MATH 281X: ARAKELOV THEORY OF ARITHMETIC SURFACES
THE ARITHMETIC RIEMANN-ROCH THEOREM

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1. The arithmetic Riemann-Roch theorem

1.1. **Arithmetic Euler characteristic.** Consider an exact sequence

$$\mathbb{E}: \quad 0 \to M_0 \to M_1 \to \ldots \to M_n \to 0$$

of finitely generated $\mathbb{Z}$-modules. Suppose that each M_j has volumes, so that we have measurable $\mathbb{Z}$-modules $\hat{M}_j$, and this data is denoted by $\hat{\mathbb{E}}$.

The determinant of $\mathbb{E}$ is

$$\det(\mathbb{E}) = \bigotimes_{j=1}^{n} \det(M_j \otimes \mathbb{C})^{\otimes(-1)^j}$$

and each factor has a metric induced by the corresponding volumes. By exactness we have a canonical isomorphism $\det(\mathbb{E}) \simeq \mathbb{C}$, thus $\hat{\mathbb{E}}$ determines a metric $\| - \|_{\hat{\mathbb{E}}}$ on $\mathbb{C}$. We say that $\mathbb{E}$ is *volume exact* if $\|1\|_{\hat{\mathbb{E}}} = 1$.

We define the *Euler characteristic of a measurable $\mathbb{Z}$-module* $\hat{M}$ by the formula

$$\chi(\hat{M}) = -\log \mathrm{Vol}\,(M \otimes_{\mathbb{Z}} \mathbb{R}/M) + \log \# M_{tor}.$$

Theorem 1.1. *$\chi(-)$ is additive on short volume exact sequences.*

Proof. Routine verification. The main point is to understand that in a short exact sequence, torsion is related to index of sublattices. Details left to the reader. □

It might seems surprising that the condition of volume exactness does not make assumptions on torsion, yet the Euler characteristic (which is defined using torsion) is compatible with volume exactness in the sense of the previous result. So, the following example can be instructive.

Example. Consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{F}_2 \to 0$$

given by the maps $n \mapsto (0, 2n)$ and $(a, b) \mapsto (a, \bar{b})$. After tensoring with $\mathbb{C}$ we get the exact sequence

$$0 \to \mathbb{C} \to \mathbb{C}^2 \to \mathbb{C} \to 0$$

given by $z \mapsto (0, 2z)$ and $(w_1, w_2) \mapsto w_1$. We put the standard euclidean volume μ_1 on the first $\mathbb{C}$ and the euclidean volume μ_2 on $\mathbb{C}^2$. Let $\| - \|_*$ be the unique metric on $\det \mathbb{C} = \mathbb{C}$ so that the second $\mathbb{C}$ gets a Haar measure that makes the sequence volume exact. Let's compute $\| - \|_*$: Take the basis vector 1 for both $\mathbb{C}$'s, then $2e_2$

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This is v.1 —not fully written up in TeX, but still might be useful.

is the image of 1 under the first map, and e_1 is a lifting of 1 in the second map. The corresponding dual basis for $(\mathbb{C}^2)^\vee$ is $e_1^*, \frac{1}{2}e_2^*$ and the volume exactness condition is

$$1 = \|1 \otimes (e_1^* \wedge \frac{1}{2}e_2^*) \otimes 1\| = 1 \cdot \frac{1}{2} \cdot \|1\|_*$$

so that $\|-\|_* = 2|-|$ on our second $\mathbb{C}$ which corresponds to $2\mu_1$. With these Haar measures we have:

$$\chi((\mathbb{Z}, \mu_1)) = -\log \mathrm{Vol}_{\mu_1}(\mathbb{R}/\mathbb{Z}) + \log \#(0) = 0$$
$$\chi((\mathbb{Z}, \mu_2)) = -\log \mathrm{Vol}_{\mu_2}(\mathbb{R}^2/\mathbb{Z}^2) + \log \#(0) = 0$$
$$\chi((\mathbb{Z}, 2\mu_1)) = -\log \mathrm{Vol}_{2\mu_1}(\mathbb{R}/\mathbb{Z}) + \log \#\mathbb{F}_2 = 0$$

More generally we define the *Euler characteristic of a measurable O_K-module $\hat{M}$* by the formula

$$\chi_K(\hat{M}) = -\log \mathrm{Vol}\,(M \otimes_{\mathbb{Z}} \mathbb{R}/M) + \log \#M_{tor} + \mathrm{rank}_{O_K}(M) \log |D_K|^{1/2}.$$

so that $\chi_{\mathbb{Q}} = \chi$.

Corollary 1.2. *$\chi_K(-)$ is additive on short volume exact sequences.*

Proof. This follows from the previous result because the rank is additive on short exact sequences. □

The additional discriminant term has been introduced because the results of the previous section suggest that this normalization is well-behaved. More precisely, from propositions **??** and **??** we deduce

Proposition 1.3. *Let $\overline{M} = (M, \{\|-\|_\sigma\}_\sigma)$ be a metrized projective O_K-module of rank 1 and let $\hat{M}$ be the associated measurable O_K-module. Then*

$$\chi_K(\hat{M}) = \widehat{\deg}_K \overline{M}.$$

Proposition 1.4. *Let $\hat{M}$ be a measurable O_K-module and let F/K be a finite extension. For the base change $\hat{M}_{O_F}$ we have*

$$\chi_F(\hat{M}_{O_F}) = [F : K] \cdot \chi_K(\hat{M}).$$

1.2. **Cohomology modules.** Let $\hat{\mathscr{X}} = (\mathscr{X}, \pi, \{d\mu_\sigma^{Ar}\}_\sigma)$ be a regular, semi-stable arithmetic surface over $S = \mathrm{Spec}\, O_K$ endowed with the Arakelov canonical p.v.f's at each $\sigma : K \to \mathbb{C}$.

In order to correctly formulate the arithmetic Riemann-Roch theorem, we need to address the following crucial technicality:

For $\overline{\mathscr{L}} = (\mathscr{L}, \{\|-\|_\sigma\}_\sigma)$ a metrized line sheaf and for $j \geq 0$, we would like to consider certain measurable O_K-modules $H^j(\mathscr{X}, \overline{\mathscr{L}})$. The arithmetic Riemann-Roch theorem will concern such objects.

As an O_K-module, $H^j(\mathscr{X}, \overline{\mathscr{L}})$ oughts to be simply the cohomology O_K-module $H^j(\mathscr{X}, \mathscr{L})$, and the question is how to put Haar measures at each σ. We make the following remarks.

- The volumes should depend in a nice way on the metrics of $\overline{\mathscr{L}}$.
- The volumes should be expected to be well-behaved with respect to basic constructions involving metrized line sheaves.

- By flat base change, we have a canonical isomorphism $H^j(\mathscr{X},\mathscr{L})\otimes_\sigma \mathbb{C} \simeq H^j(X_\sigma,\mathscr{L}_\sigma)$. So it suffices to address the problem purely in the context of compact Riemann surfaces.

In addition, it suffices to consider $j = 0, 1$ due to the following

Lemma 1.5. *For $j \geq 2$ and for any line sheaf $\mathscr{L}$ on $\mathscr{X}$, we have $H^j(\mathscr{X},\mathscr{L}) = 0$.*

Proof. The proof is a standard verification using Cech cohomology and the fact that $\pi : \mathscr{X} \to S$ is flat and projective (the latter because $\mathscr{X}$ is regular), and has relative dimension 1. $\square$

Remarks.

- At this point, it is appropriate to observe that H^0 in this context is always torsion free, but H^1 can very well only consist of torsion, for instance, when $\mathscr{L}|_X$ is ample on $X = \mathscr{X}_\eta$. Thus, torsion does play a role in this matter, which is the reason why it was considered in our previous study of measurable O_K-modules.
- As in the case of the Riemann-Roch theorem for curves over a field, the theorem we are aiming to obtain does not concern H^0 or H^1 individually, but rather, the difference of certain invariants of them —in the geometric case, the difference of dimension, and in the arithmetic case, the difference of Euler characteristics. Thus, for our purposes it is perfectly fine if we determine the volumes on H^0 and H^1 only up to a common scalar multiple (suitably weighted by dimension).

After all this preliminary comments, we can present the result due to Faltings that will provide us with the necessary volumes

Let Y be a compact Riemann surface of genus $g \geq 1$. For a line sheaf $\mathscr{L}$ on Y define the *determinant of cohomology* by

$$\lambda(Y,\mathscr{L}) := det_{\mathbb{C}} H^0(Y,\mathscr{L}) \otimes det_{\mathbb{C}} H^1(Y,\mathscr{L})^\vee.$$

This is a 1-dimensional $\mathbb{C}$-vector space. It is a good approximation of $det_{\mathbb{C}} H^0(Y,\mathscr{L})$ in the following sense

Lemma 1.6. *If $\deg\mathscr{L} > 2g-2$, then we have a canonical isomorphism*

$$\lambda(Y,\mathscr{L}) \simeq det_{\mathbb{C}} H^0(Y,\mathscr{L}).$$

In particular, if $\deg\mathscr{L} > 0$ then for large enough n we have a canonical isomorphism

$$\lambda(Y,\mathscr{L}^{\otimes n}) \simeq det_{\mathbb{C}} H^0(Y,\mathscr{L}^{\otimes n}).$$

Proof. By Serre duality $h^1(\mathscr{L}) = h^0(\Omega^1_Y \otimes \mathscr{L}^\vee) = 0$ for $\deg\mathscr{L} > 2g-2$, hence the result. $\square$

We will encounter the following setup: Let D be a divisor on Y, let $P \in Y$ and write $D' = D + P$. Then the sequence

$$0 \to \mathscr{O}_Y(D) \to \mathscr{O}_Y(D') \to \mathscr{O}_P \otimes \mathscr{O}_Y(D') \to 0 \tag{1.1}$$

is exact. The sheaf $\mathscr{O}_P \otimes \mathscr{O}_Y(D')$ is a skyscraper sheaf at P whose global sections are canonically identified with the fibre $\mathscr{O}_Y(D')|_P$. Furthermore $H^1(Y,\mathscr{O}_P \otimes \mathscr{O}_Y(D')) = 0$ as for any skyscraper sheaf. Hence the long exact sequence in cohomology is

$$\begin{aligned} 0 &\to H^0(Y,\mathscr{O}_Y(D)) \to H^0(Y,\mathscr{O}_Y(D')) \to \mathscr{O}_Y(D')|_P \\ &\to H^1(Y,\mathscr{O}_Y(D)) \to H^0(Y,\mathscr{O}_Y(D')) \to 0. \end{aligned}$$

By exactness we get the canonical isomorphism

$$\mathbb{C} \simeq \big(det_{\mathbb{C}} H^0(Y, \mathscr{O}_Y(D))\big) \otimes \big(det_{\mathbb{C}} H^0(Y, \mathscr{O}_Y(D'))\big)^{\vee} \otimes \big(det_{\mathbb{C}} \mathscr{O}_Y(D')|_P\big) \otimes \big(det_{\mathbb{C}} H^1(Y, \mathscr{O}_Y(D))\big)^{\vee} \otimes \big(det_{\mathbb{C}} H^1(Y, \mathscr{O}_Y(D'))\big)$$

Note that $\dim_{\mathbb{C}} \mathscr{O}_Y(D+P)|_P = 1$. We deduce:

Lemma 1.7 (Exactness isomorphism)**.** *In the previous setting, the exact sequence* (1.1) *induces a canonical isomorphism*

$$\lambda(Y, \mathscr{O}_Y(D+P)) \simeq \lambda(Y, \mathscr{O}_Y(D)) \otimes \mathscr{O}_Y(D+P)|_P.$$

As a preparation for Faltings's theorem on volumes, let us first address a rather natural case

Lemma 1.8. *Let Y be a compact Riemann surface of genus $g \geq 1$. We have a canonical isomorphism*

$$\lambda(Y, \mathscr{O}_Y) \simeq det_{\mathbb{C}} H^0(Y, \Omega^1)^{\vee}.$$

Hence, the inner product

$$\langle \alpha, \beta \rangle = \frac{i}{2} \int_Y \alpha \wedge \overline{\beta}$$

on $H^0(Y, \Omega^1_Y)$ induces a unique hermitian norm $\| - \|_{F,can}$ on $\lambda(Y, \mathscr{O}_Y)$.

Proof. We have the canonical isomorphism $H^0(Y, \Omega^1_Y)^{\vee} \simeq H^1(Y, \mathscr{O}_Y)$, hence the result. □

We are now in a position to state Faltings's theorem on existence of convenient Haar measures in cohomology.

Theorem 1.9 (Faltings's volumes on cohomology)**.** *Let Y be a compact Riemann surface of genus $g \geq 1$. There is a unique way to associate, to each $d\mu^{Ar}$-admissible metrized line sheaf $\overline{\mathscr{L}}$, a hermitian norm $\| - \|_{F,\overline{\mathscr{L}}}$ on $\lambda(Y, \mathscr{L})$ in such a way that the following holds:*

- *(Isometry) An isometric isomorphism $\overline{\mathscr{L}}_1 \simeq \overline{\mathscr{L}}_2$ induces an isomorphism $\lambda(Y, \mathscr{L}_1) \simeq \lambda(Y, \mathscr{L}_2)$ that respects the metrics $\| - \|_{F,\overline{\mathscr{L}}_j}$.*
- *(Scaling) For $\overline{\mathscr{L}} = (\mathscr{L}, \| - \|)$ and $a \in \mathbb{R}_{>0}$ write $a \cdot \overline{\mathscr{L}} = (\mathscr{L}, a\| - \|)$. Then*
$$\| - \|_{F,a\cdot\overline{\mathscr{L}}} = a^{h^0(\mathscr{L}) - h^1(\mathscr{L})} \cdot \| - \|_{F,\overline{\mathscr{L}}}$$
- *(Exactness) Let D be a divisor on Y and $P \in Y$. Recall that there is a unique μ_Y^{Ar}-admissible and normalized metric on $\mathscr{O}_Y(D)$and on $\mathscr{O}_Y(D+P)$, and in particular this induces a hermitian norm on $\mathscr{O}_Y(D+P)|_P$. With respect to these norm and the corresponding norms $\| - \|_{F,-}$ on the determinant of cohomology, the exactness isomorphism*
$$\lambda(Y, \mathscr{O}_Y(D+P)) \simeq \lambda(Y, \mathscr{O}_Y(D)) \otimes \mathscr{O}_Y(D+P)|_P$$
becomes an isometry.
- *(Normalization) The metric $\| - \|_{F,\overline{\mathscr{O}}_Y}$ on $\lambda(Y, \mathscr{O}_Y)$ is equal to $\| - \|_{F,can}$.*

From now on, for a metrized line sheaf $\overline{\mathscr{L}}$ we write $\lambda(Y, \overline{\mathscr{L}})$ for the determinant of cohomology with Faltings's norm $\| - \|_{F,\overline{\mathscr{L}}}$.

We postpone the (outline of the) proof of this important theorem to a later section.

1.3. **The arithmetic Riemann-Roch theorem.** Fix $\hat{\mathscr{X}} = (\mathscr{X}, \pi, \{d\mu_\sigma^{Ar}\}_\sigma)$ a regular semi-stable arithmetic surface with the Arakelov canonical p.v.f.'s as before.

For $\overline{\mathscr{L}}$ an admissible line sheaf on $\hat{\mathscr{X}}$, Faltings theorem on the existence of volumes in cohomology allows us to endow $H^0(\mathscr{X}, \mathscr{L})$ and $H^0(\mathscr{X}, \mathscr{L})$ with Haar measures at every σ, uniquely defined up to a common scalar multiple on the associated norms on determinants. The respective cohomology O_K-modules with these induced Haar measures will be denoted $H^j(\mathscr{X}, \overline{\mathscr{L}})$, and the only ambiguity in this definition is a common scalar multiple of the respective covolumes for $j = 0, 1$.

The ambiguity of the common scalar multiple of Faltings volumes in cohomology is resolved by considering the following quantity

$$\chi_K(\hat{\mathscr{X}}, \overline{\mathscr{L}}) := \chi_K(H^0(\mathscr{X}, \overline{\mathscr{L}})) - \chi_K(H^1(\mathscr{X}, \overline{\mathscr{L}}))$$

which is therefore a well-defined real number attached to $\overline{\mathscr{L}}$. We call this quantity the *Euler characteristic* of $\overline{\mathscr{L}}$.

Theorem 1.10 (Faltings's arithmetic Riemann-Roch theorem)**.** *Let $\overline{\mathscr{L}}$ be an admissible line sheaf on $\hat{\mathscr{X}}$. Then we have*

$$\chi_K(\hat{\mathscr{X}}, \overline{\mathscr{L}}) = \frac{1}{2}\left(\overline{\mathscr{L}}\,.\,\overline{\mathscr{L}} \otimes \hat{\omega}^\vee\right) + \chi_K(\hat{\mathscr{X}}, \overline{\mathscr{O}}_{\mathscr{X}}).$$

Proof. We may assume that $\overline{\mathscr{L}} = \overline{\mathscr{O}}(D)$ for some Arakelov divisor D.

For $D = 0$ the assertion is trivial.

We can add to D divisors of the form αF_σ ($\alpha \in \mathbb{R}$) and both sides of the claimed equation change by

$$\alpha \cdot (h^0(D|_X) - h^1(D|_X)) = \alpha \deg_X(D) + 1 - g = \alpha \cdot ((F_\sigma.D) - (F_\sigma.\hat{\omega}))$$

where we used the scaling property of Faltings volumes for the scalar $e^{-\alpha}$. So we may assume that $D = D_{fin}$.

It now suffices to check that both sides of the claimed equality change in the same way when we add to D a (classical) irreducible divisor of $\mathscr{X}$.

We may replace K by any finite extension F/K. Indeed, $\mathscr{X}/S$ is regular semi-stable and we are attempting to replace $\mathscr{X}$ by $\mathscr{Y}'/S_F$ ($S_F = \operatorname{Spec} O_F$) where $\mathscr{Y}'$ is the minimal desingularization of $\mathscr{Y} = \mathscr{X} \otimes_S S_F$. Despite the desingularization, this construction affects cohomology only by base change, affects the Haar measures only by the base-change construction, and replaces $\hat{\omega}$ by the corresponding canonical sheaf of $\mathscr{Y}'/S_F$ (cf. the section *The base change trick*). Furthermore, the Arakelov intersection pairing and the Euler characteristics get multiplied by a factor $[F : K]$. Therefore, we can indeed replace K by F without affecting the truth of the claimed equality.

Let's now check that the truth of the equality is unaffected by adding to D an irreducible finite divisor C.

Suppose first that C is horizontal. After possibly enlarging K we may assume that C is a section (degree 1 over S), say $C = \nu(S)$ with $\nu : S \to \mathscr{X}$ section of π. Since S is affine we have $H^1(S, \nu^*\mathscr{O}(D + C)) = 0$, hence we get the exact sequence of O_K-modules

$$\begin{aligned} 0 &\to H^0(\mathscr{X}, \mathscr{O}(D)) \to H^0(\mathscr{X}, \mathscr{O}(D + C)) \to H^0(S, \nu^*\mathscr{O}(D + C)) \\ &\to H^1(\mathscr{X}, \mathscr{O}(D)) \to H^1(\mathscr{X}, \mathscr{O}(D + C)) \to 0. \end{aligned}$$

Using the normalized admissible metrics and the corresponding Faltings volumes on cohomology, this sequence is volume exact. Hence

$$\chi_K(\hat{\mathscr{X}}, \overline{\mathscr{O}}(D)) + \chi_K(H^0(S, \nu^*\mathscr{O}(D+C))) = \chi_K(\hat{\mathscr{X}}, \overline{\mathscr{O}}(D+C))$$

where $H^0(S, \nu^*\mathscr{O}(D+C))$ is the projective rank 1 O_K-module with the metrics induced from $\overline{\mathscr{O}}(D+C)$. By adjunction for sections we have an isometric isomorphism

$$\nu^*\overline{\mathscr{O}}(D+C) \simeq \nu^*\left(\overline{\mathscr{O}}(D)\otimes\hat{\omega}^\vee\right).$$

Hence, by our covolume computation in the case of projective rank 1 modules, we have

$$\begin{aligned}\chi_K(H^0(S, \nu^*\mathscr{O}(D+C))) &= \chi_K(H^0(S, \nu^*\left(\overline{\mathscr{O}}(D)\otimes\hat{\omega}^\vee\right)))\\ &= \widehat{\deg}_K \nu^*\left(\overline{\mathscr{O}}(D)\otimes\hat{\omega}^\vee\right) = (\overline{\mathscr{O}}(C).\overline{\mathscr{O}}(D)\otimes\hat{\omega}^\vee)\end{aligned}$$

This is the amount by which the left hand side changes. On the other hand, using the properties of the Arakelov intersection pairing along with adjunction again, we see that the change of the right hand side is:

$$\begin{aligned}&\frac{1}{2}\left(\overline{\mathscr{O}}(D+C).\overline{\mathscr{O}}(D+C)\otimes\hat{\omega}^\vee\right) - \frac{1}{2}\left(\overline{\mathscr{O}}(D).\overline{\mathscr{O}}(D)\otimes\hat{\omega}^\vee\right)\\ &= \frac{1}{2}\left((D.C) + (C.D+C) - (C.\hat{\omega})\right)\\ &= (C.D) - \frac{1}{2}\left(-C^2 + (C.\hat{\omega})\right) = (C.D) - (C.\hat{\omega})\end{aligned}$$

which is the same.

Suppose C is a fibre component at $s \in S$. The Haar measures and ranks remain exactly the same. The computation is now similar to the horizontal case but easier, since one uses geometric adjunction for C rather than the arithmetic adjunction that we just applied. Details left to the reader.

□

2. Metrics on the determinant of cohomology

In this brief section we outline Faltings's proof of existence of suitable metrics on determinant of cohomology. Technical details will be omitted, but we expect that this sketch can help to understand the main ideas. Everything in this section is over the complex numbers.

2.1. **Setup and preliminary reductions.**

Let Y be a compact Riemann surface. For each line sheaf $\mathscr{L}$ on Y we consider the determinant of cohomology

$$\lambda(Y, \mathscr{L}) = \det H^0(Y, \mathscr{L}) \otimes \det H^1(Y, \mathscr{L})^\vee$$

For a metrized line sheaf $\overline{\mathscr{L}} = (\mathscr{L}, \|-\|)$ with a $d\mu^{Ar}$-admissible metric, we aim to construct Faltings's metric $\|-\|_{F,\overline{\mathscr{L}}}$ on $\lambda(Y, \mathscr{L})$ satisfying the required properties: Isometry compatibility, scaling, exactness, and normalization.

First, we only need to consider sheaves of the form $\mathscr{L} = \mathscr{O}(D)$ for D a divisor on Y. This is because if we succeed in this case, then the metrics are compatible with isometric isomorphisms so we obtain the required metrics in all cases.

Secondly, we already know what metric to take for $D = 0$ and the trivial metrics on $\mathscr{O}_Y$, by the normalization property.

Thirdly, we only need to consider the case of the unique admissible normalized metrics on $\mathscr{O}(D)$, thanks to the scaling property.

Lemma 2.1. *Under the conditions of "normalization" and "exactness", every line sheaf $\mathscr{O}(D)$ with the unique admissible normalized metric, receives a unique and well-defined metric on $\lambda(Y, \mathscr{O}(D))$.*

Proof. By normalization this is the case for $D = 0$. By exactness we know how to put metrics on $\lambda(Y, \mathscr{O}(D))$ for every D in a unique way, possibly depending on the order in which we add/subtract points to go from the divisor 0 to D.

Finally, using the fact that the Green function is symmetric, it is straightforward to check that the order in which we add/subtract points does not matter. □

Therefore, now it suffices to show that the metric on $\lambda(Y, \mathscr{O}(D))$ only depends on the linear equivalence class of D (metrics are the unique admissible normalized ones on each $\mathscr{O}(D)$)). For this, we can furthermore only focus on the case

$$\deg D = g - 1$$

which is convenient for technical reasons.

Furthermore, fixing a divisor E of degree $r + g - 1$ for an arbitrarily large integer r, it suffices to consider divisors of the form

$$D = E - (P_1 + ... + P_r)$$

with $P_j \in Y$ (varying r and E covers all cases). With no loss of generality, we can assume that r is very large (we'll later see how large).

Here is the sketch of how, in this setting, one shows that the metric on $\lambda(Y, \mathscr{O}(D))$ only depends on the linear equivalence class of D.

- There is a "universal determinant of cohomology": Certain line sheaf $\mathscr{N}$ on Y^r whose fibre at $P = (P_1, ..., P_r) \in Y^r$ has a canonical isomorphism
$$\mathscr{N}|_P \simeq \lambda(Y, \mathscr{O}(E - P_1 - ... - P_r)).$$
- The metrics constructed for each $\lambda(Y, \mathscr{O}(E - P_1 - ... - P_r))$ vary nicely in terms of Green functions. This variation is quite explicit and one can write down its curvature.
- There is a natural map $\phi : Y^r \to \mathrm{Pic}_{g-1}(Y)$ given by $\phi(P) = [\mathscr{O}(E - P_1 - ... - P_r)]$. In $\mathrm{Pic}_{g-1}(Y)$ there is a distinguished divisor Θ consisting of effective classes. Using a degeneration argument, one checks that
$$\phi^* \mathscr{O}(-\Theta) \simeq \mathscr{N}$$
- Finally, by comparing the curvature of the metric on $\mathscr{N}$ (coming from the metrics on the λ's) and the pull-back by ϕ of the curvature of certain metric on $\mathscr{O}(-\Theta)$ (which needs to be explicitly constructed on $\mathscr{O}(-\Theta)$ once and for all —this part is difficult), one sees that the metric on $\mathscr{N}$ comes for $\mathrm{Pic}_{g-1}(Y)$ via the isomorphism $\phi^* \mathscr{O}(-\Theta) \simeq \mathscr{N}$.

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