1 Basics of Smooth Manifolds

A Manifold is a topological space with a collection of charts, i.e. collections of maps $\psi_i$ from open sets $U_i$ covering $M$ to open subsets of $\mathbb{R}^n$ for some $n$ (called the dimension of $M$), such that $\psi_j \psi_i^{-1}$ is always smooth (on the open set where it is defined). This allows us to talk about $C^\infty(M)$, i.e. smooth functions on $M$, which is a map $f : M \to \mathbb{R}$ such that $f \psi_i^{-1}$ is always smooth.

**Exercise.**

- Show that the set of points in the complex plane of norm 1 is a manifold in such a way that every smooth function on the complex plane pulls back to a smooth function on this set of points.
- Construct a smooth structure on $\mathbb{R}$ so that $x^{1/3}$ is smooth.
- Show that a given collection of charts is contained in a “maximal” collection of charts (ignoring the boring issue that an open set can have two charts), and that two collections of charts define the same $C^\infty(M)$ if and only if the two collections are the same.

Given two manifolds, and a map between them $f : M \to N$, $f$ is said to be smooth if for any $g \in C^\infty(N)$, we have $gf \in C^\infty(M)$. A Diffeomorphism is a smooth bijection with a smooth inverse.

**Exercise.**

- Show smooth maps from $M$ to $\mathbb{R}$ are the same as elements of $C^\infty(M)$.
- Construct a diffeomorphism between the smooth structure on $\mathbb{R}$ constructed in the previous exercise (making $x^{1/3}$ smooth) to the usual one.
- Show that the inclusion of manifolds from $S^1$ to $\mathbb{C}$ is smooth.
Define the tangent space of a manifold $M$ at a point $x \in M$ to be the equivalence class of smooth maps $(-1, 1) \to M$ such that 0 is sent to $x$, and two such maps being equivalent if they have the same tangent vector at the point corresponding to $x$ in some (and hence any) chart containing $x$. This can be identified with a vector space of the same dimension as $M$ (choosing some chart with $x$ mapped to 0, we can do addition and scalar multiplication there). Again, one should check this doesn’t depend on the choice of chart used.

Alternatively, we can define it to be the set of Derivations at $x$, $D: \mathcal{C}^\infty(M) \to \mathbb{R}$, satisfying $D(fg) = D(f)g(x) + f(x)D(g)$. This is actually a local property at $x$ because of the existence of bump functions, which are functions in $\mathcal{C}^\infty(M)$ smooth outside of an arbitrarily small open set around $x$. This perspective is more typically associated with vector fields, which is a smoothly varying collection of vectors on $M$ (again, the notion of vector fields make sense in a single chart, and then use the derivative of the map $\psi_j^{-1}\psi_i$ at a point as the compatibility condition of the vectors between charts).

Given a smooth $f: M \to N$ (everything will always be assumed to be smooth from now on), and a point $x \in M$, we get an associated map from $df_x: T_x(M) \to T_{f(x)}(N)$ (which are the associated tangent spaces). Again, check this makes sense...

**Theorem.** (Inverse Function Theorem) If $df_x: T_x(M) \to T_{f(x)}(N)$ is an isomorphism between tangent spaces (which forces $M$ and $N$ to have the same dimension), then $f$ is a diffeomorphism between an open set around $x$ and an open set around $f(x)$.

**Proof.** This follows from the corresponding statement for maps $\mathbb{R}^n \to \mathbb{R}^n$ by looking at charts. 

**Theorem.** (Implicit Function Theorem) Given a smooth $f: M \to N$, then there exists charts around $x$ and $f(x)$ so that the map between charts $\mathbb{R}^k \to \mathbb{R}^l$ is given by projection onto the first $r$ coordinates, where $r$ is the dimension of the image of $df_x$, followed by inclusion of the subspace as the first $r$ coordinates.

**Theorem.** (Corollary of the Implicit Function Theorem) Given a smooth $f: M \to N$, if $df_x$ is surjective (i.e. $x$ is a regular point), then the inverse of $f(x)$ in $M$, when intersected with a sufficiently small ball around $x$, is canonically a submanifold. If the surjectivity is true for all points in $f^{-1}(f(x))$, then $f^{-1}(x)$ is canonically a submanifold.

**Exercise.** Go on Wikipedia and prove their version from this one.

A submanifold $N$ of $M$ is a subset which is a manifold such that the inclusion is a map of manifolds. This is unique.

To integrate on manifolds, we need a differential form. On $\mathbb{R}^n$, this is a formal sum of terms of the form $g dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ (for some fixed $k$ between 1 and $n$), under the rule that swapping the order of two of the $dx_i$ negates the term (so if two $dx_i$’s appear in the same term, the term is 0). If we had used a different set of coordinates $y_i$, then writing
Given a map $f : \mathbb{R}^n \to \mathbb{R}^m$, we can pull back a differential $k$-form in $\mathbb{R}^m$ to $f^*(\omega)$ by taking each monomial $gdx_{i_1} \wedge \ldots \wedge dx_{i_k}$, and replacing it with $(g \circ f)dx_{i_1 \circ f} \wedge \ldots \wedge dx_{i_k \circ f}$. We could have done a similar procedure even if we had used a not-fully expanded out version of the differential form, the non-trivial content is that it doesn’t matter how you do this. The special case of change of using new coordinates to define a map $\mathbb{R}^n \to \mathbb{R}^n$.

Anyway, this lets you define a differential $k$-form on a manifold in the obvious way (requiring it to transform as via the above procedure between charts), and allows pullback of differential forms from $M$ to $N$ given a map $f : M \to N$ (denoted $f^*$). Also, for a fixed manifold $M$, there’s a map $d$ which takes functions to differential forms, acting as $g \to dg$, and in general giving a map from $k$-forms to $k + 1$-forms via $gdx_{i_1} \wedge \ldots \wedge dx_{i_k} \mapsto dg \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$. Checking that this doesn’t depend at all on how you represent the form is an ambitious exercise.

Now, there are two upshots to this: Given $k$ (locally defined) vector fields on a manifold $M$ with a $k$-form $\omega$, we can create a function in $C^\infty(M)$ by taking $(v_1, \ldots, v_k, gdx_{i_1} \wedge \ldots \wedge dx_{i_k}) \mapsto gv_1(x_1) \cdots v_k(x_{i_k})$, where a vector field acts on a function by taking the directional derivative (in a chart). Actually, we didn’t need vector fields around $x$, but rather just vectors at $x$ — in this case we would just get a single value in $\mathbb{R}$ rather than a smooth function on $M$.

We are now ready to integrate! Given a top-dimensional differential form on $\mathbb{R}^n$, we can integrate it by putting a little $\int$ symbol in front of it. On $M$, we could do this on any individual chart — to patch together, we need a partition of unity, which is a collection of functions $\rho_i$ such that around any point in $M$, there is an open set around the point so that only finitely many of the $\rho_i$ are non-zero on this set, the closure of the set of the set of points of $M$ where each $\rho_i$ takes non-zero values (called its support) is contained in the defining open set of some chart (which can differ for different $i$), and $\sum \rho_i = 1$. Given such functions (which always exist for any manifold), we can take for $f \in C^\infty(M)$ the integral $\int f = \sum \int (\psi_i^{-1})^*(f\rho_i)$.

That this doesn’t depend on the choice of partition of unity is again taxing to check. The main reason that everything works out great is that using the formal $d$-rules, the change of coordinates map between charts is the Jacobian, so it doesn’t matter which charts we use!

Typically, we’re given a sub-manifold of dimension $k$ $N \subset M$, in which case we can integrate a $k$-form from $M$ on $N$ by first pulling back to $N$.

2 Basics of matrix Lie Groups over $\mathbb{R}$ and $\mathbb{C}$

A matrix Lie Group is a smooth manifold $G$ which has a group structure so that multiplication is smooth from $G \times G \to G$, and the inversion map $G \to G$ is also smooth.
1. $GL(n)$ are $n \times n$ matrices of non-zero determinant. This is an open subset of $\mathbb{R}^{n^2}$, so is a manifold.

2. $SL(n)$ are $n \times n$ matrices of determinant 1. Here, we have to check that this is a manifold, as it is a closed subset of $\mathbb{R}^{n^2}$. The trick is to apply the corollary of the implicit function theorem to the determinant map, which one trivially checks has surjective differentials at all points in the preimage of 1.

3. $O(n)$ are $n \times n$ real matrices with orthonormal column vectors.

4. $SO(n)$ are the matrices in $O(n)$ of determinant 1.

5. $U(n)$ are $n \times n$ complex matrices with orthonormal column vectors (with respect to the hermitian inner product $\langle v, w \rangle = v^T \overline{w}$).

6. $SU(n)$ are the matrices in $U(n)$ of determinant 1.

The latter three have a submanifold structure coming from their embeddings into $\mathbb{R}^{n^2}$ or $\mathbb{C}^{n^2}$. Also, the first two have analogous versions over the complex numbers which are defined in exactly the same way.

Given a Lie-group $G$, we can identify the tangent space to the identity quite explicitly, as $n \times n$ matrices $X$ such that $I + \epsilon X$ is a solution up to first order of the defining equation of the matrix group (i.e. plug it into all the equations, and let $\epsilon^2 = 0$).

1. $T_eGL(n)$ is just all $n \times n$ matrices.

2. $T_eSL(n)$ is the set of trace zero matrices as $det(I + \epsilon X) = I + \text{tr}X$

3. $T_eO(n)$ is the set of skew-symmetric matrices as the defining equation is $QQ^T = I$.

4. $T_eSO(n)$ is the set of trace zero matrices in the previous item.

5. $T_eU(n)$ is the set of skew-hermitian matrices.

6. $T_eSU(n)$ is the set of trace zero matrices in the previous item.

The left-translation map $\rho_g : G \to G$ (smooth as its $\{g\} \times G \to G \times G \to G$), gives an isomorphism of tangent spaces $T_eG \cong T_0G$. Using this, we can create a left-invariant vector field associated to any vector $v \in T_eG$ by using the various $\rho_g$ to move $v$ around to each point. If we had used right-multiplication, then we would have a completely different vector field — by convention, we consider only the left ones. By abuse of notation, we identify a vector in $\mathfrak{g} = T_eG$ with the left-invariant vector field on $G$. By using bump functions, we can write any vector field on $G$ uniquely as a $C^\infty(G)$-linear combination of the elements of $\mathfrak{g}$ (this uses the fact that the elements of $\mathfrak{g}$ form a basis of the tangent space of $G$ at every point), so $\Gamma(TM) = \mathfrak{g} \otimes C^\infty(G)$ (so $TG$ is a trivial vector bundle over $G$).
Given two elements \( X, Y \in \mathfrak{g} \), we consider \([X,Y] = XY - YX\). This is a second-order differential operator on \( C^\infty(G) \), but luckily the second order terms cancel out so it just becomes a derivation (i.e. \([X,Y](fg) = g([X,Y]f) + f([X,Y]g)\)), so we get an associated vector field. More explicitly, \( \sum f_i \frac{\partial}{\partial x_i}, \sum g_i \frac{\partial}{\partial x_i} = \sum (f_i \frac{\partial}{\partial x_i} - g_i \frac{\partial}{\partial x_i}) \frac{\partial}{\partial x_i} \). This works for any vector fields on any manifold, but what we get in this situation is that \([X,Y]\) is also left-invariant (check this!), so \([X,Y] \in \mathfrak{g}\). This bracket satisfies the properties that it is linear in each variable, \([X,Y] = -[Y,X]\), and \([X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0\) (actually, these are true for any vector fields on any manifold), which by definition makes \( \mathfrak{g} \) into a lie-algebra.

Similarly, we could pull back differential forms at the origin using \( \rho_{g^{-1}}^* \), so we get \( \Gamma(\Lambda^k T^*M) \cong \Lambda^k \mathfrak{g} \otimes C^\infty(M) \), where \( \Lambda^k \mathfrak{g} \) are left-invariant differential forms.

Given ANY metric at the identity (i.e. some element of \( Sym^2 \mathfrak{g}^* \), taking two vectors in \( T_eG \) and spitting out a real number in a bilinear symmetric way), we can do the same thing to get a left-invariant metric on \( G \) (i.e. for two vectors \( v \) and \( w \) at \( x \), \( (v,w) = ((dp_g)_x v, (dp_g)_x w) \)).

The last thing we need to know here is the exponential map. This is the map \( exp_v : \mathbb{R} \to G \) for \( v \) a vector in \( T_eG \) such that the derivative at each point \( g \) is given by \( (dp_g)_v \) (i.e. it agrees with the value of the left-invariant vector field generated by \( v \)). This is called being an integral curve. To get this map, the easiest way is to literally take the map \( t \mapsto e^{tv} = I + tv + \frac{(tv)^2}{2!} + \frac{(tv)^3}{3!} + \ldots \).

### 3 Real and Complex Vector Bundles

We used \( TM \) and \( T^*M \) a lot, so maybe we should define some things. A real vector bundle on \( M \) is a manifold \( X \) with a smooth surjection to \( M \) such that it “locally looks like \( M \times \mathbb{R}^k \)” (where \( k \) is the rank of the vector bundle). More precisely, the fiber over any point \( x \in M \) has the structure of a vector space, and for every point \( x \in M \), there exists an open neighborhood \( U \) of \( x \) such that the inverse image in \( X \) is diffeomorphic to \( U \times \mathbb{R}^k \) in a way that identifies the vector space on each fiber with the corresponding \( \{x\} \times \mathbb{R}^k \).

An open set \( U \) which allows us to carry out this procedure is said to trivialis the vector bundle. Replacing \( \mathbb{R} \) with \( \mathbb{C} \) everywhere gives a complex vector bundle.

We can glue together vector bundles over charts with given trivializations of the vector bundle if we’re given maps \( \phi_{ij} \) between a copy of the trivial bundle over \( U_j \) and a copy of the trivial bundle over \( U_i \) which are linear maps over each fiber, provided that they satisfy the compatibility requirements \( \phi_{ij} \phi_{ji} = Id \) and \( \phi_{ij} \phi_{jk} \phi_{ki} = Id \) for all \( i,j,k \) (this would also work if we were given non-trivial vector bundles as well).

To this end, we define the tangent bundle by taking the trivial bundle of the same dimension as \( M \) on each chart, with transition map between the fiber over \( x \) and the fiber over \( \psi_j \psi_i^{-1}(x) \) given by the differential (i.e. Jacobian) of \( \psi_j \psi_i^{-1} \). The fibers over points of \( M \) of this bundle are then canonically the same as the tangent spaces at these points.
A section of a vector bundle \( X \) is a map \( M \to X \) such that \( M \to X \to M \) is the identity map. Note that we always have the zero section. We denote \( \Gamma(X) \) the set of sections of \( X \), which is naturally a \( C^\infty(M) \)-module. For example, an element of \( \Gamma(TM) \) gives a smooth vector field. Given an open set \( U \), a section over \( U \) is the same thing, a map \( U \to X \) such that \( U \to X \to U \) is the identity.

Given a vector bundle \( X \), we define \( X^* \) by taking the transition maps defining \( X \) to be the inverse transpose of the maps used to define \( X \). Then given a section of \( X \) and a section of \( X^* \), we can pair them off to give an element of \( C^\infty(M) \) by taking dot products in each fiber (which is well-defined precisely because of the choice of transition maps of \( X^* \)).

In particular, applying this construction to \( TM \), we get \( T^*M \). Elements of \( \Gamma(T^*M) \) are the 1-forms on \( M \).

Now, given any natural operation on vector spaces (such as taking duals), we can lift the construction to a construction on vector bundles, and the transition maps modify in the expected way (just like with the construction of the dual vector bundle above). For example, we can define \( X \otimes Y \), \( \Lambda^k X \), and \( X \oplus Y \) from the analogous construction on vector spaces.

Remark. It is NOT true that an element of \( \Gamma(X \otimes Y) \) is given by an element of \( \Gamma(X) \otimes \Gamma(Y) \), just like it is not true for vector spaces. Similar considerations apply for all other constructions, even for the dual — we only have locally that it is defined in this way, globally we might need to patch things together on overlaps.

Finally, given a map of manifolds \( f : M \to N \), and a vector bundle \( X \) on \( N \), we have a pullback bundle \( f^*N \), morally defined by taking the vector space over \( x \) to be \( X_{f(x)} \) (to define it properly, we define it on trivializations as before, in the obvious way).

4 Differential forms and stuff

So we defined differential forms already in an ad-hoc way, the easiest way to define a \( k \)-form is just as an element of \( \Gamma(\Lambda^k M) \). From the natural operation of wedging two wedges of a vector space, we get natural maps \( \Lambda^k M \otimes \Lambda^l M \to \Lambda^k M \otimes \Lambda^l M \to \Lambda^{k+l} M \). On sections, this takes \((g_1 dx_{i_1} \wedge \ldots dx_{i_k}, g_2 dx_{j_1} \wedge \ldots dx_{j_l}) \mapsto g_1 g_2 dx_{i_1} \wedge \ldots dx_{i_k} \wedge dx_{j_1} \otimes \ldots \otimes dx_{j_l} \). This, along with the map \( d : \Gamma(\Lambda^k M) \to \Gamma(\Lambda^{k+1} M) \) which satisfies \( d^2 = 0 \) defines all the structure we need to define deRham cohomology.

We have a sequence of maps \( 0 \to \mathbb{C}^r \to C^\infty(M) \to \Gamma(\Lambda^1 M) \to \Gamma(\Lambda^2 M) \to \ldots \Gamma(\Lambda^n M) \to 0 \), where \( n \) is the dimension of \( M \), \( r \) is the number of connected components, and the maps after the first two are all given by \( d \) (so \( d^2 = 0 \)). In particular, the image of one list inside the kernel of the next. If you know anything about sheaf cohomology, we resolved the constant sheaf with some flabby, and hence acyclic sheaves, so this complex calculates the cohomology of the manifold. If you don’t know anything about sheaf cohomology, we define \( H^k_{dR}(M) = \ker(\Gamma(\Lambda^k M) \to \Gamma(\Lambda^{k+1} M))/\text{Im}(\Gamma(\Lambda^{k-1} M) \to \Gamma(\Lambda^k M)) \), and \( H^0_{dR} = \mathbb{C}^r \).
Then $H_{dR}^* = \oplus_k H_{dR}^k(M)$ is a ring, with the commutativity relation $xy = (-1)^{kl}yx$ if $x$ is in $H^k$ and $y$ is in $H^l$. This is the deRham cohomology of $M$. Because each of the components of the ring can be pulled back, given a map $f : M \to N$, we have a map $H_{dR}^*(N) \to H_{dR}^*(M)$.

**Exercise.** Calculate the deRham cohomology of $\mathbb{R}$ and $S^1$ from first principles. In general, this is just the usual cohomology, so feel free in the future to use any algebraic topology methods you want to accomplish this.

## 5 Metrics on Vector Bundles and Riemannian Metrics

A metric on a vector bundle $E$ is an element of $\text{Sym}^2(E^*)$ which is non-degenerate in each fiber. An example of this is a Riemannian Metric, which is a metric on the tangent bundle $TM$. This allows us to compute lengths of curves, by taking $\int_a^b \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} \, dt$. Also, the dot product of two vectors $v, w$ at a point $x$ is just the value under the metric tensor at that point. We can pull back elements of $\Gamma(\text{Sym}^2(T^*N))$ along a map $f : M \to N$, so metrics pull back. More explicitly, given two vectors at a point in $M$, we can use $df_x$ to push them to $N$, and then use the metric there.

Metrics allow us also to do non-oriented volume integrals: To integrate $f$ over $M$ using the metric, we do the partition of unity trick if need be, then in each chart calculate the integral as $\int_{\mathbb{R}^n} f \sqrt{|\det g|} \, dx_1 \cdots dx_n$, where $g$ is represented in $\mathbb{R}^n$ as the matrix $g_{ij} = g(\partial_i, \partial_j)$. If the manifold is oriented, then we can create the canonical top dimensional form $\sqrt{|\det g|} \, dx_1 \wedge \cdots \wedge dx_n$ (assuming $x_1, \ldots, x_n$ gives a positively oriented coordinate system).

As an example, suppose we have a surface $f(u, v) : \mathbb{R}^2 \to \mathbb{R}^3$. Then to calculate the surface integral of a function $\psi$, we need a metric on the surface — we can get this via the embedding of the graph of the surfaces into $\mathbb{R}^3$, pulling back along the inclusion. This is an inconvenient place to do an integral however as we all know, so we further pull back to $\mathbb{R}^2$ to do the integral (this is the most convenient chart you will ever find in your life). To represent the metric here, note we have coordinates $u$ and $v$, so we need to know how the matrix of the metric with basis $\partial_u$ and $\partial_v$. But by definition of the pullback, this can be found by pushing the vectors forward into the surface, and then into $\mathbb{R}^3$, and calculating the dot product there. The image of $\partial_u$ under $df$ is $(\partial_u f_1) \partial_1 + (\partial_u f_2) \partial_2 + (\partial_u f_3) \partial_3$ and similarly for $\partial_v$. Using that the metric in $\mathbb{R}^k$ gives $(\partial_i, \partial_j) = \delta_{ij}$, we can find the dot products to get $2 \times 2$ metric matrix, which is just $J^t J$ where $J$ is the Jacobian of the map. In conclusion, we get the familiar formula $\int \int \sqrt{EG - F^2} \, dA$ for the surface area.

## 6 Geodesics

Suppose we have a manifold $M$ with a Riemannian metric $g$ (also called a Riemannian Manifold). A geodesic between two points is a critical point of the length functional for paths between the two points (in particular, length minimizing and maximizing paths are
always geodesics). In particular, it only depends on the metric, not on any embedding of the manifold if this respects the metric (so e.g. a cone and a cone cut open and laid flat on the table have the same geodesics because they have the same metric, and geodesics of flat things are given by lines). If the manifold is embedded in an ambient vector space with the metric being the pullback metric, then this is equivalent to saying $\gamma''(t)$ is perpendicular to the surface of the manifold at all times. In general, if we take the Levi-Cevita connection (defined later), then being a geodesic is saying that $\nabla_{\gamma'(t)} \gamma'(t) = 0$. Given a point $p$, there exists a unique geodesic leaving $p$ with derivative at $p$ a given vector in $T_p M$. As an example, the exponential map provides all of the geodesics for a Lie group.

7 Principal Lie group bundle and Associated Vector Bundles

Given a Lie group $G$, a principal $G$-bundle $X$ over a manifold $M$ is a manifold $X$ with a map $X \to M$ with a right action of $G$ on $X$ (i.e. $X \times G \to X$) which acts on the individual fibers of the map $X \to M$ as well, such that for any point in $M$, there’s an open neighborhood $U$ of the point such that there is a diffeomorphism of the inverse of $U$ in $X$ with $U \times G$ which doesn’t move points between different fibers, and such that the obvious right action of $G$ on $G \times U$ is the same as the action on the inverse of $U$ in $X$.

Given a representation of $G$ on a vector space $V$, $G \to \text{End}(V)$ (which happens if for example $G$ is a matrix Lie group lie in the previous sections), we get the associated vector bundle of $X$ is the quotient of $X \times V$ by the equivalence $(x,v) \sim (xg, gv)$. Given a vector bundle $E$ on $M$, we need more information to recover the associated $G$-bundle, unless $G$ is $\text{GL}(n)$. Over an open set $U$ over which the vector bundle is trivial, we create a space on which $G$ acts on the right by taking the space of isomorphisms from $U \times \mathbb{R}^n$ to the vector bundle over $U$, with $g$ acting on a map $\rho$ via $\rho \to \rho \circ g$. This is equivalently the frame bundle which to each point in the base space associates the set of frames in the fiber (which is isomorphic to $\text{GL}(n)$). In general, to recover the principal $G$-bundle, we need the additional data of a collection of frames which is acted simply transitively by $G$. For example, to recover an $O(n)$-bundle, we need the data of the notion of an “orthonormal frame”, which can be accomplished by equipping the vector bundle with a metric for example. For an $\text{SL}(n)$-bundle, we need a global trivialization of the top exterior power of the bundle, etc.

How do we glue together a principal G-bundle by gluing together principal $G$-bundles on charts? Well, given 2 charts $U_1$ and $U_2$ over which the $G$-bundle trivializes, we need to glue $U_1 \times G$ and $U_2 \times G$. This is done in the intersection $(U_1 \cap U_2) \times G$, with a bijective map $\phi_{21} : (U_1 \cap U_2) \times G \to (U_1 \cap U_2) \times G$ mapping the copy in $U_1 \times G$ to the copy in $U_2 \times G$, in a way that preserves the fibers and commutes with the right multiplication of $G$. If under $\phi_{21}$, a section $\sigma_1$ gets mapped to a section $\sigma_2$, then by using the right-multiplication of $G$, this determines where every point gets mapped to. Consider $\sigma_2 \sigma_1^{-1}$ as a section of $(U_1 \cap U_2) \times G$. Then left multiplication by this
SECTION COMMUTES WITH THE RIGHT MULTIPLICATION ON \( G \) AND ACTS CORRECTLY ON \( \sigma_1 \). This should set off warning bells in your head, but forgetting about the base \( U_1 \cap U_2 \), just look at what happens if we have a map \( G \to G \) which respects the right multiplication. If it sends \( g_1 \mapsto g_2 \), then it will send \( g_1 h \mapsto g_2 h \) as expected. However, if I want to know where \( g \) gets mapped to, I have to set \( h = g_1^{-1} g \), and then I get \( g \mapsto g_2 g_1^{-1} g \), which is indeed left-multiplication. Hence, bizarrely, the gluing maps are given by left-multiplication by maps from the base space to \( G \) (satisfying the usual gluing conditions like with vector bundles gluing).

8 Covariant Derivatives for vector bundles and connections on Principal bundles

A covariant derivative \( \nabla \) on a vector bundle \( E \) over \( M \) is an assignment to every vector field \( X \) a covariant derivative along \( X \) \( \nabla_X : \Gamma(E) \to \Gamma(E) \) which is \( C^\infty(M) \)-linear in the \( X \) part (i.e. \( \nabla_{fX_1 + gX_2} = f \nabla_X \sigma_1 + g \nabla_X \sigma_2 \)), \( \nabla_X (\sigma_1 + \sigma_2) = \nabla_X \sigma_1 + \nabla_X \sigma_2 \), and \( \nabla_X (f \sigma) = f \nabla_X \sigma + X(f) \sigma \). Equivalently, it is a map \( \nabla : \Gamma(E) \to \Gamma(T^*M \otimes E) \) such that \( \nabla(f \sigma) = f(\nabla \sigma) + \sigma \otimes df \).

Note that the difference of two covariant derivatives will cancel out the \( \sigma \otimes df \) term, so will be an element of \( \Gamma(T^*M \otimes \text{End}(E)) \).

Covariant derivatives allow horizontal sections: a section of \( E \) is said to be horizontal if \( \nabla \) sends it to 0. Along any path, there is a basis of horizontal sections, and writing an arbitrary section in terms of these makes \( \nabla_X \) simply differentiate each coefficient in the \( X \)-direction. It is important to note that horizontal sections only exist in general along paths — given two different paths between two points, the horizontal sections will end up at different places. If for every simply connected region we have a horizontal section around the loop giving the identity endomorphism in the fiber, then we say the connection is flat. We will return to this notion later.

Now we will give a seemingly unrelated notion — the definition of a connection on a principal \( G \)-bundle. Connecting the two notions requires an unbelievable amount of mental fortitude, we will hopefully get through this together. There are many equivalent notions of connection on a principal \( G \)-bundle \( X \to M \) on a manifold \( M \) of dimension \( n \). The easiest to state is that it is a smoothly varying collection of horizontal subspaces of \( TX \) of dimension \( n \) (i.e. a subbundle \( H \) of \( TX \)) so that at each point \( x \in X \), \( T_xX \) is the direct sum of the subspace at \( x \) and the “vertical tangent vectors”, i.e. those killed by \( d\pi_x \), and such that the collection is \( G \)-equivariant in the sense that the subspace at \( x \), when pushed forward by the right action \( (d\rho_y)_x \) gives the subspace at \( gx \). From this notion, we obtain the notion of a horizontal section over an open set \( U \), which is a section such that \( \gamma'(t) \) always lies in the connection subspace at \( \gamma(t) \). The direct sum condition ensures that every vector in \( TM \) at the point \( m \) has a lift to a unique vector specified by the connection in \( T^*E \) at each point \( x \) mapping to \( m \), and that we can actually lift any curve in a horizontal
way uniquely. The equivariance condition ensures that if we have a horizontal lift of a curve, then right translation by \( g \) gives another horizontal lift of the same curve.

Now, what information is actually needed to specify such a connection? Well, we have a splitting \( TX = H \oplus V \), where \( V \) is the vertical subbundle (those vectors killed by \( d\pi \)). \( H \) is completely specified from the vector bundle map \( TX \to V \) as the kernel of the map. As luck would have it, we have a very explicit description of \( V \). Suppose I have a trivialization of \( X \) over a neighborhood \( U \) of a point \( m \), \( U \times G \), with a vector field \( \tilde{\nu} \) given by taking an element \( \nu \in \mathfrak{g} \), and producing the LEFT-invariant vector field on \( U \times G \) in the natural way (we get the vector field lying naturally inside \( V \)). NOTE THAT \( U \times G \) DOES NOT HAVE A CANONICAL LEFT ACTION SO THIS SHOULD SCARE YOU A BIT!!! The reason we use the left invariant field is that if I have a different trivialization of \( U \times G \), i.e. a \( G \)-equivariant map \( U \times G \to U \times G \) (i.e. commuting with the action of \( G \)) respecting the fibers, such a map is given by LEFT-MULTIPLICATION by a section of \( U \times G \to U \), just like in the previous section. Under such a map, the left-invariant vector fields will be preserved by definition (as they are preserved in each fiber). Hence, in what can only be described as very confusing, left-invariant vector fields generated by elements of \( \mathfrak{g} \) make sense on \( TX \). Moreover, they happen to lie in \( V \), and a basis of \( \mathfrak{g} \) gives a basis of sections of \( V \), so we get in this way that \( V \) is the trivial vector bundle \( \mathfrak{g} \times M \to M \). A map \( TX \to V = \mathfrak{g} \times M \) is the same thing though as an element \( \omega \) of \( \Gamma(T^*X \otimes \mathfrak{g}) \). This element has two properties:

1. It acts as the identity on sections of \( V \), or equivalently \( \omega(\tilde{\nu}) = \nu \) (note that the \( \nu \) on the right hand side is really the constant function \( \nu \), which under the identification \( P \cong \mathfrak{g} \times M \) represents \( \tilde{\nu} \)).

2. \( Ad_g(R_g^*\omega) = \omega \).

Let's explain this second item, which encodes the \( G \)-invariance of \( H \). Here \( R_g \) denotes right-translation by \( g \), and \( Ad_g \) is the action of \( G \) on \( \mathfrak{g} \) by conjugation \( \nu \mapsto g\nu g^{-1} \). This should really by called \( d(Ad_g)_e \), since it really comes from \( G \) acting on \( \mathfrak{g} \) by conjugation, but we abuse this notation (we will see exactly where in the proof this notation gets abused). Note that the \( G \)-invariance of \( H \) implies the \( G \)-invariance of the map \( TX = H \oplus V \to V \), and vice versa (as \( V \) is \( G \)-invariant), so we have to understand the \( G \)-invariance of the map \( TX \to V \) in terms of \( \omega \). Given a section \( \sigma \) of \( TX \), if I apply right-translation and then map it to a section of \( V = \mathfrak{g} \times M \), we get \( \omega(d(R_g)\sigma) \), which is by definition the same as \((R_g^*\omega)\sigma\). If I do the mapping first and THEN right-translate, then we have to figure out how the vector field on \( V \) given by \( \omega(\sigma) \) behaves under right-translation. Work in a local trivialization \( U \times G \), and ask ourselves what vector ends up at \((h, u)\). Before the right translation, at the element \( hg^{-1} \) above \( u \), the vector field was \( d(L_{hg^{-1}})e\omega(\sigma) \), and right-translating this by \( g \) gives the vector at \( h \) is now \( d(R_g)hgd(L_{hg^{-1}})e\omega(\sigma)(u) = d(R_gL_{hg^{-1}})e\omega(\sigma)(u) = d(L_hR_gL_{g^{-1}})e\omega(\sigma(u)) = d(L_h)\sigma d(Ad_{g^{-1}})e\omega(\sigma(u)) = d(L_h)e(Ad_{g^{-1}}\omega(\sigma(u))) \), which in
terms of the identification of the vectors at the various tangent spaces with \( g \) via left-translation is just \( Ad_{g^{-1}}\omega(\sigma(u)) \). Hence, the section gets mapped to \( Ad_{g^{-1}}\omega(\sigma(u)) \), and equating, we get \( R^*_g\omega = Ad_{g^{-1}}\omega \). Applying \( Ad_g \) to both sides gives the second item as desired.

Now, we have to relate this to the notion of connection on a vector bundle! It is easy to do this in a very high brow way, by identifying horizontal sections, but the associated vector bundle is never presented in quotient form, so this is unhelpful.

Suppose we have a principal \( G \)-bundle \( X \), which was defined by gluing together trivializations \( U_i \times G \) using left multiplication of maps \( \rho_{ji} : U_i \cap U_j \to G \). Then as \( G \) lies inside endomorphisms of a vector space \( V \), we get maps \( U_i \cap U_j \to End(V) \), which is precisely what is needed to glue a vector bundle.

What does a \( G \)-bundle connection look like here? Well, on \( U \times G \), the value is completely determined by what happens on \( U \times \{ e \} \), i.e. if we pull back the connection 1-form via the section \( U \to U \times \{ e \} \), we don’t lose any information (but this section isn’t a natural section, it depended greatly on the choice of trivialization). Then we get a 1-form \( \omega \in \Gamma(T^*U) \otimes g \), and what we need now is to make sure the 1-forms for different \( U \) are compatible. This says that if \( U_1 \) has \( A_1 \) and \( U_2 \) has \( A_2 \), then \( A_1 = \rho^{-1}_{21}d\rho_{21} + \rho^{-1}_{21}A_2\rho_{21} \). Here \( \rho^{-1}_{21}d\rho_{21} \) is the pullback via \( \rho_{21} \) of the Maurer-Cartan form \( g^{-1}dg \) on \( G \) which lies in \( \Gamma(T^*G) \otimes g \), which eats a vector field, and spits out the corresponding element of \( g \) at each point that the vector at that point corresponds to.

What does a vector bundle with connection look like here? Well, on a trivialization \( U \times \mathbb{R}^k \) (standard basis of sections \( e_1, \ldots, e_k \), \( \nabla \) looks like \( d + A \), where \( d \) takes \( \sum f_i e_i \mapsto e_i \otimes df_i \) (i.e. the \( e_i \) are horizontal), and \( A \) is a matrix of \( \text{End}(\mathbb{R}^k) \)-valued 1-forms, i.e. an element of \( \Gamma(T^*U) \otimes \text{End}(\mathbb{R}^k) \), which eats a vector field on \( U \) and a section of \( E \) and spits out a section of \( E \). The compatibility is again \( A_1 = \rho^{-1}_{21}d\rho_{21} + \rho^{-1}_{21}A_2\rho_{21} \). Here the \( \rho_{ji} \) map \( U_i \cap U_j \to \text{End}(\mathbb{R}^k) \).

Then for the connection to arise from a connection on \( G \), it is equivalent to saying that the connection form always takes values in \( \Gamma(T^*U) \otimes g \subseteq \Gamma(T^*U) \otimes \text{End}(\mathbb{R}^k) \), and the equivalence of the two definitions is now trivial.

9 Curvature, Flat connections, Parallel Transport

Given a vector bundle or a principal bundle, parallel transport along a path is the map between fibers induced by horizontal sections. A connection is said to be flat if this parallel transport doesn’t depend on which path is chosen for any two paths surrounding a simply connected region. Given a flat connection and a basepoint \( x \), we get a representation \( \pi_1(X) \to G \) given by looking at what automorphism we get in the fiber when we travel around a loop (for vector bundle connections, this maps to \( \text{End}(V) \), where \( V \) is the vector space over the fiber), and flat connections are in one-to-one correspondence with conjugacy classes of such maps. Consequently, if \( G \) is for example \( S^1 \), then this is a map.
$H_1(X) \to S^1$, which by algebraic topology nonsense (universal coefficient theorem, etc.) is $H^1(X; \mathbb{R})/H^1(X; \mathbb{Z})$.

The Holonomy group for a non-flat connection is the subgroup of endomorphisms of the fiber we can get by parallel transport. One can show that for a connection induced from $G$, this is a subgroup of $G$ (I think?).

How do we determine if a connection is flat? Well, the curvature operator, as defined two sections from now, must vanish identically. We can check this locally in charts, so we can apply the same procedure for a principal $G$-bundle with connection.

10 How to efficiently compute Christoffel Symbols

Recall.

\[
\Gamma^k_{ij} = \frac{1}{2} g^{ka} (\partial_i g_{aj} + \partial_j g_{ai} - \partial_a g_{ij})
\]

Where $g^{ka}$ corresponds to $M^{-1}$, and we sum over repeated indices (i.e. $\alpha$) as per Einstein summation convention.

The Christoffel symbols $\Gamma^k_{ij}$ are defined so that the associated connection $\nabla$ with

\[
\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} = \Gamma^k_{ij} \frac{\partial}{\partial x_k}
\]

satisfies the following two conditions:

1. $\nabla$ is Torsion-Free, i.e. $\nabla_X Y - \nabla_Y X = [X, Y]$ for all vector fields $X$ and $Y$

2. $\nabla$ preserves the metric, i.e. $\nabla g = 0$, i.e. $\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

(where the derivative on the left hand side is the usual directional derivative of a function).

Exercise.

1. Show the first condition is equivalent to $\Gamma^k_{ij} = \Gamma^k_{ji}$

2. Show the second condition is equivalent to $\partial_k g_{ij} = g_{ji} \Gamma^l_{ik} + g_{il} \Gamma^l_{jk}$

3. Solve for $\Gamma^k_{ij}$ using the previous two parts and verify it agrees with the formula provided.

The resulting connection is the Levi-Cevita Connection.

Remark. Given vector fields $\{v_i\}$ which locally form a basis of all tangent spaces near a fixed point (i.e. a local frame), we could ask what the coefficients $\Gamma^k_{ij}$ are such that

$\nabla_{v_i} v_j = \Gamma^k_{ij} v_k$. One cannot use the formula for Christoffel symbols given at the beginning, because in order for $\Gamma^k_{ij} = \Gamma^k_{ji}$ to be true, we need $[v_i, v_j] = 0$ (locally, if we have $[v_i, v_j] = 0$ for all $i$ and $j$, then we say that $\{v_i\}$ are holonomic, and then $v_i = \frac{\partial}{\partial x_i}$ for some local coordinate functions $x_i$).
Exercise. If we write $[v_i, v_j] = c^k_{ij}v_k$, then solve for $\tilde{\Gamma}_{ij}^k$.

Now, we note the following amazing fact about the Levi-Cevita connection. If $M$ is immersed inside $\mathbb{R}^n$ (or any $N$ equipped with the Levi-Cevita connection), then if $A, B$ are vector fields on $M$, and $\tilde{A}, \tilde{B}$ local extensions of these vector fields, then $\nabla_A B = \text{orthogonal projection of } \nabla_{\tilde{A}} \tilde{B}$ to $M$.

From this fact, we see that parallel transport along a curve in $M \subseteq \mathbb{R}^n$ depends only of the Levi-Cevita connection along the curve, so if one finds $M' \subseteq \mathbb{R}^n$ tangent to $M$ along the curve, then one can compute it along $M'$ instead.

Exercise. Compute the parallel transport around some circle on the unit sphere (hint: take a cone which is tangent to this circle and unfold the cone).

It appears that computing Christoffel symbols is laborious. This is the fastest way in my opinion: let $M$ be the matrix associated to $g_{ij}$, let $c_1, \ldots, c_n$ be its columns, and $r_1, \ldots, r_n$ be its rows. Then for a fixed $i$,

$$\Gamma_{ij}^k = \frac{1}{2} M^{-1}(\partial_i M + (\partial_1 c_i \partial_2 c_i \ldots \partial_n c_i) - \left( \begin{array}{c} \partial_1 r_i \\ \partial_2 r_i \\ \vdots \\ \partial_n r_i \end{array} \right))$$

where the matrix on the right is indexed by $k, j$. In the case of surfaces, then by using just one $i$, using $\Gamma_{ij}^k = \Gamma_{ji}^k$ we get $\frac{3}{4}$ of the symbols immediately!

11 Curvatures

Let $R(u,v) = \nabla_u \nabla_v - \nabla_v \nabla_u - [u,v]$ be the curvature operator. Riemannian Curvature is the tensor $(X,Y,Z,W) \mapsto \langle R(X,Y)Z,W \rangle$, i.e. $R_{ijkl} = \langle (\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\partial_j} \nabla_{\partial_i})\partial_k, \partial_l \rangle$. Note that $R_{ijkl} = R_{klij}$. We will use $\Gamma$ to represent global sections. Recall that a tensor is defined to be any $C^\infty(M)$-linear map $\Gamma(TM)^a \otimes \Gamma(TM)^b \rightarrow C^\infty(M)$, i.e. a map $\Gamma(TM) \times \Gamma(TM) \times \ldots \times \Gamma(TM) \times \Gamma(TM) \times \ldots \times \Gamma(TM) \rightarrow C^\infty(M)$ which is linear in each variable, and multiplying one factor by a $C^\infty(M)$ function multiplies the output by the same function.

Exercise.

1. Why is $(X,Y) \mapsto \nabla_X Y$ not a tensor? (note it is tensorial in $X$ if we fix $Y$).
2. Show Riemannian Curvature is in fact a tensor.
3. Show that if two elements of $\Gamma(TM) \times \Gamma(TM) \times \ldots \times \Gamma(TM) \times \Gamma(TM) \times \ldots \times \Gamma(TM)$ agree at a point of $M$, then the values of the function in $C^\infty(M)$ they map to at that point is the same (so a tensor only see $0$th order information).
Anyway, the Ricci tensor applied to the vectors $X$ and $Y$ is the trace of the map $Z \mapsto R(X, Z)Y$ on $T_xM$. Call this tensor $Ric_{ab}$.

**Exercise.** $Ric_{ij} = g^{al}R_{ijal}$

For a surfaces, $Ric_{ab} = K g_{ab}$ for a constant called the Gaussian Curvature.

**Exercise.**

1. Check that this constant $K$ agrees with the sectional curvature of $X$ and $Y$ (which is a function of the 2-plane spanned by $X$ and $Y$) defined by

   $$K(X, Y) = \frac{\langle R(X, Y)Y, X \rangle}{\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2}.$$ 

2. Check that the scalar curvature is $2K$, where the scalar curvature is $g^{ij}Ric_{ij} = g^{ik}g^{jl}R_{ijkl}$.

To compute the Gaussian Curvature $K$, use the first exercise on the basis $\{\partial_1, \partial_2\}$ to get $K = \frac{\langle \nabla_{\partial_1} \nabla_{\partial_2} \partial_2, \partial_1 \rangle}{\det g_{ij}}$. To simplify calculation, one can use the metric compatibility of $\nabla$ to for example get $\langle \nabla_{\partial_1} \nabla_{\partial_2} \partial_2, \partial_1 \rangle = -\langle \nabla_{\partial_2} \partial_2, \nabla_{\partial_1} \partial_1 \rangle + \partial_1 \langle \nabla_{\partial_2} \partial_2, \partial_1 \rangle$, and similarly for the other term. If you want a formula purely in terms of Christoffel symbols, feel free to use:

$$K = -\frac{1}{g_{11}} (\partial_1 \Gamma_{22}^2 - \partial_2 \Gamma_{11}^2 + \Gamma_{12}^1 \Gamma_{11}^1 - \Gamma_{11}^1 \Gamma_{12}^1 + \Gamma_{12}^2 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{12}^2).$$

**12 Method of Moving Frames**

We can drastically simplify computations if one can produce a nice orthonormal frame (not necessarily satisfying $[e_i, e_j] = 0$).

Let $\{e_i\}$ be an orthonormal frame, and $\omega_i$ the dual basis in $T^*M$. Then there exists a unique matrix of 1-forms $\omega_{ij}$ such that $d\omega_i = \omega_{ij} \wedge \omega_j$, $\omega_{ij} = -\omega_{ji}$, $\omega_{ij} = g(\nabla_{e_k} e_i, e_j) \omega_k$, don’t bother remembering that though, just solve for it on the spot). There’s an important reason that this works — because our connection is compatible with the metric, we can define what it means to be an orthonormal frame, which gives us a principal $O(n)$-bundle. Then this data is locally the data of an $o(n)$-valued 1-form in a trivialization, which gives the continuously varying skew-symmetric matrix. The fact that we have $d\omega_i = \omega_{ij} \wedge \omega_j$ is equivalent to vanishing torsion. I chose a convention with a different sign than on Wikipedia because if you do it their way, the way the $\omega_{ij}$ wedge together is weird and corresponds to matrix multiplication in the wrong direction.

Then we can compute $\Omega_{ij} = -\frac{1}{2} R_{ijkl} \omega_k \wedge \omega_l$ by $\Omega_{ij} = d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} = d\omega - \omega \wedge \omega$. For a surface, $\Omega_{12} = -K \omega_1 \wedge \omega_2$ (this is almost always easier to compute than using Christoffel symbols).
13 Identities you should know

1. $R_{abcd} = -R_{ba cd} = -R_{abdc}$

2. $R_{abcd} = R_{cdab}$

3. First Bianchi: $R_{abcd} + R_{acdb} + R_{adbc} = 0$

4. Second Bianchi: Define $R_{abcde} = \partial_e R_{abcd} - R(\nabla_e \partial_a, \partial_b, \partial_c, \partial_d) - R(\partial_a, \nabla_e \partial_b, \partial_c, \partial_d) - \ldots$, then we have $R_{abcde} + R_{abecd} + R_{abdec} = 0$. 
