Applications of Combinatorics to Problems of Geometry

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Introduction

In this thesis we solve four geometric problems using combinatorial techniques. The intersection of combinatorics and geometry is the study of discrete, finitary processes which govern continuous, dynamical behaviour. Here, we will focus on exploiting these processes to solve problems in Euclidean and algebraic geometry. My goal in research is not only to solve concrete problems, but also to create frameworks which capture the key aspects of these problems and how they were solved. This tension between creating and contextualizing solutions yields new, unexpected bridges between different areas of research in mathematics and physics. These connections cement the importance of each of these projects in their respective fields, not just in the context of the original problems that they solve, but also through the novel techniques that are developed along the way.

Euclidean Geometry

Euclidean geometry is the study of regions in Euclidean $n$-dimensional space $\mathbb{R}^n$. With so few constraints placed on the objects under consideration, this area is the home to some of the most notorious outstanding problems in all of mathematics. For example, the Hadwiger-Nelson problem asks for the smallest number of colors needed to color the plane $\mathbb{R}^2$ so that no two points at distance 1 apart have the same color. It is humbling that we have not been able to narrow down which of 5, 6 or 7 the answer might be despite nearly a half-century of effort. I believe that the best combinatorics problems are ones that anyone can understand but resist many good attempts at them. The Euclidean problems in this thesis certainly have this flavour, and one of them in particular lies at the center of an extremely active area of research surrounding stability of geometric inequalities.

Local Maxima of Quadratic Boolean Functions, a single-author project published in Combinatorics, Probability, and Computing, solves a conjecture of Ron Holzman from the 1980s generalizing Kleitman’s foundational result on the Littlewood-Offord problem on concentration of sums of Bernoulli random variables. Littlewood-Offord theory, and inverse Littlewood-Offord theory, are active areas of research, and this problem alludes to the possibility that certain central results in the areas can be strengthened in unexpected ways. Our result also generalizes Sperner’s century old result bounding the size of a family
of subsets of \{1, \ldots, n\} where no set is contained in another. Recently, for his PhD thesis at Cambridge, Stefan David re-interpreted this generalization in the context of studying metastable states of very general models of spin glasses, modelling how properties of objects made up from disordered magnetic pieces change when energy is added or removed.

**Sharp Stability of Brunn-Minkowski for Homothetic Regions**, joint with Peter van Hintum and Marius Tiba, resolves a conjecture of David Jerison and recent Field’s medalist Alessio Figalli, concerning the sharp stability of the Brunn-Minkowski inequality. The Brunn-Minkowski inequality \(|A + B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}\) for \(A, B \subset \mathbb{R}^n\) is one of the most fundamental geometric inequalities, with numerous direct applications to physics. This particular problem, asking about sharp stability in the particular case \(A = B\), was not amenable to well-established techniques of convex geometry, which require one of \(A\) and \(B\) to be convex. We were able to resolve this problem by constructing a fractal-like structure which implies a sharp homogeneity result for how sets \(A\) sit inside their convex hulls when \(\frac{1}{2}(A + A)\) is not too much larger than \(A\). In work in progress, we use related techniques to resolve in two dimensions the sharp stability question for the Brunn-Minkowski inequality for general \(A, B\).

**Algebraic Geometry**

Algebraic geometry is the study of regions cut out by polynomial equations in \(\mathbb{C}^n\). Such regions display certain rigidities that arbitrary surfaces do not have, but they still vary continuously in families by varying the underlying equations. There are many profound open problems in the field, such as the Hodge conjecture and the Jacobian conjecture. At the same time, there are more basic questions about computing invariants of algebraic varieties which are still open and have extremely important enumerative consequences — we will focus on this latter type of problem here.

**GL_{r+1}-orbits in \((\mathbb{P}^r)^n\) and quantum cohomology**, to appear in *Advances in Mathematics*, joint with Mitchell Lee, Anand Patel, and Dennis Tseng, is a marriage of 3 almost completely disjoint fields of mathematics: quantum cohomology (which has applications to mirror symmetry and string theory), matroids, and the lattice point theory of polytopes. Using this connection, we resolve in an equivariant setting an important programme of Kapranov, and of Aluffi and Faber, to compute cohomology classes of \(GL_{r+1}\)-orbit closures in \((\mathbb{P}^r)^n\). The connection between matroids and algebraic geometry was solidified by the recent startling proof by June Huh of the log concavity of coefficients of matroid characteristic polynomials using the Hodge index theorem. Our connection appears in an entirely different way, while only vague shadows of the full picture have appeared scattered throughout the literature to this point. The results (heavily excerpted in this thesis) would be daunting to summarize here, but we note that this project accomplishes the most comprehensive bridge-building of the ones appearing in this thesis.
Incidence strata of affine varieties with complex multiplicities, joint with Dennis Tseng, builds an algebraic model for weighted $k$-point configuration spaces. Configuration spaces are important in physics, modelling for example, the positions of (indistinguishable) electrons, and these particular spaces appear topologically as models of configuration spaces with summable labels. Analyzing the singularities of this model, we answer a question of Farb and Wolfson concerning whether two natural moduli spaces are isomorphic. Our construction appears to be the first concrete application we are aware of for the exciting new field of Deligne tensor categories interpolating the representation theory of $S_n$ with $n \in \mathbb{C}$, and sharply contrasts with the explicit combinatorial techniques we use to establish quantitative elimination results for the finite-typeness of the resulting construction. This construction is especially interesting from the perspective of the burgeoning field of Arithmetic Topology, because this opens the possibility that the topology of these interpolated spaces can relate seemingly different point-counting results over finite fields.

**Conclusion**

Like all fields of science, mathematics is being driven in an interdisciplinary direction. By relating different areas of research, results become more accessible to a broader range of scientists, who can collaborate and explore these new ideas. Making these connections is not just an end, but a means to develop novel tools and ideas for solving outstanding open problems. The connection between combinatorics and geometry outlined in this thesis is not yet fully developed, and there are many directions that can be pursued. It is my hope that this thesis might inspire the reader to look for their own problems in geometry and see if they can be better understood through their underlying combinatorial processes.

**Future Work**

In the next few years, I plan on pursuing a number of directions directly and indirectly generalizing the work above. Below I have outlined some works in progress, as well as some projects which are still in their early stages.

1. Joint with Peter van Hintum and Marius Tiba, we have shown the sharp stability question for the planar Brunn-Minkowski inequality for general $A, B \subset \mathbb{R}^2$. This makes important progress on the stability question for $A, B \subset \mathbb{R}^n$, one of the most important open questions in mathematics, and is an area of active current interest.

2. Joint with Dennis Tseng and Andrew Berget, we are extending our computations to arbitrary $G/P$ using recent results of Berget and Fink. We are still pursuing how the connections to quantum cohomology can be generalized to this much broader setting.
3. Joint with Dennis Tseng and Andrew Berget, we are computing intersections arising from intersecting other divisors on the Bergman fan of a matroid than the one that June Huh used to prove log concavity of the characteristic polynomial coefficients, and interpreting the log concavity results in terms of certain matroid invariants.

4. Joint with Dori Ali-Bejleri, generalizing work of Getzler and Pandharipande, we have developed a new recursion for the Grothendieck class of the Kontsevich stable map compactification $\overline{M}_{0,n}(\mathbb{P}^r, d)$, and are using these results to count the number of points on these stacks over arbitrary finite fields.

5. Currently, I am investigating the relationships between point counts on the interpolated strata constructed in this final project.

**Joint Authorship Statement:** I claim equal parts in all of my projects (listed below).

**Published and Accepted Works**

1. Orbits in $(\mathbb{P}^r)^n$ and small equivariant quantum cohomology
   (with Mitchell Lee, Anand Patel, Dennis Tseng), *Advances in Mathematics* 2019+

2. Judiciously 3-partitioning 3-uniform hypergraphs
   (with Marius Tiba), *Random Structures and Algorithms*.

3. Orthogonal symmetric chain decompositions of hypercubes
   2019
   *Siam Journal of Discrete Mathematics*.

4. Symmetric chain decompositions of products of posets with long chains
   (with Marius Tiba and Stefan David), *Electronic Journal of Combinatorics*.

5. Local maxima of quadratic boolean functions
   2016
   *Combinatorics Probability and Computing*.

**ArXiv Preprints**

1. Sharp stability of Brunn-Minkowski for homothetic regions
   (with Peter van Hintum and Marius Tiba)

2. Multicolour chain avoidance in the boolean lattice
   (with Marius Tiba)

3. Incidence strata of affine varieties with complex multiplicities
   (with Dennis Tseng)

4. Modified diagonals and linear relations between small diagonals
   2018

5. $PGL_2$-equivariant strata of point configurations in $\mathbb{P}^1$
   2018
   (with Dennis Tseng)