1 Previous Material

(Definition of reductive group, existence of maximal compact, unique up to conjugacy, other structural properties.)

To do Figure out what happened the first two days. (1)

2 Continuous Representations

Let $G$ be a locally compact Hausdorff group and $V$ a complete locally convex Hausdorff topological vector space. Suppose we have a homomorphism $\pi : G \to \text{Aut}(V)$ where the morphisms in $V$ are the continuous linear maps. There are a few notions of continuity:

- $\pi$ is continuous if the action map

$$G \times V \overset{a}{\longrightarrow} V$$

$$(g, v) \mapsto \pi(g)v$$

(2.1)
is continuous.

Suppose $g, g_0 \in G$ and $v, v_0 \in V$. Then

\[
\pi(g)v - \pi(g_0)v_0 = \pi(g)v - \pi(g)v_0 + \pi(g)v_0 - \pi(g_0)v_0
\]

\[
= \pi(g_0)\left[\pi(g_0^{-1})v - \pi(g_0^{-1})v_0\right]
\]

(2.2)

(2.3)

Hence continuity is equivalent to:

(a) $a$ being continuous at $(e, 0)$.
(b) $g \mapsto \pi(g)v_0$ is continuous at $g = e$ for every $v_0 \in V$.

Also, (b) is equivalent to (b'): for $g \in G$, $g \mapsto \pi(g)v_0$ is continuous for all $v_0 \in V$.

- $\pi$ is strongly continuous if for every $v \in V$, $g \mapsto \pi(g)v$ is continuous as a map from $G$ to $V$.
- $\pi$ is weakly continuous if for every $v \in V$ and every $\varphi \in V'$ (the space of continuous linear functionals from $V$ to $C$), $g \mapsto \langle \varphi, \pi(g)v \rangle$ is a continuous function on $G$.

We see that continuity implies (!) strong continuity, which in turn implies weak continuity.

**Lemma 2.1.** If $V$ is a Banach space, then strong continuity implies continuity.

**Proof.** Let $C \subseteq G$ be a compact neighborhood of $e$ in $G$. For each $v \in V$, the set

\[
\left\{ \pi(g)v \middle| g \in C \right\}
\]

(2.4)

is bounded. By the uniform boundedness principle, this implies that there exists a constant $M$ such that $\|\pi(g)\| \leq M$ for every $g \in C$. Hence $a$ is continuous at $(e, v)$.

In fact:

**Theorem 2.2.** If $V$ is a Banach space, then weak continuity implies continuity.

This follows from the Krein-Sinnlian theorem (see G. Warner for reference).

Note that weak continuity is symmetric in $V$ and $V'$, since

\[
\langle \varphi, \pi(g)v \rangle = \langle \pi'(g^{-1})\varphi, v \rangle.
\]

(2.5)

Hence if $V$ is a reflexive Banach space (such as a Hilbert space), then $\pi$ is continuous if and only if $\pi'$ is continuous.

As a non-example, $(\mathbb{R}, +)$ acts on $L^1(\mathbb{R})$ by translation, and this representation is continuous (because $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$); the same argument works for $L^p$ for $1 \leq p < \infty$. But the dual representation of $(\mathbb{R}, +)$ on $L^\infty(\mathbb{R})$ is not continuous!

**Remark.** There are more general cases of continuity implying continuity of the dual representation.
If $V$ is a Banach space, one could also require that $\pi : G \to \text{Aut}(V)$ (with $\text{Aut}(V)$ now equipped with the sup norm). But this is much too restrictive unless $V$ is finite dimensional.

Let $G$ be locally compact, Hausdorff, and unimodular. Then the left and right Haar measures coincide.

Remark. Reductive Lie groups are always unimodular. This is because the adjoint action $\text{Ad}$ of $G$ on $\mathfrak{g}$, raised to the top exterior power, is trivial when $G$ is semisimple, and when $G$ is abelian. If $G$ is reductive, then $G$ has a finite cover by a direct product of an abelian group and a semisimple group.

Also let $V$ be a complete locally convex Hausdorff topological vector space and $\pi$ a representation of $G$ on $V$. From now on, this always means $\pi$ is continuous. For $f \in C_c(G)$ and $v \in V$, define

$$\pi(f)v = \int_G f(g)\pi(g)v \, dg \in V. \quad (2.6)$$

The map $\pi(f) : V \to V$ is continuous.

We also have representations $\ell$ and $r$ of $G$ on $C_c(G)$ by $(\ell(g)f)(h) = f(g^{-1}h)$ and $(r(g)f)(h) = f(hg)$.

**Proposition 2.3.** The following hold for $f, f_1, f_2 \in C_c(G)$ and $g \in G$:

1. $\pi(\ell(g)f) = \pi(g)\pi(f)$.
2. $\pi(r(g)f) = \pi(f)\pi(g)$.
3. $\pi(f_1)\pi(f_2) = \pi(\ell(f_1)f_2)$. Here $\ell(f_1)f_2 = f_1 \star f_2$ is the convolution:

$$\pi(f_1 \star f_2)(h) = \int_G f_1(g)f_2(g^{-1}h) \, dg. \quad (2.7)$$

Now suppose $G$ is a connected reductive linear Lie group, and $(\pi, V)$ is a representation of $G$ on a complete locally convex Hausdorff topological vector space over $\mathbb{C}$. We say that $v \in V$ is a $C^\infty$ vector if $g \mapsto \pi(g)v$ is a $C^\infty$ map from $G$ to $V$.

Observe that for $f \in C_c^\infty(G)$ and $v \in V$, $\pi(f)v$ is a $C^\infty$ vector. For applying $\pi(g)$ gives $\pi(\ell(g)f)v$. To see that this map is $C^\infty$, we just need to differentiate $f$.

**Theorem 2.4.** The space $V^\infty$ of $C^\infty$ vectors in $V$ is dense in $V$, and $\mathfrak{g} = \mathbb{C} \otimes T_eG = \mathbb{C} \otimes \mathfrak{g}_\mathbb{R}$, by differentiation, acts on $V^\infty$. That is,

$$\pi(\zeta)v = \left. \frac{d}{dt} \pi(\exp t\zeta)v \right|_{t=0}. \quad (2.8)$$

Thus $V^\infty$ is a $\mathfrak{g}$-module.

**Proof.** Let $\{f_n\} \subseteq C_c^\infty(G)$ be an approximate identity. That is, $f_n \geq 0$, $\int_G f_n(g) \, dg = 1$, and $\text{supp}(f_n) \setminus \{e\}$. Then $\pi(f_n)v \to v$ for each $v \in V$. As $\pi(f_n)v \in V^\infty$ for every $n$, density follows.

The fact that $\pi(\zeta_1)\pi(\zeta_2) - \pi(\zeta_2)\pi(\zeta_1) = \pi([\zeta_1, \zeta_2])$ follows from Campbell-Hausdorff. \qed

3
Hence, to each representation of $G$ on $V$, we get an infinitesimal representation of $\mathfrak{g}$ on $V^\infty$. $V^\infty$ is called the G\"arding space. However, if $\pi$ is irreducible (meaning there is no nontrivial closed $G$-invariant subspace), then $V^\infty$ does not have to be algebraically irreducible!

Remark. We can write $V^\infty = \{ F : G \xrightarrow{C^\infty} V | F(g_1g_2) = \pi(g_1)F(g_2) \}$. This is a closed subspace of $\{ F : G \xrightarrow{C^\infty} V \}$, which is also a complete locally convex Hausdorff topological vector space.

### 3 Admissible Representations

Let $\pi_K = \pi|_K$ for $K \subseteq G$ a maximal compact subgroup. Write $\hat{K}$ for the set of isomorphism classes of irreducible finite dimensional (over \(\mathbb{C}\)) representations of $K$. For $i \in \hat{K}$, choose a representative $(\tau_i, U_i)$. Define $\varphi_i = (\dim U_i)\chi_i$ where $\chi_i$ is the character of $\tau_i$,

$$\chi_i(g) = \text{tr} \tau_i(g). \quad (3.1)$$

The Schur orthogonality relations imply $\tau_i(\varphi_j) = \delta_{ij}1_{U_i}$ and $\ell(\varphi_i)\varphi_j = \delta_{ij}\varphi_i$.

We say that:

1. $v \in V$ is $K$-finite if $v$ is contained in a finite dimensional $K$-invariant subspace of $V$. Write $V_{K-\text{fin}}$ for the linear span of all finite dimensional $K$-invariant subspaces.

2. $v \in V$ is $i$-isotypic, for $i \in \hat{K}$, if $v$ lies in the linear span of the images of all $K$-invariant continuous linear maps $T_i : U_i \to V$. Write $V(i)$ for the space of $i$-isotypic vectors.

**Theorem 3.1.**

1. $\pi_K(\varphi_i)\pi_K(\varphi_j) = \delta_{ij}\pi(\varphi_i)$.

2. $V(i)$ is the image of $\pi_K(\varphi_i)$.

3. $V_{K-\text{fin}} = \bigoplus_{i \in \hat{K}} V(i)$ (as an algebraic direct sum).

4. $V_{K-\text{fin}}$ is dense in $V$.

5. $V(i) \cap V^\infty$ is dense in $V(i)$ for every $i$.

**Proof.** $\pi_K(\varphi_i)\pi_K(\varphi_j) = \pi_K(\tau_i(\varphi_i)\varphi_j)$, which equals $\delta_{ij}\pi(\varphi_i)$.

Now we have $\varphi_i \in L^2(K)$, and Peter-Weyl implies

$$L^2(K) = \bigoplus_{i \in \hat{K}} (U_i \otimes U_i^*) \quad (3.2)$$

By definition, $\varphi_i$ lies in the image of $U_i \otimes U_i^*$, which is $i$-isotypic in $C(K)$. Therefore, for each $v \in V$, $\pi(\varphi_i)v \in V(i)$. We find that $\text{im} \pi(\varphi_i) \subseteq V(i)$.

Suppose $T_i : U_i \to V$ is $K$-invariant. Then $\pi_K(\varphi_i) \circ T_i = T_i \circ \tau_i(\varphi_i) = T_i$. So $\pi_K(\varphi_i)$ acts as the identity on $V(i)$, implying (2).

(3) follows from complete reducibility of finite dimensional representations of $K$.

$C^\infty(K)$ is dense in $L^2(K)$, and $\ell(K)$ acts on $C^\infty(K)$. The space of $K$-finite vectors is the algebraic direct sum of the $U_i \otimes U_i^*$, and therefore contained in $C^\infty(K)$.
Now choose an approximate identity \( \{ f_n \} \subseteq C^\infty(K) \), and approximate each \( f_n \) arbitrarily closely by an \( \ell(K) \)-finite, hence \( C^\infty \), function. We see that \( \pi(C^\infty(K))v \), with \( v \in V \), is dense in \( V \), implying (4).

Now suppose \( v \in V(i) \), and let \( \{ f_n \} \subseteq C^\infty(K) \) be an approximate identity. Then \( \pi_K(f_n)v \to v \), and therefore

\[
\pi_K(\ell(\varphi_i)f_n)v = \pi_K(\varphi_i)\pi_K(f_n)v \to \pi_K(\varphi_i)v = v \tag{3.3}
\]

showing (5).

**Corollary 3.2.** \( V(i) \) is closed in \( V \).

*Proof.* By (1) and (2), \( V(i) = \ker \left( \pi_K(\varphi_i) - 1 \right) \).

**Corollary 3.3.** Any irreducible representation of a compact Lie group on a complete locally convex Hausdorff topological vector space is finite dimensional.

A representation \((\pi, V)\) of \( G \) on a complete locally convex Hausdorff topological vector space is said to be admissible if \( V(i) \) is finite dimensional for every \( i \in \hat{K} \).

**Theorem 3.4** (Harish-Chandra). Any irreducible unitary representation is admissible.

(To be proven next time.)

**Remark.** 1. Harish-Chandra suggested (but did not formally conjecture) that any irreducible representation on a Banach space is admissible.

2. Soegel produced a counterexample.

In the following, \((\pi, V)\) will be assumed to be admissible.

**Corollary 3.5.** \( V_{K\text{-fin}} \subseteq V^\infty \). That is, every \( K \)-finite vector is also \( C^\infty \).

We had seen that \( g \) acts on \( V^\infty \) by a Lie algebra representation, so that \( V^\infty \) obtains the structure of a \( U(g) \)-module.

**Claim.** \( V_{K\text{-fin}} \) is a \( U(g) \)-module.

*Proof.* The map

\[
g \otimes V^\infty \xrightarrow{\mu} V^\infty
\]

\[
\zeta \otimes v \mapsto \frac{d}{dt} \pi(\exp t\zeta)v \bigg|_{t=0}
\tag{3.4}
\]

is \( G \)-invariant:

\[
\pi(g) \frac{d}{dt} \pi(\exp t\zeta)v \bigg|_{t=0} = \frac{d}{dt} \pi(g)\pi(\exp(t\zeta))\pi(g^{-1})\pi(g)v \bigg|_{t=0}
\tag{3.5}
\]

\[
= \frac{d}{dt} \pi(\exp(t(\text{Ad}g)\zeta))\pi(g)v \bigg|_{t=0}
\tag{3.6}
\]
which is the image under our map of $(\text{Ad} \, g) \zeta \otimes \pi(g)v = g(\zeta \otimes v)$.

In particular, this map is $K$-invariant. Now suppose $v$ is such that $\pi(G)v \subseteq W$ for $W$ a finite dimensional subspace of $V$. Then for $\zeta \in \mathfrak{g}$, $K\mu(\zeta \otimes v)$ lies in $\mu(\mathfrak{g} \otimes W)$, which is finite dimensional.

$V_{K\text{-fin}}$ is a $U(\mathfrak{g})$-module. Also $K$ acts, and $V_{K\text{-fin}}$ is a direct sum of irreducible finite dimensional representations of $K$, each occurring finitely often. So $K_C$ acts, by an “algebraic representation”, with each irreducible representation occurring only finitely often. These two actions are compatible in the following sense:

(i) The action of $\mathfrak{k}$ by differentiation of the $K_C$-action agrees with the restriction to $\mathfrak{k}$ of the $\mathfrak{g}$-action.

(ii) $\pi(k)\pi(\zeta)v = \pi(\text{Ad} \, k\zeta)\pi(k)v$. (This was previously proven even for $G$.)

Remark. If $G$, and hence $K$, is connected, then (i) implies (ii). But if $G$, and hence $K$, has finitely many connected components, it does not.

A $(\mathfrak{g}, K_C)$-module $M$ is a $U(\mathfrak{g})$-module, equipped with an algebraic action of $K_C$, such that the $\mathfrak{g}$ and $K_C$ actions are compatible in the sense of (i) and (ii). A $(\mathfrak{g}, K_C)$-module is said to be admissible if each irreducible $K_C$-module occurs at most finitely often.

We’ve shown that if $(\pi, V)$ is admissible, then $V_{K\text{-fin}}$ is an admissible $(\mathfrak{g}, K_C)$-module.

Let $(\pi, V)$ be a representation of $G$ (a complete locally convex Hausdorff topological vector space).

(a) If $V$ is a Banach space, we say $v \in V$ is an analytic vector if $g \mapsto \pi(g)v$ is a real analytic map from $G$ to $V$.

(b) In general, $v \in V$ is a weakly analytic vector if, for every $\varphi \in V'$, $g \mapsto \langle \varphi, \pi(g)v \rangle$ is a real analytic function.

**Theorem 3.6.** If $(\pi, V)$ is admissible, then every $v \in V_{K\text{-fin}}$ is weakly analytic. Here $G$ is reductive, with finitely many connected components, and usually connected.

**Proof.** Let $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the trace form of a faithful finite dimensional representation of $G$. Then:

(a) $B$ is defined over $\mathbb{R}$ with respect to $\mathfrak{g}_\mathbb{R}$ and $u_\mathbb{R}$ (a compact real form).

(b) $B$ is negative definite on $\mathfrak{k}_\mathbb{R}$ and positive definite on $p_\mathbb{R}$.

(c) $\mathfrak{g}_\mathbb{R} = \mathfrak{k}_\mathbb{R} \oplus p_\mathbb{R}$ as an orthogonal direct sum with respect to $B$. In particular, $B$ is nondegenerate.

(d) $B$ is $(\text{Ad} \, G_C)$-invariant.

Let $\{X_i\}$ be an orthonormal basis of $\mathfrak{g}$. Then

$$\Omega = \sum X_i^2 \in U(\mathfrak{g})$$

(3.7)
is independent of the particular choice of orthonormal basis. So for \( g \in G_C \), \((\text{Ad} g)\Omega = \Omega\). In particular, \((\text{ad} \zeta)\Omega = \Omega\) for \( \zeta \in \mathfrak{g} \), so \( \Omega \) lies in the center of \( U(\mathfrak{g}) \). \( \Omega \) is called the Casimir operator.

Analogously, we can define \( \Omega_K \in U(\mathfrak{t}) \subseteq U(\mathfrak{g}) \) (using an orthonormal basis of \( \mathfrak{t}_C \)), and \( \Omega_p \) (using an orthonormal basis of \( \mathfrak{p}_R \)). We have \( \Omega = \Omega_K + \Omega_p \). Now let

\[ D = \Omega - 2\Omega_K = \Omega_p - \Omega_K \]  

(3.8)

View \( U(\mathfrak{g}) \) as the algebra of left invariant linear differential operators on \( G \). At \( e \), the symbol of \( D \) is the image in \( S(\mathfrak{g}_R) \) (the associated graded algebra) under \( U(\mathfrak{g}_R) \rightarrow S(\mathfrak{g}_R) \). This symbol is positive definite. We conclude that \( -D \) is a strongly elliptic operator with real analytic coefficients.

Now suppose \( v \in V(i) \). \( \Omega_K \) acts on \( V(i) \) by a scalar, by Schur’s lemma. \( \Omega \) is \((\text{Ad} G_C)\)-invariant, so in particular \((\text{Ad} K)\)-invariant. This implies \( \Omega V(i) \subseteq V(i) \). Therefore \( D \) maps \( V(i) \) into \( V(i) \), finite dimensional vector spaces. Let \( q_i \) be the minimal monic polynomial for \( D \) acting on \( V(i) \). Then \( q_i(D) \) is an elliptic operator and \( q_i(D)v = 0 \). Therefore the function \( f : g \mapsto \langle \varphi, \pi(g)v \rangle \), for \( \varphi \in V' \), is such that

\[ q_i(D)f(g) = \langle \varphi, \pi(g)q_i(D)v \rangle = 0. \]  

(3.9)

By the regularity theorem for elliptic differential operators, \( f \in C^\omega(G) \).

\[ \square \]

**Theorem 3.7** (about Banach spaces). For functions defined on open subsets of \( \mathbb{R}^n \) (or a real analytic manifold), any weakly analytic function with values in a Banach space is analytic.

**Corollary 3.8.** If \( V \) is a Banach space and \((\pi,V)\) is admissible, then \( V_{K,\text{fin}} \subseteq V^\omega \), the space of analytic vectors.

See Lang, \( SL_2(\mathbb{R}) \), for proof.

Nelson (1959) proved that for any irreducible unitary representation \((\pi,V)\) of any Lie group \( G \), \( V^\omega \subseteq V \) is dense. Around the same time, for \( G \) reductive and \((\pi,V)\) a Banach representation which is irreducible and admissible, \( V^\omega \) was shown to be dense in \( V \).

If \((\pi,V)\) is a Banach representation, then

\[ V^\omega = \left\{ F : G \xrightarrow{C^\omega} V \mid F(g_1g_2) = \pi(g_1)F(g_2) \forall g_1, g_2 \in G \right\} \subseteq \left\{ F : G \xrightarrow{C^\omega} V \right\} \]  

(3.10)

as a closed subspace. The second subspace has a natural structure as a complete locally convex topological vector space.

**Theorem 3.9.** Suppose that \((\pi,V)\) is admissible and that \( G = G^0 \). Let \( M \subseteq V_{K,\text{fin}} \) be a \( U(\mathfrak{g})\)-invariant subspace. Then the closure of \( M \) in \( V \) is \( G\)-invariant.

**Proof.** Let \( M^\perp \) denote the annihilator of \( M \) in \( V' \), and \( M^{\perp\perp} \) denote the annihilator of \( M^\perp \) in \( V \). Then by the Hahn-Banach theorem, \( M^{\perp\perp} \) is the closure of \( M \). Now for \( v \in M \) and \( \varphi \in M^\perp \), the derivatives of all orders of the function \( g \mapsto \langle \varphi, \pi(g)v \rangle \) at \( g = e \) vanish. Therefore this function vanishes. Therefore \( \pi(g)v \in M^{\perp\perp} \).

\[ \square \]

**Corollary 3.10** (even for disconnected groups). If \( M \subseteq V_{K,\text{fin}} \) is a \((\mathfrak{g},K_C)\)-submodule, then the closure of \( M \) is \( G\)-invariant.
Proof. $K$ meets every connected component of $G$. \hfill \Box

**Corollary 3.11.** $M \leadsto \overline{M}$ (the closure of $M$) establishes a bijection between closed invariant subspaces of $V$ and $(\mathfrak{g}, K_C)$-submodules of $V_{K\text{-fin}}$, with inverse $W \leadsto W_{K\text{-fin}}$.

**Proof.** $W_{K\text{-fin}}$ is dense in $W$, therefore $\overline{W_{K\text{-fin}}} = W$.

Now suppose $i \in \hat{K}$. Then if $M \subseteq W_{K\text{-fin}}$ is a $(\mathfrak{g}, K_C)$-submodule of $V_{K\text{-fin}}$,

$$\pi_K(\varphi_i)(\overline{M}) = \pi_K(\varphi_i)(M). \quad (3.11)$$

Then upon taking the algebraic direct sum over all $i \in \hat{K}$, we find that

$$M = \bigoplus \pi_K(\varphi_i) \quad (3.12)$$

$$= (\overline{M})_{K\text{-fin}}. \quad (3.13)$$

We say that $(\pi, V)$ has **finite length** if every chain of closed $G$-invariant subspaces

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V \quad (3.14)$$

breaks off after finitely many steps.

**Corollary 3.12.** $(\pi, V)$ has finite length if and only if $V_{K\text{-fin}}$ is finitely generated over $U(\mathfrak{g})$.

**Proof.** $G/G^0$ is finite, so without loss of generality we may assume that $G = G^0$.

By Poincaré-Birkhoff-Witt, $U(\mathfrak{g})$ is a Noetherian ring. Therefore if $V_{K\text{-fin}}$ is finitely generated over $U(\mathfrak{g})$, it has a finite composition series. Now we may apply the previous corollary.

Conversely, suppose $(\pi, V)$ has finite length, say $\ell$. We proceed by induction on $\ell$.

For $\ell = 1$, take any nonzero $v_0 \in V_{K\text{-fin}}$. Then $U(\mathfrak{g}) \cdot v_0$ is a $U(\mathfrak{g})$-submodule, so its closure is $G$-invariant, and therefore must be $V$.

For the inductive step, suppose $V_{\ell-1}$ is a subspace of $V$ where $V_{\ell-1}$ has length $\ell-1$ and $V/V_{\ell-1}$ is irreducible. Then $(V_{\ell})_{K\text{-fin}}$ is finitely generated over $U(\mathfrak{g})$. Now choose $v_0 \in V_{K\text{-fin}}$ which does not lie in $(V_{\ell-1})_{K\text{-fin}}$. Then

$$(V_{\ell-1})_{K\text{-fin}} + U(\mathfrak{g}) \cdot v_0 \quad (3.15)$$

is a $U(\mathfrak{g})$-submodule, whose closure is strictly bigger than $V_{\ell-1}$. This closure must then be all of $V$. \hfill \Box

A **Harish-Chandra module** $M$ is a $(\mathfrak{g}, K_C)$-module which is:

(a) finitely generated over $U(\mathfrak{g})$.

(b) admissible, meaning that each $i \in \hat{K}$ occurs only finitely often in $M$.  

8
Morphisms of Harish-Chandra modules are \((g, K_C)\)-invariant linear maps.

So if \((\pi, V)\) is an admissible representation of \(G\), then \(V_{K_{\text{fin}}}\) is a Harish-Chandra module.

**Corollary 3.13.** \(V \mapsto V_{K_{\text{fin}}}\) is a covariant, faithful, exact functor from the category of admissible representations of finite length (and \(G\)-invariant continuous maps between them) and Harish-Chandra modules.

**Proof.** The functor is clearly covariant, and \(V_{K_{\text{fin}}} \subseteq V\) being dense implies it is faithful.

For exactness, suppose

\[
0 \to V_1 \to V_2 \to V_3 \to 0 \tag{3.16}
\]

is an exact* sequence of admissible representations of finite length. By universality of the projection operators \(\pi_K(\varphi_i)\), we obtain exact sequences

\[
0 \to \pi_{1,K}(\varphi_i)V_1 \to \pi_{2,K}(\varphi_i)V_2 \to \pi_{3,K}(\varphi_i)V_3 \to 0. \tag{3.17}
\]

Hence summing over \(i\),

\[
0 \to (V_1)_{K_{\text{fin}}} \to (V_2)_{K_{\text{fin}}} \to (V_3)_{K_{\text{fin}}} \to 0 \tag{3.18}
\]

is exact.

(*) By the open mapping theorem, which applies in our context, the quotient topology on \(V_3\) must agree with any intrinsic topology that \(V_3\) might have. So exactness is equivalent to topological exactness.

If \(M\) is a Harish-Chandra module and \(i \in \widehat{K}\), we define

\[
M(i) = \text{(integrated version of } \varphi_i)M \tag{3.19}
\]

which is finite dimensional. Let \(T : M_1 \to M_2\) be a morphism of Harish-Chandra modules. Then, for each \(i \in \widehat{K}\), \(T\) maps \(M_1(i)\) into \(M_2(i)\). Therefore we get a version of Schur’s lemma for Harish-Chandra modules:

**Corollary 3.14** (Schur’s lemma). *If \(M\) is an irreducible Harish-Chandra module and \(T : M \to M\) is a morphism, then \(T\) is a scalar multiple of the identity.*

**Proof.** \(T : M(i) \to M(i)\), for appropriate \(i \in \widehat{K}\), has an eigenvalue.

Define \(Z(g)\) to be the center of \(U(g)\). Recall that the Casimir operator \(\Omega\) lies in \(Z(g)\).

We will make a general hypothesis that even if \(G \neq G^0\), we have \(G_C = G_C^0\). That is, for \(g \in G\), \(\text{Ad } g : g \to g\) is an element of \(G_C^0/(\text{center of } G_C^0)\), the adjoint group of \(G_C\). In this case,

\[
Z(g) = \left\{ \zeta \in U(g) \middle| (\text{Ad } g)\zeta = \zeta \forall g \in G \right\}. \tag{3.20}
\]

**Corollary 3.15.** *If \(M\) is an irreducible Harish-Chandra module, then \(Z(g)\) acts by a character \(\chi : Z(g) \to \mathbb{C}\). \(\chi\) is called the infinitesimal character.*
In particular, irreducible admissible representations of $G$ have infinitesimal characters.

Two admissible representations of finite length are called *infinitesimally equivalent* if their Harish-Chandra modules are isomorphic.

As an example, $G = SU(1, 1)$ acts by fractional linear transformations on the unit disc $\Delta$, and hence acts on the space $\mathcal{O}(\Delta)$ of holomorphic functions on $\Delta$. $\mathcal{O}(\Delta)$ has a natural topology of a complete locally convex Hausdorff space (for example, by uniform convergence on compacts). Also holomorphic functions with boundary values being $C^k$, $C^\infty$, or $L^p$ for $p \geq 1$ have such topologies, and $G$ acts continuously in all cases. In all of these cases, the Harish-Chandra module is $\mathbb{C}[z]$. So all of these spaces are infinitesimally equivalent.

**Corollary 3.16.** Let $(\pi_1, V_1)$ and $(\pi_2, V_2)$ be two irreducible, admissible, unitary representations. If they are infinitesimally equivalent, then they are unitarily equivalent.

**Remark (preliminary).** Let $M$ be a Harish-Chandra module. Define $M'$ to be the $K$-finite part of the algebraic dual of $M$. Then $M'$ is a Harish-Chandra module.

**Proof.** Let $M = (V_1)_{K\text{-fin}}$. Using infinitesimal equivalence, we obtain two inner products on $M$, namely $(\ , \ )_1$ and $(\ , \ )_2$. These inner products are $K$-invariant, and $g$-invariant in the sense that

$$(\zeta u, v) + (u, \overline{\zeta} v) = 0$$

where the bar denotes complex conjugation with respect to $g_\mathbb{R}$. We know that the $K$-isotypic spaces are perpendicular with respect to both, so they are determined by their restrictions to the $M(i)$ for $i \in \hat{K}$. By the “spectral theorem”, there exists $T \in \text{End}(M)$ (that is, $TM(i) = M(i)$) such that $(u, v)_2 = (Tu, v)_1$. Now $T$ is $K$-invariant because the inner products are. For $u, v \in M$ and $\zeta \in g$, we have

$$(T \zeta u, v)_1 = (\zeta u, v)_2$$

$$= -(u, \overline{\zeta} v)_2$$

$$= -(Tu, \overline{\zeta} v)_1$$

$$= (\zeta Tu, v)_1$$

so $T$ is $g$-invariant as well. Hence $T$ must be a scalar. □

**Theorem 3.17.** If $(\pi, V)$ is irreducible and unitary, then it is admissible. More precisely, for each $i \in K$,

$$\dim V(i) \leq (\deg i)^2.$$  \hfill (3.26)

Suppose $V$ is a Banach space. The *strong topology* on $\text{End}(V)$ is characterized by $T_n \to T$ if and only if $T_n v \to T v$ for every $v \in v$.

**Remark.** We can define this topology on $\text{End}(V)$ for every complete locally convex Hausdorff topological vector space using nets.

A representation $(\pi, V)$ of $G$ is strongly irreducible if $\pi(C_c(G))$ is strongly dense in $\text{End}(V)$. Any strongly irreducible representation will be irreducible.

Theorem 3.17 follows from the following:
Proposition 3.18. Irreducible and unitary implies strongly irreducible.

and

Proposition 3.19. Strongly irreducible implies admissible and that for each \( i \in \hat{K} \),
\[
\dim V(i) \leq (\deg i)^2. \tag{3.27}
\]

Proof of Proposition 3.18. Let \( V \) be a Hilbert space. Then \( \text{End}(V) \) is a *-algebra, meaning there exists a conjugate linear anti-automorphism \( T \mapsto T^* \) of order 2. Also \( C_c(G) \) is a *-algebra under convolution, with \( f^*(g) = \overline{f(g^{-1})} \).

Claim. \( f \mapsto \pi(f) \) is a *-homomorphism.

Proof. We know it is an algebra homomorphism. Also,
\[
\begin{align*}
\left( \pi(f^*)u, v \right) &= \int_G \overline{f(g^{-1})} \left( \pi(g)u, v \right) dg \\
&= \int_G \overline{f(g)} \left( \pi(g^{-1})u, v \right) dg \\
&= \int_G \left( u, f(g)\pi(g)v \right) dg \\
&= \left( u, \pi(f)v \right)
\end{align*}
\]
so it is a *-homomorphism. \( \square \)

Now suppose \( A \subseteq \text{End}(V) \) is a subalgebra. Then
\[
A^c = \left\{ T \in \text{End}(V) \middle| Ta = aT \forall a \in A \right\} \tag{3.32}
\]
is called the commutant of \( A \). Observe that the strong closure of \( A \) has the same commutant as that of \( A \).

Theorem 3.20 (von Neumann). Suppose \( A \subseteq \text{End}(V) \) is a *-subalgebra. Then \( A^{cc} \) is the strong closure of \( A \).

Now suppose \( g \in G \) is given. Then let \( \{f_n\} \) be an approximate identity at \( g \), meaning \( \pi(f_n)v \rightarrow \pi(g)v \). So \( \{\pi(g)|g \in G\} \) is contained in the strong closure of \( \pi(C_c(G)) \). Suppose \( T \in A^c \). Then \( \pi(g)T = T\pi(g) \) for every \( g \in G \). Write
\[
T = \frac{1}{2}(T + T^*) - \frac{i}{2}(iT - iT^*) \tag{3.33}
\]
which expresses \( T \) as the sum of two self-adjoint operators.

So assume that \( T \) is self-adjoint. By the spectral theorem, we can write \( T = \int_\mathbb{R} \lambda dp_\lambda \) where each \( p_\lambda \in \text{End}(V) \) is an orthogonal projection, and \( \text{im} p_\lambda \supseteq \text{im} p_\mu \) if \( \lambda \geq \mu \). Any operator that commutes with \( T \) commutes with all of the \( p_\lambda \). Thus \( \pi(g)p_\lambda = p_\lambda \pi(g) \) for each \( \lambda \) and each \( g \), so \( \text{im} p_\lambda \) is a closed \( G \)-invariant subspace. Therefore there exists \( \lambda_0 \) such that
\( p_\lambda = \begin{cases} 
  0 & \lambda < \lambda_0 \\
  \text{id} & \lambda \geq \lambda_0.
\end{cases} \) \hspace{1cm} (3.34)

We find that \( T = \lambda_0 \cdot 1 \).

Hence any \( T \in \pi(C_c(G))^c \) is a multiple of the identity. Therefore the strong closure of \( \pi(C_c(G)) \) is the commutant of the set of scalars, which is all of \( \text{End}(V) \).

\[ \square \]

Proof of Proposition 3.19. Fix some \( i \in \hat{K} \). Then for \( f \in C_c(G) \),

\[
\pi_K(\varphi_i)\pi(f)\pi_K(\varphi_i) = \int_K \int_G \int_K \varphi_i(k_1)\varphi_i(k_2)f(g)\pi(k_1gk_2)\,dk_1\,dg\,dk_2 \tag{3.35}
\]

\[
= \int_K \int_G \int_K \varphi_i(k_1)f(k_1^{-1}gk_2^{-1})\varphi_i(k_2)\pi(g)\,dk_1\,dg\,dk_2 \tag{3.36}
\]

\[
= \pi\left( \ell(\varphi_i)r(\varphi_i)\right) f. \tag{3.37}
\]

We define

\[ C_c(G)_{i,i^*} = \left\{ f \in C_c(G) \mid \ell(\varphi_i)f = f = r(\varphi_i)f \right\}. \tag{3.38} \]

Suppose \( f_1, f_2 \in C_c(G)_{i,i^*} \). Then

\[
\ell(\varphi_i)\left( f_1 \ast f_2 \right) = \left( \ell(\varphi_i)f_1 \right) \ast f_2 \tag{3.39}
\]

\[
r(\varphi_i^*)\left( f_1 \ast f_2 \right) = f_1 \ast \left( r(\varphi_i^*)f_2 \right) \tag{3.40}
\]

so \( C_c(G)_{i,i^*} \) is a subalgebra of \( \text{End}(V) \).

For \( f \in C_c(G)_{i,i^*} \), \( f = \ell(\varphi_i)f \) implies \( \pi(f) = \pi_K(\varphi_i)\pi(f) \). Also \( f = r(\varphi_i^*) \) implies \( \pi(f) = \pi(f)\pi_K(\varphi_i) \). Hence \( \pi(C_c(G)_{i,i^*}) \) can be thought of as a subalgebra of \( \text{End}(V(i)) \).

Claim. If \( \pi \) is strongly irreducible, then \( \pi(C_c(G)_{i,i^*}) \) is strongly dense in \( \text{End}(V(i)) \).

Proof. Suppose \( T \in \text{End}(V(i)) \). By trivial extension to the other \( K \)-types, we can view \( T \) as lying in \( \text{End}(V) \). By strong irreducibility, there exists \( \{ f_n \} \subseteq C_c(G) \) such that \( \pi(f_n) \to T \) strongly. Therefore

\[
\pi_K(\varphi_i)\pi(f_n)|_{V(i)} \to \pi_K(\varphi_i)T = T \in \text{End}(V(i)). \tag{3.41}
\]

\[ \square \]

If \( A \) is an associative algebra over \( \mathbb{C} \) and \( \pi \) is a representation of \( A \) on a Banach space \( V \), then we say that \( \pi \) is strongly irreducible if \( \pi(A) \) is dense in \( \text{End}(V) \) in the strong topology. Then Proposition 3.19 follows from:

Lemma 3.21. Let \( A \) be an associative algebra over \( \mathbb{C} \), and \( \{ \pi_\sigma \mid \sigma \in S \} \) be a family of finite dimensional representations of \( A \) such that:
(i) \( \bigcap_{\sigma \in S} \ker \pi_\sigma = \{0\} \).

(ii) \( \dim \pi_\sigma \leq n \forall \sigma \).

Then any strongly irreducible representation of \( A \) is finite dimensional, of dimension at most \( n \).

**Lemma 3.22.** Suppose \( f \in C_c(G) \) and \( f \neq 0 \). Then there exists an irreducible finite dimensional representation \( (\pi, V) \) such that \( \pi(f) \neq 0 \).

**Lemma 3.23.** Any irreducible finite dimensional representation can be realized as a subrepresentation of a principal series representation.

**Lemma 3.24.** Let \( (\pi, V) \) be a principal series representation and \( i \in \hat{K} \). Then \( V(i) \) has dimension at most \( (\deg i)^2 \).

Applying these four lemmas to \( A = C_c(G)_{i,i^*} \) implies Proposition 3.19. The family \( S \) taken in Lemma 3.21 will be the family of principal series representations. \( \square \)

**Proof of Lemma 3.21.** For \( r > 0 \), define a polynomial \( P_r \) in non-commuting, associative variables \( X_1, \ldots, X_r \) by

\[
P_r(X_1, \ldots, X_r) = \sum_{\sigma \in S_n} \text{sgn}(\sigma)X_{\sigma_1}X_{\sigma_2} \cdots X_{\sigma_r}.
\]

We can evaluate \( P_r \) on any \( r \) elements of any associative ring.

Suppose \( r > n^2 \). Then for any \( T_1, \ldots, T_r \in \text{End}(\mathbb{C}^n) \), \( P_r(T_1, \ldots, T_r) = 0 \), because this is an alternating \( r \)-linear function on an \( n^2 \)-dimensional vector space. Let

\[
r(n) = \min \left\{ r \mid P_r(T_1, \ldots, T_r) \equiv 0 \text{ on } \text{End}(\mathbb{C}^n) \right\}.
\]

**Claim.** \( r(n) \) is strictly increasing.

**Proof.** Induct on \( n \) and let \( r = r(n) \). We know that there exist endomorphisms \( T_1, \ldots, T_{r-1} \in \text{End}(\mathbb{C}^n) \) such that \( P_{r-1}(T_1, \ldots, T_{r-1}) \neq 0 \). Pick \( S \in \text{Hom}(\mathbb{C}^n, \mathbb{C}) \) such that \( SP_{r-1}(T_1, \ldots, T_{r-1}) \neq 0 \). Identify \( \mathbb{C}^{n+1} \) with \( \mathbb{C}^n \oplus \mathbb{C} \) and define \( \tilde{T}_j \in \text{End}(\mathbb{C}^{n+1}) \) by

\[
\tilde{T}_j = \begin{cases} 
\begin{pmatrix} T_j & O \\ O & 0 \end{pmatrix} & 1 \leq j \leq r-1 \\
\begin{pmatrix} O & O \\ O & S \end{pmatrix} & j = r
\end{cases}
\]

where we view \( S \) as a \( 1 \times n \) matrix. Now \( P_r(\tilde{T}_1, \ldots, \tilde{T}_r) \neq 0 \), so \( r(n + 1) > r \). \( \square \)

Now suppose \( \dim V(i) > n \) (or is infinite). Then:

**Claim.** \( \text{End}(V(i)) \) contains a subalgebra isomorphic to \( \text{End}(\mathbb{C}^{n+1}) \).
Proof. This is clear if \( \dim V(i) \) is finite. Choose a linearly independent subset \( \{ \varphi_\alpha \}_{1 \leq \alpha \leq n+1} \subseteq V(i)' \), and choose \( v_1, \ldots, v_{n+1} \) such that \( \langle \varphi_\alpha, v_\beta \rangle = \delta_{\alpha\beta} \). Then \( V(i) = U_{n+1} \oplus U \) (topologically) with \( U_{n+1} \) equal to the linear span of the \( v_\alpha \) and \( U \) equal to the intersection of the kernels of the \( \varphi_\beta \). Now \( \text{End}(V(i)) \supseteq \text{End}(U_{n+1}) \).

Take the polynomial \( P_r \) for \( r = r(n) \). There exist \( T_1, \ldots, T_r \in \text{End}(\mathbb{C}^{n+1}) \) (viewed as elements of \( \text{End}(V(i)) \)) such that

\[
P_r(T_1, \ldots, T_r) \neq 0. \tag{3.45}
\]

Therefore, by density, there exist \( f_1, \ldots, f_r \in C_c(G)_{i,is} \) such that

\[
P_r(\pi(f_1), \ldots, \pi(f_r)) \neq 0. \tag{3.46}
\]

Remark. If \( T_n \to T \) and \( S_n \to S \) in the strong topology, then

\[
\lim_{m \to \infty} \left( \lim_{n \to \infty} S_m T_n \right) = ST. \tag{3.47}
\]

Now \( 0 \neq \pi[P_r(f_1, \ldots, f_r)] \). On the other hand,

\[
\pi_\sigma \left[ P_r(f_1, \ldots, f_r) \right] = P_r(\pi_\sigma(f_1), \ldots, \pi_\sigma(f_r)) \tag{3.48}
\]

\[
= 0 \tag{3.49}
\]

since each \( \pi_\sigma(f_i) \) lies in \( \text{End}(\mathbb{C}^m) \) for some \( m \leq n \). Thus \( P_r(f_1, \ldots, f_r) \) is an element of \( \bigcap_\sigma \ker \pi_\sigma \), a contradiction.

Remark. Suppose \( V \) is a Banach space and \( \{ T_n \} \subseteq \text{End}(V) \) is a strongly compact sequence. Then the uniform boundedness principle implies \( \{ \| T_n \| \} \) is bounded. Hence composition of operators is continuous in the strong topology.

If \( V \) is a complete locally convex Hausdorff topological vector space, the argument given above works, but we need to require that the \( P_r(X_1, \ldots, X_r) \) are linear in each variable.

Proof of Lemma 3.22. \( G \subseteq G_C \subseteq GL(N, \mathbb{C}) \subseteq \text{End}(\mathbb{C}^N) \). By restriction, we can view the matrix entries of \( \text{End}(\mathbb{C}^N) \) as functions on \( G \). Let \( A \) be the algebra generated by the matrix entries and their complex conjugates. Then \( A \) is an algebra of continuous functions on \( G \) which separates points and is closed under complex conjugation.

By Stone-Weierstrass, \( f \) can be approximated arbitrarily closely on its support by functions in \( A \). Hence there exists a function \( h \in A \) such that \( h \) is very close to \( f^* \). (Recall that \( f^*(g) = \overline{f(g^{-1})} \).) By making \( h \) close enough, we can arrange that
\[
0 \neq \int_G f(g) h(g^{-1}) \, dg \\
= \int_G f(g) h(g^{-1}k) \, dg \bigg|_{h=e} \\
= \ell(f)h(e)
\]

so that \(\ell(f)h \neq 0\). But \(h \in A(d)\), the space of polynomials in the matrix entries and their complex conjugates of degree at most \(d\). This is a finite dimensional algebraic representation of \(G\). As \(G\) is reductive, this representation is a direct sum of irreducibles. So at least one of these must be nonzero on \(f\). \(\square\)

### 4 Iwasawa Decomposition

Recall that we may write \(g_R = t_R \oplus p_R\), and that \(G\) is diffeomorphic to \(K \times p_R\), via

\[
k \exp \zeta \leftrightarrow (k, \zeta)
\]

This is the "Cartan decomposition".

Now let \(a_R \subseteq p_R\) be a maximal abelian subspace, i.e. \(a_R\) is maximal subject to \([a_R, a_R] = 0\). We have \(a_R \subseteq p_R \subseteq iu_R\), so \(a_R\) acts semisimply with real eigenvalues. Then \(A = \exp a_R\) is a closed, connected, abelian, diagonalizable subgroup of \(G\).

Consider \(Z_{g_R}(a_R)\). We claim that

\[
Z_{g_R}(a_R) = Z_{t_R}(a_R) \oplus a_R
\]

because \(a_R\) is \(\theta\)-invariant. Let \(m_R = Z_{t_R}(a_R)\), so that \(Z_{g_R}(a_R) = m_R \oplus a_R\). Now let

\[
M = Z_K(a_R) = Z_K(A)
\]

a (possibly not connected) group with Lie algebra \(m_R\). \(M\) and \(A\) commute, \(M\) lies in \(K\), and \(A\) lies in \(\exp p_R\), so \(M \cap A = \{e\}\). Again by \(\theta\)-invariance, if \(k \exp \zeta\) commutes with \(A\), then so do \(k\) and \(\zeta\), so \(Z_G(A) = MA\), with \(MA\) a direct product.

We know that \(a_R\), and hence \(a = \mathbb{C} \otimes_R a_R\), acts, via \(\text{ad}\), semisimply (and with real eigenvalues in the case of \(a_R\)), so

\[
g = m \oplus a \oplus \bigoplus_{\alpha \in \Phi(a,g)} g^\alpha
\]

where \(\Phi(a,g)\) is the root system of \((a,g) \subseteq a_R^* \setminus \{0\}\), which in turn sits inside \(a^* \setminus \{0\}\). Here

\[
g^\alpha = \{ \xi \in g \mid [\eta, \xi] = \langle \alpha, \eta \rangle \xi \forall \eta \in a \}
\]
is the $\alpha$-root space for $\mathfrak{a}$ acting on $\mathfrak{g}$. These are restricted roots. $\dim \mathfrak{g}^\alpha$ may be greater than 1, and if $\alpha$ is a root, $\pm 2\alpha$ may also be roots (in addition to $\pm \alpha$, but there will not be any other roots proportional to $\alpha$).

Other properties:

1. $\Phi(\mathfrak{a}, \mathfrak{g}) = -\Phi(\mathfrak{a}, \mathfrak{g})$, because $\theta = -1$ on $\mathfrak{a}$.

2. If $\alpha, \beta \in \Phi(\mathfrak{a}, \mathfrak{g})$, then $2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$. (To get $(\beta, \beta)$, take the form $B$, restrict to $\mathfrak{a}_R$, and dualize to get an inner product on $\mathfrak{a}_R^*$.)

3. If $\alpha, \beta \in \Phi(\mathfrak{a}, \mathfrak{g})$, then $\beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha$, the reflection of $\beta$ about $\alpha^\perp$, is also an element of $\Phi(\mathfrak{a}, \mathfrak{g})$.

Choose a connected component $C^+$ of $\mathfrak{a}_R^* \setminus \bigcup_{\alpha \in \Phi(\mathfrak{a}, \mathfrak{g})} \alpha^\perp$. $C^+$ will be called the positive Weyl chamber. Pick $\zeta_0 \in C^+$, and define

$$\Phi^+(\mathfrak{a}, \mathfrak{g}) = \left\{ \alpha \in \Phi(\mathfrak{a}, \mathfrak{g}) \mid \langle \alpha, \zeta_0 \rangle > 0 \right\}$$

(4.6)

the set of positive roots. These have the following properties:

1. $\Phi(\mathfrak{a}, \mathfrak{g}) = \Phi^+(\mathfrak{a}, \mathfrak{g}) \sqcup -\Phi^+(\mathfrak{a}, \mathfrak{g})$.

2. If $\alpha, \beta \in \Phi^+(\mathfrak{a}, \mathfrak{g})$ and $\alpha + \beta \in \Phi(\mathfrak{a}, \mathfrak{g})$, then $\alpha + \beta \in \Phi^+(\mathfrak{a}, \mathfrak{g})$.

3. $\Phi^+$ depends only on the choice of $C^+$, not on the choice of $\zeta_0 \in C^+$.

Then, with

$$\mathfrak{n}_+ = \bigoplus_{\alpha \in \Phi^+(\mathfrak{a}, \mathfrak{g})} \mathfrak{g}^\alpha$$

(4.7)

$$\mathfrak{n}_- = \bigoplus_{\alpha \in \Phi^+(\mathfrak{a}, \mathfrak{g})} \mathfrak{g}^{-\alpha}$$

(4.8)

(observe that $\mathfrak{n}_- = \theta(\mathfrak{n}_+)$), we have

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-.$$  

(4.9)

Also $MA$ acts on each root space, hence $\mathfrak{n}_+$ and $\mathfrak{n}_-$. $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are nilpotent subalgebras of $\mathfrak{g}$, defined over $\mathbb{R}$. Let $N_{\pm} = \exp(\mathfrak{n}_{\pm} \cap \mathfrak{g}_\mathbb{R})$, closed connected unipotent subgroups of $G$.

In the following, we will take $N = N_-$. (Many authors use $N_+$ instead.) Then:

**Proposition 4.1** (Iwasawa Decomposition). The multiplication map

$$K \times A \times N \to G$$

(4.10)

is a diffeomorphism.
Remark. $K$ is unique up to $G$-conjugacy. Once $A$ is known, $N$ is unique up to the action of $W$. $A$ corresponds bijectively with $\mathfrak{a}_\mathbb{R}$, which is determined by $\mathfrak{p}_\mathbb{R}$, and is unique up to $K/M$-conjugacy. This is because $G = KAK$.

As an example, take $G = GL(n, \mathbb{R})$, $K = O(n)$, and

$$
A = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \middle| a_j > 0 \right\} \quad (4.11)
$$

$$
M = \left\{ \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix} \middle| \epsilon_j = \pm 1 \right\} \quad (4.12)
$$

$$
N = \left\{ \begin{pmatrix} 1 \\ \ast \\ \vdots \\ \ast \end{pmatrix} \right\} \quad (4.13)
$$

The proof of the Iwasawa decomposition for $G$ can be reduced to the case of $GL(n, \mathbb{R})$.

Remark. We have similar results for algebraic groups defined over $\mathbb{Q}$, but then $M$, rather than being compact, is anisotropic.

Let $W = N_K(A)/Z_K(A)$, a finite group called the Weyl group. $W$ acts on $\mathfrak{a}_\mathbb{R}$, on $\Phi(\mathfrak{a}, g)$, and permutes the Weyl chambers simply transitively. So the choice of $C^+$ was immaterial modulo the action of $W$.

$K$ acts by conjugation on $\mathfrak{p}_\mathbb{R}$, and $M$ centralizes $\mathfrak{a}_\mathbb{R}$, so we get a map

$$
K/M \times \mathfrak{a}_\mathbb{R} \longrightarrow \mathfrak{p}_\mathbb{R}
$$

$$(kM, \zeta) \longmapsto k\zeta k^{-1} \quad (4.14)
$$

This map is surjective, and generically $\#W$-to-one.

**Corollary 4.2.** $G = KAK = KA^+K$, where

$$
A^+ = \exp C^+ \ast \quad (4.15)
$$

$$
C^+ \ast = \left\{ \zeta \in \mathfrak{a}_\mathbb{R} \middle| \langle \alpha, \zeta \rangle \geq 0 \forall \alpha \in \Phi^+(\mathfrak{a}, g) \right\} \quad (4.16)
$$

The decomposition $G = KAK$ is generically $\#W$-to-one modulo $M$, and $G = KA^+K$ is generically one-to-one modulo $M$.

**Proof.** $G \cong K \times \mathfrak{p}_\mathbb{R}$. □
Now suppose that \( H \subseteq G \) is a closed subgroup. Let \((\tau, E)\) be a representation of \( H \), either finite dimensional, or on a complete locally convex Hausdorff topological vector space. Define

\[
\mathcal{E} = G \times_H E = \left( G \times E \right) / \left( (gh, e) \sim (g, \tau(h)e) \right) \quad (4.17)
\]

We get a projection map \( \mathcal{E} \to G/H \). This is a vector bundle over \( G/H \) with typical fiber \( E \), and a \( G \)-action compatible with the \( G \)-action on \( G/H \). Such a bundle is called a homogeneous vector bundle with fiber \( E \) at the identity coset \( eH \). Upon unraveling the definition, we find that

\[
C^\infty(G/H, E) \cong \left\{ F : G \xrightarrow{C^\infty} E \mid F(gh) = \tau(h^{-1})F(g) \forall g \in G, h \in H \right\} \quad (4.18)
\]

\[
\cong \left( C^\infty(G) \otimes \mathbb{C} \right)^H \quad (4.19)
\]

where \( H \) acts on \( E \) via \( \tau \) and an \( C^\infty(G) \) via \( r \).

Here are some special cases:

(a) If \( E = \mathfrak{g}/\mathfrak{h} \), the complexified tangent space to \( G/H \) at \( eH \), then the associated homogeneous vector bundle is the complexified tangent bundle, which has a real structure.

(b) \( \Lambda^{\top}(\mathfrak{g}/\mathfrak{h})^* \) is the homogeneous vector bundle whose sections are \( C^\infty \) forms of top degree. Again, this carries a real structure.

(c) Suppose \( \Lambda^{\top}(\mathfrak{g}_R/\mathfrak{h}_R)^* \) has a square root. The associated real line bundle is the bundle of "half-forms".

In this case, complexifying gives a 1-dimensional representation of \( H \) on \( \mathbb{C}_{hf} \). Let \( \mathcal{L}_{hf} \) be the associated homogeneous line bundle. Then, if \((\tau, E)\) is a unitary representation of \( H \), we get a \( G \)-invariant inner product on \( C^\infty(G/H, \mathcal{E} \otimes \mathcal{L}_{hf}) \) by integration. The completion is \( L^2(G/H, \mathcal{E} \otimes \mathcal{L}_{hf}) \), the space of \( L^2 \)-sections, and \( G \) acts unitarily.

Now suppose the role of \( H \) is played by \( P = MAN \), a minimal parabolic subgroup. Note that \( MA \) acts on the various root spaces \( \mathfrak{g}^\alpha \), hence on \( \mathfrak{n} \), hence normalizes \( N \). So \( P \) is the semidirect product of the reductive group \( MA \) with the normal unipotent subgroup \( N \).

Recall that \( G = KAN \). Then \( G/P \cong K/M \) as spaces with \( K \)-action. Let \((\tau, E)\) be an irreducible (hence finite dimensional) representation of \( M \). It has an essentially unique \( M \)-invariant inner product. Take \( \mathcal{E} \to G/P \) the associated hermitian vector bundle, with \( A \) and \( N \) acting trivially on \( E \), by definition.

Now choose \( \lambda \in \mathfrak{a}^* \). Then \( e^\lambda : A \to \mathbb{C}^* \), where for \( \zeta \in \mathfrak{a}_R \),

\[
e^\lambda(\exp \zeta) = e^{(\lambda, \zeta)} \quad (4.20)
\]

If \( \lambda \in i\mathfrak{a}_R^* \), then \( e^\lambda \) is unitary, as \( |e^\lambda| = 1 \). Let \( \mathcal{L}_\lambda \to G/P \) be the associated line bundle.

Define \( \rho \in \mathfrak{a}^* \) by

\[
2\rho = \sum_{\alpha \in \Phi^+(\alpha, \mathfrak{g})} (\dim \mathfrak{g}^\alpha) \alpha. \quad (4.21)
\]
Then $e^{2\rho} : A \to \mathbb{C}^*$ is the action of $A$ on $\Lambda^\top \mathfrak{n}_+ = \Lambda^\top \mathfrak{g}/\mathfrak{p}$. So the line bundle $\mathcal{L}_{-2\rho}$ is the line bundle of forms of top degree. So $\mathcal{L}_{-\rho}$, defined in complete analogy to $\mathcal{L}_\lambda$, is the line bundle of half-forms. Implicitly, we have extended $e^\rho$ from $A$ to $MA$ with $M$ acting trivially. But traditionally, for very good reason, $e^\rho$ has a canonical meaning as a character of $MA$ which is trivial only on $M^0$, and possibly not trivial on other components of $M$. In the following, we’ll use the notation $\mathcal{L}_{-\rho}$ in the traditional sense.

For example, consider $G = SL(2, \mathbb{R})$, $K = SO(2)$, and

$$a_{\mathbb{R}} = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \big| x \in \mathbb{R} \right\} \quad (4.22)$$

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \big| a \in \mathbb{R}^* \right\} \quad (4.23)$$

$$M = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \quad (4.24)$$

$$N = \left\{ \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \right\} \quad (4.25)$$

Then $\rho$ takes $\left( \begin{smallmatrix} x & 0 \\ 0 & -x \end{smallmatrix} \right)$ to $\frac{x}{2}$. Now the naive definition of $e^\rho$ would map $\left( \begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix} \right)$ to $|a|$, while a better definition would map it to $a$.

Define the space

$$V_{E,\lambda}^\infty = C^\infty(G/P, \mathcal{E} \otimes \mathcal{L}_{\lambda-\rho}) \quad (4.26)$$

on which $G$ acts. If $\lambda \in i\mathfrak{a}_{\mathbb{R}}^*$, we get a $G$-invariant inner product, and hence can then define

$$V_{E,\lambda} = L^2(G/P, \mathcal{E} \otimes \mathcal{L}_{\lambda-\rho}). \quad (4.27)$$

This is the principal series, respectively the unitary principal series if $\lambda \in i\mathfrak{a}_{\mathbb{R}}^*$.

Recall that $G/P \cong K/M$. We have

$$(V_{E,\lambda}^\infty)_{K\text{-fin}} = C^\infty(K/M, \mathcal{E} \otimes \mathcal{L}_{\lambda-\rho})_{K\text{-fin}} \quad (4.28)$$

$$= \left( C^\infty(K) \otimes E \otimes \mathbb{C}_{-\rho} \right)_{K\text{-fin}}^M \quad (4.29)$$

$$= \bigoplus_{i \in \hat{K}} V(i) \otimes \left( V(i)^* \otimes E \otimes \mathbb{C}_{-\rho} \right)^M. \quad (4.30)$$

We would like to know how often $V(i)$ occurs in this Harish-Chandra module. It occurs no more often than the multiplicity of the representation $E \otimes \mathbb{C}_{-\rho}$ in $V(i)^*|_M$, hence at most with multiplicity $\dim V(i)$. This implies Lemma 3.24.

Let $W$ be an irreducible finite dimensional representation space for $G$. Then $A$ acts diagonalizably, so we may write
where $a$ acts on $W^\mu$ by $\mu$. $\mu$ is a (restricted) weight if $W^\mu \neq 0$. The action map $g \otimes W \to W$ is $G$-invariant, so $g^a \otimes W^\mu \to W^{\mu+\alpha}$. The roots lie in an open convex cone, so there must exist a $\mu_0$ such that $W^{\mu_0} \neq 0$ but $W^{\mu_0+\alpha} = 0$ for every $\alpha \in \Phi^+(a,g)$. Such a $\mu_0$ is a highest weight, and is unique by irreducibility. In fact, $\mu_0$ is the restriction to $a$ of the highest weight of $W$ with respect to a Cartan containing $a$ and a positive root system which extends $\Phi^+(a,g)$.

$M$ and $A$ commute, hence $W^{\mu_0}$ is $MA$-invariant. Let $E = W^{\mu_0}$. (Notation is slightly different from the previous notation because $E$ is now an $MA$-representation.) Let $p : W \to E$ be the unique $MA$-invariant projection. For $v \in W$, define $F_v : G \to E$ by $F_v(g) = p(g^{-1}v)$. Then

\[
F_v(g_{man}) = p((ma)^{-1}g^{-1}v) = p((ma)^{-1}g^{-1}v) = (ma)^{-1}F_v(g)
\]  (4.32)

(since the action of $n$ will lower the weights and hence lie in $\ker p$). So $F_v$ defines a section of $E \to G/P$. $v \rightsquigarrow F_v$ defines $W \to C^\infty(G/P,E)$. Meanwhile, for $g, h \in G$ and $v \in W$,

\[
F_{gv}(h) = p((ma)^{-1}g^{-1}v) = p((g^{-1}h)^{-1}v) = F_v(g^{-1}h) = (\ell(g)F_v)(h)
\]  (4.33)

so the map $W \hookrightarrow C^\infty(G/P,E)$ is $G$-invariant. (It is injective by irreducibility and the fact that it is obviously nonzero.) We obtain Lemma 3.23.

Now consider $G$, $P$, and $E$ as usual, and $V^\infty_E$ the corresponding induced representation, but this time without the $\rho$ shift. Let $W$ be the associated Harish-Chandra module.

**Theorem 4.3** (Casselman Reciprocity). $\text{Hom}_{(g,K\mathbb{C})}(W,(V^\infty_E)_{K\text{-fin}}) \cong \text{Hom}_{(m@a,M)}(W/nW,E)$.

**Proof.** This is the same as the proof of the embedding of finite dimensional irreducible representations into principal series, except with the infinitesimal actions of $G$ and $P$. \(\square\)

**Theorem 4.4** (Casselman-Osborne). Let $W$ be a Harish-Chandra module. Then the $n$-homology groups of $W$, in particular $H_0(n,W) = W/nW$, are finite dimensional.

The proof of this theorem is algebraic.

**Theorem 4.5** (Casselman). Let $(\pi,V)$ be an irreducible admissible representation. Then $H_0(n,V_{K\text{-fin}}) \neq 0$.

This proof is analytic.
Corollary 4.6. Let $(\pi, V)$ be irreducible and admissible, and $i \in \hat{K}$. Then $\dim V(i) \leq (\deg i)^2$.

Proof. More generally, the Harish-Chandra module of an irreducible admissible representation can be embedded into the Harish-Chandra module of a principal series representation.

Now suppose $E$ is irreducible. Then there exists an $M$-invariant inner product. Recall that $G/P \cong K/M$ and that

$$V_E^\infty \cong \left(C^\infty(K) \otimes E\right)^M$$

(4.39)

where $M$ acts on $C^\infty(G)$ via $r$. We can complete to get $\left(L^2(K) \otimes E\right)^M$. For $g \in G$ “small”, $\ell(g)$ distorts the $L^2$ norm by a “small” amount, so $G$ acts on this space continuously, although not unitarily in general.

Corollary 4.7. Any irreducible admissible representation of $G$ is infinitesimally equivalent to a representation on a Hilbert space.

We shall define the character of any irreducible admissible representation of $G$ on a Hilbert space, which will depend only on the infinitesimal equivalence class. Hence this defines the character of any irreducible admissible representation.

5 Hilbert Space Operators

Suppose $V$ is a Hilbert space, initially separable. $T \in \text{End}(V)$ is of Hilbert-Schmidt class if for some orthonormal basis $\{v_k\}$,

$$\sum_{k,\ell} \left| (Tv_k, v_\ell) \right|^2 < \infty.$$  

(5.1)

For an equivalent definition, consider $T \in \text{End}(V)$ with finite dimensional image. Then $\ker T$ has finite codimension, so there exists a finite dimensional subspace $V_0 \subseteq V$ such that $T = 0$ on $V_0^\perp$ and $T(V_0) \subseteq V_0$. Then the sum $\sum_{k,\ell} \left| (Tv_k, v_\ell) \right|^2$ does not depend on the choice of orthonormal basis. So we may define the Hilbert-Schmidt class to be the completion with respect to the Hilbert-Schmidt inner product of finite rank operators. This class is independent of choice of orthonormal basis.

Remark. We have

$$\sum_{k,\ell} \left| (Tv_k, v_\ell) \right|^2 = \sum_k \left( \sum_\ell \left| (Tv_k, v_\ell) \right|^2 \right)$$

(5.2)

$$= \sum_k \|Tv_k\|^2$$

(5.3)

So the Hilbert-Schmidt inner product is
\[(T_1, T_2)_{HS} = \sum_k (T_1 v_k, T_2 v_k) \] (5.4)

which converges absolutely when \(T_1\) and \(T_2\) are Hilbert-Schmidt.

**Lemma 5.1.**

1. \((T_1, T_2)_{HS} = (T_2^*, T_1^*)_{HS}.

2. For \(T \in \text{End}(V)_{HS}\) and \(A \in \text{End}(V)\), \(AT\) and \(TA\) are Hilbert-Schmidt operators, with
\[\|AT\|_{HS} \leq \|A\|\|T\|_{HS} \text{ and } \|TA\|_{HS} \leq \|T\|_{HS}\|A\|\).

**Proof.**

1. We have
\[(T_1, T_2)_{HS} = \sum_k (T_1 v_k, T_2 v_k) \] (5.5)
\[= \sum_{k, \ell} (T_1 v_k, v_\ell)(v_\ell, T_2 v_k) \] (5.6)
\[= \sum_{k, \ell} (v_k, T_1^* v_\ell)(T_2^* v_\ell, v_k) \] (5.7)
\[= \sum_{\ell} (T_2^* v_\ell, T_1^* v_\ell). \] (5.8)

2. We have
\[\|AT\|_{HS}^2 = \sum_k \|ATv_k\|^2 \] (5.9)
\[\leq \|A\|^2 \sum_k \|Tv_k\|^2. \] (5.10)

For \(TA\), \(\|TA\|_{HS}^2 = \|A^* T^*\|_{HS}^2\).

We say that \(T \in \text{End}(V)\) is a **trace class** operator if it is a finite linear combination of operators which can be expressed as a composition \(S_1 S_2\) with \(S_1\) and \(S_2\) both Hilbert-Schmidt. If \(T = S_1 S_2\) with \(S_1\) and \(S_2\) Hilbert-Schmidt, then we define
\[\text{tr} T = (S_2, S_1^*)_{HS}. \] (5.11)

For \(T\) in trace class, we extend the trace by linearity. Without loss of generality, suppose \(T = S_1 S_2\). Then
\[ \text{tr} T = (S_2, S_1^*)_{HS} \]
\[ = \sum_k (S_2 v_k, S_1^* v_k) \]
\[ = \sum_k (S_1 S_2 v_k, v_k) \]
\[ = \sum_k (Tv_k, v_k). \]

**Lemma 5.2.** Suppose \( T \) is of trace class and \( A \in \text{End}(V) \). Then \( AT \) and \( TA \) are of trace class, and \( \text{tr} AT = \text{tr} TA \).

*Proof.* Without loss of generality, we may suppose \( T = S_1 S_2 \). Then \( AT = (AS_1) S_2 \), and

\[ \text{tr} AT = (S_2, (AS_1)^*)_{HS} = (AS_1, S_2^*)_{HS} = \text{tr}(S_2 AS_1). \]

Meanwhile \( TA = S_1(S_2A) \), and a similar argument as above shows that \( \text{tr} TA = \text{tr}(S_2 AS_1) \).

**Corollary 5.3.** If \( T \) is of trace class and \( A \in \text{End}(V) \) is such that \( A \) has a bounded inverse, then \( \text{tr} ATA^{-1} = \text{tr} T \).

*Remark.* If \( T \in \text{End}(V) \) has the property that for every orthonormal basis \( \{v_k\} \), \( \sum(T v_k, v_k) \) converges absolutely, then \( T \) is of trace class. However, this is far less useful as a potential definition.

Suppose \( (\pi, V) \) is an irreducible admissible representation on a Hilbert space. By the uniform boundedness principle, there exists a constant \( C \) such that \( \parallel \pi(k) \parallel \leq C \) for every \( k \in K \). But also

\[ 1 = \parallel \pi(1_V) \parallel = \parallel \pi(k) \pi(k^{-1}) \parallel \leq \parallel \pi(k) \parallel \parallel \pi(k^{-1}) \parallel \]
\[ \parallel \pi(k) \parallel \geq C^{-1} \]

so \( \parallel \pi(k) \parallel \geq C^{-1} \) as well. So by averaging the inner product over \( K \), we get a new inner product which is \( K \)-invariant, but which defines the same topology on \( V \). So we may (and will) suppose that \( K \) acts unitarily on \( V \).

We know that \( \dim V(i) \leq (\deg i)^2 \). From the theory of finite dimensional representations of \( K \), we know:

1. \( \Omega_K \) acts by a scalar \( \omega_i \) on \( i \in \hat{K} \).
2. \( \omega_i = (\mu_i + \rho_K, \mu_i + \rho_K) - (\rho_K, \rho_K) \) where \( \rho_K \) is half the sum of the positive roots and \( \mu_i \) is the highest weight. We have

\[ (\mu_i + \rho_K, \mu_i + \rho_K) - (\rho_K, \rho_K) = (\mu_i, \mu_i + 2\rho_K) \geq \parallel \mu_i \parallel^2. \]

3. \( \deg i \) is bounded by a polynomial in \( \parallel \mu_i \parallel \) (by the Weyl dimension formula).
4. For \( k \gg 0 \),

\[ \sum_{i \in \hat{K}} \parallel \mu_i \parallel^{-k} < \infty. \]
(1) through (4), along with the unitarity of the $K$-action, imply that there exists a positive integer $r$ such that

$$
\sum_k \left\| \left( 1 + \pi(\Omega_K) \right)^{-r} v_k \right\|^2 < \infty. \tag{5.20}
$$

Here $\{v_k\}$ is an orthonormal basis obtained by putting together orthonormal bases of the various $V(i)$. Therefore $\left( 1 + \pi(\Omega_K) \right)^{-r}$ is Hilbert-Schmidt.

Suppose $f \in C_c^\infty(G)$, $v \in V^\infty$, and $\zeta \in g$. Then

$$
\pi \left( \ell(\zeta) f \right) v = \pi(\zeta) \pi(f) v \tag{5.21}
$$

and so

$$
\pi(f) v = \left( 1 + \pi(\Omega_K) \right)^{-2r} \pi \left( \left( 1 + \pi(\Omega_K) \right)^{2r} \ell(f) \right) v. \tag{5.22}
$$

However, the right hand side forms a trace class operator (a product of two Hilbert-Schmidt operators and a bounded operator). Therefore $\pi(f)$ is of trace class. This proves that for $f \in C_c^\infty(G)$, $\text{tr} \pi(f)$ is well-defined, and the map $f \mapsto \text{tr} \pi(f)$ is a distribution on $G$. This is because $\text{tr} \pi(f)$ can be bounded in terms of the support of $f$ and the derivatives of $f$ up to order $4r$.

We say a distribution on a $C^\infty$ manifold is a continuous linear function on the space of smooth compactly supported measures, i.e. a “generalized function”.

Letting $\Theta_\pi(f) = \text{tr} \pi(f)$, we have

$$
\text{tr} \pi(f) = \int_G \Theta_\pi(f) \, dg \tag{5.23}
$$

with $f \, dg$ a compactly supported Haar measure, and $\Theta_\pi$ independent of the choice of normalization of Haar measure. We have

$$
\text{tr} \pi(f) = \sum_{\ell} \left( \pi(f) v_\ell, v_\ell \right) \tag{5.24}
$$

$$
= \sum_{\ell} \int_G \left( \pi(g) v_\ell, v_\ell \right) f(g) \, dg. \tag{5.25}
$$

Informally, we write

$$
\Theta_\pi(g) = \sum_{\ell} \left( \pi(g) v_\ell, v_\ell \right) \tag{5.26}
$$

and this is precise in the sense of weak* convergence.

**Theorem 5.4.** $\Theta_\pi$ is an “invariant eigendistribution”, i.e. is conjugation invariant and an eigendistribution under the action of $Z(g)$. Moreover, $\Theta_\pi$ depends only on the infinitesimal equivalence class of $\pi$. 

24
Proof. Let \( \{v_\ell \} \) be an orthonormal basis obtained by combining orthonormal bases of the various isotypic spaces \( V(i) \). Then

\[
\Theta_\pi = \sum_{i \in \hat{K}} \sum_{v_\ell \in V(i)} \left( \pi(g)v_\ell, v_\ell \right) \tag{5.27}
\]

\[
= \sum_{i \in \hat{K}} \sum_{v_\ell} \left( \pi_K(\varphi_i)\pi(g)\pi_K(\varphi_i)v_\ell, v_\ell \right) \tag{5.28}
\]

\[
= \sum_{i \in \hat{K}} \text{tr} \left( \pi_K(\varphi_i)\pi(g)\big|_{V(i)} \right) \tag{5.29}
\]

which depends only on the \((g, K)\)-action, because the functions in each such matrix block have Taylor series at the identity computable in terms of the Harish-Chandra module.

For \( h \in G \), we have

\[
\int_G \left( \ell(h)r(h)\Theta_\pi \right)(g)f(g) \, dg = \int_G \Theta_\pi(g)f(h^{-1}gh) \, dg \tag{5.30}
\]

\[
= \text{tr} \pi \left( \ell(h^{-1})r(h^{-1})f \right) \tag{5.31}
\]

\[
= \text{tr} \left( \pi(h^{-1})\pi(f)\pi(h) \right) \tag{5.32}
\]

\[
= \text{tr} \pi(f) \tag{5.33}
\]

\[
= \int_G \Theta_\pi f \, dg \tag{5.34}
\]

so that \( \Theta_\pi \) is conjugation invariant.

Recall that \( Z(g) \) acts on \( V_{K\text{-fin}} \) by a character \( \chi_\pi \), the infinitesimal character. Choose \( \{v_\ell\} \) as in the previous argument. From

\[
\sum_\ell \left( \pi(gh)v_\ell, v_\ell \right) = \sum_\ell \left( \pi(g)\pi(h)v_\ell, v_\ell \right) \tag{5.35}
\]

on the infinitesimal level, for \( \zeta \in U(g) \), we find

\[
r(\zeta)\Theta_\pi = \sum_\ell \left( \pi(g)\zeta v_\ell, v_\ell \right). \tag{5.36}
\]

In particular, for \( \zeta \in Z(g) \), we get

\[
\chi_\pi(\zeta) \sum_\ell \left( \pi(g)v_\ell, v_\ell \right) = \chi_\pi(\zeta)\Theta_\pi. \tag{5.37}
\]

\begin{theorem}
Let \( \{\pi_\alpha | \alpha \in I \} \) be a collection of irreducible admissible representations, no two of which are infinitesimally equivalent. Then the \( \{\Theta_{\pi_\alpha} | \alpha \in I \} \) are linearly independent.
\end{theorem}
Proof. Suppose not. Then consider a minimal nontrivial linear relation, and replace $I$ by the set of $\pi_\alpha$ which enter this relation nontrivially. Then $I$ is finite, and the relation is

$$\sum_\alpha c_\alpha \Theta_\pi = 0 \quad (5.38)$$

with $c_\alpha \neq 0$ for every $\alpha \in I$. Choose $i \in \hat{K}$ such that $V_\alpha(i) \neq 0$ for at least one $\alpha$.

Let $A = C_\infty^G(G)_{i,i^*}$. Then $A$ acts on $V_\alpha(i)$ for every $\alpha$ by $\pi_\alpha$.

Lemma 5.6. For each $\alpha$, $V_\alpha(i)$ is a simple $A$-module (or zero).

Proof. Suppose not. Then there exists an $\alpha$ and a nontrivial $A$-invariant $U_\alpha \subseteq V_\alpha(i)$. Then

$$\left\{ \pi(f)U_\alpha \middle| f \in C_\infty^G(G) \right\} \quad (5.39)$$

is nonzero (containing at least $U_\alpha$, by letting $f$ run over an approximate identity) and $G$-invariant, therefore dense in $V$. So given $v \in V_\alpha(\alpha)$ with $v \notin U_\alpha$, there exists a sequence $\{f_n\} \subseteq C_\infty^G(G)$ and $\{u_n\} \subseteq U_\alpha$ such that $\pi_\alpha(f_n)u_n \to v$. But then

$$\pi_K(\varphi_i)\pi_\alpha(f_n)\pi_K(\varphi_i)u_n \to v \quad (5.40)$$

$$\pi(\ell_K(\varphi_i)r_K(\varphi_i^*)f_n)u_n \to v \quad (5.41)$$

so $v$ is in the closure of $U_\alpha$. But this closure is $U_\alpha$, a contradiction. \qed

Suppose $M_\alpha$ is a finite collection of simple $A$-modules (eventually we'll take $M_\alpha = V_\alpha(i)$). Then $M = \bigoplus_\alpha M_\alpha$ is a semisimple $A$-module. Let

$$R = \text{End}_A(M) = \left\{ T \in \text{End}_C(M) \middle| Ta = aT \forall a \in A \right\} \quad (5.42)$$

which is the commutant of $A$. We can view $M$ as an $R$-module, and get a topological morphism $A \to \text{End}_R(M)$.

Theorem 5.7 (Jacobson Density). For any $T \in \text{End}_R(M)$ and any finite set $m_1, \ldots, m_k \in M$, there exists $a \in A$ such that $am_i = Tm_i$ for $1 \leq i \leq k$. In particular, if $M$ is finitely generated over $R$, then $A \to \text{End}_R(M)$ is surjective.

Suppose the $M_\alpha$ are pairwise non-isomorphic. Then $\text{End}_R(M) \cong \mathbb{C}^N$ is the ring of diagonal $N \times N$ matrices, where $N$ is the number of $M_\alpha$.

Remark. If $N = 1$, then the Jacobson Density Theorem implies $A$ maps onto $\text{End}_C(M)$. This is one way of proving irreducible admissible representations are strongly irreducible.

In the present situation, for each $\alpha$, there exists $a_\alpha \in A$ such that $a_\alpha|_{M_\beta} = \delta_{\alpha\beta}$.

In the original setting, this means either there exists $\alpha, \beta$ with $\alpha \neq \beta$ such that $V_\alpha(i) \cong V_\beta(i)$ as $A$-modules, or for every $\alpha$ such that $V_\alpha(i) \neq 0$, there exists $f_\alpha \in C_\infty^G(G)_{i,i^*}$ such that $\pi_\beta(f_\alpha) = \delta_{\alpha\beta}1_{V_\beta(i)}$. 26
We will show that the second option is incompatible with the nontrivial linear relation among the 
\( \Theta_{\pi_\alpha} \). Since \( \pi_\alpha(f_\alpha) \) maps \( V_\alpha(i) \) to \( V_\alpha(i') \) and \( V_\alpha(i') \) to 0 for \( i' \neq i \), we find \( \text{tr} \pi_\alpha(f_\alpha)\big|_{V_\alpha(i)} = \text{tr} \pi_\alpha(f_\alpha) \). Hence the linear functions \( f \mapsto \text{tr} \pi_\alpha(f)\big|_{V_\alpha(i)} \) must be linearly dependent. But then the second option would imply that each \( \text{tr} \pi_\alpha(f)\big|_{V_\alpha(i)} \) is zero, a contradiction.

We can conclude there exist \( \alpha \neq \beta \) such that \( V_\alpha(i) \cong V_\beta(i) \) as \( C_c^\infty(G)_{i,i^*} \)-modules. Now given \( k \in K \), we can choose an approximate identity \( \{ f_n \} \subseteq C_c^\infty(G) \) at \( k \). Considering the action on \( V_\alpha(i) \), this means that for \( v \in V_\alpha(i) \),

\[
v \leftarrow \pi_{\alpha,K}(\varphi_i)\pi_\alpha(f_n)v = \pi_{\alpha,K}(\varphi_i)\pi_\alpha(f_n)\pi_{\alpha,K}(\varphi_i)v \quad (5.43)
\]

and \( \ell_K(\varphi_i)r_K(\varphi_i^*)f \in C_c^\infty(G)_{i,i^*} \). So the isomorphism \( V_\alpha(i) \cong V_\beta(i) \) is \( K \)-invariant. Now it only remains to prove:

**Claim.** The isomorphism is also \( U(\mathfrak{g}) \)-invariant.

Choose a nonzero \( v_\alpha \in V_\alpha(i) \) and the corresponding vector \( v_\beta \in V_\beta(i) \). \( U(\mathfrak{g})v_\alpha \) plus \( K \)-translates gives all of \( V_{K-f_\text{in}} \). Since we know \( K \)-invariance, it suffices to show that \( U(\mathfrak{g})v_\alpha \cong U(\mathfrak{g})v_\beta \) as a \( U(\mathfrak{g}) \)-module.

As \( U(\mathfrak{g}) \)-modules, \( U(\mathfrak{g})v_\alpha \cong U(\mathfrak{g})/U(\mathfrak{g})I_\alpha \), where

\[
I_\alpha = \left\{ \zeta \in U(\mathfrak{g}) \big| \zeta v_\alpha = 0 \right\}. \quad (5.46)
\]

To complete the argument, it suffices to show that the \( C_c^\infty(G)_{i,i^*} \)-action on \( V_\alpha(i) \) determines \( I_\alpha \). We have

\[
\zeta \in I_\alpha \iff \zeta v_\alpha = 0 \quad (5.47)
\]
\[
\iff \pi_\alpha(f)\zeta v_\alpha = 0 \forall f \in C_c^\infty(G) \quad (5.48)
\]
\[
\iff \pi_\alpha(r(\zeta)f)\pi_{\alpha,K}(\varphi_i)v_\alpha = 0 \forall f \in C_c^\infty(G) \quad (5.49)
\]
\[
\iff \pi_{\alpha,K}(\varphi_i)\pi_\alpha\left( r(\zeta)f \right)\pi_{\alpha,K}(\varphi_i)v_\alpha = 0 \forall f \quad (5.50)
\]
\[
\iff \pi_\alpha(\ell_{K}(\varphi_i)r_{K}(\varphi_i^*)(r(\zeta)f))v_\alpha = 0 \forall f. \quad (5.51)
\]

In the last line, \( \ell_{K}(\varphi_i)r_{K}(\varphi_i^*)(r(\zeta)f) \in C_c^\infty(G)_{i,i^*} \). So now we just have to prove the one reverse implication.

Suppose there exists \( f \in C_c^\infty(G) \) such that \( \pi_\alpha\left( r(\zeta)f \right)\pi_{\alpha,K}(\varphi_i)v_\alpha \neq 0 \). Then \( \{ \pi_\alpha(h)(\cdots) \} \) is \( G \)-invariant and nonzero, hence dense in \( V \). But

\[
\pi_\alpha(h)\pi_\alpha\left( r(\zeta)f \right)\pi_{\alpha,K}(\varphi_i)v_\alpha = \pi_\alpha\left( r(\zeta)(h*\tilde{f}) \right)\pi_{\alpha,K}(\varphi_i)v_\alpha \quad (5.52)
\]
is dense. So for some $h$, $\pi_{a,K}(\varphi_i)(\cdots)v_\alpha \neq 0$.

6 Invariant Eigendistributions

Suppose $G_\mathbb{C}$ is a complex, reductive, connected Lie group. We say $H_\mathbb{C} \subseteq G_\mathbb{C}$ is a Cartan subgroup if:

(a) It is abelian.
(b) It acts semisimply on $g_\mathbb{C}$, via $\text{Ad}$.
(c) It is maximal with respect to (a) and (b).

Cartan subgroups exist and are unique up to conjugacy.

A Cartan subalgebra of $g_\mathbb{C}$ means a Lie algebra of a Cartan subgroup.

Let $h_\mathbb{C} \subseteq g_\mathbb{C}$ be a Cartan subalgebra. Let $\Phi = \Phi(h_\mathbb{C}, g_\mathbb{C})$ and choose a system of positive roots $\Phi^+ \subseteq \Phi$. Set

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

(6.1)

(in this case, $\dim g^\alpha = 1$ for every $\alpha \in \Phi$).

We wish to understand the center $Z(g_\mathbb{C})$ of $U(g_\mathbb{C})$. We can express this as $U(g_\mathbb{C})^{hc}$, the space of $g_\mathbb{C}$-invariants in $U(g_\mathbb{C})$.

We will use Poincaré-Birkhoff-Witt: choose a basis

$$\zeta_1, \ldots, \zeta_r, \eta_1, \ldots, \eta_s, \xi_1, \ldots, \xi_t$$

(6.2)

where each $\zeta_i$ lies in some $g^\alpha$ for $\alpha \in \Phi^+$, each $\xi_i$ lies in some $g^\alpha$ for $\alpha \in -\Phi^+$, and each $\eta_i$ lies in $h_\mathbb{C}$. Then the monomial

$$\xi_{i_1} \cdots \xi_{i_k} \eta_{i_1} \cdots \eta_{i_s} \zeta_{i_1} \cdots \zeta_{i_t}$$

(6.3)

cannot be ad $h_\mathbb{C}$-invariant if $k > 0$ and $m = 0$, or if $k = 0$ and $m > 0$. So we have

$$Z(g_\mathbb{C}) = U(g_\mathbb{C})^{hc} \subseteq U(g_\mathbb{C})^{hc} = U(h_\mathbb{C}) \oplus \left(n_- U(g_\mathbb{C})n_+\right)^{hc}.$$  

(6.4)

Define $q : Z(g_\mathbb{C}) \rightarrow U(h_\mathbb{C})$ by projection onto the $U(h_\mathbb{C})$ summand. As $h_\mathbb{C}$ is abelian, we have $U(h_\mathbb{C}) = S(h_\mathbb{C})$, the space of polynomial functions on $h_\mathbb{C}^*$.

Remark. $q$ is an algebra homomorphism because $U(h_\mathbb{C})n_+ = n_+ U(h_\mathbb{C})$.

Define $t_{-\rho} : U(h_\mathbb{C}) \rightarrow U(h_\mathbb{C})$ by

$$t_{-\rho} \zeta = \zeta - \langle \rho, \zeta \rangle 1$$

(6.5)
for \( \zeta \in \mathfrak{h}_C \). Equivalently, if we view \( U(\mathfrak{h}_C) \) as the algebra of polynomials on \( \mathfrak{h}_C^* \), then \( t_{-\rho} \) shifts the argument by \(-\rho\).

**Theorem 6.1** (Harish-Chandra). \((t_{-\rho}q) : Z(\mathfrak{g}_C) \to U(\mathfrak{h}_C) \) takes values in \( U(\mathfrak{h}_C)^W \), the space of Weyl group invariants. Also, the map \( t_{-\rho}q : Z(\mathfrak{g}_C) \to U(\mathfrak{h}_C)^W \) is an isomorphism, called the Harish-Chandra isomorphism. It is independent of the choice of \( \Phi^+ \).

As an example, let \( B \) be the trace form of a finite dimensional faithful representation. By duality, we get a bilinear form on \( \mathfrak{h}_C^* \) which is positive definite on the linear span of the weight lattice. Choose an orthonormal basis \( \eta_1, \ldots, \eta_r \) of \( \mathfrak{h}_C^* \), and generators \( \zeta_\alpha \in \mathfrak{g}^\alpha \) and \( \zeta_{-\alpha} \in \mathfrak{g}^{-\alpha} \) for \( \alpha \in \Phi^+ \), such that \( B(\zeta_\alpha, \zeta_{-\beta}) = \delta_{\alpha\beta} \). Under this basis,

\[
\Omega = \sum_i \eta_i^2 + \sum_{\alpha \in \Phi^+} \left( \zeta_\alpha \zeta_{-\alpha} + \zeta_{-\alpha} \zeta_\alpha \right) \\
= \sum_i \eta_i^2 + \sum_{\alpha \in \Phi^+} \left[ \zeta_\alpha, \zeta_{-\alpha} \right] + 2 \zeta_{-\alpha} \zeta_\alpha \\
q\Omega = \sum_i \eta_i^2 + \sum_{\alpha \in \Phi^+} \left[ \zeta_\alpha, \zeta_{-\alpha} \right].
\]

By the structure theory of compact or complex semisimple groups, \( \left[ \zeta_\alpha, \zeta_{-\alpha} \right] = \eta_\alpha \in \mathfrak{h}_C \) where \( \eta_\alpha \leftrightarrow \alpha \) via \( B \), because \( B(\zeta_\alpha, \zeta_{-\beta}) = 1 \). We find that

\[
q\Omega = \sum_i \eta_i^2 + 2\eta_\rho.
\]

So now we can determine what the Harish-Chandra isomorphism does to \( \Omega \):

\[
(t_{-\rho}q)\Omega = \sum_i \left( \eta_i - \langle \rho, \eta \rangle 1 \right)^2 + 2\left( \eta_\rho - \langle \rho, \eta_\rho \rangle 1 \right) \\
= \left( \sum_i \eta_i^2 \right) - \|\rho\|^2.
\]

**Theorem 6.2** (Chevalley, for invariants under finite reflection groups). \( S(\mathfrak{h}_C)^W \) is a polynomial algebra of rank equal to \( \dim \mathfrak{h}_C \) (which is the rank of \( \mathfrak{g}_C \) by definition).

For \( \lambda \in \mathfrak{h}_C^* \), define a character \( \chi_\lambda : Z(\mathfrak{g}_C) \to \mathbb{C} \) by \( \chi_\lambda(\zeta) = \langle (t_{-\rho}q)\zeta, \lambda \rangle \), where the angle brackets indicates evaluation. Then \( \chi_\lambda = \chi_{w\lambda} \) for \( w \in W \) because \( Z(\mathfrak{g}_C) \cong S(\mathfrak{h}_C)^W \).

**Corollary 6.3.** \( W \backslash \mathfrak{h}_C^* \) is isomorphic to the space of maximal ideals in \( Z(\mathfrak{g}_C) \).

As another example, choose \( \Phi^+ \subseteq \Phi \) as before, and suppose \( V_\lambda \) is the irreducible finite dimensional representation of \( G_C \) of highest weight \( \lambda \). Then \( Z(\mathfrak{g}_C) \) acts by a character on \( V_\lambda \), which we called the infinitesimal character.

Choose \( v_\lambda \in V_\lambda \) which generates the highest weight space. Then \( g^\alpha v_\lambda = 0 \) for \( \alpha \in \Phi^+ \). So for \( \zeta \in Z(\mathfrak{g}_C) \), we have
\[ \zeta v_\lambda = \langle q\zeta, \lambda \rangle v_\lambda \quad (6.12) \]
\[ = \langle t_{\rho - \rho q\zeta}, \lambda \rangle v_\lambda \quad (6.13) \]
\[ = \langle t_{-\rho} q\zeta, \lambda + \rho \rangle v_\lambda \quad (6.14) \]
\[ = \chi_{\lambda + \rho}(\zeta) v_\lambda \quad (6.15) \]

implying \( V_\lambda \) has infinitesimal character \( \chi_{\lambda + \rho} \).

Now suppose \( f \) is a conjugation invariant algebraic function on \( G_\mathbb{C} \). Then \( f \) is completely determined by \( f|_{H_\mathbb{C}} \). (Unlike for real compact groups, the conjugates of a complex semisimple group do not cover \( G_\mathbb{C} \), but they do up to codimension 1.)

**Theorem 6.4** (Harish-Chandra). For \( \zeta \in Z(\mathfrak{g}_\mathbb{C}) \),

\[ \zeta f|_{H_\mathbb{C}} = \Delta^{-1} \left( (t_{-\rho} q\zeta) f|_{H_\mathbb{C}} \Delta \right) \quad (6.16) \]

where we have

\[
\Delta = \prod_{\alpha \in \Phi^+} \left( e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}} \right) \quad (6.17)
\]
\[
= e^\rho \prod_{\alpha \in \Phi^+} \left( 1 - e^{-\alpha} \right) . \quad (6.18)
\]

Observe that the \( 1 - e^{-\alpha} \) are well-defined algebraic functions on \( H_\mathbb{C} \). \( e^\rho \) is a well-defined algebraic function on \( H_\mathbb{C} \) if and only if \( \rho \) is a weight, and \( e^{2\rho} \) is always a well-defined algebraic function. \( \rho \) will be a weight if \( G_\mathbb{C} \) is semisimple and simply connected. More generally, \( \rho \) can be made a weight on an appropriate double cover of \( G_\mathbb{C} \). Alternatively, \( e^\rho \) is an algebraic function up to sign.

Let \( \varphi_\lambda \) be the character of \( V_\lambda \), i.e., of the finite dimensional irreducible representation of highest weight \( \lambda \). Then the **Weyl character formula** is

\[
\varphi_\lambda|_{H_\mathbb{C}} = \frac{\sum_{w \in W} \epsilon(w) e^{w(\lambda + \rho)}}{\Delta}. \quad (6.19)
\]

In particular, we see that

\[
\zeta \varphi_\lambda|_{H_\mathbb{C}} = \Delta^{-1} \left( t_{-\rho} Q\zeta \right) \sum_{w \in W} \epsilon(w) e^{w(\lambda + \rho)} \quad (6.20)
\]
\[
= \Delta^{-1} \sum_{w} \epsilon(w) \chi_{w(\lambda + \rho)}(\zeta) e^{w(\lambda + \rho)} \quad (6.21)
\]
\[
= \Delta^{-1} \sum_{w} \epsilon(w) \chi_{\lambda + \rho}(\zeta) e^{w(\lambda + \rho)} \quad (6.22)
\]
\[
= \chi_{\lambda + \rho}(\zeta) \varphi_\lambda|_{H_\mathbb{C}} \quad (6.23)
\]
so \( Z(\mathfrak{g}_C) \) acts on \( \varphi_\lambda \) by \( \chi_{\lambda+\rho} \), which we already knew. The fact that we get the same formula in two different ways is the crucial part of the proof of Theorems 6.1 and 6.4.

Let \( G \) be a reductive linear Lie group, not necessarily connected, but with \( G \subseteq G_C \) and \( G_C \) connected. Let \((\pi, V)\) be an irreducible admissible representation of \( G \), \( \Theta_\pi \) its character, and \( \chi_\lambda \) the infinitesimal character of \( \pi \). We had seen that for \( \zeta \in Z(\mathfrak{g}_C) \), we have the system of differential equations

\[
\zeta \Theta_\pi = \chi_\lambda(\zeta) \Theta_\pi. \tag{6.24}
\]

We would like to know what they mean for conjugation invariant distributions.

We say that \( H \subseteq G \) is a Cartan subgroup if \( H^0_C \) is a Cartan subgroup of \( G_C \), and \( H \) is maximal abelian in \( G \).

**Theorem 6.5** (Harish-Chandra, Sugiura). *Up to conjugacy, there exist only finitely many Cartan subgroups. Each conjugacy class contains a \( \theta \)-invariant Cartan. Every Cartan subgroup \( H \) can be expressed as a direct product \( H = TA \), where \( T \) is compact and abelian with \( T^0 \) a torus, and \( A \) connected and semisimple with real eigenvalues. Up to conjugacy, there exists a unique one for which \( \dim T \) is maximal, called a fundamental Cartan, and a unique one for which \( \dim T \) is minimal, called a maximally split Cartan. Every other Cartan, up to conjugacy, can be constructed in finitely many steps by a concrete recipe starting from either a fundamental Cartan or a maximally split Cartan.*

Some, but not all, groups \( G \) contain a compact Cartan, which is then fundamental. \( G \) contains a compact Cartan if and only if \( K \) and \( T \) have the same rank. Therefore if \( G \) is connected, then so are \( K \) and \( T \), so \( T \) is a torus.

Suppose \( G = KAN \) and \( P = MAN \). Take \( T \) to be the centralizer in \( M \) of \( A \). Then \( TA \) is a maximally split Cartan.

In the case \( G = SL(2, \mathbb{R}) \) (or \( SU(1, 1) \)), there are two Cartans up to conjugacy. The maximally split Cartan is the diagonal subgroup of \( SL(2, \mathbb{R}) \), which is \( \{ \pm \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} | a \in \mathbb{R}_{>0} \} \). The compact Cartan is the diagonal subgroup of \( SU(1, 1) \), which is connected.

Now suppose \( \Theta \) is an invariant eigendistribution and \( \lambda \in \mathfrak{h}_C^* \) such that \( Z(\mathfrak{g}) \) acts on \( \Theta \) via \( \chi_\lambda \). For example, \( \Theta \) could be the character of an irreducible admissible representation.

We say that \( g \in G_C \) (or \( g \in G \)) is regular semisimple if \( \dim Z_g \) is minimal and \( g \) acts semisimply in any (or some faithful) finite dimensional representation. The set of regular semisimple elements is open, and the complement is an algebraic variety. For \( g \) is regular semisimple if and only if \( \text{Ad} \ g \) has eigenvalue 1 with minimal multiplicity. Let \( G_{rs} \) be the set of regular semisimple elements; the complement of \( G_{rs} \) in \( G \) is a real algebraic variety.

Let \( H \subseteq G \) be a Cartan subgroup, and \( H_{rs} \subseteq H \) be the set of regular semisimple elements in \( H \). The centralizer of any \( h \in H_{rs} \) is \( H \). Therefore, locally near \( h \), we get a fibration of \( G_{rs} \) over \( H_{rs} \) with fibers that are (neighborhoods of \( eH \)) in \( G/H \). \( \Theta \) is conjugation invariant, so by this observation, \( \Theta|_{H_{rs}} \in C^{-\infty}(H_{rs}) \) is well-defined.

Complex algebraic systems of linear differential equations on \( G_C \) and \( H_C \) can be restricted to real analytic systems on \( G \) and \( H \), so the system

\[
\zeta \Theta = \chi_\lambda(\zeta) \Theta \tag{6.25}
\]
for $\zeta \in Z(g)$ gives a system

$$\chi_\lambda(\zeta)\Theta|_{H_{rs}} = \left(\zeta\Theta\right)|_{H_{rs}} = \Delta^{-1}\left(t_{-\rho q}\zeta(\Theta|_{H_{rs}}\Delta)\right). \quad (6.26)$$

We have $t_{-\rho q}Z(g) = U(h)^W = S(h)^W$. So our system of equations becomes

$$\zeta\left(\Delta\Theta|_{H_{rs}}\right) = \langle \zeta, \lambda \rangle\Delta\Theta|_{H_{rs}} \quad (6.27)$$

for all $\zeta \in S(h)^W$. These $\zeta$ may be viewed as $W$-invariant linear differential operators on $H$, or on $\mathfrak{h}$ by $\exp: \mathfrak{h} \to H$ which covers $H^0$.

$S(h)^W$ contains elliptic differential operators, for example $(t_{-\rho q})\Omega$.

**Corollary 6.6.** $\Theta$ is real analytic on $G_{rs}$, and satisfies the system of differential equations (6.27) in the strong sense.

The system (6.27) is a system of linear partial differential equations which behaves like a system of linear ordinary differential equations.

**Proposition 6.7** (Addendum to Chevalley’s theorem). Let $\mathcal{H}(W) \subseteq S(\mathfrak{h})$ be the space of $W$-harmonic polynomials on $\mathfrak{h}^*$, i.e. the polynomials which are annihilated by the augmentation ideal in $S(\mathfrak{h}^*)^W$, viewed as linear differential operators on $\mathfrak{h}^*$. Then:

(a) $\dim \mathcal{H}(W) = \#W$.

(b) $S(\mathfrak{h}) \cong \mathcal{H}(W) \otimes S(\mathfrak{h})^W$ via multiplication.

**Corollary 6.8.** The augmentation ideal in $S(\mathfrak{h})^W$ generates an ideal in $S(\mathfrak{h})$ which has codimension $\#W$, and $\mathcal{H}(W)$ is a linear complement.

Viewed as the vector space of polynomial functions on $\mathfrak{h}^*$, $\mathcal{H}(W)$ is translation invariant because it is defined by a system of constant coefficient operators. Therefore for any $\lambda \in \mathfrak{h}^*$, the set

$$\left\{ \zeta \in S(\mathfrak{h})^W \mid \langle \zeta, \lambda \rangle = 0 \right\} \quad (6.28)$$

generates an ideal of codimension $\#W$ in $S(\mathfrak{h})$, and $\mathcal{H}(W)$ is a complementary subspace.

Let $w = \#W$ and $\{\eta_1, \ldots, \eta_w\}$ be a basis of $\mathcal{H}(W)$, and let $\{\zeta_1, \ldots, \zeta_r\}$ be a basis of $\mathfrak{h}$. Then we may write

$$\zeta_k\eta_j = \sum_i a_{ij}^k \eta_i + \xi_{kj} \quad (6.29)$$

$$\xi_{kj} \in S(\mathfrak{h}) \cdot \left\{ \zeta \in S(\mathfrak{h})^W \mid \langle \zeta, \lambda \rangle = 0 \right\} \quad (6.30)$$

for $a_{ij}^k \in \mathbb{C}$ depending on $\lambda$. Then the system

$$\zeta f = \langle \zeta, \lambda \rangle f \quad \forall \zeta \in S(\mathfrak{h})^W \quad (6.31)$$

for $w = \#W$ and $\{\eta_1, \ldots, \eta_w\}$ be a basis of $\mathcal{H}(W)$, and let $\{\zeta_1, \ldots, \zeta_r\}$ be a basis of $\mathfrak{h}$. Then we may write

$$\zeta_k\eta_j = \sum_i a_{ij}^k \eta_i + \xi_{kj} \quad (6.29)$$

$$\xi_{kj} \in S(\mathfrak{h}) \cdot \left\{ \zeta \in S(\mathfrak{h})^W \mid \langle \zeta, \lambda \rangle = 0 \right\} \quad (6.30)$$

for $a_{ij}^k \in \mathbb{C}$ depending on $\lambda$. Then the system

$$\zeta f = \langle \zeta, \lambda \rangle f \quad \forall \zeta \in S(\mathfrak{h})^W \quad (6.31)$$

32
is equivalent to
\[ \zeta_k \eta_j f = \sum_{i} a_{ij}^k(\lambda) \eta_i f. \] (6.32)

For example, in the case of $SL(2, \mathbb{R})$, $\mathfrak{h}^* = \mathbb{C}$ and $S(\mathfrak{h}) = \mathbb{C} \left[ \frac{d}{dx} \right]$. $W = \pm$ so $S(\mathfrak{h})^W = \mathbb{C} \left[ \frac{d^2}{dx^2} \right]$.

Now $\mathcal{H}(W) = \mathbb{C} \cdot 1 \oplus \mathbb{C} \cdot \frac{d}{dx}$. Now the system (6.31) becomes
\[ \left( \frac{d^2}{dx^2} - \lambda^2 \right) f = 0 \] (6.33)

while (6.32) becomes
\[ \frac{d}{dx} \left( \frac{f}{\frac{df}{dx}} \right) = \begin{pmatrix} 0 & 1 \\ \lambda^2 & 0 \end{pmatrix} \frac{f}{\frac{df}{dx}} \] (6.34)

This is a system of first order equations (“separation of variables”). We conclude that the (local) space of solutions of (6.31) has dimension equal to $\# W$. If $\lambda \in \mathfrak{h}^*$ is regular, then on $\mathfrak{h}_\mathbb{R}$, the vector space of solutions of (6.31) is spanned by the $e^{w\lambda}$ over $w \in W$. If $\lambda$ is singular, the space of solutions is spanned by the $p_w e^{w\lambda}$ with $w \in W$ and $p_w \in \mathcal{H}(W_{w\lambda})$. Here $W_{w\lambda}$ is the stabilizer of $w\lambda$ in $W$.

Let $\Theta$ be an invariant eigendistribution, acted on by $Z(\mathfrak{g})$ via $\chi_\lambda$. Let $H \subseteq G$ be a Cartan subgroup, and $U \subseteq H_{rs}$ a connected component. Then we have proved:

**Proposition 6.9.** We have
\[ \Theta|_U = \sum_{w \in W} p_w e^{w\lambda} \] (6.35)

where $p_w \in \mathcal{H}(W_{w\lambda})$ are polynomials on $\mathfrak{h}_\mathbb{R}$, or “logarithmic polynomials on $H$”, and the $p_w$ are constant if $\lambda$ is regular.

**Remark.**
1. If $\Delta$ is not single-valued (as may be the case), this does not matter since the restriction to any connected component of $H_{rs}$ can be thought of as well-defined. Or else, we can make $\Delta$ well-defined on a twofold cover of $G$.
2. $\Theta$ is conjugation invariant, so $\Theta|_U$ must be invariant under those elements of $W(H,G)$ that fix $U$ (if any). This does not imply $p_{vw} = e(v)p_w$ for $v \in W(H,G)$.
3. If $\Theta = \Theta_\pi$ is an irreducible character, then $Z_G$ acts by scalars via $\pi$, hence acts by scalars on $\Theta$ via translation. This imposes conditions on the local formulas.
4. The local formulas for various choices of $H$ and $U$ satisfy compatibility conditions (called “matching conditions”) which we’ll discuss later.
5. If $\Theta = \Theta_\pi$ is an irreducible character, the $p_w$ are constants, even when $\lambda$ is singular. This is a consequence of the “Jantien-Zuckerman translation principle” and will be discussed later.

We have shown that $\Theta|_{G_{rs}}$ is real analytic and, transversely to conjugacy classes, is given by explicit expressions with (so far unknown) coefficients $p_w$. This implies $|\Delta|\Theta|_{G_{rs}}$, as a “function” on $G$, is locally bounded.
Theorem 6.10. \( \frac{1}{|\Delta|} \in L^1_{\text{loc}}(G) \).

Sketch of proof. We will prove this for \( G = \text{SL}(2, \mathbb{R}) \), and sketch a proof in general. Define

\[
\tilde{G}_C = \{(g, z) \in G_C \times \mathbb{P}^1 \mid gz = z\} \tag{6.36}
\]

Let \( p_1 \) and \( p_2 \) be the projections of \( \tilde{G}_C \) onto \( G_C \) and \( \mathbb{P}^1 \). Then both \( p_1 \) and \( p_2 \) are \( G_C \)-equivariant when \( G_C \) acts on \( G_C \times \mathbb{P}^1 \) as conjugation on the \( G_C \) factor and the natural action on the \( \mathbb{P}^1 \) factor. \( G_C \) acts transitively on \( \mathbb{P}^1 \), and its isotropy group at 0 is

\[
B_C = \left\{ \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \middle| a \in \mathbb{C}^*, c \in \mathbb{C} \right\} \tag{6.37}
\]

Now \( \tilde{G}_C \overset{p_2}{\to} \mathbb{P}^1 \) is a principal bundle with structure group \( B_C \). We have \( \tilde{G}_C \cong G_C \times_{B_C} B_C \), where \( B_C \) acts on itself by conjugation. In particular, \( \tilde{G}_C \) is smooth.

Now analyzing \( p_1, g \in G_C \) has 2 fixed points on \( \mathbb{P}^1 \) if \( g \) is regular semisimple, 1 fixed point if \( \pm g \) is regular unipotent, and fixes all of \( \mathbb{P}^1 \) if \( g = \pm e \). So \( p_1 : \tilde{G}_C \to G_C \) is a twofold branched cover, branched over \( G_C \setminus (G_C)_{rs} \).

For \((g, z) \in \tilde{G}_C \), \( g \) acts on \( T_z \mathbb{P}^1 \) by the inverse of the eigenvalue \( \lambda \) by which \( g \) operates on the line in \( \mathbb{C}^2 \) determined by \( z \). Now \( \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \) has roots \( \lambda \pm 2 \), so “\( \lambda = \pm \frac{2}{3} \)” is well-defined holomorphic function on \( \tilde{G}_C \). Then \( p_2 : \tilde{G}_C \to \mathbb{P}^1 \) is the branched cover determined by the multi-valued function \( \Delta \) on \( G_C \).

Define \( \tilde{G} = (G \times \mathbb{P}^1) \cap \tilde{G}_C \). Then \( \tilde{G} \subseteq \tilde{G}_C \) is a real analytic subvariety. It is not smooth since the isotropy subgroups of \( g \) jump as \( g \) goes from “elliptic” to “hyperbolic” via “unipotent”. Now let \( \omega \) be the \( G_C \)-invariant holomorphic 3-form on \( \tilde{G}_C \) such that \( \omega|_{\tilde{G}} \) is the Haar measure on \( G \). Then \( p_1^* \omega \) is a holomorphic 3-form on \( \tilde{G}_C \) which vanishes to first order along \( p_1^{-1}(G_C \setminus (G_C)_{rs}) \) because \( (p_1)_* \) vanishes over the branch locus. Also \( \Delta \) vanishes to first order over the branch locus of \( p_1 \). Therefore \( \omega_{\tilde{G}} \) is holomorphic on \( \tilde{G}_C \) (except possibly in codimension 2, which is fine).

So the theorem follows from:

Lemma 6.11. Let \( Z \) be an \( n \)-dimensional real analytic manifold, \( Y \) a \( k \)-dimensional real analytic subvariety, and \( \omega \) a smooth \( k \)-form on \( Z \). Then \( |\omega|_{Y_{\text{reg}}} \in L^1_{\text{loc}}(Y) \).

Proof. This is a local problem, so we may suppose \( Z \) is the unit cube in \( \mathbb{R}^n \). Cover \( Y_{\text{reg}} \) by \( \binom{n}{k} \) open subsets such that for each of them, the projection to one of the \( k \)-dimensional coordinate hyperplanes is locally invertible with inverse whose differential is bounded. Instead of integrating \( |\omega| \) over \( Y_{\text{reg}} \), integrate over the various subsets, and do the integration over the image, where it is finite. Then by real analyticity, this process is finite.

To generalize the proof of this theorem, replace \( \mathbb{P}^1 \) by the flag variety of \( G_C \).

Now we can view \( \Theta \) as a distribution in two, a priori different, ways:

(i) As \( \Theta \in C^{-\infty}(G) \).
(ii) As a locally $L^1$, conjugation invariant function on $G$, real analytic on $G_{rs}$, which therefore defines a distribution.

Their difference is supported on the singular set. But in fact:

**Theorem 6.12.** Any invariant eigendistribution is a locally $L^1$ function which is real analytic on $G_{rs}$.

That is, their difference is 0 everywhere.

*Proof for $SL(2,\mathbb{R})$.* Let $G = SL(2,\mathbb{R})$ and let $t : G \to \mathbb{R}$ be the trace map. Then $t$ is $C^\infty$ and conjugation invariant, hence constant on conjugacy classes. Now suppose $g_0 \in G$.

(a) If $|t(g_0)| < 2$, then $g$ is regular elliptic with eigenvalues $e^{\pm i\theta}$.
(b) If $|t(g_0)| > 2$, then $g$ is regular hyperbolic with eigenvalues $e^{\pm \lambda}$ with $\lambda$ real.
(c) If $t(g_0) = 2$, then either $g_0$ is conjugate to $(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix})$ for some $x \neq 0$, or $g_0 = 1$.
(d) If $t(g_0) = -2$, then $t(-g_0) = 2$, so $g_0$ is the negative of one of the above matrices.

Suppose $g_0 \neq \pm 1$. Then, locally near $g_0$, $G$ is a product of “pieces of conjugacy classes”, interval in $\mathbb{R}$. So locally we can choose coordinates $t, u, v$ such that $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ are tangential to conjugacy classes. So $t$ is a local coordinate transverse to the conjugacy classes. Then we may write

$$\Omega = D_t + D_u \frac{\partial}{\partial u} + D_v \frac{\partial}{\partial v}. \quad (6.38)$$

**Lemma 6.13.** We have

$$D_t = (t^2 - 4) \frac{\partial^2}{\partial t^2} + 3t \frac{\partial}{\partial t}. \quad (6.39)$$

*Proof.* Let $\varphi_n$ be the character of the $n$-dimensional irreducible representation of $SL(2,\mathbb{C})$. We have $\varphi_1 = 1$, $\varphi_2 = t$, and (since the second tensor power of the standard is $V_3 \oplus V_1$) $\varphi_3 = t^2 - 1$.

In general, $\Omega$ acts on the irreducible representation of highest weight $\lambda$ by $|\lambda + \rho|^2 - |\rho|^2$. In our case, it is $n^2 - 1$. We see that these scalars are 0, 3, and 8 for $n = 1, 2, 3$. If we have

$$D_t = a_2(t) \frac{\partial^2}{\partial t^2} + a_1(t) \frac{\partial}{\partial t} + a_0(t) \quad (6.40)$$

then these cases uniquely determine $a_0(t), a_1(t), a_2(t)$.

Let $F = \Theta|_{G_{rs}}$. We have $\chi_{\lambda}(\Omega) = \lambda^2 - 1$. So

$$\left(D_t - (\lambda^2 - 1)\right) \Theta = 0. \quad (6.41)$$

So the same will be true for $F$ on $G_{rs}$. Now
\[
(D_t - (\lambda^2 - 1)) F = (D_t - (\lambda^2 - 1)) (F - \Theta) \tag{6.42}
\]

with the difference \( F - \Theta \) supported on \( G \setminus G_{rs} \). \( D_t \) is elliptic away from \( t = \pm 2 \). This implies that \( \Theta|_{G_{rs}} \) is real analytic. Next, look near \( g_0 \) such that \( t(g_0) = \pm 2 \) but \( g_0 \neq \pm 1 \). We will show that \( \Theta = F \) near any such point. For simplicity, suppose that \( t(g_0) = 2 \).

Change coordinates, letting \( x = t - 2 \), so we are looking near \( x = 0 \). In these new coordinates,

\[
D_t = x(x + 4) \frac{\partial}{\partial x^2} + 3(x + 2) \frac{\partial}{\partial x}. \tag{6.43}
\]

Now \( F - \Theta \) is supported at \( x = 0 \), so we can write

\[
F - \Theta = \sum_{k=0}^{N} c_k \frac{\partial^k}{\partial x^k} \delta_0. \tag{6.44}
\]

for uniquely determined \( c_k \). We say this is a distribution of order \( N \) at 0 if \( c_N \neq 0 \).

Observe that a distribution \( \tau \) supported at \( x = 0 \) has order at most \( N \) if, for every \( \varphi \in C_\infty(\mathbb{R}) \), the limit

\[
\lim_{h \to 0} h^N \int_{\mathbb{R}} \tau(x) \varphi(h^{-1}x) \, dx \tag{6.45}
\]

exists. If in addition, the limit is nonzero for some \( \varphi \), then \( \tau \) has order exactly \( N \). Now consider a differential of the form

\[
D = xa(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} + c(x). \tag{6.46}
\]

Lemma 6.14. If \( \tau \) has order exactly \( N \), then \( D\tau \) has order at most \( N + 1 \), with equality if and only if

\[
-(N + 2)a(0) + b(0) \neq 0. \tag{6.47}
\]

Proof. For \( d(x) \in C_\infty(\mathbb{R}) \), \( d(x) \frac{d^k}{dx^k} \equiv \frac{d^k}{dx^k} d(x) \) modulo lower order terms. This implies the first statement. We also have

\[
x \frac{d^k}{dx^k} = \frac{d^k}{dx^k} x - k \frac{d^{k-1}}{dx^{k-1}}. \tag{6.48}
\]

and this will imply the second statement. \( \square \)

We apply this lemma to our situation, where

\[
D = x(x + 4) \frac{d}{dx^2} + 3(x + 2) \frac{d}{dx} - (\lambda^2 - 1) \tag{6.49}
\]

to find that if \( F - \Theta \) is not identically zero and has order \( N \), \( D(F - \Theta) \) will have order exactly \( N + 1 \).
Lemma 6.15. With $D$ as before and $f \in L^1_{\text{loc}}(\mathbb{R})$, if $\text{supp}(Df) \subseteq \{0\}$, then $Df$ has order 0.

Proof. By our observation, it's enough to show that for every $\varphi \in C_\infty_c(\mathbb{R})$,

$$\lim_{h \to 0} h \int_{\mathbb{R}} Df(x) \varphi(h^{-1}x) \, dx = 0. \quad (6.50)$$

Now we have (for $\psi(x) = x \varphi(x)$)

$$h \int_{\mathbb{R}} Df(x) \varphi(h^{-1}x) \, dx = h \int_{\mathbb{R}} f(x) D^* \varphi(h^{-1}x) \, dx = h \int_{\mathbb{R}} f(x) \frac{d^2}{dx^2} \left(xa(x) \varphi(h^{-1}x)\right) dx - \frac{d}{dx} \left(b(x) \varphi(h^{-1}x)\right) dx + c(x) \varphi(h^{-1}x) \, dx \quad (6.51)$$

$$= h^2 \int_{\mathbb{R}} f(x) \frac{d^2}{dx^2} \left(a(x) \psi(h^{-1}x)\right) dx - h \int_{\mathbb{R}} f(x) \frac{d}{dx} \left(b(x) \varphi(h^{-1}x)\right) dx + h \int_{\mathbb{R}} f(x) c(x) \varphi(h^{-1}x) \, dx \quad (6.52)$$

$$= \int_{\mathbb{R}} f(x) a(x) \psi''(h^{-1}x) \, dx \quad (6.53)$$

$$+ 2h \int_{\mathbb{R}} f(x) a'(x) \psi'(h^{-1}x) \, dx \quad (6.54)$$

$$+ h^2 \int_{\mathbb{R}} f(x) a''(x) \psi(h^{-1}x) \, dx \quad (6.55)$$

$$- \int_{\mathbb{R}} f(x) b(x) \varphi'(h^{-1}x) \, dx \quad (6.56)$$

$$- h \int_{\mathbb{R}} f(x) b'(x) \varphi(h^{-1}x) \, dx \quad (6.57)$$

$$+ h \int_{\mathbb{R}} f(x) c(x) \varphi(h^{-1}x) \, dx \quad (6.58)$$

and this tends to 0. \hspace{1cm} \Box

We get a contradiction. So $\Theta$ is locally $L^1$ except possibly near $\pm 1$.

With $F = \Theta|_{G_{rs}}$ as before, we now know supp$(F - \Theta) \subseteq \{\pm 1\}$. Consider this problem at 1. Choose coordinates $x_1, x_2, x_3$ centered at 1 such that

$$F - \Theta = \sum_{k=0}^{N} P_k \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \delta_0 \quad (6.62)$$

where the $P_k$ are homogeneous polynomials, of degree $k$, with $\mathbb{C}$-coefficients. We say that this distribution has order $N$ if $P_N$ is not identically 0.
The symbol of $\Omega$ at 1 is $Q_2\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ where $Q_2$ is homogeneous of degree 2 and not identically zero, because $\Omega$ is left (and right) invariant, of order 2. So if $F - \Theta$ has order exactly $N$, then $(\Omega - (\lambda^2 - 1))(F - \Theta)$ has order exactly $N + 2$. So to prove $\Theta = F$, it suffices to show:

**Lemma 6.16.** $(\Omega - (\lambda^2 - 1))F$ has order at most 1.

**Proof.** We need to show that for any $\varphi \in C^\infty_c(\mathbb{R}^3)$,

$$
\lim_{h \to 0} h^2 \int_{\mathbb{R}^3} (\Omega - (\lambda^2 - 1))F(x)\varphi(h^{-1}x) \, dx = 0.
$$

(6.63)

We can argue as in the previous lemma. □

**Remark.** This is simpler than the previous case, because here the symbol is not zero. Whereas before, we considered $\Omega$ as an operator transverse to the unipotent stratum, and in that sense, the symbol vanished on the unipotent stratum.

For a general group $G$, if $\Theta$ is an invariant eigendistribution on $G$, assuming $\Theta$ is $L^1_{loc}$ (as has been proven for $G = SL(2, \mathbb{R})$), we had seen that on any connected component of $H_{rs} = H \cap G_{rs}$, for any of the finitely many (up to conjugacy) Cartan subgroups, $\Delta \Theta|_{H_{rs}}$ is a finitely linear combination of exponential terms with complex coefficients if $\lambda$ is regular, and logarithmically polynomial coefficients of bounded degree of $\lambda$ is singular.

**Corollary 6.17.** The space of invariant eigendistributions on $G$, which are acted on via $\chi_\lambda$ by $Z(\mathfrak{g})$, is finite dimensional.

**Corollary 6.18.** Up to infinitesimal equivalence, there exist only finitely many irreducible admissible representations with a given infinitesimal character.

## 7 Matching Conditions

Now suppose $F \in C^\infty(G_{rs})$ is conjugation invariant and satisfies $\zeta F = \chi_\lambda(\zeta)F$ for every $\zeta \in Z(\mathfrak{g})$. We would like to know the conditions on $F$ that make it an invariant eigendistribution. The answer is provided by the “Harish-Chandra matching conditions”.

Let $H \subseteq G$ be a Cartan subgroup. We know that, up to conjugacy, we can arrange that $\theta H = H$. Then $H = TA$, where $T \subseteq K$ is such that $T^0$ is a torus, and $A = A^0 \subseteq \text{exp}\mathfrak{p}_\mathbb{R}$, so $\theta(a) = a^{-1}$ for $a \in A$. Consider $\alpha \in \Phi(\mathfrak{h}, \mathfrak{g})$. Then:

- $\alpha$ is a real root if $\alpha|_t = 0$. Equivalently, $\theta \alpha = -\alpha$.
- $\alpha$ is a complex root if $\theta \alpha \neq \pm \alpha$.
- $\alpha$ is an imaginary root if $\alpha|_a = 0$, which is equivalent to $\theta \alpha = \alpha$.

Complex conjugation acts on roots and weights, as $-1$ on $\mathfrak{t}_\mathbb{R}^*$ and $+1$ on $\mathfrak{a}_\mathbb{R}^*$ (the opposite of $\theta$). This explains the terminology.
Suppose $\alpha$ is an imaginary root. Set

$$s_\alpha = g^\alpha \oplus g^{-\alpha} \oplus [g^\alpha, g^{-\alpha}] \cong \mathfrak{sl}(2, \mathbb{C})$$  \hspace{1cm} (7.1)$$

which is closed under complex conjugation, so

$$s_\alpha = \mathbb{C} \otimes_{\mathbb{R}} (s_\alpha \cap g_{\mathbb{R}})$$  \hspace{1cm} (7.2)$$

and $s_\alpha \cap g_{\mathbb{R}}$ could either be $\mathfrak{su}(2)$ or $\mathfrak{su}(1,1)$. In these cases, we say $\alpha$ is *compact imaginary* or *noncompact imaginary*.

Given $\alpha$, $e^\alpha : H \to \mathbb{C}^*$ is real-valued if $\alpha$ is a real root, and takes values in $S^1$ if $\alpha$ is an imaginary root. If $\alpha$ is a complex root, both $\Re e^\alpha$ and $\Im e^\alpha$ are not identically zero. Now

$$H_{rs} = \left\{ h \in H \left| e^\alpha(h) \neq 1 \forall \alpha \in \Phi(h, g) \right. \right\}.$$  \hspace{1cm} (7.3)$$

So for $\alpha \in \Phi(h, g)$, ker $e^\alpha$ has codimension 1 in $H$ if $\alpha$ is real or imaginary, and codimension 2 in $H$ if $\alpha$ is complex. Informally, “complex roots do not separate components of $H_{rs}$”, and so they do not enter the matching conditions.

Consider the case $G = SL(2, \mathbb{R})$. Up to conjugacy, there are two Cartans:

$$T = \left\{ k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \left| \theta \in \mathbb{R}/2\pi \mathbb{Z} \right. \right\}$$  \hspace{1cm} (7.4)$$

$$H = \left\{ \pm a_t = \pm \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \left| t \in \mathbb{R} \right. \right\}.$$  \hspace{1cm} (7.5)$$

Fix $\lambda \in t^* \cong \mathfrak{h}^*$. We shall consider an invariant eigendistribution $\Theta$ such that:

1. $\Omega \Theta = (\lambda^2 - 1) \Theta$.
2. $\Theta(-g) = (-1)^\epsilon \Theta(g)$ for $\epsilon \in \mathbb{Z}/2\mathbb{Z}$.
3. If $\lambda = 0$, there are still only complex coefficients.

((2) and (3) automatically hold for irreducible characters.) Then

$$\Theta(k_\theta) = \frac{a_+ e^{i\lambda \theta} + a_- e^{-i\lambda \theta}}{e^{i\theta} - e^{-i\theta}}$$  \hspace{1cm} (7.6)$$

$$\Theta(a_t) = \frac{b_+ e^{\lambda t} + b_- e^{-\lambda t}}{e^t - e^{-t}}$$  \hspace{1cm} (7.7)$$

for suitable constants $a_+, a_-, b_+, b_-$ on each connected component.

Here are the matching conditions for $SL(2, \mathbb{R})$:

1. The numerator of the expression (7.6) holds globally on $T$. So $\lambda \in \mathbb{Z}$, and $\lambda \equiv 1 + \epsilon \pmod{2}$ since $k_{\theta + \pi} = -k_\theta$. 

39
2. The numerator of the expression (7.7) may have jumps, but $b_+$ and $b_-$ are well-defined for $a_t$ with $t > 0$. Then $\Theta(a_{-t})$ is determined by Weyl invariance*. Also $\Theta(-a_t) = (-1)^s \Theta(a_t)$, so $b_+$ and $b_-$ determine $\Theta|_H$ globally.

3. The matching conditions (necessary and sufficient for $\Theta$ to be an invariant eigendistribution) is that

$$a_+ - a_- = b_+ - b_-.$$  \hspace{10cm} (7.8)

*W(H, G) \cong W(H_\mathbb{C}, G_\mathbb{C}) \cong \mathbb{Z}/2\mathbb{Z}$. Meanwhile $W(T, G) = \{1\}$.

**Remark.** 1. The formula for $\Theta(a_t), t > 0$ implies an expression for $\Theta(a_{-t})$, by Weyl invariance of $\Theta|_A$.

2. Locally near 0, respectively $e$, $\mathfrak{g}_\mathbb{R} \cong G$ via the exponential map which preserves the conjugation action. The $G$-orbits in $\mathfrak{g}_\mathbb{R}$, with appropriate linear coordinates, are given by $x^2 + y^2 - z^2 = C$. ($C = 0$ corresponds to two orbits, $\{0\}$ and the cone of nonzero nilpotents.) If $C < 0$, this is a hyperboloid of one sheet, and these are the regular elliptic orbits. If $C > 0$, this is a hyperboloid of two sheets, and these are the regular hyperbolic orbits. Therefore the set of regular elliptic element is locally connected near 0.

3. Choose constants $a_+, a_-, b_+, b_-$ in the same fashion, to define a conjugation invariant function $F$ on $G_{rs}$ which, by construction, satisfies

$$\Omega F = (\lambda^2 - 1)F$$  \hspace{10cm} (7.9)

on $G_{rs}$. Now suppose either $b_+ = b_- = 0$ or $a_+ = a_- = 0$. Then $(\Omega - (\lambda^1 - 1))F$ has support in the nilpotent cone, and is a conjugation invariant distribution on the nilpotent cone. This distribution involves one normal derivative. This is the reason for seeing $a_+ - a_-$ and $b_+ - b_-$ with multiplicative factor $\lambda$. This explains also:

4. If $\lambda = 0$, then the matching conditions become vacuous, because we can let $a_+ = a_-$ and $b_+ = b_-$. 

We now describe the matching conditions in general. Let $G$ be a reductive linear group with $G_\mathbb{C}$ connected. By going to a 2-fold cover of both $G$ and $G_\mathbb{C}$, we can arrange that $\rho \in \Lambda$. We’ll assume this. Then the Weyl denominator $\Delta$ is well-defined (once the sign has been fixed) on each Cartan.

Recall that any conjugacy class of Cartans contains one or more which is $\theta$-invariant. Then such a Cartan has the form $H = TA$ with $T \subseteq K$, $T^0$ a torus, and $A \subseteq \exp \mathfrak{p}_\mathbb{R}$. Up to conjugacy, there exists only one fundamental Cartan (that is, with $\dim T$ maximal) and one maximally split Cartan (one with $\dim A$ maximal).

Suppose $H = TA$ as above and $\alpha_0$ is a real root, so $\alpha_0|_t = 0$. Then $\mathfrak{s}_{\alpha_0}$ is defined over $\mathbb{R}$, and $\mathfrak{s}_{\alpha_0} \cong \mathfrak{sl}(2, \mathbb{R})$. Let $A_0$ denote the image in $A$ of \((\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}) \in SL(2, \mathbb{R}), a > 0\). Then $\alpha_0 = [\mathfrak{g}^{\alpha_0}, \mathfrak{g}^{-\alpha_0}]$ and $T_0$ is the image of \((\begin{smallmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{smallmatrix})\). We then have

$$\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a} = \left\{ \eta \in \mathfrak{h} \middle| \langle \alpha_0, \eta \rangle = 0 \right\} \oplus \mathfrak{a}_0.$$  \hspace{10cm} (7.10)

The first summand contains all of $\mathfrak{t}$ and is a codimension 1 subspace of $\mathfrak{a}$. In $SL(2, \mathbb{C})$, consider
\[ c = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \] 

(7.11)

In the action of \( SL(2, \mathbb{C}) \) on \( \mathbb{P}^1 \), \( c \cdot 0 = i \) and \( c \cdot \infty = -i \). Therefore \( \text{Ad} \ c \) maps

\[ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad \text{for} \quad t \in \mathbb{C}. \]

(7.12)

Define \( c_{\alpha_0} \in \text{Aut}(g) \), by \( c_{\alpha_0} \) being \( \text{Ad} \) of the image of \( c \) in \( G_{\mathbb{C}} \). Then

\[ c_{\alpha_0} \mathfrak{h} = \left\{ \eta \in \mathfrak{h} \mid \langle \alpha_0, \eta \rangle = 0 \right\} \oplus \mathfrak{t}_0 \]

(7.13)

is \( \theta \)-invariant, defined over \( \mathbb{R} \), contains \( T \) and the 1-dimensional toroidal group \( T_0 \), plus a codimension 1 subgroup of \( A \). So \( c_{\alpha_0} \mathfrak{h} = \mathfrak{h}_{\alpha_0} \), the complexified Lie algebra of a \( \theta \)-stable, defined over \( \mathbb{R} \), Cartan subgroup \( H_{\alpha_0} \). This process, called the \emph{Cayley transform about a real root}, increases the dimension of the compact part of \( H \) by 1.

To go in the other direction, \( (c_{\alpha_0}^{-1})^* \alpha_0 \) is a noncompact imaginary root for \( H_{\alpha_0} \).

We say that two roots \( \alpha \) and \( \beta \) which are not proportional are \emph{strongly orthogonal} if neither \( \alpha + \beta \) nor \( \alpha - \beta \) is a root. Recall that the set of \( k \in \mathbb{Z} \) such that \( \beta + k\alpha \) is a root is an uninterrupted string of integers \( \{ p \leq k \leq q \} \) with \( p + q = 2 \frac{(\alpha, \beta)}{(\alpha, \alpha)} \). So if \( \alpha \) and \( \beta \) are strongly orthogonal, then \( \alpha \) must be orthogonal to \( \beta \). It is easy to see that if \( \alpha \) and \( \beta \) are strongly orthogonal, then \( s_\alpha \) and \( s_\beta \) commute. This implies:

1. If \( \beta \) is either a real root, or compact imaginary, or noncompact imaginary, and \( \alpha \) is strongly orthogonal to \( \beta \), then \( (c_{\alpha}^{-1})^* \beta \) has the same character as \( \beta \).
2. \( c_\alpha \) and \( c_\beta \) commute.

\textbf{Theorem 7.1 (Sugiura).} 1. \emph{Up to conjugacy, there exists only one maximally split Cartan and one fundamental Cartan.}

2. \emph{Let} \( H \) \emph{be a maximally split Cartan. Then given any conjugacy class of Cartans} \( \mathcal{C} \), \emph{there exists a set of strongly orthogonal real roots} \( \alpha_1, \ldots, \alpha_s \) \emph{for} \( H \) \emph{such that} \( c_{\alpha_1} \cdots c_{\alpha_s} \mathfrak{h} \) \emph{is defined over} \( \mathbb{R} \), \emph{and the corresponding Cartan subgroup belongs to} \( \mathcal{C} \).

Let \( H \) be a \( \theta \)-stable Cartan subgroup of \( G \), \( \alpha \) a real root, and \( H_\alpha \) the Cartan subgroup obtained by applying \( c_\alpha \) to \( H \). Let \( \Theta \) be an invariant eigendistribution, with \( \zeta \Theta = \chi_{\lambda}(\zeta)\Theta \) for every \( \zeta \in \mathbb{Z}(g) \). Also assume \( \rho \in \Lambda \). The matching conditions are:

(A) \( \Delta \Theta \) is real analytic on the set of \( h \in H \) such that \( e^{\beta}(h) \neq 1 \) for all real roots \( \beta \).
(B) \emph{Let} \( U \) \emph{be a connected component of this set. Define}

\[ U_\alpha = \left\{ h \in \text{clos}(U) \mid e^{\beta}(h) \neq 1 \text{ for } \beta \text{ real, strongly orthogonal to } \alpha, e^\alpha(h) = 1 \right\} \times T_0 \]

(7.14)

Suppose \( e^\alpha > 1 \) on \( U \). Then
Then for each $w$, $a_w - a_{s_{\alpha}w} = b_w - b_{s_{\alpha}w}$, for $s_{\alpha}$ the reflection about $\alpha^\perp$.

**Remark.** The above statement assumes that the $a_w$ and $b_w$ are constants, as they are for characters. The statement can be modified for polynomial coefficients.

Here is some (convenient, but not standard) terminology. An invariant eigendistribution $\Theta$ is \textit{bounded at $\infty$} (as half-form, transversely to conjugacy classes) if for any Cartan subgroup $H$ (or equivalently, for one from each conjugacy class), $\Delta \Theta|_{H_{rs}}$ is bounded. $\Theta$ \textit{vanishes at $\infty$} if $\Delta \Theta|_{H_{rs}} \to 0$ outside of compact subsets of $H$.

**Remark.** Harish-Chandra and others call “bounded at $\infty$” \textit{tempered}.

Here are some consequences of the matching conditions.

Suppose $\Theta$ is an invariant eigendistribution with $\zeta \Theta = \chi_\lambda(\zeta) \Theta$ for every $\zeta \in Z(g)$. Assume that no polynomial coefficients occur in the Weyl numerators. (We shall see that this holds for characters.) Suppose $H = TA$ is a $\theta$-stable Cartan. Then:

1. $\Theta|_H = 0$ unless for some $w \in W(h, g)$, $w\lambda$ lifts to a character of $T^0$. For we know that $\Delta \Theta$ is smooth on each connected component $U$ of the set of $g \in g$ such that $e^\alpha(g) \neq 1$ for all real roots $\alpha$. Suppose we have

$$\Delta \Theta|_U = \sum_{w \in W(h, g)} a_w e^{w\lambda} \tag{7.17}$$

on some connected component $U$, with $a_w \neq 0$. Then $e^{w\lambda}$ is well-defined on $U$. Choose $h_0 \in U$; then $h_0 T^0 \subseteq U$, so $e^{w\lambda}$ must be well-defined on $T^0$.

2. Suppose $H = TA$ is a fundamental Cartan. Then there do not exist any real roots in $\Phi(h, g)$. Therefore $\Delta \Theta$ is smooth on any connected component of $H$, and $a_w \neq 0$ for some connected component implies $w\lambda$ lifts to a character of $T^0$.

3. Suppose $H$ is a compact Cartan subgroup, and $G$ connected. Up to conjugacy, we can arrange that $\theta H = H$, so $H \subseteq K$ is a Cartan subgroup of $K$. $G$ connected implies $K$ connected, and so $H$ is connected. Then we may write

$$\Delta \Theta|_H = \sum_{w \in W(h, g)} a_w e^{w\lambda} \tag{7.18}$$

and $a_w \neq 0$ implies $w\lambda \in \Lambda$, the weight lattice of $H_\mathbb{C}$ in $G_\mathbb{C}$. This weight lattice is invariant under the action of $W(h, g)$, so if $\Theta|_H$ is not identically zero, then $\lambda \in \Lambda$.

**Remark.** In the preceeding argument, we have assumed implicitly that $\rho \in \Lambda$. Without this assumption, we instead conclude that $\lambda + \rho \in \Lambda$. Also observe that $\rho \in \Lambda$ if and only if $w\rho \in \Lambda$ for every $w \in W(h, g)$.
Theorem 7.2. Suppose $\Theta$ vanishes at $\infty$. Then $G$ must contain a compact Cartan subgroup $H$, and $\Theta|_H$ is not identically zero. Moreover, if
\[
\Delta \Theta|_H = \sum_{w \in W(h, g)} a_w e^{w\lambda}
\]
then $a_w \neq 0$ implies $(w\lambda, \alpha) \neq 0$ for all compact (imaginary) roots $\alpha$.

Proof. Among all $\theta$-stable Cartan subgroups $H = TA$, choose one such that:

(i) $\Theta|_H$ is not identically zero.

(ii) $\dim T$ is maximal among all Cartans satisfying (i).

We must prove that $H = T$, or equivalently that $A = \{e\}$. Let $U$ be a connected component of the set of $h \in H$ such that $e^{\alpha}(h) \neq 1$ for all real roots $\alpha$. By (i) and (ii) plus the matching conditions, if
\[
\Delta \Theta|_H = \sum_{w \in W(h, g)} a_w e^{w\lambda}
\]
then for every real root $\alpha$, $a_w = a_{s_\alpha w}$. Hence $|\Delta| \Theta$ is invariant under all $s_\alpha$ for $\alpha$ real, and hence invariant under the subgroup of $W(h, g)$ generated by the $s_\alpha$ for $\alpha$ real. Hence $|\Delta| \Theta$ is continuous on $H$, and therefore $|\Delta| \Theta \to 0$ outside of compact subsets of $H$.

We may, and shall, suppose that $a_w = a_{s_\beta w}$ whenever $\beta$ is a root orthogonal to $w\lambda$. This implies the $w\lambda$ corresponding to those $w$ such that $a_w \neq 0$ are linearly independent. So if $a_w \neq 0$, then $e^{w\lambda} \to 0$ outside of compact subsets of $H$. This is so because by linear independence of $A$, for each $\mu \in a^*$,
\[
\sum_{w \in W(h, g)} a_w e^{w\lambda} = 0
\]
as it tends to 0 in all directions. Apply the same argument to any $t_0$-translate of $\Delta \Theta|_H$ and use linear independence. Then (7.21) implies $e^{w\lambda} \to 0$ outside compact subsets whenever $a_w \neq 0$.

But now $e^{w\lambda} \to 0$ outside compact subsets of $H$ forces $A$ to be trivial. It remains to show that $a_w \neq 0$ implies $(w\lambda, \alpha) \neq 0$ for all compact roots $\alpha$. If $\alpha$ is compact, then
\[
s_\alpha \in W(h, g) \subseteq W(h, g) \subseteq W(H, G) = N_G(H)/H
\]
but $N_G(H) \subseteq K$, so these inclusions are actually equalities. Now $\Theta|_H$ is fixed by $s_\alpha$, but $s_\alpha \Delta = -\Delta$. Therefore $s_\alpha$ acts by $-1$ on $\Delta \Theta|_H$. So if $(\alpha, w\lambda) = 0$, we find $a_{s_\alpha w} = a_w = -a_{s_\alpha w} = 0$.

Corollary 7.3. Suppose $H \subseteq G$ is a compact Cartan subgroup, and $\Theta$ an invariant eigendistribution which vanishes at $\infty$. Then $\Theta$ is completely determined by its restriction to $H$. 

43
8 Discrete Series

Let $G$ be as usual, and let $\hat{G}$ be the set of isomorphism classes of irreducible unitary representations of $G$. For each $i \in \hat{G}$, choose $(\pi_i, V_i)$ which represents $i$.

**Theorem 8.1** (Abstract Plancherel Theorem). We have

$$L^2(G) \cong \int_{i \in \hat{G}} \text{End}(V_i)_{HS} d\mu(i) \tag{8.1}$$

where $\mu$ is a measure on $\hat{G}$. The action of $G \times G$ is invariant under this isomorphism and preserves the inner product.

As a formal consequence, we get:

**Theorem 8.2.** The following conditions on an irreducible unitary representation $(\pi, V)$ are equivalent:

(a) $V \hookrightarrow L^2(G) \ell(G)$-invariantly and unitarily.

(b) $V \hookrightarrow L^2(G) \ell(G)$-invariantly and unitarily.

(c) $\text{End}(V)_{HS} \hookrightarrow L^2(G) (\ell \times r)(G)$-invariantly and unitarily.

(d) $\mu(\{\pi\}) > 0$.

(e) For some nonzero $u, v \in V$, the function $g \mapsto (\pi(g)u, v)$ lies in $L^2(G)$.

(f) For all $u, v \in V$, the function $g \mapsto (\pi(g)u, v)$ lies in $L^2(G)$.

We say that $\pi$ is square-integrable if it satisfies these conditions. The discrete series of $G$ is the set of isomorphism classes of irreducible, square-integrable, unitary representations of $G$.

Suppose for simplicity that $G$ is connected, so $K$ is connected as well. Then:

**Theorem 8.3** (Harish-Chandra). 1. The discrete series is nonempty if and only if $G$ contains a compact Cartan subgroup.

2. Suppose $H \subseteq K$ is a theta-stable Cartan subgroup of $G$. Then for each regular $\lambda \in \Lambda + \rho$ (meaning $(\alpha, \lambda) \neq 0$ for every root $\alpha$), there exists a unique invariant eigendistribution $\Theta_\lambda$ on $G$ which vanishes at $\infty$, such that

$$\Theta_\lambda|_{H_{rs}} = (-1)^q \sum_{w \in W(H,K)} e^{w \lambda} \epsilon(w) e^{\alpha/2} - e^{-\alpha/2}$$

(Unless $\rho \in \Lambda$ already, the numerator and denominator become well-defined only after multiplying by $e^\rho$.) Here $q = \frac{1}{2} \dim(G/K)$, which also equals $\frac{1}{2} \dim(g/t)$, which is half the number of noncompact roots. This is an integer.

3. Each $\Theta_\lambda$ is the character of a discrete series representation, and all discrete series characters arise in this way.
Remark. \( \Theta_\lambda = \Theta_{\tilde{\lambda}} \) if and only if \( \tilde{\lambda} = w \lambda \) for some \( w \in W(H, K) \). This means the discrete series is parameterized by
\[
W(H, K) \setminus \{ \lambda \in \Lambda + \rho | \lambda \text{ regular} \}. \tag{8.3}
\]
The \( k \)th left Sobolev space on \( G \) is the space
\[
\left\{ f \in L^2(G) \big| \ell(\zeta)f \in L^2(G) \forall \zeta \in U_k(g) \right\}. \tag{8.4}
\]

**Lemma 8.4.** Let \((\pi, V)\) be an irreducible unitary square-integrable representation, and \(\Theta_\pi\) its character. Then for \( k \gg 0 \), the map
\[
f \mapsto \int_G \Theta_\pi f \, dg \tag{8.5}
\]
(on \( f \in C^\infty_c(G) \)) is continuous with respect to the \( k \)th Sobolev norm of \( f \).

**Proof.** We had seen that for any test function \( \varphi \in C^\infty_c(G) \) and \( r \gg 0 \),
\[
\pi(\varphi) = \pi((1 + \Omega_K)^{-r}) \pi((1 + \Omega_K)^{-r}) \pi((1 + \Omega_K)^{-r}) \pi(\ell(1 + \Omega_K)^{2r}\varphi) \tag{8.6}
\]
\[
= \pi((1 + \Omega_K)^{-r}) \pi(\ell(1 + \Omega_K)^{r}\varphi) \tag{8.7}
\]
is a product of two Hilbert-Schmidt operators. So we have
\[
\left| \int_G \Theta_\pi \varphi \, dg \right| = |\text{tr } \pi(\varphi)| \leq \|\pi(1 + \Omega_K)^{-r}\|_{HS} \|\pi(\ell(1 + \Omega_K)^{r}\varphi)\|_{HS}. \tag{8.8}
\]
Now by the Plancherel theorem,
\[
\|\ell((1 + \Omega_K)^r)\varphi\|_{L^2(G)}^2 = \int_{i \in \hat{G}} \|\pi_i(\ell((1 + \Omega_K)^r)\varphi)\|_{HS}^2 d\mu(i) \tag{8.9}
\]
\[
\geq \|\pi(\ell(1 + \Omega_K)^r\varphi)\|_{HS}^2 \mu(\{i\}). \tag{8.10}
\]
So we find that
\[
\left| \int_G \Theta_\pi \varphi \, dg \right| \leq \frac{C}{\sqrt{\mu(\{\pi\})}} \|\ell(1 + \Omega_K)^r\varphi\|_{L^2(G)} \tag{8.11}
\]
which is bounded by the \( 2r \)th left Sobolev norm. \( \square \)

**Lemma 8.5.** Suppose \( \Theta \) is an invariant eigendistribution which, in the local expressions for the Weyl denominator, involves only constant coefficients. Then if \( \Theta \) is continuous in the \( r \)th Sobolev norm for some \( r > 0 \), then \( \Theta \) vanishes at \( \infty \).

**Corollary 8.6.** Discrete series characters are completely determined by their restrictions to the compact Cartan subgroup \( H \).
Sketch for $SL(2, \mathbb{R})$, which generalizes. Up to conjugacy, $G = SL(2, \mathbb{R})$ contains only one noncompact Cartan,

$$H = \left\{ \pm a_t = \pm \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix} \bigg| t \in \mathbb{R} \right\}. \quad (8.12)$$

We may suppose that $\Theta(-g) = \pm \Theta(g)$. Fix $\delta > 0$ with $\delta \ll 1$. Then define

$$G_\delta = \left\{ g \in G_{rs} \big| |e^\alpha(t) - 1| > \delta \text{ for both roots } \alpha \right\} \quad (8.13)$$

$$H^0_\delta = \left\{ a_t \big| t > \log(1 + \delta) \right\}. \quad (8.14)$$

We now have a map

$$G/H \times H_\delta \xrightarrow{F} G_\delta \quad (8.15)$$

$$\begin{array}{c}
(gH, h) \\
\end{array} \xrightarrow{F} \begin{array}{c}
ghg^{-1} \\
\end{array}$$

which is a diffeomorphism onto an open subset of $G_\delta$. We shall use it only on

$$\left\{ \text{small open neighborhood of } eH \text{ in } G/H \right\} \times H_\delta. \quad (8.16)$$

Now given $\psi \in C^\infty_c(H_\delta)$, we associate $\varphi \in C^\infty_c(G_\delta)$ whose support is contained in the image of the above set, where

$$F^*\varphi = \frac{\alpha}{e^t - e^{-t}} \psi \quad (8.17)$$

for $\alpha \in C^\infty_c(G/H)$ with supp($\alpha$) near $\{eH\}$ and $\int \alpha \, dg = 1$. By the Weyl integration formula,

$$\int_{G_\delta} \Theta \varphi \, dg = \int_{H_\delta} \Theta|_H(e^t - e^{-t})\psi \, dh. \quad (8.18)$$

**Fact.** The map $\psi \mapsto \varphi$ is continuous in Sobolev norm. This is left as an exercise.

Then the map

$$\psi \mapsto \int \Theta|_H(e^t - e^{-t})\psi \, dh \quad (8.19)$$

is continuous in the $r$th Sobolev norm for $r \gg 0$. Now

$$\Theta|_H(e^t - e^{-t}) = c_+ e^{\lambda t} + c_- e^{-\lambda t} \quad (8.20)$$

where $c_+ = c_-$ if $\lambda = 0$. Here we assume $\Re \lambda \geq 0$. Then define

$$\psi_n = \frac{1}{n} f \left( \frac{t}{n} - n \right) \quad (8.21)$$

46
where \( f \in C^\infty(\mathbb{R}) \) is supported on \((0, 1)\), positive on \((0, 1)\), and \( \int f \, dx = 1 \). The support of \( \psi_n \) is then contained in \([n^2, n^2 + n]\), and \( \int_H \psi_n \, dt = 1 \). Meanwhile the \( r \)th Sobolev norm of \( \psi_n \) tends to 0 as \( n \to \infty \), for every \( r \geq 0 \). Now

\[
\int_{H^s} \left(c_+ e^{\lambda t} + c_- e^{-\lambda t}\right) \psi_n e^{-\lambda t} \, dt \to \begin{cases} c_+ & \lambda \neq 0 \\ c_+ + c_- & \lambda = 0 \end{cases}
\]

but the limit must be 0. So \( c_+ = 0 \).

Now suppose \( H \subseteq G \) is a compact Cartan, so a torus. Assume \( H \subseteq K \). Define \( \tau_\pi \in C^{-\infty}(K) \), the \( K \)-character, by

\[
\int_K \tau_\pi \psi \, dk = \text{tr} \pi |_K (\psi).
\]

for \( \psi \in C^\infty(K) \).

**Remark.** The bounds on the \( K \)-multiplicities along with the Weyl dimension formula imply that \( \pi |_K (\varphi) \) is of trace class.

**Theorem 8.7** (Harish-Chandra). \( \tau_\pi \) is a function on \( K \cap G_{rs} \) and, as such a function, coincides with \( \Theta_\pi |_{K \cap G_{rs}} \).

This is for \((\pi, V)\) irreducible and admissible on a Hilbert space, with \( K \) acting unitarily.

**Proof.** Choose an orthonormal basis \( \{v_n\} \) of \( V \), obtained by taking the union of the orthonormal bases of the various \( K \)-isotypic subspaces. Then, with \( f_{m,n}(g) = (\pi(g)v_m, v_n) \), we have

\[
\Theta_\pi = \sum_n f_{n,n} \quad \text{(8.24)}
\]

\[
\tau_\pi = \sum_n f_{n,n} |_K. \quad \text{(8.25)}
\]

In greater detail, fix \( k_0 \in K \cap G_{rs} \), and consider the map

\[
p_{\mathbb{R}} \times (K \cap G_{rs}) \xrightarrow{F} G_{rs} \quad \text{(8.26)}
\]

\[
(\zeta, k) \mapsto \exp(\zeta)k \exp(-\zeta)
\]

At \((0, k_0)\), for \( \zeta_1 \in p_{\mathbb{R}} \) and \( \zeta_2 \in K \),

\[
F_*(\zeta_1, \zeta_2) = \zeta_2 + (\text{Ad} k_0^{-1} - 1)\zeta_1. \quad \text{(8.27)}
\]

For \( k_0 \) regular, \( \text{Ad} k_0^{-1} : p_{\mathbb{R}} \xrightarrow{\sim} p_{\mathbb{R}} \). So \( F \) is a diffeomorphism at least locally near \((0, k_0)\). It identifies products of (open neighborhood of 0 in \( p_{\mathbb{R}} \) \times \( (K\)-conjugacy classes) with \( G \)-conjugacy classes in \( G \), but locally near \((0, k_0)\).
Identify \( p_\mathbb{R} \) with \( G/K \) via exp. Let \( \{ \alpha_n \} \) be a sequence of smooth measures on \( p_\mathbb{R} \cong G/K \) supported near 0, which is an approximate identity at 0 in \( p_\mathbb{R} \). In particular, \( \text{supp}(\alpha_n) \setminus \{0\} \). Given \( \psi \in C^\infty(K) \) with support a small neighborhood of \( k_0 \) in \( K \cap G_{rs} \), define \( \varphi_n = (F^{-1})^*(\alpha_n \psi) \). Here \( \psi \) and \( \varphi_n \), in addition to \( \alpha_n \), are smooth measures. Then, by conjugation invariance of \( \Theta \),

\[
\int_K \Theta|_K \psi = \int_G \Theta \varphi_m \tag{8.28}
\]

\[
= \sum_n \int_G f_{nn} \varphi_m \tag{8.29}
\]

\[
\to \sum_n \int_K f_{nn} \psi \tag{8.30}
\]

The justification of switching limit and integration relies on the uniform boundedness principle. \( \square \)

With \( G \) having a compact Cartan \( H \subseteq K \), all roots in \( \Phi(h, g) \) are imaginary. We decompose \( \Phi = \Phi_c \amalg \Phi_n \), where \( \Phi_c = \Phi(h, \mathfrak{k}) \) is the set of compact roots and \( \Phi_n \) is the set of noncompact roots. Let \( p = \bigoplus_{\beta \in \Phi_n} g^\beta \). Then \( \#\Phi_n = \text{dim} G/K \) is even; let this number be \( 2q \).

The adjoint action of \( K \) on \( p \) preserves \( B(\cdot, \cdot) \), which is positive definite on \( p_\mathbb{R} \), so we get a map \( K \hookrightarrow SO(p_\mathbb{R}) = SO(2q) \). Now \( SO(2q) \) has a twofold cover \( \text{Spin}(2q) \), which is the universal cover if \( q \geq 3 \). \( \text{Spin}(2q) \) has two irreducible representations \( S_+ \) and \( S_- \) which do not descend to \( SO(2q) \), such that:

(i) \( S_+ \) and \( S_- \) have no weights in common.

(ii) \( \text{ch}(S_+) - \text{ch}(S_-) = \prod_i (e^{x_i/2} - e^{-x_i/2}) \) where \( \pm x_1, \ldots, \pm x_q \) are the weights of the standard representation of \( SO(2q) \).

(iii) \( \text{Hom}_{\text{Spin}(2q)}(\mathbb{C}^{2q} \otimes S_\pm, S_\pm) \) has dimension 1.

\( S_\pm \) are called the half-spin representations. For reference on the Clifford Algebra derivation, see Atiyah-Bott-Shapiro.

We can either lift the representations \( S_+ \) and \( S_- \) back to \( K \) or to a twofold cover \( \tilde{K} \) of \( K \). In the following, we’ll write \( \tilde{K} \) either way, even if we may use \( \tilde{K} = K \).

*Remark.* \( \pi_1(K) \cong \pi_1(G) \), so if we need to pass to a twofold cover of \( K \), this cover will live in a twofold cover of \( G \). However, that cover need not be linear, even if \( G \) is.

Choose a positive root system \( \Phi^+ \subseteq \Phi \), and set \( \Phi_n^+ = \Phi_n \cap \Phi^+ \). If we regard \( S_+ \) and \( S_- \) as representations of \( \tilde{K} \), they need not be irreducible, but
\[ ch_\tilde{K}(S_+) - ch_\tilde{K}(S_-) = \prod_{\beta \in \Phi^+_K} (e^{\beta/2} - e^{-\beta/2}) \]  
\[ = \prod_{\alpha \in \Phi^+}(e^{\alpha/2} - e^{-\alpha/2}) \]  
\[ = \sum_{w \in W(\mathfrak{h}, \mathfrak{g})} e(w)e^{\omega_\rho} \]  
\[ \prod_{\beta \in \Phi^+_K} (e^{\beta/2} - e^{-\beta/2}) \]  
\[ = \sum_j e(w_j) \sum_{w \in W(\mathfrak{h}, \mathfrak{g})} e(w)e^{\omega_{w_j \rho}} \]  
\[ \prod_{\alpha \in \Phi^+_K} (e^{\alpha/2} - e^{-\alpha/2}) \]  
(8.31)

(where \( w_1, \ldots, w_N \) is a complete set of coset representatives for \( W(\mathfrak{h}, \mathfrak{k}) \setminus W(\mathfrak{h}, \mathfrak{g}) \) chosen such that \( w_j \rho \) is \( \Phi^+_c \)-dominant for each \( j \))

\[ = \sum_j e(w_j) \cdot (\text{character of irreducible } \tilde{K}\text{-module of highest weight } w_j \rho - \rho_c) \]  
(8.32)

where \( \rho_c = \frac{1}{2} \sum_{\alpha \in \Phi^+_c} \alpha \).

Claim. There is no cancellation in this formula. That is,

\[ ch_\tilde{K}(S_+) = \sum_{e(w_j)=1} (\text{character of highest weight } w_j \rho - \rho_c) \]  
(8.33)

and similarly for \( ch_\tilde{K}(S_-) \).

It suffices to show that the sum of an even number of (distinct) positive noncompact roots cannot be equal to the sum of an odd number of (distinct) positive noncompact roots. Every positive root \( \alpha \) can be expressed uniquely as \( \alpha = \sum n_j \alpha_j \) where \( \alpha_1, \ldots, \alpha_r \) are the simple roots and the \( n_j \) are nonnegative integers. \( \alpha \) is noncompact if and only if the sum of the \( n_j(\alpha) \) over noncompact \( \alpha_j \) is odd. This is because \( [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}, \) and \( [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k} \).

We know that there are unique (up to scaling) \( \text{Spin}(\mathfrak{p})\)-invariant linear maps \( \mathfrak{p} \otimes S_\pm \rightarrow S_{\mp} \). We use the notation \( \zeta \otimes s \mapsto c(\zeta)s \). Here we view \( c(\zeta) \) as maps \( S_\pm \rightarrow S_{\mp} \). \( c \) is called Clifford multiplication.

Fact. \( c \) can be normalized so that

\[ c(\zeta)c(\eta) = -B(\zeta, \eta) \cdot 1. \]  
(8.34)

Now suppose \( (\pi, V) \) is an irreducible admissible representation of \( G \) with character \( \Theta_\pi \) and \( K \)-character \( \tau_\pi \). We had seen that

\[ \Theta_\pi \cdot \prod_{\beta \in \Phi^+_K} (e^{\beta/2} - e^{-\beta/2}) \bigg|_{H_{rs}} = \tau_\pi |_{H_{rs}}. \]  
(8.35)

Consider the maps
\[
V_{K\text{-fin}} \otimes S_\pm \xrightarrow{D_\pm} V_{K\text{-fin}} \otimes S_\mp \\
\quad v \otimes s \quad \mapsto \sum_i \zeta_i v \otimes e(\zeta_i) s
\]

where \(\{\zeta_1, \ldots, \zeta_{2q}\}\) is an orthonormal basis of \(p\).

**Theorem 8.8.** Letting \(s\) be the action of \(\tilde{K}\) on \(S_\pm\),

\[
D_- D_+ = D_+ D_- = \pi \otimes s(\Omega_K) - \pi(\Omega) \otimes 1 - (\rho, \rho) 1 + (\rho_c, \rho_c) 1.
\]  

(8.40)

**Corollary 8.9.** \(D_+\) and \(D_-\) have finite dimensional kernel and cokernel.

For formal reasons,

\[
\chi_{\tilde{K}} \ker D_+ - \chi_{\tilde{K}} \text{coker } D_+ = \chi_{\tilde{K}}(V_{K\text{-fin}} \otimes S_+) - \chi_{\tilde{K}}(V_{K\text{-fin}} \otimes S_-)
\]

\[
= \chi_{K}(V_{K\text{-fin}}) \cdot (\chi_{\tilde{K}} S_+ - \chi_{\tilde{K}} S_-)
\]

\[
= \tau_{\pi} \prod_{\beta \in \Phi^+_n} (e^{\beta/2} - e^{-\beta/2}).
\]

(8.41)

(8.42)

**Corollary 8.10.** The product

\[
\tau_{\pi} \prod_{\beta \in \Phi^+_n} (e^{\beta/2} - e^{-\beta/2})
\]

(8.43)

is a finite dimensional virtual character of \(\tilde{K}\).

We now have

\[
\Theta_{\pi} \prod_{\beta \in \Phi^+_n} \left( e^{\beta/2} - e^{-\beta/2} \right) \bigg|_{H^r s} = \tau_{\pi} \prod_{\beta \in \Phi^+_n} \left( e^{\beta/2} - e^{-\beta/2} \right) \bigg|_{H^r s}
\]

(8.44)

(8.45)

with the right hand side coming from a smooth function on \(K\).

Let \(M\) be a complete Riemannian manifold and \(\Gamma\) the group of isometries which acts properly discontinuously on \(M\). So \(\Gamma\backslash M\) has the structure of a Riemannian manifold, and \(M \to \Gamma\backslash M\) is an isometric covering map. We want \(\Gamma\backslash G\) to be compact. Let \(\mathcal{E}_+ \to M\) and \(\mathcal{E}_- \to M\) be hermitian vector bundles such that the action of \(\Gamma\) on \(M\) lifts to hermitian bundle maps. So \(\mathcal{E}_+\) and \(\mathcal{E}_-\) descend to hermitian vector bundles on \(\Gamma\backslash M\). We also want \(\Gamma\)-invariant linear differential operators

\[
D_\pm : C^\infty(M, \mathcal{E}_\pm) \to C^\infty(M, \mathcal{E}_\mp)
\]

(8.46)

which are elliptic, and with \(D_- = D_+^*\). Define

\[
\mathcal{H}_\pm = \ker D_\pm \cap L^2(M, \mathcal{E}_\pm).\]

(8.47)
By ellipticity, the orthogonal projection $L^2(M, \mathcal{E}_\pm) \to \mathcal{H}_\pm$ is an integral kernel operator with $C^\infty$ kernel $\Phi(x, y)$ on $M \times M$ with values in $\mathcal{E}_\pm \otimes \mathcal{E}_\pm^* = \mathcal{E}_\pm \otimes \overline{\mathcal{E}}_\pm$. These are $\Gamma$-invariant. We now define

$$\dim_\Gamma \mathcal{H}_+ = \int_{\Gamma \setminus M} \text{tr} \Phi(x, x) \, dm \in \mathbb{R} \quad (8.48)$$

$$\text{ind}_\Gamma D_+ = \dim_\Gamma \mathcal{H}_+ - \dim_\Gamma \mathcal{H}_- \in \mathbb{R} \quad (8.49)$$

**Claim.** $\dim_\Gamma \mathcal{H}_+ \geq 0$, and is positive if and only if $\mathcal{H}_+ \neq 0$.

**Reason.** Let $\{s_i\}$ be an orthonormal basis of $\mathcal{H}_+$. Then $\Phi = \sum s_i(x)s_i(y)$.

In particular, if $\text{ind}_\Gamma D_+ > 0$, then $\mathcal{H}_+ \neq 0$.

**Theorem 8.11** (Atiyah’s $L^2$-index Theorem). $\text{ind}_\Gamma D_+$ equals the index of $D_+$ on $\Gamma \setminus M$.

Returning to our situation, we can define differential operators

$$D_\pm : C^\infty(\tilde{G}/\tilde{K}, \mathcal{S}_\pm) \to C^\infty(\tilde{G}/\tilde{K}, \mathcal{S}_\mp) \quad (8.50)$$

and more generally,

$$D_\pm : C^\infty(\tilde{G}/\tilde{K}, \mathcal{V} \otimes \mathcal{S}_\pm) \to C^\infty(\tilde{G}/\tilde{K}, \mathcal{V} \otimes \mathcal{S}_\mp) \quad (8.51)$$

where $\mathcal{S}_\pm = \tilde{G} \times_\tilde{K} \mathcal{S}_\pm$ and $\mathcal{V} = \tilde{G} \times_\tilde{K} \mathcal{V}$, for $\mathcal{V}$ a finite dimensional representation of $\tilde{K}$. So $\mathcal{S}_+$, $\mathcal{S}_-$, and $\mathcal{V}$ are $\tilde{G}$-invariant vector bundles on $\tilde{G}/\tilde{K}$. $D_+$ and $D_-$ are called Dirac operators and their kernels are called twisted spinors.

To define $D_\pm$, we have

$$C^\infty(\tilde{G}/\tilde{K}, \mathcal{V} \otimes \mathcal{S}_\pm) \cong \left( C^\infty(\tilde{G}) \otimes \mathcal{V} \otimes \mathcal{S}_\pm \right) \tilde{K} \quad (8.52)$$

with $\tilde{K}$ acting on $C^\infty(\tilde{G})$ by $r$. Under this identification, we define

$$C^\infty(\tilde{G}) \otimes \mathcal{V} \otimes \mathcal{S}_\pm \xrightarrow{D_\pm} C^\infty(\tilde{G}) \otimes \mathcal{V} \otimes \mathcal{S}_\mp \quad (8.53)$$

$$f \otimes v \otimes s \xrightarrow{\sum r(\zeta_i) \otimes 1 \otimes c(\zeta_i)} \quad (8.54)$$

with $\{\zeta_i\}$ an orthonormal basis of $\mathfrak{p}$. These maps commute with the right action of $\tilde{K}$. $D_+$ and $D_-$ are elliptic because

$$D_-D_+ = D_+D_- = (r \otimes 1 \otimes s)(\Omega) - r(\Omega) \otimes 1 \otimes 1 - (\rho, \rho) 1 + (\rho_c, \rho_c) 1 \quad (8.54)$$

and $\Omega$ as a differential operator on $G/K$ is strongly elliptic.
In the following, either $V, S_+, S_-$ will either be representations of $K$, or $V$ and $S_+ \oplus S_-$ will be “genuine” representations of a twofold cover $\tilde{K}$ of $K$. (Here “genuine” means these representations do not descend to $K$.) In both cases, $V \otimes S_+$ and $V \otimes S_-$ are $K$-representations.

$D_+$ and $D_-$ are not only $\Gamma$-invariant, but $G$-invariant, and so $\Phi_\pm$ are $G$-invariant. This allows us to define

$$\dim_G \mathcal{H}_+ = \text{tr} \Phi_+ (X, X) \quad (8.55)$$

for any $X \in G/K$, and similarly define the $G$-index. We find that

$$\text{ind}_G D_+ = \text{vol}(\Gamma \backslash G/K) \text{ind}_G D_+. \quad (8.56)$$

Hence the index of $D_+$ on $\Gamma \backslash G/K$ equals the product of the volume of $\Gamma \backslash G/K$ and $\text{ind}_G(D_+)$. To compute the left hand side, let $U \subseteq G_\mathbb{C}$ be the compact real form such that $U \cap G = K$. Then in complete analogy to $D_+$ and $D_-$ on $G/K$, we get maps

$$\begin{array}{c}
C^\infty(U/K, V \otimes S_+) \\
\downarrow^{D_+} \\
C^\infty(U/K, V \otimes S_-)
\end{array} \quad (8.57)$$

Proposition 8.12 (Hirzebruch Proportionality Principle). Normalize the Haar measure on $U$ such that $\text{vol}(U/K) = 1$, and use the same normalization to define Haar measure on $G$. Then the index of $D_+$ on $\Gamma \backslash G/K$ is equal to

$$(-1)^q \text{vol}(\Gamma \backslash G/K) \text{ind}(D_+ \text{ on } U/K). \quad (8.58)$$

Idea of proof. The characteristic classes for both $G/K$ and $U/K$ are invariant and given by universal formula as elements of $\wedge P^*$. Here $\wedge P^*$ is the ring of $G$-invariant forms on $G/K$ and $U$-invariant forms on $U/K$. We have $T_{eK}G/K \cong \mathfrak{p}_R$ and $T_{eK}G/K \cong \mathfrak{p}_R^*$. The integrand for Hirzebruch-Riemann-Roch is a form of top degree, hence related on the two spaces by the factor $\text{dim } p = (-1)^q$. \hfill \square

We see that the index of $D_+$ on $U/K$ differs from the $G$-index of $D_+$ on $G/K$ by a factor of $(-1)^q$. It remains to compute the index of $D_+$ on $U/K$. By Peter-Weyl,

$$L^2(U) = \bigoplus W_i \otimes W_i^* \quad (8.59)$$

for $W_i$ irreducible (finite dimensional) $U$-modules. Then

$$L^2(U/K, V \otimes S_+) = \bigoplus W_i \otimes \left(W_i^* \otimes V \otimes S_+\right)^K \quad (8.60)$$

$$= \bigoplus W_i \otimes \text{Hom}_K(V, W_i \otimes S_+^*)^* \quad (8.61)$$

and $\text{Hom}_K(V, W_i \otimes S_+^*)$ has dimension equal to the multiplicity of $V$ in $W_i \otimes S_+^*$, which equals the multiplicity of $\text{ch}_K V|_H$ in $\text{ch}_K(W_i \otimes S_+^*)$. We find that the index of $D_+$ on $U/K$ is equal to
\[ \sum_{i \in \mathcal{U}} \dim W_i \cdot (\text{mult. of } V \text{ in } W_i \otimes (S_+^* - S_-)) = \sum_{i \in \mathcal{U}} \dim W_i \cdot (\text{mult. of } \text{ch}_K V \text{ in } \text{ch}_K(W_i) \text{ch}_K(S_+^* - S_-)). \]  

(8.62)

Now we have

\[ \text{ch}_K(S_+ - S-) = \prod_{\beta \in \Phi^+} (e^{\beta/2} - e^{-\beta/2}) \]  

(8.63)

for \( \Phi^+ \) a positive root system, which gives meaning to \( S_+ \) and \( S_- \). On dualizing, we find

\[ \text{ch}_K(S^*_+ - S^*) = (-1)^q \prod_{\beta \in \Phi^+} (e^{\beta/2} - e^{-\beta/2}). \]  

(8.64)

Let \( \lambda_i \) be \( \rho \) plus the highest weight of \( W_i \) with respect to \( \Phi^+ \). We find that our index is equal to

\[ (\dim \text{ of irreducible } G_\mathbb{C}-\text{module of highest weight } \lambda_i - \rho)(-1)^q \]  

(8.65)

times

\[ \text{mult of } \sum_{w \in W(H,K)} \epsilon(w) e^{w(\mu + \rho_c)} \prod_{\alpha \in \Phi^+_\mathbb{C}} (e^{\alpha/2} - e^{-\alpha/2}) \text{ in } \prod_{\beta \in \Phi^+} (e^{\beta/2} - e^{-\beta/2}) \sum_{w \in W(H_\mathbb{C},G_\mathbb{C})} \epsilon(w) e^{w \lambda_i} \prod_{\alpha \in \Phi^+_\mathbb{C}} (e^{\alpha/2} - e^{-\alpha/2}) \]  

(8.66)

which equals

\[ (-1)^q \dim(\cdots \lambda_i - \rho) \cdot \text{mult of } \sum_{w \in W(H,K)} \epsilon(w) e^{w(\mu + \rho_c)} \prod_{\alpha \in \Phi^+_\mathbb{C}} (e^{\alpha/2} - e^{-\alpha/2}) \text{ in } \sum_{w \in W(H_\mathbb{C},G_\mathbb{C})} \epsilon(w) e^{w \lambda_i} \prod_{\alpha \in \Phi^+_\mathbb{C}} (e^{\alpha/2} - e^{-\alpha/2}) \]  

(8.67)

where \( \mu \) is the highest weight of \( V \). This equals \( \epsilon(w_j) \) if \( \mu + \rho_c = w_j \lambda \) for some \( w_j \in W(H_\mathbb{C},G_\mathbb{C}) \), and 0 otherwise.

To compute the right hand side (that is, the index of \( D_+ \) on \( G/K \)), we use the Plancherel decomposition

\[ L^2(G) = \int_G W_i \hat{\otimes} W_i^* \ d\mu(i). \]  

(8.68)

**Claim.** The \( G \)-dimension of the \( L^2 \)-kernel of \( D_+ \) is equal to

\[ \int_G \dim \ker \left( D_+ : (W_i^* \otimes V \otimes S_+)^K \to (S_i^* \otimes V \otimes S_-)^K \right) d\mu(i). \]  

(8.69)

**Reason.** The \( L^2 \)-kernel of \( D_+ \) is

53
\[
\int_G W_i \otimes \left( \ker \left( D_+ : (W^*_i \otimes V \otimes S^+)^K \to (W^*_i \otimes W \otimes S^-)^K \right) \right) \, d\mu.
\] (8.70)

Now choose an orthonormal basis of this kernel so that the first few basis elements evaluated at \( e \) span \( \ker D_+ \), and the others vanish at \( e \).

By the same argument as in the \( U \)-case,

\[
\text{ind}_G(D_+) = \int_G \left( \dim(W^*_i \otimes V \otimes S^+)^K - \dim(W^*_i \otimes V \otimes S^-)^K \right) \, d\mu(i).
\] (8.71)

The integrand equals

\[
\text{mult of } V \text{ in } \left( (W_i)_{K\text{-fin}} \otimes S^+ - (W_i)_{K\text{-fin}} \otimes S^- \right)
\] (8.72)

which equals

\[
\text{mult of } \text{ch}_K(V) \text{ in } (-1)^q \prod_{\beta \in \Phi_+^c} (e^{\beta/2} - e^{-\beta/2}) \cdot \tau_i
\] (8.73)

for \( \tau_i \) the \( K \)-character of \( W_i \). This equals

\[
\text{mult of } \frac{\sum_{w \in W(H,K) \epsilon(w)e^{w(\mu+\rho)\frac{\lambda}{\rho}}}{\prod_{\alpha \in \Phi_+^c} (e^{\alpha/2} - e^{-\alpha/2})} \text{ in } (-1)^q \prod_{\beta \in \Phi_+^c} (e^{\beta/2} - e^{-\beta/2}) \cdot \Theta_i|_{H_{rs}}.
\] (8.74)

We know that

\[
\prod_{\beta \in \Phi_+^c} (e^{\beta/2} - e^{-\beta/2}) \cdot \Theta_i|_{H_{rs}} = \sum a_j \frac{\sum_{w \in W(H,K) \epsilon(w)e^{w\lambda_i}}}{\prod_{\alpha \in \Phi_+^c} (e^{\alpha/2} - e^{-\alpha/2})}
\] (8.75)

with the sum over \( w_j \in W(H_C,G_C) \) such that \( w_j \lambda_i \) is \( \Phi_+^c \)-dominant.

Now fix \( \lambda \in \Lambda + \rho \), with \( \lambda \) regular, and let \( \Phi_+^c \) consist of the positive roots with respect to \( \lambda \) (the roots \( \alpha \) such that \( \langle \alpha, \lambda \rangle > 0 \)). Let \( V \) be the finite dimensional irreducible \( "K" \)-module of highest weight \( \lambda - \rho_c \). Observe that since \( \lambda \) is \( \Phi_+^c \)-dominant and regular, \( \lambda - \rho_c \) is \( \Phi_+^c \)-dominant.

\( S^+_+ S^- \) contains an irreducible submodule of highest weight \( \rho_n \). So \( V \otimes (S^+_+ S^-) \) exists as a \( K \)-representation if and only if \( \lambda - \rho_c + \rho_n \in \Lambda \). But \( \lambda - \rho_c + \rho_n = \lambda - \rho + 2\rho_n \), which is in \( \Lambda \). Hence \( V \otimes (S^+_+ S^-) \) is a \( K \)-representation. The discussion above implies the dimension of the irreducible \( G_C \)-representation of highest weight \( \lambda - \rho \) is equal to

\[
\int_{\tilde{G}} \left( \dim(V^*_i \otimes V \otimes S^+)^K - \dim(V^*_i \otimes V \otimes S^-)^K \right) \, d\mu(i)
\] (8.76)

and the integrand is equal to
\[ (-1)^g \cdot \text{mult of } \frac{\sum_{w \in W(H,K)} \epsilon(w)e^{w\lambda}}{\prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2})} \in \Theta_{H,s} \prod_{\beta \in \Phi^+} (e^{\beta/2} - e^{-\beta/2}) \quad (8.77) \]

where the right hand expression is the restriction to \( H \) of a virtual finite dimensional \( K \)-module.

We shall sketch proofs of the following facts:

(a) There exists at most one irreducible unitarizable Harish-Chandra module \( M \) such that

\[ \dim \text{Hom}_K(M, V \otimes S_+) - \dim \text{Hom}_K(M, V \otimes S_-) \neq 0. \quad (8.78) \]

(Here unitarizable means there exists an infinitesimally invariant inner product.) For this \( M \), we have \( \dim \text{Hom}_K(M, V \otimes S_+)^K = 1 \) and \( \dim \text{Hom}_K(M, V \otimes S_-)^K = 0 \).

(b) If \( \tilde{V} \) is an irreducible \( K \)-module of highest weight \( w\lambda - \rho_c \) with \( w \in W(H_C, G_C) \) and \( w\lambda \Phi^+_c \)-dominant, then \( \dim \text{Hom}_K(M, \tilde{V} \otimes (S_+ + S_-)) = 0 \) unless \( w = e \), in which case \( \tilde{V} = V \).

(c) If \( \Theta \) is the character of a discrete series representation with infinitesimal character \( \chi_{\lambda} \), then \( \lambda \) is regular.

(d) Outside the discrete series, the set of \( i \in \hat{G} \) such that

\[ D_+ : (V_i^* \otimes V \otimes S_+)^K \rightarrow (V_i^* \otimes V \otimes S_-)^K \quad (8.79) \]

is not an isomorphism has measure zero.

We’ll assume these facts hold for now. The first formula implies that the set of \( i \in \hat{G} \) such that

\[ \dim(\cdots S_+)^K - \dim(\cdots S_-)^K \neq 0 \]

must have positive measure, and the difference must have a positive total integral. Up to a set of measure zero, this can happen only on a finite set in the discrete series. For this set, by (a), \( \dim(V_i^* \otimes V \otimes S_+)^K = 1 \) and \( \dim(\cdots S_-)^K = 0 \), so the first integral shows that \( \dim(\cdots S_+)^K - \dim(\cdots S_-)^K \) is nonzero for exactly one discrete series representation. For this class \( i \) in the discrete series, \( \mu(\{i\}) \) is equal to the dimension of the irreducible \( G_C \)-module of highest weight \( \lambda - \rho \). The quantity \( \mu(\{i\}) \) is called the formal degree. Going back to the meaning of the integral,

\[ \ker D_+ : L^2(G/K, V \otimes S_+) \rightarrow L^2(G/K, V \otimes S_-) \quad (8.80) \]

is the discrete series representation in question. Letting \( \Theta \) be its character, we would like to know \( \Theta \) (or equivalently, \( \Theta|_H \)). We know so far that the infinitesimal character must be \( \chi_{\lambda} \), and that

\[ \Theta|_{H,s} = \sum_{w_j} a_{w_j} \sum_{w \in W(H,K)} \epsilon(w)e^{w_{w_j}\lambda} \prod_{\beta \in \Phi^+} (e^{\beta/2} - e^{-\beta/2}) \quad (8.81) \]

with the sum over \( w_j \in W(H_C, G_C) \) such that \( w_j\lambda \) is \( \Phi^+_c \)-dominant. We must have \( a_e = (-1)^g \). Now given \( w_j \),
\[ a_{w_j} = (-1)^q \left( \dim(V_j^* \otimes \tilde{V} \otimes S^+)^K - \dim(V_j^* \otimes \tilde{V} \otimes S^-)^K \right) \quad (8.82) \]

for \( \tilde{V} \) having highest weight \( w_j \lambda - \rho_c \). By (b), this is zero unless \( w_j = e \). We end up with

\[ \Theta|_H = (-1)^q \sum_{w \in W(H,K)} \epsilon(w)e^{w\lambda} \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2}) \quad (8.83) \]

We know the restrictions of discrete series characters having infinitesimal character \( \chi_\lambda \) to \( H \) must be linearly independent, and all of the form

\[ \sum_{w_j} a_{w_j} \sum_{w \in W(H,K)} \epsilon(w)e^{w\lambda} \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2}) \quad (8.84) \]

By (c), since \( \lambda \in \Lambda + \rho \) was arbitrary except for being regular, we have constructed all discrete series representations, and described their characters restricted to \( H \).

Fix \( \Phi_e^+ \), but not \( \Phi^+ \). Let \( \mu \in \Lambda \) and \( V_\mu \) the irreducible \( K \)-module of highest weight \( \mu \). Then we have determined the kernel and cokernel of

\[ D_+: L^2(G/K, V_\mu \otimes \tilde{S}^+) \to L^2(G/K, V_\mu \otimes \tilde{S}^-) \quad (8.85) \]

provided \( \mu + \rho_c \) is \( \Phi \)-regular. Now suppose \( \mu + \rho_c \) were singular. In this case, \( \ker D_+ \) and \( \text{coker} D_+ \) will both vanish.

Consider \( \lambda \in \Lambda + \rho \) regular, \( \Phi^+ \) as before, and \( \Theta_\lambda \) the discrete series character with

\[ \Theta_\lambda|_H = (-1)^q \sum_{w \in W(H,K)} \epsilon(w)e^{w\lambda} \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2}) \quad (8.86) \]

\[ = (-1)^q \sum_{w \in W(H,K)} \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2}) e^{w\lambda} \quad (8.87) \]

\[ = (-1)^q \sum_{w \in W(H,K)} \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2}) e^{w(\lambda - \rho)} \quad (8.88) \]

(we now give a formal argument)

\[ = \sum_{w \in W(H,K)} \sum_{n_1, \ldots, n_q \geq 0} e^{w(\lambda + \rho_n + n_1 \beta_1 + \cdots + n_q \beta_q)} \prod_{\alpha \in \Phi_e^+} (e^{\alpha/2} - e^{-\alpha/2}) \quad (8.89) \]

(where \( \Phi_n^+ = \{ \beta_1, \ldots, \beta_q \} \))

\[ = \sum_{n_1, \ldots, n_q \geq 0} \sum_{w \in W(H,K)} \epsilon(w)e^{w(\lambda + \rho_n + n_1 \beta_1 + \cdots + n_q \beta_q)} \prod_{\alpha \in \Phi_e^+} (e^{\alpha/2} - e^{-\alpha/2}) \quad (8.90) \]

Each term is either \( \pm \) an irreducible character of \( K \), or zero. So our sum is an integer linear combination of irreducible characters of \( K \).
Theorem 8.13 (Blattner’s Conjecture). This is the $K$-character of the discrete series representation with character $\Theta_\lambda$.

We’ll suppose this holds.

Claim. $\lambda + \rho_n - \rho_c$ is $\Phi_c^+$-dominant, because $\lambda - \rho_c$ is the highest weight of “$V$”, and $\rho_n$ is the highest weight of $S_+ \oplus S_-$ (in fact the highest weight of one irreducible constituent of $S_+$), so $\rho_n$ is $\Phi_c^+$-dominant.

So $\lambda + \rho_n - \rho_c$ is the highest weight of an irreducible $K$-module, and in fact is the highest component in $V \otimes S_+$.

Our sum lies in the cone spanned by $\Phi_\rho^+$ with vertex $\lambda + \rho_n - \rho_c$, which is $\Phi_c^+$-dominant. The “minimal highest weight” of $K$-types in the discrete series representation in question occurs with multiplicity 1.

Let $M$ be the irreducible Harish-Chandra module of the discrete series representation in question, with $K$-multiplicity as predicted by Blattner’s Conjecture.

Claim. $\dim \text{Hom}_K(M, V \otimes (S_+ \oplus S_-)) = 1$ and comes from the irreducible $K$-type of highest weight $\lambda - \rho_c + \rho_n$.

Reason. Suppose $\lambda - \rho_c + \rho_n + n_1\beta_1 + \cdots + n_q\beta_q = \lambda - \rho_c + \rho_n$ minus a nonzero sum of distinct positive noncompact roots. Then the right hand side is a potential highest weight of a $K$-type in $V \otimes (S_+ \oplus S_-)$ other than $\lambda - \rho_c + \rho_n$. But this would imply $\lambda = \lambda$ plus a nonzero sum of positive roots, which is a contradiction by comparing their lengths.

This argument depended on the following: if $V_1$ and $V_2$ are irreducible $K$-modules of highest weight $\mu_1$ and $\mu_2$, respectively, then:

1. The irreducible $K$-module of highest weight $\mu_1 + \mu_2$ occurs in $V_1 \otimes V_2$ exactly once.

2. Any irreducible constituent of $V_1 \otimes V_2$ has a highest weight expressible as $\mu_1 + r$, where $r$ is a weight of $V_2$, and the multiplicity of this constituent is less than or equal to the multiplicity of the weight $r$.

3. Some $W_K$-translate of $\mu_1 + (\text{lowest weight of } V_2)$ is the highest weight of an irreducible constituent of $V_1 \otimes V_2$ with multiplicity one (this is difficult). In the case where this sum is $\Phi_c^+$-dominant, however, this follows from (1).

\[\square\]

Claim. Suppose $w_j \in W(\mathfrak{h}, \mathfrak{g})$ is such that $w_j \lambda$ is $\Phi_c^+$-dominant, but $w_j \neq e$. Let $\tilde{V}$ be the irreducible $K$-module of highest weight $w_j \lambda - \rho - c$. Then

$$\text{Hom}_K(M, \tilde{V} \otimes (S_+ \oplus S_-)) = 0.$$  (8.91)

Reason. Suppose not. Then $M$ must contain an irreducible $K$-constituent of highest weight

$$w_j \lambda - \rho_c + (\text{some weight of } S_+ \oplus S_-) = w_j \lambda - \rho_c + \sum_{j=1}^{q} \epsilon_j \cdot \frac{\beta_j}{2}$$  (8.92)
(for $\Phi^+_n = \{\beta_1, \ldots, \beta_q\}$ and $\epsilon_j = \pm 1$)

$$w_j \lambda - \rho_c - \rho_n + \sum_{\epsilon_j = 1}^{\beta_j}. \quad (8.93)$$

This must equal $\lambda - \rho_c + \rho_n + \sum n_j \beta_j$ with $n_j \geq 0$. So we have

$$w_j \lambda = \lambda + \left(2\rho_n - \sum \beta_j\right) + \sum n_j \beta_j. \quad (8.94)$$

and the right hand side is a sum of positive noncompact roots. However, $w_j \lambda$ must be of the form $\lambda$ minus a nonempty sum of positive roots. This is a contradiction. \qed

We use the following about $M$:

1. The irreducible $K$-module of highest weight $\lambda - \rho_c + \rho_n$ occurs in $M$ with multiplicity one.

2. All of the $K$-multiplicities are at most those predicted by Blattner’s conjecture.

**Suppose $G$ is a Lie group and $H \subseteq G$ is a closed subgroup. Let $\mathfrak{g}_\mathbb{R}$ and $\mathfrak{h}_\mathbb{R}$ be their Lie algebras, and $\mathfrak{g}$ and $\mathfrak{h}$ their complexifications.**

**Theorem 9.1 (Nijenhuis).** The $G$-invariant structures of complex manifold on $G/H$ correspond bijectively to Lie subalgebras $\mathfrak{q} \subseteq \mathfrak{g}$ such that:

(i) $\mathfrak{q} + \overline{\mathfrak{q}} = \mathfrak{g}$, with the complex conjugation relative to $\mathfrak{g}_\mathbb{R}$.

(ii) $\mathfrak{q} \cap \overline{\mathfrak{q}} = \mathfrak{h}$.

(iii) $\mathfrak{q}$ is invariant under $\text{Ad} H$.

This correspondence is given by

$$\mathfrak{q}/\mathfrak{h} \subseteq \mathfrak{g}/\mathfrak{h} = T^c_{eH}G/H \sim T^{0,1}_{eH}G/H. \quad (9.1)$$

**Remark.** If $H$ is connected, then (ii) implies (iii).

Now suppose $G$ is connected as before, and $H \subseteq G$ is a compact Cartan (that is, a torus). Let $\Phi^+$ be a positive root system, and define

$$n = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}^{-\alpha}. \quad (9.2)$$

$$\mathfrak{q} = \mathfrak{h} \oplus n. \quad (9.3)$$
Let \( p : G \to G/H \) be the projection, and \( U \subseteq G/H \) open. Then
\[
\mathcal{O}_U \cong \left\{ f \in C^\infty(p^{-1}U) \mid r(h \oplus n)f = 0 \right\}. \tag{9.4}
\]

Now suppose \( \mu \in \Lambda \). Let \( \mathbb{C}_\mu \) be the one-dimensional \( H \)-module such that \( H \) acts by \( e^\mu \). Now define
\[
\mathcal{L}_\mu = G \times_H \mathbb{C}_\mu, \tag{9.5}
\]
a \( G \)-invariant \( C^\infty \) line bundle over \( G/H \), whose restriction to \( eH \) is \( \mathbb{C}_\mu \) as an \( H \)-module. With \( U \) and \( p \) as above,
\[
C^\infty(U, \mathcal{L}_\mu) \cong \left\{ f \in C^\infty(p^{-1}U) \mid f(gh^{-1}) = e^\mu(h)f(g) \forall h \in H, g \in G \right\}. \tag{9.6}
\]

**Fact.** \( \mathcal{L}_\mu \) has a unique structure of a \( G \)-invariant holomorphic line bundle such that
\[
\mathcal{O}_U(\mathcal{L}_\mu) \cong \left\{ f \in C^\infty(p^{-1}U) \mid f(gh^{-1}) = e^\mu(h)f(g), r(n)f = 0 \right\}. \tag{9.7}
\]

**Remark.** Let \( X \) be the flag variety of \( G_\mathbb{C} \). Consider the \( G \)-orbit of \( h \oplus n \). It is open in \( X \), and as such, isomorphic as complex manifolds to \( G/H \) with the invariant complex structure corresponding to \( h \oplus n \).

To prove this, we just need to identify the Cauchy-Riemann equations.

Let \( D = G/H \) with this invariant complex structure, and let \( S = K/H \subseteq D \).

**Claim.** This is a closed complex submanifold.

**Reason.** We have
\[
\mathfrak{k} = \left( \mathfrak{h} \oplus (\mathfrak{k} \cap \mathfrak{n}) \right) + \left( \mathfrak{h} \oplus (\mathfrak{k} \cap \mathfrak{n}) \right) \]
and the intersection of the two summands is \( \mathfrak{h} \). \( \square \)

**Theorem 9.2.** There exists an “exhaustion function” \( \varphi \) on \( D \), meaning \( \varphi : D \to \mathbb{R}_{\geq 0} \) is \( C^\infty \) with \( \varphi \to \infty \) outside of compact subsets, such that the matrix \( \left( \frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_j} \right) \) has everywhere at least
\[
q = \dim_\mathbb{C} D - \dim_\mathbb{C} S = \frac{1}{2} \dim G/K \tag{9.9}
\]
strictly positive eigenvalues.

The matrix \( \left( \frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_j} \right) \) is called the Levi form of \( \varphi \).

**Corollary 9.3** (consequence of theorem of Andreotti-Grauert). For any coherent sheaf \( \mathcal{F} \) on \( D \) and any \( p > s = \dim_\mathbb{C} S \),
\[
H^p(D, \mathcal{F}) = 0. \tag{9.10}
\]
Suppose $\mu \in \Lambda$ is such that $\mu + \rho_c$ is $\Phi_c$-regular and $\Phi_c^+$-antidominant. Recall:

**Theorem 9.4** (Borel-Weil-Bott for $S$). Suppose $\lambda \in \Lambda$ is arbitrary. Then

$$H^p(S, \mathcal{O}_S(L_\lambda)) = \begin{cases} 0 & \lambda + \rho_c \text{ is } \Phi_c\text{-singular} \\ 0 & \lambda + \rho_c \text{ is } \Phi_c\text{-regular, } p \neq k(\lambda + \rho_c) \\ \star & \lambda + \rho_c \text{ is } \Phi_c^+\text{-regular, } p = k(\lambda + \rho_c). \end{cases} \quad (9.11)$$

Here

$$k(\lambda + \rho_c) = \#\{\alpha \in \Phi_c^+ | (\lambda + \rho_c, \alpha) < 0\}. \quad (9.12)$$

And $\star$ is irreducible of highest weight $w(\lambda + \rho_c) - \rho_c$, where $w$ is chosen so that $w(\lambda + \rho_c)$ is $\Phi_c^+$-dominant.

Now $H^p(S, \mathcal{O}_S(L_\mu)) = 0$ unless $p = s = \dim C_S$, and is irreducible of highest weight $w_\ell(\mu + \rho_c) - \rho_c$ if $p = s$, where $w_\ell \in W(H, K)$ is the longest element.

Let $\mathcal{I}_S$ be the ideal sheaf of $S$, a subsheaf of $\mathcal{O}_D$. Then

$$0 \to \mathcal{I}_S \cdot \mathcal{O}_D(L_\mu) \to \mathcal{O}_D(L_\mu) \to \mathcal{O}_S(L_\mu) \to 0 \quad (9.13)$$

is an exact sequence of sheaves. So upon taking cohomology,

$$H^s(D, \mathcal{O}_D(L_\mu)) \to H^s(S, \mathcal{O}_S(L_\mu)) \to 0. \quad (9.14)$$

Let $N_S = T^{1,0}_S D$ denote the (holomorphic) normal bundle of $S \subseteq D$, and $(N^*_S)^{(k)}$ be the $k$th symmetric power of the dual bundle $N^*_S$. Then we get the exact sequence

$$0 \to T^{k+1}_S \to T^k_S \to \mathcal{O}_S((N^*)^k) \to 0. \quad (9.15)$$

Tensoring over $\mathcal{O}_D$ with $\mathcal{O}_D(L_\mu)$, we see that

$$0 \to H^s(D, T^{k+1}_S \mathcal{O}(L_\mu)) \to H^s(D, T^k_S \mathcal{O}(L_\mu)) \to H^s(S, \mathcal{O}_S(L_\mu \otimes (N^*)^k)) \to 0. \quad (9.16)$$

This gives us a decreasing filtration of $H^s(D, \mathcal{O}_D(L_\mu))$ by $H^s(D, T^k_S \mathcal{O}_D(L_\mu))$ with successive quotients $H^s(S, \mathcal{O}_S(L_\mu \otimes (N^*)^k))$.

**Theorem 9.5.** $H^s(D, \mathcal{O}(L_\mu))$ has a natural Frechét structure, and the resulting representation of $G$ is admissible, with $K$-multiplicities

$$\bigoplus_{k \geq 0} H^s(S, \mathcal{O}_S(L_\mu \otimes (N^*)^k)). \quad (9.17)$$

Also $H^p(D, \mathcal{O}_D(L_\mu)) = 0$ if $p \neq s$.

From now on, suppose $(\mu + \rho_c, \alpha) \ll 0$ for all $\alpha \in \Phi_c^+$. 
Remark. Eventually this can be relaxed.

This implies that $H^k(S, \mathcal{O}_S((N^*)^{(k)} \otimes L_\mu)) = 0$ for $k < s$. (We shall give the reason later.) So the successive quotients of the filtration are, as $K$-modules, $H^*(S, \mathcal{O}_S((N^*)^{(k)} \otimes L_\mu))$.

Let $V$ be the irreducible $K$-module $H^*(S, \mathcal{O}_S(L_\mu))$. By Borel-Weil-Bott, the highest weight is $w_\ell(\mu + \rho_c) - \rho_c$, where $w_\ell \in W(\mathfrak{h}, \mathfrak{k})$ is the longest element. Also observe that $w_\ell(\mu + \rho_c) - \rho_c = w_\ell - 2\rho_c$. The lowest weight is $\mu + 2\rho_c$.

Consider the fibration $D \cong G/H \to G/K$. This is $G$-equivariant, and its fibers are the $G$-translates of $S = K/H$. This is not holomorphic, since in general, $G/K$ has no $G$-invariant complex structure. However, if $p = p_+ \oplus p_-$, where $p_\pm = \bigoplus_{\beta \in \Phi^\pm} g^\beta$ are both $K$-invariant (so that $p = p_+ \oplus p_-$ is preserved by the $K$-action), then

$$g = (p_- \oplus \mathfrak{f}) + (p_- \oplus \mathfrak{f})$$

with the intersection of the summands being $\mathfrak{f}$. So by Nijenhuis, there exists an invariant complex structure such that $T_{eK}^0(G/K) \cong p_-$. These give hermitian symmetric spaces, also called bounded symmetric domains. Suppose this is the case. Then $G/H \to G/K$ is holomorphic if and only if $n \supset p_-$. The positive root systems that make this come about are of hermitian type, which means that if $\beta_1, \beta_2 \in \Phi^+_n$, then $\beta_1 + \beta_2 \notin \Phi^+$.

We return to the general case (so $G/K$ may not have an invariant complex structure). Let $\mathcal{V} \to G/K$ be a $G$-invariant $C^\omega$ vector bundle modeled on $V$. By restriction of cohomology to fibers, we get a $G$-invariant, continuous* map

$$H^*(D, \mathcal{O}(L_\mu)) \to C^\infty(G/K, \mathcal{V}).$$

Consider the map $(\ast) : V \otimes p \to V^+$, the $K$-invariant subspace consisting of all irreducible constituents of lowest weight $\mu + 2\rho_c + \beta$ for $\beta \in \Phi^+_n$. The map $(\ast)$ is the symbol of a unique first order $K$-invariant differential operator

$$D_+ : C^\infty(G/K, \mathcal{V}) \to C^\infty(G/K, \mathcal{V}^+)$$

with $\mathcal{V}^+$ modeled on $V^+$. The “very antidominant” hypothesis implies $D_+$ is elliptic, and has real analytic coefficients.

**Theorem 9.6.** We get an exact sequence

$$0 \to H^*(D, \mathcal{O}(L_\mu)) \to C^\infty(G/K, \mathcal{V}) \xrightarrow{D_+} C^\infty(G/K, \mathcal{V}^+)$$

and $H^k(D, \mathcal{O}(L_\mu)) = 0$ for $k < s$.

*Proof.* Use the “Leray spectral sequence”, plus the vanishing theorem for $H^p(S, \mathcal{O}(\wedge^k N^* \otimes L_\mu))$ for $p < s$. □

Under this map, the image of $H^*(D, \mathcal{T}^k_S \mathcal{O}(L_\mu))$ consists of sections in the kernel of $D_+$ which vanish to order $k$ at $eK$. 

61
Corollary 9.7.
\[
\bigcap_{k \geq 0} \text{im} H^k(D, \mathcal{T}^k_S \mathcal{O}(\mathcal{L}_\mu)) = 0. \tag{9.22}
\]

Corollary 9.8. The natural topology on \( H^*(D, \mathcal{O}(\mathcal{L}_\mu)) \) is Hausdorff.

We had seen (modulo vanishing theorems) that the Frechét representation \( H^*(D, \mathcal{O}(\mathcal{L}_\mu)) \) has a formal \( K \)-decomposition \( H^*(S, \mathcal{O}_S((N^*)^k \otimes \mathcal{L}_\mu)) \). We have \( T_{eH}^1 D \cong g/\mathfrak{h} \oplus \mathfrak{n} \), and \( T_{eH}^1 S \cong \mathfrak{k}/\mathfrak{h} \oplus (\mathfrak{n} \cap \mathfrak{k}) \). Therefore the normal space to \( S \) in \( D \) at \( eH \) is isomorphic to \( g/(\mathfrak{k} + \mathfrak{n}) \). The dual of this, via the Killing form \( B \), is \( \mathfrak{n} \cap \mathfrak{p} \). So \( (N^*)^k \), viewed as a \( K \)-invariant holomorphic vector bundle, is modeled on \((\mathfrak{p} \cap \mathfrak{n})^k \) as \( \mathfrak{h} \oplus (\mathfrak{k} \cap \mathfrak{n}) \)-module.

(The action of \( \mathfrak{g} \cap \mathfrak{n} \) on \( C^\infty(K) \otimes (\text{module in question}) \) by infinitesimal right translation on \( C^\infty(K) \), by \( \mathfrak{k} \cap \mathfrak{n} \)-action on the module in question, is the Cauchy-Riemann equations. This is similar to the case of line bundles.)

We want a vanishing theorem, below the top degree \( s \), for the holomorphic vector bundles, \( K \)-equivariant, modeled on \((\mathfrak{p} \cap \mathfrak{n})^k \otimes \mathcal{C}_\mu\).

Fact (important). Any \( K \)-invariant holomorphic vector bundle on \( K/H \), modeled on a finite dimensional \( K \)-representation (thought of as an \( \mathfrak{h} \oplus (\mathfrak{k} \cap \mathfrak{n}) \)-module), is holomorphically, but not equivariantly, trivial.

Proof. Let \( V \) be the \( K \)-module. Then for any nonzero \( v \in V \), \( k \mapsto k^{-1}v \) is a section which is everywhere nonzero. This implies \( \mathfrak{p} \cap \mathfrak{n} \) is a subbundle of a trivial vector bundle. Standard vanishing theorems, plus the assumption on \( \mu \), then imply \( H^p(S, \mathcal{O}_S((N^*)^k \otimes \mathcal{L}_\mu)) = 0 \) for \( p < 0 \). \( \square \)

\( \mathfrak{p} \cap \mathfrak{n} \), as a \( \mathfrak{h} \oplus (\mathfrak{k} \cap \mathfrak{n}) \)-module, has a composition series with one-dimensional quotients, all of the form \( \mathfrak{g}^{-\beta} \) with \( \beta \in \Phi_n^+ \). Now calculate \( \bigoplus_{k=0}^\infty H^*(S, \mathcal{O}_S((N^*)^k \otimes \mathcal{L}_\mu)) \) in Euler characteristic, and use Borel-Weil-Bott plus the vanishing theorems. We conclude that the formal \( K \)-character of \( H^*(D, \mathcal{O}(\mathcal{L}_\mu)) \) is

\[
\epsilon(w_\ell) \sum_{n_1, \ldots, n_q \geq 0} \frac{\sum_{w \in W(\mathfrak{h}, \mathfrak{g})} \epsilon(w) w^{(\mu + \rho_c - \sum n_j \beta_j)} \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2})}{\prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2})} = (-1)^s \sum_{w \in W(\mathfrak{h}, \mathfrak{g})} \epsilon(w) e^{w(\mu + \rho_c + \rho_\mathfrak{n})}. \tag{9.23}
\]

\( \mu + \rho_c + \rho_\mathfrak{n} \) is dominant for \( -\Phi^+_c \). Let \( \lambda = \mu + \rho \in \Lambda + \rho \). Then \( \lambda \) is \( -\Phi^+_c \)-dominant, but assume that it is \( -\Phi^+ \)-dominant. Then the above becomes the restriction to \( K_{rs} \) of \( \Theta_\lambda \).

Proposition 9.9. \( H^*(D, \mathcal{O}(\mathcal{L}_\mu)) \) contains a unique nonzero \( G \)-invariant closed subspace which contains the \( K \)-type \( V \).

Proof. Let \( M \) be a nonzero \( G \)-invariant closed subspace. Then \( M \hookrightarrow \ker D_+ \), and there exists \( m \in M \) which takes a nonzero value somewhere. By translation, we may assume there exists \( m \in M \) such that \( m|_{eK} \in V \) is nonzero. Now \( \ker D_+ \rightarrow V = \mathcal{V}_{eK} \) has \( s \mapsto s(eK) \), is \( K \)-invariant, so \( M \) contains the \( K \)-type \( V \). \( \square \)

This implies that the intersection of all nonzero \( G \)-invariant closed subspaces is nonzero, irreducible, and contains the \( K \)-type \( V \).
Returning to the original setting, with $\lambda \in \Lambda + \rho$, $\lambda \Phi^+_c$-dominant, regular, and (for now) sufficiently $\Phi^+_c$-regular. Then using the proposition, we can state what we have proven as follows: let $M_\lambda$ be the Harish-Chandra module of the unique irreducible subrepresentation of $H^s(D, O(\mathcal{L}_{\lambda-\rho}))$. Then:

1. It contains the $K$-module of highest weight $\lambda - \rho + \rho_n$ once.

2. All $K$-types occurring in $M_\lambda$ have multiplicities less than or equal to the multiplicities in Blattner’s Conjecture. In particular, they have highest weights of the form

$$\lambda - \rho + \rho_n + (\text{sum of positive noncompact roots}).$$  

This implies:

3. $M_\lambda$ contains no $K$-type of highest weight $\lambda - \rho + \rho_n - \beta$ with $\beta \in \Phi^+_c$.

Conditions (1) and (3) determine the irreducible Harish-Chandra module $M_\lambda$ uniquely.

Remark. We shall show that $M_\lambda$ is unique in the above sense among those irreducible Harish-Chandra modules which arise from irreducible admissible representations. In arguments which will appear later, this will be enough. But the “Casselman embedding theorem” implies the general statement.

Proof. Let $\pi$ be an irreducible admissible representation of $G$ satisfying (1) and (3). As before, let $V$ be the irreducible $K$-module of lowest weight $\mu + 2\rho_c$, and let $V^+$ be as before. If $v$ is a $C^\infty$ vector for $\pi$, and $p$ is the $K$-invariant projection from the representation space of $\pi$ to $V$ (which exists and is surjective by (1)), then define

$$F_v(g) = p(\pi(g)^{-1}v) \in V.$$  

(9.25)

For $g \in G$ and $k \in K$, we have $F_v(gk) = \pi(k^{-1})F_v(g)$. So $F_v$ can be viewed as an element of $C^\infty(G/K, \mathcal{V})$. Also $F_{\pi(h)v}(g) = \ell(h)F_v(g)$. So the map $v \mapsto F_v$ is a $G$-invariant continuous map from the space of $C^\infty$ vectors for $\pi$ to $C^\infty(G/K, \mathcal{V})$. The image lies in the kernel of $D_+$ because $\pi$ satisfies (3), and the $K$-types of $V^+$ are exactly the ones that (3) prevents from occurring for $\pi$. So the space of $C^\infty$ vectors of $\pi$ has as its Harish-Chandra module the unique irreducible subrepresentation of $H^s(D, O(\mathcal{L}_\mu))$.  

At this point, we have proved that for any $\lambda \in \Lambda + \rho$ which is $\Phi^+_c$-dominant, regular, and (for now) very $\Phi_c$-regular, then there exists an irreducible representation of $G$ on a Frechét space which:

1. contains the $K$-type of highest weight $\lambda - \rho + \rho_n$.

2. The multiplicities are bounded by those predicted by Blattner’s Conjecture. In particular:

3. it contains no $K$-type of highest weight $\lambda - \rho + \rho_n - B$ for $B$ a nonempty sum of distinct positive noncompact roots.
Let $V_{\lambda - \rho_c}$ be the irreducible $K$-module (or $\tilde{K}$-module) of highest weight $\lambda - \rho_c$. Now suppose $N_\lambda$ is the Harish-Chandra module of an irreducible unitary representation such that $\Omega$ acts by $(\lambda, \lambda) - (\rho, \rho)$ and $V_{\lambda - \rho_c} \hookrightarrow N_\lambda \otimes (S_+ \oplus S_-)$. Recall that we have maps $D_\pm : N_\lambda \otimes S_\pm \to N_\lambda \otimes S_\mp$. If $N_\lambda$ occurs in the $L^2$ kernel or cokernel of

$$D_+ : L^2(G/K, V_{\lambda - \rho_c} \otimes S_+) \to L^2(G/K, V_{\lambda - \rho_c} \otimes S_-)$$  \hspace{1cm} (9.26)

then, as we had shown, $N_\lambda$ must contain a $K$-type in $V_{\lambda - \rho_c} \otimes (S_+ \oplus S_-)$ and it must lie in the kernel or cokernel of $D_+ : N_\lambda \otimes S_+ \to N_\lambda \otimes S_-$. So because $D_+ D_- = D_- D_+$ must be less than or equal to 0 on a $K$-type in $N_\lambda$ of highest weight $\lambda - \rho_c + \rho_n - B$ for $B$ a sum of distinct positive noncompact roots, we find

$$0 \geq \Omega - \Omega_K + (\rho, \rho) - (\rho_c, \rho_c)$$  \hspace{1cm} (9.27)

acting on any $K$-type in $N_\lambda \otimes (S_+ \oplus S_-)$, in particular the one of highest weight $\lambda - \rho_c - B$, with $B$ as above. This equals $(\lambda - \rho_c + \rho_n - B) - \rho_n$. This is $\Phi_{\rho_c}^+$-dominant because of the hypothesis that $\lambda$ was very $\Phi_{\rho_c}^+$-regular. We obtain, in this case,

$$0 \geq \left( (\lambda, \lambda) - (\rho, \rho) \right) - \left( (\lambda - B, \lambda - B) - (\rho_c, \rho_c) \right) + (\rho, \rho) - (\rho_c, \rho_c)$$

$$= (\lambda, \lambda) - (\lambda - B, \lambda - B)$$

$$\geq 2(\lambda, B).$$ \hspace{1cm} (9.30)

$\lambda$ being very dominant implies $B$ must be 0. This proves what we want, and also that $\Omega$ acts on $N_\lambda$ by $(\lambda, \lambda) - (\rho, \rho)$. The same argument also shows that if $N_\lambda \subseteq \ker D_+$, then $N_\lambda$ cannot contain any $K$-type of highest weight $\lambda - \rho_c + \rho_n - \beta$ for $\beta \in \Phi_{\rho_c}^+$. These inequalities are called the Dirac inequalities on unitarizable Harish-Chandra modules. The lowest $K$-type, i.e. the one of highest weight $\lambda - \rho_c + \rho_n$, can occur only once in the $L^2$-kernel of the Dirac operator, by calculation of the character. This proves:

**Theorem 9.10.** Suppose $\lambda \in \Lambda + \rho_c$ is $\Phi^+$-dominant, $\Phi$-regular, and (as proved so far) very regular. Then the kernel of

$$D_+ : L^2(G/K, V_{\lambda - \rho_c} \otimes S_+) \to L^2(G/K, V_{\lambda - \rho_c} \otimes S_-)$$  \hspace{1cm} (9.31)

is irreducible, unitary, square-integrable, with character $\Theta_\lambda$ which vanishes at $\infty$ and satisfies

$$\Theta_\lambda|_{H_{rs}} = (-1)^q \sum_{w \in W(h, \beta)} \epsilon(w) e^{w\lambda} \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2}).$$ \hspace{1cm} (9.32)

The $K$-type of highest weight $\lambda - \rho_c + \rho_n$ occurs exactly once, and all other $K$-multiplicities are less than or equal to those predicted by Blattner.

A proof of Blattner’s conjecture would be equivalent to having equality in the $K$-multiplicities. Here is what remains to be done:

(A) Remove the “sufficiently regular” hypothesis (the translation principle).
(B) Show that there do not exist $L^2$-spinors where $\lambda$ is $\Phi$-singular.

First, we shall discuss the relationship between the Dolb. cohomology of line bundles and the $L^2$-cohomology.
To do...

- 1 (p. 1): Figure out what happened the first two days.