MODIFIED DIAGONALS AND LINEAR RELATIONS BETWEEN SMALL DIAGONALS

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Abstract. Let \((X, pt)\) be a pointed smooth projective variety. We prove that the vanishings of the modified diagonal cycles of Gross and Schoen govern the \(\mathbb{Z}\)-linear relations between small \(m\)-diagonals \(pt^{1,\ldots,n}\setminus A \times \Delta_A\) in the rational Chow ring of \(X^n\) for \(A\) ranging over \(m\)-element subsets of \(\{1, \ldots, n\}\).

The combinatorial heart of this paper, which may be of independent interest, is showing the \(\mathbb{Z}\)-linear relations between elementary symmetric polynomials \(e_k(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]\) are generated by the \(S_n\)-translates of a certain alternating sum over the facets of a hyperoctahedron.

1. Introduction

Let \((X, pt)\) be a pointed smooth projective variety and \(n \in \mathbb{N}\). We denote by \(\Delta_A \subset X^A\) the diagonal cycle for any \(A \subset \{1, \ldots, n\}\), and denote the small diagonal
\[
\Delta_A(A) := pt^{1,\ldots,n} \setminus A \times \Delta_A \subset X^n
\]
the locus of \((x_1, \ldots, x_n) \in X^n\) where \(x_i = pt\) if \(i \notin A\) and \(x_i = x_j\) if \(i, j \in A\).

In this paper we give a new interpretation of the vanishings of the “modified diagonal cycles” of Gross and Schoen [GS95] in the rational Chow ring \(A^\bullet(X^n)_{\mathbb{Q}}\). We will show that they classify the group of \(\mathbb{Z}\)-linear relations between the classes of the \(\binom{n}{m}\) small diagonals \(\Delta_A(A)\) with \(|A| = m\) in \(A^\bullet(X^n)_{\mathbb{Q}}\) for all \(m, n\).

Recall that the modified diagonal cycle on \(X^k\) is defined to be
\[
\Delta'_k := \sum_{\emptyset \neq B \subset \{1, \ldots, k\}} (-1)^{k-|B|} \Delta_B(B) \in A^\bullet(X^k)_{\mathbb{Q}}.
\]
In general, the class depends on the choice of \(pt \in X\), and its vanishing has been intensely studied in the context of diagonal decompositions in Chow groups. In particular, we have the following list of vanishing results.

- In [GS95], Gross and Schoen showed that
  - \(\Delta'_k = 0\) precisely if \(k \geq 2\) for \(X = \mathbb{P}^1\),
  - \(\Delta'_k = 0\) precisely if \(k \geq 3\) for \(X\) of genus 1, and
  - \(\Delta'_k = 0\) for \(X\) a hyperelliptic curve with \(pt \in X\) a Weierstrass point.

- In [BV04], Beauville and Voisin showed that
  - \(\Delta'_k = 0\) on a \(K3\)-surface \(X\) if \(pt \in X\) lies on a rational curve.

- In [O'G14], O’Grady showed that
  - If \(\Delta'_m = 0\) then \(\Delta'_{m+s} = 0\) for all \(s \geq 0\) for \(X\) any smooth projective variety.

- In [Voi15], Voisin showed that

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Theorem 1.1. Denote by $m$.

Corollary 2.6 and the above results immediately imply the following.

1. If $X$ is a smooth projective connected variety of dimension $n$ swept out by irreducible curves of genus $g$ supporting a zero-cycle rationally equivalent to $pt \in X$, then we have $\Delta'_m = 0$ for $m \geq (n+1)(g+1)$.

- In [MY16], Moonen and Yin showed that
  - $\Delta'_m = 0$ on a $g$-dimensional abelian variety precisely when $n \geq 2g+1$, and
  - $\Delta'_m = 0$ on a curve of genus $g$ whenever $n \geq g+2$ (which is sharp for a generic pointed curve, see [Qi14]).

One of our main results (Corollary 2.6) is that the minimal $k \leq m$ such that $\Delta'_k = 0$ determines the $\mathbb{Z}$-linear relations between the $\binom{n}{m}$ classes $\Delta_A(A) \in A^*(X^n)_Q$ for $A \subset \{1, \ldots, n\}$ ranging over $m$-element subsets as generated by the $S_n$-translates of a single “$k$-hyperoctahedral $m$-relation” (see Definition 2.2). This will be a consequence of a more general statement (Theorem 2.5) with certain classes derived from an arbitrary symmetric class $\alpha \in A^*(X^m)'_Q$ in place of diagonals.

As it is known (and easy to show) that only for $X = pt$ or $\mathbb{P}^1$ do we have $\Delta'_2 = 0$, Corollary 2.6 and the above results immediately imply the following.

**Theorem 1.1.** Denote by $G_k = G_k(m, n)$ the group generated by the $S_n$-translates of “$k$-hyperoctahedral $m$-relations” (see Definition 2.2). Then for the various smooth pointed projective varieties $(X, pt)$ below, the $\mathbb{Z}$-linear relations between the classes of the $\binom{n}{m}$ small diagonals $\Delta_A(A) \in A^*(X^n)_Q$ with $A \subset \{1, \ldots, n\}$ of size $m$ are given as follows.

- $G_2$ if $X = \mathbb{P}^1$ and $G_3$ is $X$ is of genus 1.
- $G_{2g+1}$ if $X$ is a $g$-dimensional abelian variety.
- $G_{g+2}$ if $X$ is a generic pointed curve of genus $g$.
- $G_3$ if $X$ is a $K_3$ surface and $pt \in X$ lying on a rational curve.
- $G_3$ if $X$ is a hyperelliptic curve with $pt \in X$ a Weierstrass point.

Moreover, if for some $\ell$ we have $\Delta'_k = 0$, then the $\ell$-hyperoctahedral $m$-relations are a subset of the relations satisfied by the $\Delta_A(A)$ classes (which may turn out to be all such relations if $\ell$ happens to be the minimal $k$), yielding in the remaining cases a partial list of relations.

The combinatorial heart of this paper Theorem 2.3 shows that the “$(k+1)$-hyperoctahedral $m$-relations” govern the $\mathbb{Z}$-linear relations between elementary symmetric polynomials $e_k(x_{a_1}, \ldots, x_{a_m}) \in \mathbb{Z}[x_1, \ldots, x_n]$ where $\{a_1, \ldots, a_m\} \subset \{1, \ldots, n\}$ ranges over $m$-element subsets.

The geometric heart of this paper is a Chow motive computation, which allows us to extract useful information about classes in $A^*(X^m)_Q$ despite having essentially no information about the ring itself. We will decompose the diagonal class in $A^*((X^m)^2)_Q$ in such a way that convolving with the pieces yields a system of orthogonal idempotent endomorphisms of $A^*(X^m)_Q$, which consequently decomposes $A^*(X^m)_Q$ into the direct sum of the images of the idempotents. Our key insight is that we can produce such a decomposition where the non-zero components of the diagonal class in these summands govern the $\mathbb{Z}$-linear relations between the small diagonals in $A^*(X^n)_Q$.

The structure of this paper is as follows.

- In Section 2 we describe the “hyperoctahedral relations” and state our main results.
• In Section 3 we prove our main combinatorial result Theorem 2.3 classifying the $\mathbb{Z}$-linear relations between polynomials $e_k(x_{a_1},\ldots,x_{a_m}) \in \mathbb{Z}[x_1,\ldots,x_n]$ where $\{a_1,\ldots,a_m\} \subset \{1,\ldots,n\}$ ranges over $m$ element subsets.

• In Section 4 we prove our main geometric result Theorem 2.5 which specializes to the modified diagonal result mentioned previously.

2. Hyperoctahedral relations and statement of results

To state our results, the following notation will be useful.

**Definition 2.1.** Denote by $e_k(x_{a_1},\ldots,x_{a_m}) \in \mathbb{Z}[x_1,\ldots,x_m]$ the $k$'th elementary symmetric polynomial. For a symmetric polynomial $f \in \mathbb{Z}[x_1,\ldots,x_m]$ and $A = \{a_1,\ldots,a_m\} \subset \{1,\ldots,n\}$ an $m$-element set, we denote by $f(A) := f(x_{a_1},\ldots,x_{a_m}) \in \mathbb{Z}[x_1,\ldots,x_n]$.

We will be interested in the $\mathbb{Z}$-linear relations between the $e_k(A)$ for fixed $k,m,n$. To motivate the hyperoctahedral relations, we first consider the case $m = 3$ and $k = 2$. Then we can view a relation
\[
\sum_{|A|=3} \lambda_A e_2(A) = 0
\]
as describing a formal $\mathbb{Z}$-linear combination of triangles $\{i,j,k\} \subset \{1,\ldots,n\}$ such that after replacing each $\{i,j,k\}$ with the sum of its three edges $\{i,j\}+\{j,k\}+\{i,k\}$, the sum becomes zero.

If we have a triangulation of a surface such that the faces can be alternately colored black and white, then we can alternately sum the triangles on the surface to get such a relation. The smallest non-trivial instance of this occurs for an octahedron. The sum of $\{i,j,k\}$ over all dark triangles minus the sum over light triangles in the octahedron
\[
\begin{array}{c}
1 \\
2 & 5 \\
3 & 4 \\
6
\end{array}
\]
is what we will call a 3-hyperoctahedral 3-sum, yielding the relation
\[
\begin{align*}
e_2(\{1,2,3\}) + e_2(\{1,4,5\}) + e_2(\{6,3,4\}) + e_2(\{6,2,5\}) \\
= e_2(\{1,3,4\}) + e_2(\{1,2,5\}) + e_2(\{6,2,3\}) + e_2(\{6,4,5\}).
\end{align*}
\]

Theorem 2.3 specialized to this case shows that such relations generate all $\mathbb{Z}$-linear relations between the $e_2(A)$ with $|A| = 3$. In the notation of Definition 2.2, this relation is a 3-hyperoctahedral 3-sum.

**Definition 2.2.** Given a set $B \subset \{1,\ldots,n\}$ of size $m-k$ and disjoint sets $C_i = \{c_{i,0},c_{i,1}\} \subset \{1,\ldots,n\} \setminus B$ for $i = 1,\ldots,k$, we say that the element
\[
\sum_{(c_1,\ldots,c_k) \in \{0,1\}^k} (-1)^{\sum_{i=1}^k c_i} B \cup \{c_{1,c_1},\ldots,c_{k,c_k}\}
\]
is a $k$-hyperoctahedral $m$-sum in $\mathbb{Z}^m$.
Define \( G_k \subset \mathbb{Z}^{(m)} \) to be the subgroup generated by \( k \)-hyperoctahedral \( m \)-sums (we suppress the dependence on \( m \) and \( n \) in the notation).

The parameter \( k \) may be thought of as the “dimension” of the hyperoctahedron, and when \( k < m \) this expression can be thought of as an (iterated) cone with apex(es) \( B \) over a \( k \)-hyperoctahedral \( k \)-sum.

The following theorem is our central combinatorial result.

**Theorem 2.3.** Let \( k \leq m \leq n \) be integers. Then the kernel of the map \( \mathbb{Z}^{(n)} \to \mathbb{Z}[x_1, \ldots, x_n] \) given by \( A \mapsto e_k(A) \) is \( G_{k+1} \), i.e. the \( \mathbb{Z} \)-linear relations between the \( e_k(A) \)'s are generated by the \( (k+1) \)-hyperoctahedral \( m \)-sums.

Furthermore, we have the sequence of inclusions

\[
0 = G_{m+1} \subset G_m \subset \ldots \subset G_0 = \mathbb{Z}^{(n)}.
\]

**Corollary 2.4.** If \( f \in \mathbb{Z}[x_1, \ldots, x_m]^S_m \) is a symmetric polynomial, then the kernel of the map \( \mathbb{Z}^{(n)} \to \mathbb{Z}[x_1, \ldots, x_n] \) given by \( A \mapsto f(A) \) is \( G_{k+1} \), where \( k \) is the largest number of distinct \( x_i \) to appear in a non-zero monomial.

**Proof of Corollary 2.4.** For \( B = (b_1, \ldots, b_j) \) with \( b_1 \leq b_2 \leq \ldots \leq b_j \), if we let \( g_B \) be the sum of all distinct monomials \( x_1^{b_1} \cdots x_j^{b_j} \), then we can write \( f = \sum_B \lambda_B g_B \) for some coefficients \( \lambda_B \), and the \( \mathbb{Z} \)-linear relations between \( f(A) \) are the intersections of the relation groups for each \( g_B \) such that \( \lambda_B \neq 0 \). But it is easy to see that the relations between \( g_B(A) \) are identical to the relations between \( e_{|B|}(A) \), which by Theorem 2.3 is given by \( G_{|B|+1} \). As the \( G \)'s are nested, the result follows. \( \square \)

We are now in a position to state our geometric results properly. In a product space, we use \( \boxtimes \) to denote intersection product of pullbacks from disjoint factors, and we denote by \( \int \) for the pushforward map on rational Chow groups.

**Theorem 2.5.** Let \( X \) be a smooth projective variety and let \( \gamma \in A^*(X)_Q \) be a class for which there exists a \( \gamma^* \in A^*(X)_Q \) with \( \int_{X \rightarrow X^p.t} \gamma \cup \gamma^* = 1 \). Then for any \( \alpha \in A^*(X^m)_Q^S_m \), the kernel of the map \( \mathbb{Z}^{(n)} \to A^*(X^n)_Q \) given by

\[
A \mapsto \alpha(A) := \gamma \boxtimes (1, \ldots, n) \setminus A \boxtimes \alpha
\]

is of the form \( G_{k+1} \) for some \(-1 \leq k \leq m \). If \( k = 0 \) then \( k = -1 \), otherwise \( k \) is the largest number \( \leq m \) such that

\[
0 \neq \alpha_k' := \sum_{B \subset \{1, \ldots, k\}} (-1)^{k-|B|} \alpha_B(B) \in A^*(X^k)_Q
\]

where

\[
\alpha_B(B) = \gamma \boxtimes (1, \ldots, k) \setminus B \boxtimes \int_{X^m \to X^B} \left( \gamma^* \right) \boxtimes (1, \ldots, m) \setminus B \cup \alpha
\]

**Corollary 2.6.** Let \( (X, pt) \) be a pointed smooth projective variety. Then the kernel of the map \( \mathbb{Z}^{(n)} \to A^*(X^n)_Q \) given by

\[
A \mapsto \Delta_A(A)
\]

is \( G_{\ell} \) where \( \ell \leq m \) is the smallest number such that the modified diagonal \( \Delta_{\ell} \) vanishes. If \( \ell = 0 \) then there are no relations.

**Proof.** Take \( \gamma = [pt] \) and \( \alpha \) the class of the diagonal in \( X^m \). Then \( \alpha_k' \) is the modified diagonal cycle \( \Delta_k' \). By [O’G14], if \( \Delta_k' \) vanishes then \( \Delta_{k+s}' \) vanishes for all \( s \geq 0 \), so \( \ell = k + 1 \) where \( k \) is the largest number such that \( \Delta_k' \) does not vanish. \( \square \)
3. Proof of Theorem 2.3

In this section, we prove our main combinatorial result Theorem 2.3.

\textbf{Proof of Theorem 2.3.} The $e_k(A)$’s satisfy every $(k+1)$-hyperoctahedral $m$-relation because for every monomial $x_{i_1} \ldots x_{i_k}$, one of the pairs $C_i$ associated to the hyperoctahedral relation contains none of the indices $i_j$, and hence the terms in the alternating sum with $\epsilon_i = 1$ containing this monomial cancel the terms with $\epsilon_i = -1$.

We now show that there are no linear relations between $e_k(A)$ with $A$ ranging over $m$-element subsets of $\{1, \ldots, m+k\}$. Note there are $\binom{m+k}{m}$ polynomials of the form $e_k(A)$ with $A$ of size $m$ inside $\{1, \ldots, m+k\}$, and this is equal to $\binom{m+k}{k}$, the number of $k$-products of distinct monomials $x_i$ with $i \in \{1, \ldots, m+k\}$. Hence, to show the linear independence it suffices to show that we can write each of these monomials as a $\mathbb{Q}$-linear combination of the $e_k(A)$’s with $A$ of size $m$ in $\{1, \ldots, m+k\}$.

We will do this by inductively showing that all monomials of the form $x_{i_1} \ldots x_{i_k} e_{k-\ell}(B)$ lie in the $\mathbb{Q}$-linear span of the $e_k(A)$ with $A \in \{1, \ldots, m+k\}$ distinct and $B \subseteq \{1, \ldots, m+k\} \setminus \{i_1, \ldots, i_{\ell}\}$ a subset of size $m$. This is true for $\ell = 0$. Suppose the result is true for $\ell - 1$, we will show it is true for $\ell$. Indeed,

\[
(m - (k - \ell))x_{i_1} \ldots x_{i_{\ell-1}} e_{k-\ell}(B) = x_{i_1} \ldots x_{i_{\ell-1}} \sum_{b \in B} e_{k-\ell+1}(\{i_\ell\} \cup B \setminus \{b\}) - x_{1} \ldots x_{i_{\ell-1}} (m - (k - \ell + 1)) e_{k-\ell+1}(B).
\]

Thus by induction the statement is true for $\ell = k$, and this shows that each such monomial is a $\mathbb{Q}$-linear combination of the $e_k(A)$’s as desired.

Now, we prove by induction on $k$ that given a $\mathbb{Z}$-linear relation $0 = \sum A \lambda_A e_k(A)$, we can write $\sum A \lambda_A A$ as a $\mathbb{Z}$-linear combination of $(k+1)$-hyperoctahedral $m$-sums. For $k = 0$ the result is trivial, so now assume that $k > 0$. If $n \leq m + k$ then the relation must be identically zero by what we have just proved, so assume that $n > m + k$.

Our goal is to first show that we may subtract from $\sum A \lambda_A A$ a $\mathbb{Z}$-linear combination of $(k+1)$-hyperoctahedral $m$-sums involving elements in $\{1, \ldots, n\}$ so that the resulting sum involves only sets in $\{1, \ldots, n-1\}$. Write our relation between the $e_k(A)$’s as

\[
0 = \sum \lambda_A e_k(A) = x_n \sum_{A \in \mathcal{A}} \lambda_A e_{k-1}(A \setminus n) + \sum_{A \in \mathcal{A}} \lambda_A e_k(A \setminus n) + \sum_{n \notin A} \lambda_A e_k(A),
\]

we see that the $x_n$-coefficient is $\sum_{n \in A} \lambda_A e_{k-1}(A \setminus n)$, so must equal 0. By the induction hypothesis, we know that $\sum_{n \in A} \lambda_A (A \setminus n)$ is the sum of $k$-hyperoctahedral $(m-1)$-relations, so for each $i$ there exists an $m$-element set $B_i \subseteq \{1, \ldots, n-1\}$ and disjoint pairs $C_j = \{c_{j0}, c_{j1}\} \subseteq \{1, \ldots, n-1\} \setminus B_i$ for $1 \leq j \leq k$ such that

\[
\sum_{n \in A} \lambda_A (A \setminus n) = \sum_{i} \sum_{(c_{1}, \ldots, c_{k}) \in \{0,1\}^k} (-1)^{\sum c_i} B_i \bigcup \{c_{1,1}, \ldots, c_{1,k}\}.
\]

As $n - 1 \geq m + k$ and only $m + k - 1$ elements are used in the $i$’th hyperoctahedral sum, there exists an element $r_i \in \{1, \ldots, n-1\}$ not used in the $i$’th sum. Then
letting $c_{k+1,0} = n$ and $c_{k+1,1} = r_k$, we have
\[
\sum_{n \in A} \lambda_A A = \sum_{i} \sum_{(e_1, \ldots, e_k) \in \{0,1\}^{k+1}} (-1)^{\epsilon_k} B^i \bigcup \{c_{1,e_1}, \ldots, c_{k,k+1}\}
\]
\[
+ \sum_{i} \sum_{(e_1, \ldots, e_k) \in \{0,1\}^{k}} (-1)^{\epsilon_k} \{r_i\} \cup B^i \bigcup \{c_{1,e_1}, \ldots, c_{k,k+1}\},
\]
where the first term on the right hand side is a sum of $(k+1)$-hyperoctahedral $m$-sums, and the second term does not involve $n$. Hence subtracting these $(k+1)$-hyperoctahedral $m$-relations from $\sum \lambda_A A$, we obtain a sum involving only sets in $\{1, \ldots, n-1\}$.

Subtracting the corresponding sum of $(k+1)$-hyperoctahedral $m$-sums from $\sum \lambda_A e_k(A)$, we have reduced the problem to one where all of the $A$ lie in $\{1, \ldots, n-1\}$. Repeating this we eventually reduce down to where all $A \subset \{1, \ldots, m+k\}$. By what we showed earlier the resulting relation between the $e_k(A)$ must in fact be the zero relation. Hence we have written the original relation $0 = \sum \lambda_A e_k(A)$ as a $\mathbb{Z}$-linear combination of $(k+1)$-hyperoctahedral $m$-relations as desired, and our induction is complete.

Finally, the nesting of the $G_i$ follows since given a relation between $e_k(A)$'s, applying the operator $\frac{1}{m-k+1} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}$ yields the identical relation between $e_{k-1}(A)$'s.

\[\square\]

4. Proof of Theorem 2.5

In this section we prove Theorem 2.5, our main geometric result. We first discuss some generalities on Chow motives, and then proceed with the proof.

In general $A^\bullet$ does not satisfy a Künneth formula, so we only have a possibly non-injective and non-surjective map $A^\bullet(X)_Q^m \to A^\bullet(X^m)_Q$. By using Chow motives we will see that certain correspondences provide enough of a substitute for the Künneth formula for our purposes.

We say that a correspondence on $X$ is an element of $A^\bullet(X^2)_Q$. Let $\Gamma$ be a correspondence on $X$, which induces an endomorphism of $A^\bullet(X)_Q$ via

$$\Gamma : x \mapsto (\pi_x)_* (\pi^*_x \alpha) \cap \Gamma$$

where $\pi_i$ is the projection $X^2 \to X$ onto the $i$'th factor. For any space $S$, $\Gamma$ similarly induces an endomorphism of $A^\bullet(S \times X)_Q$. There is a notion of composition of correspondences, which for $\Gamma_1, \Gamma_2 \in A^\bullet(X^2)_Q$ is defined by

$$\Gamma_2 \circ \Gamma_1 = (\pi_{13})_* (\pi^*_{12} \Gamma_1 \cup \pi^*_{23} \Gamma_2)$$

where $\pi_{ij}$ is the projection $X^3 \to X^2$ onto the $i, j$ factors. On the level of functions, $\Gamma_2 \circ \Gamma_1$ induces the composite endomorphism of $A^\bullet(S \times X)_Q$. Suppose that $\Gamma$ is \textit{idempotent}, which means that

$$\Gamma = \Gamma \circ \Gamma.$$

In particular, $\Gamma \in A^{\dim(X)}(X^2)_Q$ so the associated function doesn’t shift the grading. An effective Chow motive is a pair of the form $(X, \Gamma)$ with $\Gamma$ an idempotent correspondence on $X$. The identity for composition is the idempotent $\Gamma = \Delta_2$, whose associated endomorphism on $A^\bullet(S \times X)_Q$ is the identity.

As $\Gamma$ is idempotent and $\Delta_2$ corresponds to the identity, $\Gamma, \Delta_2 - \Gamma$ are orthogonal idempotents, so the images of their associated functions $A^\bullet(X)_Q \to A^\bullet(X)_Q$ direct sum to $A^\bullet(X)_Q$. More generally, for $k \in \{m, n\}$, the $2^k$ elements $\Gamma^{\oplus (1, \ldots, k)} \setminus B$
\((\Delta_2 - \Gamma)^{\otimes B}\) form an orthogonal system of idempotent correspondences of \(X^k\) in \(A^\bullet((X^2)^k)_Q = A^\bullet((X^k)^2)_Q\) summing to the identity correspondence \(\Delta_2^{\otimes\{1,\ldots,k\}}\), so

\[
A^\bullet(X^m)_Q = \bigoplus_{B \subset \{1,\ldots,m\}} \text{Im}(\Gamma^{\otimes\{1,\ldots,m\}\setminus B} \otimes (\Delta_2 - \Gamma)^{\otimes B}), \quad \text{and}
\]

\[
A^\bullet(X^n)_Q = \bigoplus_{B \subset \{1,\ldots,n\}} \text{Im}(\Gamma^{\otimes\{1,\ldots,n\}\setminus B} \otimes (\Delta_2 - \Gamma)^{\otimes B}).
\]

We remark that on any product space \(X_1 \times X_2\) with correspondences \(\Gamma_i \in A^\bullet((X_i)^2)_Q\) we have \(\Gamma_1 \otimes \Gamma_2 = (\Gamma_1 \otimes (\Delta_2)_{X_2}) \circ ((\Delta_2)_{X_1} \otimes \Gamma_2)\). Hence for \(k \in \{m, n\}\), each of the above correspondences is the composition of \(k\) commuting correspondences on \(X^k\), with the \(i\)th correspondence inducing the endomorphism of \(A^\bullet(X^{i-1} \times X \times X^{k-i})_Q\) from the associated correspondence on \(X\).

It is clear that \(\alpha \in A^\bullet(X^m)_Q\) if and only if the component of \(\alpha\) in \(\text{Im}(\Gamma^{\otimes\{1,\ldots,m\}\setminus B} \otimes (\Delta_2 - \Gamma)^{\otimes B})\) is \(S_{(1,\ldots,m)\setminus B} \times S_B\)-invariant and depends only on \(|B|\), and hence

\[
A^\bullet(X^m)_{S_m} \cong \bigoplus_{k=0}^{m} \text{Im}(\Gamma^{\otimes m-k} \otimes (\Delta_2 - \Gamma)^{\otimes k})^{S_{m-k} \times S_k}.
\]

Note that \(\text{Im}\) does not distribute over \(\otimes\) in general because there may be classes in \(A^\bullet(X^n)_Q\) which are not in the image of \(A^\bullet(X^m)_{S_m} \to A^\bullet(X^m)_Q\).

**Proof of Theorem 2.5.** Let \(\Gamma = \gamma^* \otimes \gamma \in A^\bullet(X^2)_Q\) with \(\gamma^* \in A^\bullet(X)\). The idempotency of \(\Gamma\) is equivalent to \(\int \gamma \cup \gamma^* = 1 \in A^\bullet(\text{pt})_Q\), and the map \(\Gamma\) takes \(\alpha \mapsto \gamma \cup \int (\alpha \cup \gamma^*)\). More generally, in any space \(X \times Y\), \(\Gamma\) induces the endomorphism of \(A^\bullet(X \times Y)_Q\)

\[
\Gamma: \alpha \mapsto \gamma \otimes \int_{X \times Y \to Y} \alpha \cup (\gamma^* \otimes 1)
\]

and for any class \(\delta \in A^\bullet(Y)_Q\) we have \(\Gamma(\gamma \otimes \delta) = \gamma \otimes \delta\). Consequently, we have

\[
\text{Im}(\Gamma^{\otimes m-k} \otimes (\Delta_2 - \Gamma)^{\otimes k})^{S_{m-k} \times S_k} = \gamma^{\otimes m-k} \otimes \text{Im}((\Delta_2 - \Gamma)^{\otimes k})^{S_k}
\]

\[
\text{Im}(\Gamma^{\otimes(1,\ldots,n)\setminus B} \otimes (\Delta_2 - \Gamma)^{\otimes B}) = \gamma^{\otimes(1,\ldots,n)\setminus B} \otimes \text{Im}((\Delta_2 - \Gamma)^{\otimes B}).
\]

Under the map \(\alpha \mapsto \alpha(A)\), the \(k\)th summand \(\gamma^{\otimes m-k} \otimes \text{Im}((\Delta_2 - \Gamma)^{\otimes k})^{S_{m-k} \times S_k}\) of \(A^\bullet(X^m)_{S_m}\) maps to

\[
\bigoplus_{|B|=k} \gamma^{\otimes \{1,\ldots,n\}\setminus B} \otimes \text{Im}((\Delta_2 - \Gamma)^{\otimes B}) \subset A^\bullet(X^n)_Q,
\]

which is made up of disjoint summands for distinct \(k\)’s. This map acts on an element \(\gamma^{\otimes m-k} \otimes \beta \in \gamma^{\otimes m-k} \otimes \text{Im}((\Delta_2 - \Gamma)^{\otimes k})^{S_{m-k} \times S_k}\) in the \(k\)th summand of \(A^\bullet(X^m)_{S_m}\) via

\[
\gamma^{\otimes m-k} \otimes \beta \mapsto \sum_{|B|=k, B \subset A} \gamma^{\otimes \{1,\ldots,n\}\setminus B} \otimes \beta.
\]

Note that there is exactly one term in each summand \(\gamma^{\otimes \{1,\ldots,n\}\setminus B} \otimes \text{Im}((\Delta_2 - \Gamma)^{\otimes B}) \subset A^\bullet(X^n)_Q\) with \(|B| = k\) and \(B \subset A\), and the term depends only on \(\beta\) and not on \(A\).
We may check directly that
\[
\left( \Gamma^\boxtimes_{\{1, \ldots, m\}} B \boxtimes (\Delta(2) - \Gamma)^\boxtimes B \right) (\alpha) \\
= \sum_{A \subset B} (-1)^{|B|-|A|} \gamma^\boxtimes_{\{1, \ldots, m\}} A \boxtimes \int_{X^m \to X^A} (\gamma^* \boxtimes \{1, \ldots, m\}) A \cup \alpha \\
= \gamma^\boxtimes_{\{1, \ldots, m\}} B \boxtimes \alpha'_{|B|},
\]
where we recall that \( \alpha'_{|B|} \) was defined in the theorem statement. Thus the component of \( \alpha \) in the \( k \)’th summand \( \text{Im}(\Gamma^\boxtimes_{m-k} \boxtimes (\Delta(2) - \Gamma)^\boxtimes_{m-k} S_{m-k} \times S_k) \) is \( \gamma^{m-k} \boxtimes \alpha'_{k} \).

Hence, if we set
\[
f = e_{k_1} + \ldots + e_{k_r} \in \mathbb{Z}[x_1, \ldots, x_m]^{S_m}
\]
where the \( k_i \) are those \( k \) such that \( \gamma^\boxtimes_{\{1, \ldots, n\}} B \boxtimes \alpha'_{k} \neq 0 \) for any \( |B| = k \) (note that by symmetry either all the classes for such \( B \) vanish or none vanish), then the \( \mathbb{Z} \)-linear relations between the \( f(A) \) polynomials are identical to the \( \mathbb{Z} \)-linear relations between the \( \alpha(A) \) for \( A \) ranging over \( m \)-element subsets of \( \{1, \ldots, n\} \).

Finally,
\[
\text{Im}(\text{Im}(\Delta - \Gamma)^\boxtimes B) \cong \gamma^\boxtimes_{\{1, \ldots, n\}} B \boxtimes \text{Im}(\Delta - \Gamma)^\boxtimes B
\]
because the map \( \beta \mapsto \gamma^\boxtimes_{\{1, \ldots, n\}} B \boxtimes \beta \) has an inverse map given by \( \delta \mapsto \int_{X^n \to X^B} \delta \cup (\gamma^* \boxtimes \{1, \ldots, n\}) B \), so we have
\[
\gamma^\boxtimes_{\{1, \ldots, n\}} B \boxtimes \alpha'_{|B|} = 0 \iff \alpha'_{|B|} = 0.
\]
The result now follows by applying Corollary 2.4 to \( f \).

\[\square\]

**Remark 4.1.** We remark that there is a notion of homomorphism and tensor product for Chow motives, and if \( \gamma \in A^k(X) \mathbb{Q} \), then \( (X, \gamma^* \boxtimes \gamma) \) is isomorphic to the motive \( \mathbb{L} \otimes k \) where \( \mathbb{L} \) is the Lefschetz motive \((\mathbb{P}^1, \mathbb{P}^1 \times \{\text{pt}\})\), see [Kim05].

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**References**


