Goals and Philosophies for Teaching Mathematics
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This past summer I taught a course for elementary education majors on “Numbers and Operations,” the aim of which is for students to review the algorithms of arithmetic and to learn how to communicate the reasons these algorithms work. From the very beginning, I knew for my students to be successful teachers, it was not enough for them to be able to “do the math”, to solve a fraction word problem or divide decimal numbers. They needed to be able to understand and explain the logical reasoning behind these methods. They needed to be able to create concrete examples and problems to highlight the more tricky aspects of the subject. They needed to be comfortable in unfamiliar mathematical situations, for example when confronted with a student’s non-standard, but perhaps correct, approach to a problem. Finally, I believed they should see how the arithmetic they would be teaching developed from centuries of human creativity and logic and extended to mathematics such as algebra and number theory. Ultimately, I wanted them to experience mathematics for themselves as an exciting, useful, creative, and sense-making activity.

Reflecting on the goals I had for my elementary education students, I realize these are not so different from those I have for any undergraduate student of math. I want students to experience the utility, creativity and clarity of mathematical thinking. A key piece of this is learning the theorems that underpin the material and the tools to apply these effectively. However, just as above, this knowledge is incomplete unless students have an understanding of how it connects to the broader world, an ability to communicate why the tools they use make sense, and the confidence to explore mathematics beyond the proscribed context of the course.

To achieve this more complete knowledge, I aim to provide my students opportunities

- to connect the mathematics they are currently learning to their previous mathematical knowledge, to applications within and outside the discipline, and to the history of the subject.
- to communicate mathematics, not just in exam settings, but in both informal and formal written assignments and oral presentations.
- to explore mathematics, not just in homework problems, but in non-standard examples, projects, and guided discovery, both inside and outside of the classroom.

Connecting

Making sense of mathematics begins with connecting and organizing newly acquired knowledge into a bigger picture. In discussion sessions for Calculus courses, I often begin by asking students to summarize the key points of the previous lecture. Recording these on the board, we sort through the fundamental ideas and the new tools we have to apply them, as well as how these connect to previous knowledge. For example, I ask my Multivariable Calculus students to think about how Stokes’ theorem is a two-dimensional version of the Fundamental Theorem of Calculus.

I also believe it is important to emphasize both the history of mathematics and the power of its current applications. Whether presenting to the undergraduate math club about how Google’s ranking algorithm uses linear algebra or writing about the evolution of number systems for a chapter in an expository text, I try to emphasize the history and application. This motivates and enlivens topics which to some students may appear purely abstract or artificial.

For example, in a recent talk to math majors on using cryptography to securely transmit information, I taught students how to add points on an elliptic curve. I began by having students graph a curve on a hand-out, then experiment with what happened when they drew lines connecting two points on the curve. From this, we talked about when there would be a third point of intersection, and how to explicitly solve for it. In the framework of learning about private communication over the internet, I was able to introduce the concept
of a group and an elliptic curve, as well as refresh the concepts of tangency and implicit differentiation.

**Communicating**

One of the primary opportunities for students to practice communicating mathematics is through informal group work and class participation. In discussion sections and class, I often ask students to lead me through a homework problem, acting as scribe as they articulate their approach, while other students act as “translators.” Together the students become more adept at describing their thinking in the language and symbols of mathematics. Furthermore, as students often approach problems differently, this gives them a chance to compare and contrast methods.

In a different vein, at the last meeting of the year of the math club, I asked each member to describe the most interesting mathematical thing they had learned that semester. This evolved into a lively discussion, as students worked to articulate to each other the big picture of what they’d learned and why they personally found it interesting.

I would also like my courses to include an opportunity for students to formally communicate mathematics, as I believe this allows them to develop a greater ownership of the material. For the Numbers and Operations course, I had students design their own mixed fraction word problems and submit written solutions with diagrams, accompanied by an explanation appropriate for an elementary age student. This was followed by a round of feedback and corrections. Many of my students impressed me with the confidence and clarity of their explanations and the creativity of their designs. Even the students who regarded this as an “easy” task came to realize the importance and challenge of writing well-defined problems and mathematically precise explanations.

For an introductory or non-major course, a formal opportunity for communication might be a version of the previously mentioned math club exercise, an end of semester report on the topic that a student found most interesting. For a mid-level course, such as Linear Algebra, it might be an individual or group presentation on an application, from graphic visualization to population modeling. In an upper level course, each student might serve as “scribe” for one week of class, not only keeping notes, but summarizing the key ideas and supplementing them with their own examples and proofs.

**Exploring**

Fostering students’ desire and confidence to explore mathematics begins with creating an environment for questions. I always take students’ questions seriously and, if possible, take class time to explore them. For example, in the Numbers and Operation course, one student asked if it was always necessary to draw number lines that end with a whole number tick. Instead of answering myself, I turned the question back to the class, which led to an interesting discussion about scaling and infinity.

Another way to encourage exploration is to teach students to lead with examples. I believe examples should precede definitions whenever possible, and that after introducing a definition, students should be given time to create their own examples and counterexamples. This was the guiding principle of my first college math course for which I later served as teaching assistant. In this role, I wrote supplemental course material with specifically designed “pauses” for students to stop and generate an example or pose a question. This is how I want my students to learn to read and study mathematics, with a pencil in hand.

More formally, I believe well-designed guided discoveries can be exciting and empowering for students and are critical for every math major. The focus is not necessarily about discovering new mathematics, but about seeing familiar mathematics in a new way. This summer, I participated in a unique seminar on designing and implementing a “mathematical research” course for graduate students in mathematics education. In it, I witnessed how a seemingly simple question about how to rate rectangles in terms of “squareness”
led us, the participants, to interesting discussions of scale, measure, continued fractions, and more. These explorations frustrated, challenged and enlivened our mathematical senses. This spring I will teach such a course with a post-doctoral student in math education. The explorations will take two forms. The class as a whole will pursue multi-week investigations designed to stimulate problem posing, such as the rectangles problem. Students will also work on a semester-long individual project, based on a topic or problem that has puzzled or intrigued them in their own past math experience. I am excited about the possibilities for adapting such a course to the undergraduate curriculum. I feel it would be well-suited as a capstone class for majors, particularly for those who do not participate in honors projects.

I believe my enthusiasm for mathematics and my drive to present familiar material with “new eyes” can engage and motivate students. My patience, organization and fundamental belief in my students create a supportive environment in which to build confidence and mathematical ability. In all of this, I believe I can engage students in mathematics and enable them to experience it as a useful, creative and sense-making activity.