambda)$. We see immediately that $\dim \Lambda \leq m^2/2$.

Conversely, there exist isotropic subspaces of every dimension at most $m^2/2$. Start with a nonzero $v$ such that $Q(v, v) = 0$ ($Q(v, v)$ is a homogeneous quadratic). Then $\langle v \rangle ^\perp$ contains $\langle v \rangle$, and $Q$ induces a quadratic form on $\langle v \rangle ^\perp / \langle v \rangle$, of dimension $m - 2$.

**Theorem 13.2.** If $m = \dim V$ is odd, then for every $k$, the space of isotropic $k$-planes for $Q$ is an irreducible algebraic subvariety of $G(k, V)$. If $m = 2n$ is even, then this space is irreducible when $k < n$, but has exactly 2 connected components for $k = n$.

### 13.1 Clifford Algebras and Spin Representations

Let $V$ be a complex vector space of dimension $m$, and $Q$ a symmetric bilinear form on $V$. Recall that the tensor algebra of $V$ is

$$T(V) = \bigoplus_{n=0}^{\infty} V^\otimes n.$$  \hfill (13.21)

Modding out by the 2-sided ideal generated by $vw - wv$ yields the symmetric algebra

$$\text{Sym}^* V = \bigoplus_{n=0}^{\infty} \text{Sym}^n V,$$  \hfill (13.22)

the polynomial ring on $V$. Alternatively, modding out by the 2-sided ideal generated by $v^2$ gives the exterior algebra

$$\Lambda^* V = \bigoplus_{n=0}^{\infty} \Lambda^n V.$$  \hfill (13.23)

A third operation (involving $Q$) is to mod out by the 2-sided ideal generated by $v^2 - Q(v, v) \cdot 1$. We get the Clifford algebra, denoted $c(Q)$ or $\text{Cliff}(V, Q)$. This algebra has a filtration

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq c(Q)$$  \hfill (13.24)

where $F_k$ is the subspace spanned by $k$ or fewer factors. We have $F_k/F_{k-1} \cong \Lambda^k V$ as a vector space. In particular, as a vector space, $c(Q) \cong \Lambda^* V$ has dimension $2^m$. We also have a $\mathbb{Z}/2$-grading.
\[ c(Q) = c(Q)^{\text{even}} \oplus c(Q)^{\text{odd}}. \quad (13.25) \]

The key facts are:

1. \( \mathfrak{so}(V,Q) \hookrightarrow c(Q)^{\text{even}} \) as an inclusion of Lie algebras.
2. If \( m \) is even, then
   \[
   c(Q)^{\text{even}} = \text{End}(\Lambda^{\text{even}}W) \oplus \text{End}(\Lambda^{\text{odd}}W). \quad (13.26)
   
   \text{Meanwhile, if } m \text{ is odd, then}
   \[
   c(Q) = \text{End}(\Lambda^*W) \oplus \text{End}(\Lambda^*W'). \quad (13.27)
   \]

14 Classification Theorem

Given a (semi)simple Lie algebra, we associate a root system \( R \subseteq h^*_R \), and then a Dynkin diagram.

Let \( \mathfrak{g} \) be a simple Lie algebra, \( h \subseteq \mathfrak{g} \) a Cartan subalgebra (let its dimension be \( n \)), \( h^*_R \) the real subspace spanned by the weights. Denote the real Euclidean space \((h^*_R, B)\) by \( E \).

Let \( R \subseteq E \) be the root system. The key properties of \( R \) are:

1. \( R \) is finite and spans \( E \).
2. If \( \alpha \in R \), then \(-\alpha \in R \) but \( k\alpha \not\in R \) for \( k \in \mathbb{R} \setminus \{\pm 1\} \).

   Proof. Look at \( \bigoplus_{k \in \mathbb{R}} \mathfrak{g}_{k\alpha} \), a representation of \( \mathfrak{s}_\alpha \). Choose \( \alpha \) to be one of the smallest weights of this representation. Now we may write
   \[
   \bigoplus_{k \in \mathbb{R}} \mathfrak{g}_{k\alpha} = \mathfrak{s}_\alpha \oplus V \quad (14.1)
   
   so \( V \) has no eigenvectors for \( H_\alpha \) with eigenvalue \( \pm 1 \) or \( \pm 2 \). Hence \( V \) is trivial. \( \square \)

3. \( W_\alpha \), reflection in the hyperplane perpendicular to \( \alpha \), maps \( R \) to \( R \).
4. \( 2\frac{(\beta,\alpha)}{(\alpha,\alpha)} = \beta(H_\alpha) \in \mathbb{Z} \).

   Proof. \( W_\alpha \) can be expressed as
   \[
   \beta \mapsto \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha. \quad (14.2)
   
   But also, for \( \Omega_\alpha \subseteq h^*_R \) the annihilator of \( H_\alpha \), \( W_\alpha \) can be expressed as
   \[
   \beta \mapsto \beta - m\alpha, \quad m = \beta(H_\alpha). \quad (14.3)
   
   Hence \( \beta(H_\alpha) = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \).
   \( \square \)
5. Given $\alpha, \beta \in R$ with $\alpha \neq \pm \beta$, look at the $\alpha$-string through $\beta$, namely a maximal chain of roots of the form

$$\beta - p\alpha, \ldots, \beta - \alpha, \beta, \beta + \alpha, \ldots, \beta + q\alpha$$

**(14.4)**

**Claim.** This string has length at most 4 ($p + q \leq 3$). Specifically, $p - q = \eta_{\beta\alpha}$.

**Proof.** $W_\alpha(\beta + q\alpha) = \beta - p\alpha$. But also $W_\alpha(\beta + q\alpha) = \beta - \eta_{\beta\alpha}\alpha - q\alpha$, so $p - q = \eta_{\beta\alpha}$.

Now applying this logic to $\beta + q\alpha$, then the new $q$ is 0, so the new $p$ (now $p + q$), is $|\eta_{(\beta + q\alpha), \alpha}| \leq 3$.

6. If $\alpha, \beta \in R$ and $\alpha \neq \pm \beta$, then if $\theta(\alpha, \beta) > \frac{\pi}{2}$, then $\alpha + \beta \in R$. Meanwhile, if $\theta(\alpha, \beta) < \frac{\pi}{2}$, then $\alpha - \beta \in R$. Finally, if $\theta(\alpha, \beta) = \frac{\pi}{2}$, then $\alpha + \beta$ and $\alpha - \beta$ are either both roots or both non-roots.

7. If $\alpha$ and $\beta$ are distinct simple roots, then $\alpha - \beta$ is not a root (because either $\alpha - \beta$ or $\beta - \alpha$ must then be a positive root).

8. Simple roots are linearly independent.

We now say that a root system is a subset $R \subseteq \mathbb{E}$ satisfying conditions (1) through (4) (The further key facts are consequences of these). We will say that $R$ is reducible if $R$ is contained in $V \cup V^\perp$ for some proper nontrivial subspace $V \subseteq \mathbb{E}$. Otherwise $R$ is irreducible.

For $\alpha, \beta \in R$, let $\theta(\alpha, \beta)$ be the angle between $\alpha$ and $\beta$, so that $(\alpha, \beta) = \|\alpha\|\|\beta\|\cos \theta$. Hence

$$\eta_{\beta\alpha} = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)} = 2 \cos \theta \frac{\|\beta\|}{\|\alpha\|}$$

**(14.5)**

is an integer. But $\eta_{\beta\alpha}\eta_{\alpha\beta} = 4\cos^2 \theta$, so is one of 0,1,2,3,4. If $\alpha \neq \pm \beta$, then 4 is not a possibility.

The remaining possibilities are:

- $\theta = \frac{\pi}{2}$.
- $\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$, and $\|\alpha\| = \|\beta\|$.
- $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$, and the ratio of lengths is $\sqrt{2}$.
- $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$, and the ratio of lengths is $\sqrt{3}$.

Define $\text{rank}(R \subseteq \mathbb{E})$ to be $\dim \mathbb{E}$. There is a unique rank 1 root system (two roots which are negative of each other).

For rank 2 root systems, the angle between any two adjacent vectors are equal (since we can reflect one in the other). So giving any two adjacent roots determines the rest. We have:

- $\theta = \frac{\pi}{2}$. This is $\mathfrak{sl}_2 \times \mathfrak{sl}_2$.

\[
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\]

**(14.6)**
• \( \theta = \frac{\pi}{3} \). All roots are the same length. This is \( \mathfrak{sl}_3 \).

\[
\begin{array}{c}
\begin{array}{c}
\text{Diagram}
\end{array}
\end{array}
\]

(14.7)

• \( \theta = \frac{\pi}{4} \). The lengths alternate. This is \( \mathfrak{sp}_4 \).

\[
\begin{array}{c}
\begin{array}{c}
\text{Diagram}
\end{array}
\end{array}
\]

(14.8)

• \( \theta = \frac{\pi}{6} \). Again the lengths alternate. This turns out to be \( \mathfrak{g}_2 \).

\[
\begin{array}{c}
\begin{array}{c}
\text{Diagram}
\end{array}
\end{array}
\]

(14.9)

In rank 3, the irreducible root systems are \( \mathfrak{sl}_4 \), \( \mathfrak{sp}_6 \), and \( \mathfrak{so}_7 \).

Given a root system \( R \), decompose \( R \) as \( R^+ \| R^- \). We say that \( \alpha \in R^+ \) is simple if \( \alpha \) is not a sum of positive roots. The sets of simple roots for the classical Lie algebras are:

• \( \mathfrak{sl}_{n+1} \): \( L_1 - L_2, \ldots, L_n - L_{n-1} \) in \( \mathbb{C}(L_1, \ldots, L_{n+1})/(\sum L_i) \).

• \( \mathfrak{so}_{2n+1} \): \( L_1 - L_2, \ldots, L_{n-1} - L_n, L_n \) in \( \mathbb{C}(L_1, \ldots, L_n) \).

• \( \mathfrak{sp}_{2n} \): \( L_1 - L_2, \ldots, L_{n-1} - L_n, 2L_n \).

• \( \mathfrak{so}_{2n} \): \( L_1 - L_2, \ldots, L_{n-1} - L_n, L_{n-1} + L_n \).

We now discuss the Dynkin diagram. We will see that the angle between simple roots cannot be acute, so is either \( \frac{\pi}{2} \), \( \frac{2\pi}{3} \), \( \frac{3\pi}{4} \), and \( \frac{5\pi}{6} \). We draw one node for each simple root and draw:

• one line between them if the angle is \( \frac{2\pi}{3} \).

• two lines between them, with an arrow from the long root to the short root, if the angle is \( \frac{3\pi}{4} \).

• three lines between them, with an arrow from the long root to the short root, if the angle is \( \frac{5\pi}{6} \).
Theorem 14.1. The only Dynkin diagrams coming from irreducible root systems are:

<table>
<thead>
<tr>
<th>Name</th>
<th>Lie Algebra</th>
<th>Diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$\mathfrak{sl}_{n+1}$</td>
<td>![Diagram]</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$\mathfrak{so}_{2n+1}$</td>
<td>![Diagram]</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$\mathfrak{sp}_{2n}$</td>
<td>![Diagram]</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$\mathfrak{so}_{2n}$</td>
<td>![Diagram]</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$\mathfrak{g}_2$</td>
<td>![Diagram] (14.10)</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$\mathfrak{f}_4$</td>
<td>![Diagram]</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$\mathfrak{e}_6$</td>
<td>![Diagram]</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$\mathfrak{e}_7$</td>
<td>![Diagram]</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$\mathfrak{e}_8$</td>
<td>![Diagram]</td>
</tr>
</tbody>
</table>

The isomorphisms of Dynkin diagrams above are $B_2 \cong C_2$, $D_2 \cong A_1 \sqcup A_1$, and $D_3 \cong A_3$.

Here are properties of the exceptional Lie algebras:

<table>
<thead>
<tr>
<th>$\mathfrak{g}$</th>
<th>$\mathfrak{h}$</th>
<th>$\mathfrak{f}$</th>
<th>$\mathfrak{e}_6$</th>
<th>$\mathfrak{e}_7$</th>
<th>$\mathfrak{e}_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dim \mathfrak{h}$</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>$# R^+$</td>
<td>6</td>
<td>24</td>
<td>36</td>
<td>63</td>
<td>120</td>
</tr>
<tr>
<td>$\dim \mathfrak{g}$</td>
<td>14</td>
<td>52</td>
<td>78</td>
<td>133</td>
<td>248</td>
</tr>
</tbody>
</table>

(14.11)

Now we will show how to reconstruct a simple Lie algebra from its Dynkin diagram. We'll first reconstruct the root system $R$.

For $\alpha_1, \ldots, \alpha_n$ the simple positive roots, we need to determine which nonnegative linear combinations $\sum m_i \alpha_i$ are actually roots. As a first case, for $i \neq j$, $\alpha_i + \alpha_j$ is a root if and only if the corresponding roots of the Dynkin diagram are connected by at least one line.

In general, we say that a root $\sum m_i \alpha_i$ has level $m = \sum m_i$. Suppose we know all of the roots of level $m$; then we will try to determine all roots of level $m + 1$. Given a root $\beta = \sum m_i \alpha_i$ of level $m$ and an index $j$, we ask when $\beta + \alpha_j$ is a root. We’ll be done from this and:
Lemma 14.2. Every positive root is a sum $\beta = \alpha_{i_1} + \cdots + \alpha_{i_k}$ such that $\alpha_{i_1} + \cdots + \alpha_{i_k}$ is a root for every $k$.

Proof. If $\gamma = \sum m_i \alpha_i$, then

$$0 < (\gamma, \gamma) = \sum m_i (\alpha_i, \gamma) \quad (14.12)$$

implying $(\alpha_i, \gamma) > 0$ for some $i$. Now $\gamma - \alpha_i$ is a positive root. \qed

Now given $\beta$ and $j$, let $\alpha = \alpha_j$ and look at the $\alpha$-string through $\beta$. Then $\beta + \alpha$ is a root if and only if $q > 0$. But $q = p - \eta_{\beta \alpha}$,

$$\eta_{\beta \alpha} = \sum m_i \eta_{\alpha_i, \alpha} \quad (14.13)$$

is determined from the Dynkin diagram, and, since $\beta - p\alpha$ is a positive root (linear combinations of simple roots with mixed signs can’t be a root), $p$ is determinable from the roots of smaller level. Hence we can determine $q$.

We have just proven that the Dynkin diagram uniquely determines the root system. In fact, we get all of the roots by reflecting in hyperplanes $\Omega_{\alpha}$ perpendicular to the simple roots $\alpha$.

Now we will show that the root system uniquely determines the Lie algebra. Given a Lie algebra $\mathfrak{g}$, we have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha. \quad (14.14)$$

Given $R \subseteq E$, we have $\mathfrak{h} = E^* \cong E$, with the isomorphism given by the Killing form. We know that the simple roots $\alpha_1, \ldots, \alpha_n$ form a basis for $E = \mathfrak{h}^*$. Take $T_1, \ldots, T_n \in \mathfrak{h}$ to be the dual basis. The distinguished elements are then

$$H_i = H_{\alpha_i} = \frac{2T_i}{(\alpha_i, \alpha_i)}. \quad (14.15)$$

Now choose $X_i \in \mathfrak{g}_{\alpha_i}$, and then let $Y_i \in \mathfrak{g}_{-\alpha_i}$ be such that $[X_i, Y_i] = H_i$.

Claim. The $H_i$, $X_i$, and $Y_i$, for $i = 1, \ldots, n$, generate $\mathfrak{g}$ as a Lie algebra.

Proof.

Lemma 14.3. If $\alpha$, $\beta$, and $\alpha + \beta$ are all roots (with $\alpha \neq \pm \beta$), then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha + \beta}$.

Proof. Look at the $\alpha$-string through $\beta$. Set $W = \bigoplus_k \mathfrak{g}_{\beta + k\alpha}$, a sum of 1-dimensional subspaces, which is irreducible as a representation of $\mathfrak{s}_\alpha$. But if $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$, then we would have

$$\bigoplus_{k \leq 0} \mathfrak{g}_{\beta + k\alpha} \subseteq W \quad (14.16)$$

as a proper nontrivial subrepresentation, a contradiction. \qed
Now the result follows immediately from Lemma 14.2.

We now say that a sequence of indices $I = (i_1, \ldots, i_r)$ is admissible if $\alpha_{i_1} + \cdots + \alpha_{i_k}$ is a root for every $k$. For a sequence $I$, write $\alpha_I = \sum_{k=1}^r \alpha_{i_k}$. We saw from Lemma 14.2 that for every positive root $\alpha$, there exists an admissible sequence $I$ with $\alpha_I = \alpha$. In this case,

$$X_I = [X_{i_r}, [X_{i_{r-1}}, [\cdots [X_{i_2}, X_{i_1}] \cdots]]] \neq 0 \in g_{\alpha}. \quad (14.17)$$

Now suppose $\alpha_I = \alpha_J$; we would like to know how $X_I$ and $X_J$ are related.

**Lemma 14.4.** If $\alpha_I = \alpha_J$, then $X_J = qX_I$ where $q \in \mathbb{Q}$ is determined by $I$, $J$, and the Dynkin diagram.

**Proof.** Induct on $r$. Set $k = i_r$. If $j_r = k$, then we’re done by induction. Otherwise, we claim that $X_J = q_1 [X_k, [Y_k, X_J]]$ for $q_1 \in \mathbb{Q}$ depending on $k$, $J$, and the Dynkin diagram. Now $\alpha_J - \alpha_k$ is a root. For we know how $s_{\alpha_k} \cong sl_2$ acts on the $\alpha_k$-string through $\alpha_J$. Letting $V = \bigoplus_m g_{\alpha_j + m \alpha_k}, (14.18)$

the length of this string is determined by the Dynkin diagram. We can now relate $X_J$ to

$$[X_k, [Y_k, X_J]] = [X_k, [Y_k, [X_{j_r}, [X_{j_{r-1}}, [\cdots [X_{j_2}, X_{j_1}] \cdots]]]]. \quad (14.19)$$

Let $s$ be the largest integer such that $j_s = k$, and $K = (j_1, \ldots, j_s)$. Then

$$[X_k, [Y_k, X_J]] = [X_k, [Y_k, [X_{j_r}, [X_{j_{r-1}}, [\cdots [X_{j_{s+1}}, X_K] \cdots]]]]] (14.20)$$

For $i \neq k$, we have $[Y_k, [X_i, Z]] = [X_i, [Y_k, Z]]$, since $[Z, [X_i, Y_k]] = 0$ and then we can apply the Jacobi identity. ($[X_i, Y_k] = 0$ since $\alpha_i - \alpha_k$ has mixed signs, so cannot be a root.)

**15 Construction of $g_2$ and $G_2$**

Recall that the classical complex Lie groups arise as subgroups of $GL(V)$ preserving some tensor. We can attempt to produce the exceptional Lie groups in a similar way. Other tensors we might consider could be $\Lambda^3 V$ and $\text{Sym}^3 V$ (or other Schur functors).

However, for $n \geq 2$, $\text{Sym}^3 V$ has dimension at least that of $GL(V)$. This suggests that generic elements of $\text{Sym}^3 V$ have a trivial stabilizers, and this turns out to be true. The same happens for $\Lambda^3 V$ for $n \geq 9$. (In this case, we get a nontrivial moduli space of skew-symmetric trilinear forms. But that’s not what we’re interested in right now.)

In low dimensional cases, we can describe the groups preserving generic skew-symmetric trilinear forms as:
\[ n = \dim V \quad \dim GL(V) \quad \dim \Lambda^3 V \quad \dim GL(V) - \dim \Lambda^3 V \quad \text{Aut}(V, \varphi) : \varphi \in \Lambda^3 V \]

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>9</td>
<td>1</td>
<td>8</td>
<td></td>
<td>(SL_3)</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>4</td>
<td>12</td>
<td></td>
<td>extension of (SL_3) by a vector space</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
<td>10</td>
<td>15</td>
<td></td>
<td>extension of (Sp_4) by a vector space</td>
</tr>
<tr>
<td>6</td>
<td>36</td>
<td>20</td>
<td>16</td>
<td></td>
<td>(SL_3 \times SL_3)</td>
</tr>
<tr>
<td>7</td>
<td>49</td>
<td>35</td>
<td>14</td>
<td></td>
<td>(G_2)</td>
</tr>
<tr>
<td>8</td>
<td>64</td>
<td>56</td>
<td>8</td>
<td></td>
<td>? (no new simple parts)</td>
</tr>
</tbody>
</table>

Other Schur functors have even higher dimension. The remaining four exceptional Lie algebras must be constructed using other means.

16 Complex Lie Groups Associated to Simple Lie Algebras

For any semisimple Lie algebra \(\mathfrak{g}\) and any form \(G\), we have lattices

\[
\Lambda_R \subseteq \Lambda_W \subseteq \mathfrak{h}^* 
\]

and their duals

\[
\Lambda^*_W \subseteq \Lambda^*_R \subseteq \mathfrak{h}. \tag{16.2}
\]

Look at \(\exp |_{\mathfrak{h}} : \mathfrak{h} \to H \subseteq G\). Then we have

\[
\Lambda^*_W \subseteq \frac{1}{2\pi i} \ker(\exp |_{\mathfrak{h}}) \subseteq \Lambda^*_R. \tag{16.3}
\]

A representation \(V\) of \(\mathfrak{g}\) lifts to \(G\) if and only if the weights of \(V\) take integer values on the lattice \(\frac{1}{2\pi i} \ker(\exp |_{\mathfrak{h}})\).

16.1 Forms of \(\mathfrak{sl}_{n+1}\)

Claim. \(SL_{n+1}(\mathbb{C})\) is simply connected, hence is the simply connected form of \(\mathfrak{sl}_{n+1}\).

Proof. Look at the action of \(SL_{n+1}(\mathbb{C})\) on \(\mathbb{C}^{n+1} \setminus \{0\}\). This is transitive with stabilizer homeomorphic to \(SL_n(\mathbb{C}) \times \mathbb{C}^n \cong SL_n(\mathbb{C})\), and this action produces a fibration. The long exact sequence of homotopy groups then proves our claim by induction (for the base case, \(SL_1(\mathbb{C}) = 1\)).

The center of \(SL_{n+1}(\mathbb{C})\) is \(\mu_{n+1}\), so the adjoint form is \(PSL_{n+1}(\mathbb{C})\). Our lattice of forms corresponds to quotients by subgroups of \(\mu_{n+1}\). An interesting form besides the simply connected form or adjoint form is \(SL_4(\mathbb{C})/\pm 1 \cong SO_6(\mathbb{C})\). In fact, we will see that \(SL_4(\mathbb{C}) \cong \text{Spin}_6(\mathbb{C})\).
16.2 Forms of $\text{sp}_{2n}$

Claim. $\text{Sp}_{2n}(\mathbb{C})$ is simply connected with center $\pm 1$.

Proof. Write $\text{Sp}_{2n}(\mathbb{C})$ in the form $\text{Sp}(V,Q)$. This acts transitively on the manifold/variety

$$M = \{(v,w) \in V \times V : Q(v,w) = 1\}.$$  \hfill (16.4)

The map $(v,w) \mapsto v$ fits $M$ into a fiber bundle

$$\mathbb{C}^{2n-1} \rightarrow M \rightarrow V \setminus \{0\}$$  \hfill (16.5)

with $V \setminus \{0\} = \mathbb{C}^{2n} \setminus \{0\} \simeq S^{4n-1}$. In fact $M \simeq S^{4n-1}$. In particular, $\pi_1(M)$ and $\pi_2(M)$ are both trivial.

Now let $H$ be the stabilizer of the action of $G = \text{Sp}(V,Q)$ on $M$. Then $H \rightarrow G \rightarrow M$ is a fibration, and the base case $\text{Sp}_2(\mathbb{C}) \cong \text{SL}_2(\mathbb{C})$ is simply connected, as we’ve seen earlier, so we’re done by induction. \hfill $\Box$

So there are only two forms, $\text{Sp}_{2n}(\mathbb{C})$ and $P\text{Sp}_{2n}(\mathbb{C})$. Observe that $\Lambda_W/\Lambda_R \simeq \mathbb{Z}/2$ in this case.

16.3 Forms of $\text{so}_m$

Claim. For $m \geq 3$, $\pi_1(\text{SO}_m(\mathbb{C})) = \mathbb{Z}/2$, and

$$Z(\text{SO}_m(\mathbb{C})) = \begin{cases} 1 & m \text{ odd} \\ \pm 1 & m \text{ even} \end{cases}$$  \hfill (16.6)

Proof. $G = \text{SO}_m(V,Q)$ acts transitively on

$$M = \{v \in V : Q(v,v) = 1\}$$  \hfill (16.7)

with stabilizer $H \cong \text{SO}_{m-1}(\mathbb{C})$.

Let $Q$ be the diagonal form $Q(z,z) = \sum_{a=1}^n z_a^2$ and $z = x + iy$. Then $M$ is given by

$$\sum x_\alpha y_\alpha = 0$$  \hfill (16.8)

$$\sum (x_\alpha^2 - y_\alpha^2) = 1$$  \hfill (16.9)

The map $z \mapsto \frac{z}{\|z\|}$ gives a map $M \rightarrow S^{m-1}$ with fibers given by hyperplanes in $\mathbb{R}^m$. Hence $M \simeq S^{m-1}$.

We start with $m = 3$, in which case we know $\text{SL}_2(\mathbb{C}) \rightarrow \text{SO}_3(\mathbb{C})$ is a double cover. Hence $\pi_1(\text{SO}_3(\mathbb{C})) \cong \mathbb{Z}/2$. If $m \geq 4$, then we use the fibration $H \rightarrow G \rightarrow M$ and proceed by induction, since $\pi_1(M)$ and $\pi_2(M)$ are both trivial. \hfill $\Box$
If \( m = 2n + 1 \), then the only forms are \( \text{Spin}_{2n+1}(\mathbb{C}) \) and \( \text{SO}_{2n+1}(\mathbb{C}) \). If \( m = 2n \), then we have
\[
Z(\text{Spin}_{2n}(\mathbb{C})) = \begin{cases} 
\mathbb{Z}/4 & \text{n odd} \\
\mathbb{Z}/2 \times \mathbb{Z}/2 & \text{n even}
\end{cases}
\] (16.10)

There are at least three forms: \( \text{Spin}_{2n}(\mathbb{C}) \), \( \text{SO}_{2n}(\mathbb{C}) \), and \( \text{PSO}_{2n}(\mathbb{C}) \). If \( n \) is odd, these are all of the forms. If \( n \) is even, there are two other forms, each of index 2 in \( \text{Spin}_{2n}(\mathbb{C}) \).

### 17 Representation Rings and Characters

Given a group \( G \) with Lie algebra \( \mathfrak{g} \), the representation ring \( R(G) = R(\mathfrak{g}) \) is described as follows:

- The abelian group structure is the free abelian group generated by isomorphism classes of representations of \( G \) (or \( \mathfrak{g} \)), modulo the relations \([V] = [U] + [W]\) whenever.

\[
0 \to U \to V \to W \to 0.
\] (17.1)

If reducibility holds, then this is just the free abelian group generated by isomorphism classes of irreducible representations.

- The ring structure is defined by \([V][W] = [V \otimes W]\).

Now assume that \( \mathfrak{g} \) is a semisimple Lie algebra. Let \( \Lambda \) be the weight lattice in \( \mathfrak{h}^* \), \( \omega_1, \ldots, \omega_n \) be the fundamental weights, and \( \Gamma_i \) the irreducible representation with highest weight \( \omega_i \). Let \( \Gamma_\lambda \) be the irreducible representation with highest weight \( \lambda \). Then we have a natural map
\[
\mathbb{Z}[u_1, \ldots, u_n] \to R(\mathfrak{g})
\] (17.2)

given by \( u_i \mapsto [\Gamma_i] \).

**Claim.** This map is an isomorphism.

**Proof.** We have a filtration on \( R(\mathfrak{g}) \) by taking \( F_k \) to be the linear span of the \( [\Gamma_{i_1, \ldots, i_n}] \) such that \( i_1 + \cdots + i_n \leq k \). Now we have
\[
\Gamma_\lambda \otimes \Gamma_\mu = \Gamma_{\lambda+\mu} + (\text{terms of lower weights}).
\] (17.3)

Hence our map induces an isomorphism on the associated graded rings, and is therefore itself an isomorphism. \( \square \)

Next, we introduce the character ring. For any lattice \( \Lambda \), \( \mathbb{Z}[\Lambda] \) is the free abelian group with generators \( e(\lambda) \) over \( \lambda \in \Lambda \), with \( e(\lambda) e(\mu) = e(\lambda + \mu) \). This coincides with the group algebra, and in fact if \( \lambda_1, \ldots, \lambda_n \) form a basis for \( \Lambda \) and \( x_i = e(\lambda_i) \), then
\[
\mathbb{Z}[\Lambda] = \mathbb{Z}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]
\] (17.4)
\[
= \mathbb{Z}[x_1, \ldots, x_n, (x_1 \cdots x_n)^{-1}]
\] (17.5)
\[
= \mathbb{Z}[x_1, \ldots, x_{n+1}]/((x_1 \cdots x_{n+1}) - 1).
\] (17.6)

43
We have a ring homomorphism \( \text{char} : R(g) \to \mathbb{Z}[\Lambda] \) by

\[
[V] \mapsto \sum_{\lambda \in \Lambda} \dim V_\lambda \cdot e(\lambda).
\]  

(17.7)

\( \text{char} \) is actually a ring homomorphism since

\[
(V \otimes W)_\lambda = \bigoplus_{\mu + \nu = \lambda} V_\mu \otimes W_\nu.
\]  

(17.8)

Observe that \( \mathfrak{W} \) acts on \( \Lambda \), hence on \( \mathbb{Z}[\Lambda] \), and that \( \text{im(char)} \subseteq \mathbb{Z}[\Lambda]^{\mathfrak{W}} \).

Claim. \( R(g) \to \mathbb{Z}[\Lambda]^{\mathfrak{W}} \) is an isomorphism.

Proof. This map is clearly an isomorphism on associated graded rings.

We examine what happens in the classical Lie algebras:

1. \( \mathfrak{sl}_{n+1} \): \( h^* = \mathbb{C}\langle L_1, \ldots, L_{n+1} \rangle / \langle \sum L_i \rangle \) with \( \Lambda \) the lattice generated by the \( L_i \). For \( V \) the standard representation, the fundamental representations are \( \Lambda^i V \) for \( 1 \leq i \leq n \).

   \[
   \text{Set } x_i = e(L_i) \text{ and } x_i^{-1} = e(-L_i). \text{ Then }
   \]

   \[
   \text{char}(V) = x_1 + \cdots + x_{n+1} \quad (17.9)
   \]

   \[
   \text{char}(\Lambda^k V) = \sigma_k(x_1, \ldots, x_{n+1}). \quad (17.10)
   \]

   In particular, \( \mathbb{Z}[\Lambda]^{\mathfrak{W}} = \mathbb{Z}[A_1, \ldots, A_n] \) for \( A_i = \sigma_i(x_1, \ldots, x_{n+1}) \).

2. \( \mathfrak{sp}_{2n} \): Here \( h^* = \mathbb{C}\langle L_1, \ldots, L_n \rangle \) and \( \Lambda = \mathbb{Z}\langle L_1, \ldots, L_n \rangle \). The fundamental weights are

   \[
   L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_n. \quad (17.11)
   \]

   For \( V \) the standard representation, the fundamental representations are given by

   \[
   \Gamma_k = \ker(\Lambda^k V \to \Lambda^{k-2} V). \quad (17.12)
   \]

   \[
   \text{Set } x_i = e(L_i) \text{ and } x_i^{-1} = e(-L_i). \text{ Then }
   \]

   \[
   \text{char}(V) = x_1 + \cdots + x_n + x_1^{-1} + \cdots + x_n^{-1} \quad (17.13)
   \]

   \[
   \text{char}(\Lambda^k V) = \sigma_k(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}) \quad (17.14)
   \]

   \[
   \quad = C_k \quad (17.15)
   \]

   \[
   \text{char}(\Gamma_k) = C_k - C_{k-2}. \quad (17.16)
   \]

(Here for the cases \( k \leq 2 \), we have \( C_{-1} = 0 \) and \( C_0 = 1 \).) So

\[
R(\mathfrak{sp}_{2n}) = \mathbb{Z}[\Lambda]^{\mathfrak{W}} = \mathbb{Z}[C_k - C_{k-2}] = \mathbb{Z}[C_1, \ldots, C_n]. \quad (17.17)
\]
3. $\mathfrak{so}_{2n+1}$: We have $\Lambda = \mathbb{Z}\langle L_1, \ldots, L_n, \frac{L_1 + \cdots + L_n}{2} \rangle$, and fundamental weights

$$L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{n-1}, \frac{L_1 + \cdots + L_n}{2}$$

having fundamental representations

$$V, \Lambda^2 V, \ldots, \Lambda^{n-1} V, S$$

where $S$ is the spin representation.

Set $y_i = e\left(\frac{L_i}{2}\right)$ (so $x_i = y_i^2$). Then we have (for $1 \leq k \leq n-1$)

$$\text{char}(\Lambda^k V) = \sigma_k(y_1^2, \ldots, y_n^2, y_1^{-2}, \ldots, y_n^{-2}, 1)$$

$$= B_k$$

$$\text{char}(S) = \sum_{\pm} y_1^{\pm 1} \cdots y_n^{\pm 1}$$

$$= B$$

Then $\mathbb{Z}[\Lambda]^{\mathfrak{so}_{2n+1}} = \mathbb{Z}[B_1, \ldots, B_n, B]$.

4. $\mathfrak{so}_{2n}$: $\Lambda$ is the same as above, but the fundamental weights are now

$$L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{n-2}, \frac{L_1 + \cdots + L_{n-1} + L_n}{2}, \frac{L_1 + \cdots + L_{n-1} - L_n}{2}.$$ (17.24)

The corresponding fundamental representations are

$$V, \Lambda^2 V, \ldots, \Lambda^{n-2} V, S^+, S^-.$$ (17.25)

Now (for $1 \leq k \leq n-2$)

$$\text{char}(\Lambda^k V) = \sigma_k(y_1^2, \ldots, y_n^2, y_1^{-2}, \ldots, y_n^{-2})$$

$$= D_k$$

$$\text{char}(S^+) = \sum_{\pm} y_1^{\pm 1} \cdots y_n^{\pm 1}$$

$$= D^+$$

$$\text{char}(S^-) = \sum_{\pm \text{ even}} y_1^{\pm 1} \cdots y_n^{\pm 1}$$

$$= D^-.$$ (17.31)

So $\mathbb{Z}[\Lambda]^{\mathfrak{so}_{2n}} = \mathbb{Z}[D_1, \ldots, D_{n-2}, D^+, D^-]$. 

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18 The Weyl Character Formula

Now we introduce the special weight

\[ \rho = \frac{1}{2} \sum_{\alpha \in \mathbb{R}^+} \alpha. \] (18.1)

We claim that \( \rho \) is not only a weight, but it is the smallest weight in the interior of \( \mathcal{W} \). This also equals \( \omega_1 + \cdots + \omega_n \).

For \( \lambda \in \Lambda \), we set

\[ A_\lambda = \sum_{w \in \mathcal{W}} (-1)^w e(w(\lambda)) \] (18.2)

where \( (-1)^w \) denotes the sign of the determinant of \( w \).

**Lemma 18.1.**

\[ A_\rho = \prod_{\alpha \in \mathbb{R}^+} \left( e \left( \frac{\alpha}{2} \right) - e \left( -\frac{\alpha}{2} \right) \right) \] (18.3)

\[ = e(-\rho) \prod (e(\alpha) - 1) \] (18.4)

\[ = e(\rho) \prod (1 - e(-\alpha)). \] (18.5)

**Theorem 18.2** (Weyl Character Formula). If \( \Gamma_\lambda \) is the irreducible representation having highest weight \( \lambda \), then

\[ \text{char}(\Gamma_\lambda) = \frac{A_{\rho+\lambda}}{A_\rho}. \] (18.6)

We’ll see what this implies for \( \mathfrak{sl}_{n+1} \). Here we have

\[ \omega_i = L_1 + \cdots + L_i \] (18.7)

\[ \rho = nL_1 + (n-1)L_2 + \cdots + L_n \] (18.8)

\[ A_\rho = \sum_{\tau \in \mathcal{S}_{n+1}} (-1)^\tau ((e(nL_{\tau(1)}) + \cdots + e(L_{\tau(n)})) \] (18.9)

\[ = \sum (-1)^\tau x_{\tau(1)}^n \cdots x_{\tau(n)}. \] (18.10)

Up to sign, this equals the Vandermonde determinant, which equals

\[ \prod_{i<j} (x_i - x_j). \] (18.11)

Now suppose \( \lambda \in \mathcal{W} \) is given by
\[ \lambda = \alpha_1 \omega_1 + \cdots + \alpha_n \omega_n \tag{18.12} \]
\[ = \lambda_1 L_1 + \cdots + \lambda_n L_n \tag{18.13} \]
\[ \lambda_i = \alpha_1 + \cdots + \alpha_{n+1-i} \tag{18.14} \]

where the \( \alpha_i \) are nonnegative, so the \( \lambda_i \) are nonnegative and decreasing. Now we write

\[ \rho + \lambda = b_1 L_1 + \cdots + b_n L_n \tag{18.15} \]
\[ b_i = \lambda_i + (n + 1 - i) \tag{18.16} \]

so the \( b_i \) form a strictly decreasing sequence of positive integers. We may express \( A_{\rho+\lambda} \) as the determinant

\[ A_{\rho+\lambda} = \begin{vmatrix} x_1^{b_1} & x_1^{b_2} & \cdots & x_1^{b_n} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^{b_1} & x_n^{b_2} & \cdots & x_n^{b_n} & 1 \end{vmatrix} \tag{18.17} \]

The ratio \( S_{\lambda} = \frac{A_{\rho+\lambda}}{A_{\rho}} \) is a symmetric polynomial, called a Schur polynomial. These polynomials are indexed by partitions, and \( \deg S_{\lambda} = \lambda_1 + \cdots + \lambda_n \).

**Proof of Lemma 18.1.** We want to show that \( A_{\rho} = A \), for

\[ A = \prod_{\alpha \in R^+} \left( e \left( \frac{\alpha}{2} \right) - e \left( -\frac{\alpha}{2} \right) \right). \tag{18.18} \]

Claim. \( A \) is alternating under \( \mathfrak{W} \).

Proof. It suffices to show that if \( w = w_{\alpha_i} \) is reflection in \( \langle \alpha_i \rangle^\perp \), then \( w(A) = -A \).

**Lemma 18.3.** \( w \) carries \( \alpha_i \) to \( -\alpha_i \) and permutes the other positive roots.

Proof. If \( \beta \in R^+ \), write \( \beta = \sum m_j \alpha_j \). Then

\[ w_{\alpha_i}(\beta) = \beta - 2 \frac{(\beta, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i \tag{18.19} \]

If \( \beta \) is a root other than \( \alpha_i \), then some \( m_j \) for \( j \neq i \) is positive. But \( w_{\alpha_i}(\beta) \) has the same \( \alpha_j \) coefficient and is a root, so must be a positive root.

As a consequence, we must have \( w_{\alpha_i}(\rho) = \rho - \alpha_i \). This means that \( 2 \frac{(\rho, \alpha_i)}{(\alpha_i, \alpha_i)} = 1 \) for every \( i \). Because \( 2 \frac{(\omega_j, \alpha_i)}{(\alpha_i, \alpha_i)} = \delta_{ij} \), we get \( \rho = \omega_1 + \cdots + \omega_n \).

Now \( A \) has highest weight (with a nonzero coefficient) \( \frac{1}{2} \sum_{\alpha \in R^+} \alpha = \rho \), with coefficient 1. For \( \lambda \in \mathcal{W} \) we can associate two vector spaces: \( S_{\lambda} \) the space of symmetric characters with highest weight at most \( \lambda \), and \( A_{\lambda} \) the space of alternating ones. These vector spaces are finite dimensional, with
dim $S_\lambda$ equal to the number of weights in $W$ which are at most $\lambda$, and $\dim S_\lambda$ equal to the number of weights in the interior of $W$ which are at most $\lambda$.

Multiplication by $A_\rho$ yields an injective linear map $S_\lambda \rightarrow A_{\rho+\lambda}$. Because these vector spaces have the same dimension, this map is surjective.

In the case $\lambda = 0$, we have $A \in A_{\rho}$ and $S_0 \cong \mathbb{C}$, so $A$ and $A_\rho$ differ by a constant factor. By comparing $e(\rho)$ terms, this constant is 1.

The same argument shows that $\frac{A_{\rho+\lambda}}{A_\rho} \in \mathbb{Z}[\Lambda]^{\mathbb{M}}$.

19 Real Forms

Suppose first that $\mathfrak{g}$ is a real simple Lie algebra of dimension 3. Then with $X$ and $H$ as in Section 8, the nonzero eigenvalues of $\text{ad} \, H$ sum to zero and are either real or complex conjugates. So we can scale by a real number so that they are either $\pm 2$ or $\pm i$. In the second case, the Lie algebra is $\mathfrak{su}_2$. We can distinguish these Lie algebras by considering the image exponential map of the Cartan subalgebra in the adjoint form of the Lie algebra. For $\mathfrak{sl}_2$, we get $\mathbb{R}$, while for $\mathfrak{su}_2$, we get $S^1$.

In general, the possible real forms are:

- $\mathfrak{sl}_n(\mathbb{C})$: For $p + q = n$, the real forms are $\mathfrak{su}_{p,q}$ which preserve a hermitian form of signature $(p, q)$.

- $\mathfrak{sp}_{2n}(\mathbb{C})$: the real forms are $U_{p,q}(\mathbb{H})$, the group of automorphisms of a quaternionic vector space preserving a hermitian form of type $(p, q)$.

- $\mathfrak{so}_n(\mathbb{C})$: the real forms are $\mathfrak{so}_{p,q}$, and if $n = 2m$ is even, we also obtain $U_{m}^{*}(\mathbb{H})$, the group of automorphisms preserving a skew-hermitian form.

It may be that an indecomposable semisimple Lie algebra has a decomposable complexification. It turns out that the other simple real Lie algebras are the just simple complex Lie algebras, viewed as real Lie algebras.

We say that a semisimple real Lie algebra $\mathfrak{g}$ is a compact form if $\exp \mathfrak{h} \cong (S^1)^n$, and is a split form if $\exp \mathfrak{h} \cong \mathbb{R}^n$. 