MATH 281X: ARAKELOV THEORY OF ARITHMETIC SURFACES
WEEK 1

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1. Regular arithmetic surfaces

1.1. **Arithmetic surfaces.** For a field k, a *curve* over k is a 1-dimensional integral k-scheme of finite type.

Let A be a one dimensional Dedekind domain and $S = \operatorname{Spec} A$. An *arithmetic surface* (over S) is a pair $(\mathscr{X}, \pi)$ where $\mathscr{X}$ is an integral scheme, and $\pi : \mathscr{X} \to S$ is a flat and proper relative curve of finite type. Note that $\dim \mathscr{X} = 2$ and that $\mathscr{X}_\eta$ is an integral projective curve over the field $K = \operatorname{Frac}(A)$. The arithmetic surface is normal (resp. regular) if $\mathscr{X}$ is.

The arithmetic surface $(\mathscr{X}, \pi)$ is said to be an *integral model* over A (or S) of the K-curve $\mathscr{X}_\eta$. Note that if $\mathscr{X}$ is normal (or regular) then $\mathscr{X}_\eta$ is smooth.

More generally, an integral model for a smooth projective curve X/K is a regular arithmetic surface $(\mathscr{X}, \pi)$ over S together with a K-isomorphism $\mathscr{X}_\eta \simeq X$ —the isomorphism is part of the data but it will be omitted from the notation.

Theorem 1.1 (Lichtenbaum)**.** *Let $(\mathscr{X}, \pi)$ be a regular arithmetic surface. Then π is projective.*

Sketch. Construct an effective (cartier) divisor on $\mathscr{X}$ whose support contains no fibre component and meets every fibre component in the sense of the intersection numbers i_s defined above —here one needs the regularity assumption, as the construction of the required divisor involves some analysis of local rings $\mathscr{O}_{\mathscr{X},x}$ at various points x of $\mathscr{X}$. The associated line sheaf $\mathscr{L}$ is ample on each fibre.

Finally, one shows that this implies that $\mathscr{L}$ is relatively ample for π. In fact, one can show in more generality that if $f : X \to Y$ is a proper morphism, Y is noetherian, and $\mathscr{F}$ is a line sheaf on X whose restriction to each fibre of f is ample, then $\mathscr{F}$ is ample for f (cf. Corollary 2.6 [3]). □

Remarks.

- We will be mainly interested in the case when A is the ring of integers of a number field. For technical purposes, one can often reduce to the case when A is a DVR.
- Even if $\mathscr{X}$ is regular, for a closed point $s \in S$ the special fibre $\mathscr{X}_s$ can be singular, reducible, and it can have non-reduced components.

A common assumption in the theory is that for a given smooth projective curve X/K, the field K is algebraically closed in $K(X)$, i.e. $K = H^0(X, \mathscr{O}_X)$ (when K is a number field this is the same as requiring that X be geometrically irreducible).

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This assumption implies, in particular, that all fibres of an integral model for X are connected.

1.2. **Intersections.** Let C be a smooth projective curve over a field k, $f : C \to X$ a morphism to a regular scheme X and D a divisor on X. Then D is Cartier and $\deg_{C/k} f^*\mathscr{O}_X(D) \in \mathbb{Z}$ is defined.

Let $(\mathscr{X}, \pi)$ be a regular arithmetic surface over S, let $s \in S$ and let F be a reduced fibre component over s. Let $\nu : C \to F$ be the normalization map and let k be a finite sub-extension of the field $H^0(C, \mathscr{O}_C)$. Then C is a smooth projective curve over k and for every divisor D on $\mathscr{X}$ we can define the intersection number

$$i_{s,k}(F, D) = \deg_{C/k} f^*\mathscr{O}_{\mathscr{X}}(D).$$

For the choice $k = \kappa_s$ we simply write i_s, and observe that this defines a pairing

$$i_s : \mathrm{Div}_s(\mathscr{X}) \times \mathrm{Div}(\mathscr{X}) \to \mathbb{Z}$$

where $\mathrm{Div}_s(\mathscr{X})$ is the group of divisors of $\mathscr{X}$ supported on $\mathscr{X}_s$. The pairing has the expected properties: bilinear, symmetric on $\mathrm{Div}_s(\mathscr{X})$, compatible with linear equivalence, local expression for proper intersections. In addition we have:

Proposition 1.2. *Suppose that $\mathscr{X}_s$ is connected. Then i_s restricted to $\mathrm{Div}_s(\mathscr{X})$ is negative semi-definite. Furthermore, the divisors $D \in \mathrm{Div}_s(\mathscr{X})$ with $i_s(D, D) = 0$ are precisely the (possibly rational) multiples of the fibre $\mathscr{X}_s$.*

Proof. This is a standard computation due to Mumford.

We may assume that A is DVR with uniformizer ϖ. Write $\mathscr{X}_s = \sum_j m_j C_j$ as divisor, where C_j are the reduced fibre components and m_j the corresponding multiplicities. For simplicity, write $i_s(D_1, D_2) = D_1.D_2$. First we observe$c_j.\mathscr{X}_s = 0$ since $\mathscr{X}_s = (\varpi)$ is principal. Thus, for all j

$$m_j C_j^2 = -\sum_{i \neq j} m_i C_i.C_j.$$

Now let $D \in \mathrm{Div}_s(\mathscr{X})$ be arbitrary and let a_j be rational numbers with $D = \sum_j a_j m_j C_j$. Then expanding D^2 and using the previous expression to substitute C_j^2 we find

$$\begin{aligned}
D^2 &= \sum_j a_j^2 m_j^2 C_j^2 + \sum_{i \neq j} a_i a_j m_i m_j C_i.C_j \\
&= -\sum_j a_j^2 m_j \sum_{i \neq j} m_i C_i.C_j + \sum_{i \neq j} a_i a_j m_i m_j C_i.C_j \\
&= -\frac{1}{2} \sum_{i \neq j} (a_i^2 + a_j^2) m_i m_j C_i.C_j + \sum_{i \neq j} a_i a_j m_i m_j C_i.C_j \\
&= -\frac{1}{2} \sum_{i \neq j} (a_i - a_j)^2 m_i m_j C_i.C_j
\end{aligned}$$

Thus, $D^2 \leq 0$, and since $\mathscr{X}_s$ is connected we see that $D^2 = 0$ if and only if all a_j are equal, that is, D is a rational multiple of $\mathscr{X}_s$.

□

Another choice of k in the definition of $i_{s,k}$ that will be useful (specially when discussing self-intersections) is $k = H^0(C, \mathscr{O}_C)$. This field is a finite extension of κ_s but they are not necessarily the same.

1.3. **Existence of models.** Let X be a smooth projective curve over K. It is standard to construct *some* arithmetic surface $(\mathscr{X}, \pi)$ over S which is an integral model for X (idea: Embed X in some projective space $\mathbb{P}^n_K$ and consider the closure of X in $\mathbb{P}^n_S$.)

If $(\mathscr{X}, \pi)$ is an integral model for X, the normalization map $\mathscr{X}' \to \mathscr{X}$ must restrict to an isomorphism on the generic fibre. After checking that flatness and properness over S are preserved, we obtain that there is some normal integral model over S for X.

One can blow-up the singular locus of a normal integral model for X, then normalize again, and repeat. This process will not change the generic fibre. It is a theorem of Lipman [4] that this process is stationary and leads to a regular integral model.

Theorem 1.3 (Lipman)**.** *Let X/K be a smooth projective curve. There is an arithmetic surface $(\mathscr{X}, \pi)$ over S which is a regular integral model for X. It can be obtained as follows:*

Start with any normal integral model. Then blow up the singular locus (necessarily of dimension 0*) and then normalize. Repeat until the singular locus is empty. The process stops.*

We mention that when A is the ring of integer of a number field, the results of Abhyankar [1] suffice.

Let $\mathscr{X}$ and $\mathscr{Y}$ be regular integral models for a smooth projective K-curve X. The K-isomorphism $\mathscr{X}_\eta \simeq X \simeq \mathscr{Y}_\eta$ spreads-out to a birational map over S (not necessarily a morphism). Thus, any two regular integral models of X are birational over S. In the next section we will identify a convenient choice.

2. Minimal regular models

2.1. **Minimality.** A regular arithmetic surface $(\mathscr{X}, \pi)$ is *relatively minimal* if every proper birational S-morphism $\mathscr{X} \to \mathscr{Y}$ to a regular arithmetic surface $\mathscr{Y}$ is an isomorphism. It is *minimal* if it is relatively minimal and furthermore, for any relatively minimal $(\mathscr{X}', \pi')$, if we have a K-isomorphism $\phi : \mathscr{X}'_\eta \to \mathscr{X}_\eta$ then ϕ extends to an S-isomorphism $\mathscr{X}' \to \mathscr{X}$.

Note that if $(\mathscr{X}, \pi)$ is minimal, then all relatively minimal integral models for $\mathscr{X}_\eta$ are in fact minimal and isomorphic to $\mathscr{X}$ over S.

Example.

$\mathbb{P}^1_S$ is a relatively minimal regular model for $\mathbb{P}^1_K$, but it is not minimal. Indeed, pick a closed point P in a special fibre F, blow up P and then contract the strict transform of F—this is possible by Castelnuovo's criterion (see below). One obtains a new relatively minimal model for $\mathbb{P}^1_K$, and considerations on sections (for instance) show that it is not isomorphic to $\mathbb{P}^1_S$ as integral model.

2.2. **Proper birational morphisms.** Let $(\mathscr{X}, \pi)$ be a regular arithmetic surface. A prime divisor E on $\mathscr{X}$ is *exceptional* or *contractible* if there is a regular arithmetic surface $(\mathscr{Y}, \tau)$ and a proper birational morphism $\phi : \mathscr{X} \to \mathscr{Y}$ over S such that ϕ restricts to an isomorphism $\mathscr{X} \smallsetminus E \to \mathscr{Y} \smallsetminus \phi(E)$ and $\phi(E)$ is a closed point. Such a morphism ϕ is a *blow down.*

One can show that blow downs *are in fact blow ups at closed points.* Furthermore, as in the case of complex surfaces one has the following two crucial results on blow ups:

Theorem 2.1 (Castelnuovo criterion)**.** *Let $\mathscr{X}$ be a regular arithmetic surface over S. A prime divisor E is contractible if and only if the following conditions hold:*

(i) *E is fibral, say, contained in $\mathscr{X}_s$ for a closed point $s \in S$;*
(ii) *$H^1(E, \mathscr{O}_E) = 0$, and*
(iii) *$i_{s,k}(E, E) = -1$ where $k = H^0(E, \mathscr{O}_E)$.*

If this is the case, then we have a k-isomorphism $E \simeq \mathbb{P}^1_k$.

Theorem 2.2 (Factorization theorem)**.** *Let $\mathscr{X}$ and $\mathscr{Y}$ be regular arithmetic surfaces over S and let $\phi : \mathscr{X} \to \mathscr{Y}$ be a proper birational S-morphism. Then ϕ is the composition of a sequence of blow ups at closed points.*

2.3. **Lichtenbaum's theorem.**

Theorem 2.3 (Lichtenbaum)**.** *Let X be a smooth projective curve over $K =$ Frac(A). Assume that K is algebraically closed in $K(X)$. Then:*

(i) *If $\mathscr{X}_1/S$ and $\mathscr{X}_2/S$ are regular integral models of X, then there is a regular integral model $\mathscr{X}/S$ for X with proper birational S-morphisms $\mathscr{X} \to \mathscr{X}_i$.*
(ii) *Given any regular integral model for X over S, construct a sequence $\mathscr{X}^{(n)}$ as follows: $\mathscr{X}^{(0)} = \mathscr{X}$, and $\mathscr{X}^{(n+1)}$ is obtained from $\mathscr{X}^{(n)}$ by blowing down an exceptional curve (if any). Then the sequence is stationary.*
(iii) *X has some relatively minimal regular model over S.*
(iv) *If X has genus at least 1, then X has a minimal regular model over S.*

Sketch of proof. Spreading out, we may reduce to the case when A is a DVR, so S has a unique closed point s.

The hypothesis that K be algebraically closed in $K(X)$ implies that X is geometrically irreducible, that $\mathscr{X}_s$ is connected, and that the intersection pairing i_s on $\mathrm{Div}_s(\mathscr{X})$ is negative semi-definite and the only divisors with self intersection 0 are those that are (rational) multiples of $\mathscr{X}_s$. These facts are used at technical steps in the argument outlined below.

Item (i) is proved following ideas similar to those in the theory of complex surfaces, using the versions of the factorization theorem and Castelnuovo's numerical criterion cited above.

Using Castelnuovo's criterion and intersection theory, one checks that by blowing down an exceptional curve, no new ones can appear. The number of exceptional curves on $\mathscr{X}_0$ is finite, hence (ii).

If the sequence in (ii) stops at some $\mathscr{X}^{(n)}$ this means that no further exceptional curves remain. By the Factorization Theorem, this implies that any proper birational S-morphism $f : \mathscr{X}^{(n)} \to \mathscr{Y}$ must be an isomorphism, hence $\mathscr{X}^{(n)}$ is relatively minimal. Since regular models exist, item (iii) follows.

For (iv), we need to show (under the assumption that the genus of X is at least 1) that given any two $\mathscr{X}, \mathscr{X}'$ relatively minimal models for X over S, the isomorphism $\mathscr{X}_\eta \simeq \mathscr{X}'_\eta$ deduced from the fact that they are models for X, extends to an isomorphism $\mathscr{X} \simeq \mathscr{X}'$. Thanks to item (i) and the factorization theorem, it suffices to prove the following: Consider a regular integral model $\mathscr{X}/S$, then the limit of the sequence $\mathscr{X}^{(n)}$ from item (ii) is independent of choices. This will follow from:

Claim 1. *Under the additional assumption that X gas genus at least 1, exceptional curves in $\mathscr{X}$ do not meet each other, and blowing down one of them maps the other to exceptional curves.*

In fact, this claim allows one to rearrange the blow ups in the factorization theorem, thus proving that the limit in (ii) is independent of choices.

Now we prove the claim. Let E be an exceptional curve in $\mathscr{X}$ and let $f : \mathscr{X} \to \mathscr{Y}$ be the blow down of E. E is in $\mathscr{X}_s$, it is isomorphic to $\mathbb{P}^1_k$ with $k = H^0(E, \mathscr{O}_E)$, and $i_{s,k}(E,E) = -1$. Let $P = f(E) \in \mathscr{Y}_s$

Let $C \neq E$ be another exceptional curve in $\mathscr{X}$ and let $D \subseteq \mathscr{Y}_s$ be its image under f. If C does not meet E then one can use Castelnuovo's criterion (and the fact that f is an isomorphism away from E) to check that D is exceptional.

For the sake of contradiction, suppose that C meets E, thus $P \in D$. Let $k_1 = H^0(C, \mathscr{O}_C)$ and $k_2 = H^0(D, \mathscr{O}_D)$, then k_1/k_2 is finite and since $f^*D = C + nE$ for some $n \geq 1$ we get (after a computation)

$$\begin{aligned} i_{s,k_2}(D,D) &= [k_1 : k_2]\,(i_{s,k_1}(C,C) + n i_{s,k_1}(C,E)) \\ &\geq i_{s,k_1}(C,C) + 1 \end{aligned}$$

with equality if and only if f restricts to an isomorphism of C and D (thus, $k_1 = k_2$), and $P = f(E)$ is a regular point in D.

Since C is exceptional we also deduce $i_{s,k_2}(D,D) \geq 0$, hence $i_s(D,D) \geq 0$. Since i_s is negative-semidefinite on $\mathscr{Y}_s$, we deduce $i_s(D,D) = 0$ and thus D is a rational multiple of $\mathscr{Y}_s$, more precisely, $mD = \mathscr{Y}_s$ for some integer $m \geq 1$. We also deduce $i_{s,k_2}(D,D) = 0$ hence

$$i_{s,k_2}(D,D) = 0 = i_{s,k_1}(C,C) + 1$$

and it follows that P is a regular point in D, so D is regular, f induces an isomorphism $C \simeq D$, thus $k' := k_1 = k_2$ and $\mathbb{P}^1_{k'} \simeq C \simeq D$.

Let $\mathscr{I}$ be the ideal sheaf of D in $\mathscr{Y}$ and let $\nu : D \to \mathscr{Y}$ be the closed immersion of D. Note that for all $r \geq 0$

$$H^1(\mathscr{Y}, \mathscr{I}^r/\mathscr{I}^{r+1}) \simeq H^1(D, (\nu^*\mathscr{I})^{\otimes r}) = 0$$

because $\deg_{D/k'}(\nu^*\mathscr{I}) = -i_{s,k'}(D,D) = 0$ and $D \simeq \mathbb{P}^1_{k'}$.

The exact sequence

$$0 \to \mathscr{I}^r/\mathscr{I}^{r+1} \to \mathscr{O}_{\mathscr{Y}}/\mathscr{I}^{r+1} \to \mathscr{O}_{\mathscr{Y}}/\mathscr{I}^r \to 0$$

gives the exact sequence

$$0 = H^1(\mathscr{Y}, \mathscr{I}^r/\mathscr{I}^{r+1}) \to H^1(\mathscr{Y}, \mathscr{O}_{\mathscr{Y}}/\mathscr{I}^{r+1}) \to H^1(\mathscr{Y}, \mathscr{O}_{\mathscr{Y}}/\mathscr{I}^r)$$

and by induction on r we get that $H^1(\mathscr{Y}, \mathscr{O}_{\mathscr{Y}}/\mathscr{I}^r) = 0$ for all $r \geq 1$. In particular,

$$H^1(\mathscr{Y}, \mathscr{O}_{\mathscr{Y}}/\mathscr{I}^m) = 0$$

where we recall $mD = \mathscr{Y}_s$. Let ϖ be a uniformizer for the DVR A, then $\varpi\mathscr{O}_{\mathscr{Y}} = \mathscr{I}^m$ so we get that multiplication by ϖ induces the exact sequence

$$H^1(\mathscr{Y}, \mathscr{O}_{\mathscr{Y}}) \to_{\varpi} H^1(\mathscr{Y}, \mathscr{O}_{\mathscr{Y}}) \to H^1(\mathscr{Y}, \mathscr{O}_{\mathscr{Y}}/\mathscr{I}^m) = 0.$$

Since ϖ is a uniformizer, we get that the free part of the A-module $H^1(\mathscr{Y}, \mathscr{O}_{\mathscr{Y}})$ is trivial. Tensoring with K we get a contradiction, since $H^1(\mathscr{Y}_\eta, \mathscr{O}_{\mathscr{Y}_\eta}) \simeq H^1(X, \mathscr{O}_X) \neq 0$ by the genus assumption. This contradiction proves the claim. $\square$

3. Semi-stable models

Assumption. All Dedekind domains considered in this section have perfect residue fields. This technical assumption can be avoided, but it allows us to write "field extension" rather than "separable field extension" at various places. The case of interest for us consists of rings of integers of number fields, so this assumption is not problematic.

3.1. **Motivation.** So far, we have been concerned with good properties of arithmetic surfaces $(\mathscr{X}, \pi)$ in the absolute sense: normality and regularity are conditions on $\mathscr{X}$ rather than π. We now consider convenient conditions on fibres. Since bad reduction of an integral model can occur, we will be concerned with the mildest possible kind of bad fibres: normal crossings (to be defined below).

Let A be a complete DRV with and uniformizer ϖ. A completed local model for (the local ring of) the kind of fibre singularities that we want to consider, is given by

$$R = A[[X, Y]]/(XY - \varpi).$$

Note that this completed local ring is two-dimensional, and the completed local ring of the special fibre is $k[[X, Y]]/(XY)$ which corresponds to the transversal crossing of two lines.

Since $XY - \varpi \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ with $\mathfrak{m}$ the maximal ideal of $A[[X, Y]]$, we see that R is regular. Thus, fibre singularities of this sort are so mild that they do not give rise to surface singularities.

However, we will face two key technical obstacles:

(1) Not every smooth projective curve over $K = \mathrm{Frac} A$ admits an integral model having only singularities of this sort. At least not over the given base $S = \mathrm{Spec}\, A$.
(2) Even in the cases when this is possible, in number-theoretical applications it is often necessary to replace the field K by a finite extension L/K. Then surface singularities might occur. For instance, replacing A by $B = A[\lambda]$ with $\lambda^r = \varpi$ we find that

$$B[[X, Y]]/(XY - \varpi) = B[[X, Y]]/(XY - \lambda^r)$$

which is not regular if $r \geq 2$.

These issues lead to the study of normal crossings and semi-stable models, their base changes, and their desingularizations. It turns out that (1) is resolved upon a finite base change by results of Deligne and Mumford. The issue described in (2) can occur, but for normal crossings fibres the surface singularities are so mild that one can resolve them quite explicitly and the resulting regular surface again has fibres with only mild fibre singularities.

3.2. **Semi-stable arithmetic surfaces.** Let k be an algebraically closed field and C a 1-dimensional, connected, projective k-scheme of finite type. We say that C has normal crossings if it is reduced and all its sigular points are nodes (aka. ordinary double points; i.e. points with completed local rings of the form $k[[x, y]]/(xy)$). We say that C is semi-stable if it has normal crossings (in particular, it is reduced) and furthermore all non-singular components of C isomorphic to $\mathbb{P}^1_k$ have at least 2 intersection points with other components.

Let $(\mathscr{X}, \pi)$ be an arithmetic surface over $S = \operatorname{Spec} A$. Let $s \in S$ be a closed point. We say that the special fibre $\mathscr{X}_s$ has *normal crossings* or that it is *semi-stable* if the corresponding property holds for $\mathscr{X}_s \otimes_k k^{alg}$ where $k = \kappa_s$.

We simply say that the arithmetic surface $(\mathscr{X}, \pi)$ has normal crossings or that it is semi-stable if the corresponding property holds above each closed point $s \in S$.

Lemma 3.1. *Let $\mathscr{X}/S$ be an arithmetic surface with smooth generic fibre X/K. Assume that $K = H^0(X, \mathscr{O}_X)$ and that $\mathscr{X}$ has normal crossings. Then*

(i) *$\mathscr{X}$ is normal.*
(ii) *Let L/K be a finite extension and let $B \subseteq L$ be the integral closure of A in L. Let $T = \operatorname{Spec} B$. Then $\mathscr{X}_T/T$ is an arithmetic surface with normal crossings.*

The same holds if "normal crossings" is replaced by "semi-stable". Furthermore, in the semi-stable case we have that the genus of X is at least 1.

Proof. For item (i) it is enough to consider the normal crossings case. In fact, it suffices to assume that the special fibres are reduced.

Assume A is a DVR with maximal ideal $\mathfrak{m}$, uniformizer ϖ, and residue field $k = A/\mathfrak{m}$. Let R be a flat A-algebra. Suppose that R_K is normal and R_k is reduced. By flatness, R injects into R_K so R is an integral domain. Let $F = \operatorname{Frac}(R)$ and take any $f \in F$ integral over R, say, satisfying an integrality relation $f^r + a_{r-1}f^{r-1} + ... + a_0 = 0$ for some $r \geq 1$ and $a_i \in R$. By normality of R_K we see that $a = \varpi^n f \in R$ for some $n \in \mathbb{Z}$. We can assume that n is chosen so that $\varpi^n f \notin \varpi R$.

Suppose for a moment that $n \geq 1$. Then, from the integrality relation of f, we deduce that $a^n = (\varpi^n f)^r \in \varpi R$ so that $\bar{a}^n = 0$ in $R/\varpi R = R_k$. As R_k is reduced, we obtain that $\bar{a} = 0$, so $a \in \varpi R$ which is not possible by our choice of n. Thus, we see that $n \leq 0$. This proves that $f \in R$. Since f was arbitrary, this proves that R is normal.

These considerations apply when R is the local ring of $\mathscr{X}$ at a closed point, hence item (i) holds.

Item (ii) follows from the definition, since finite extensions are contained in an algebraic closure.

In the semi-stable case, the generic fibre cannot have genus 0, for otherwise any special fibre would fail the condition that all rational lines have at least two intersections with other components. □

Given X a smooth projective curve over K, it is not always the case that it admits a normal crossings integral model over $S = \operatorname{Spec} A$ (one can check, for instance, that this failure occurs for elliptic curves with additive reduction over a given number field). Nonetheless, semi-stable (hence, normal crossings) models do exist if a finite field extension is allowed.

Theorem 3.2. *Let X/K be a smooth projective geometrically irreducible curve of genus $g \geq 1$. There is a finite extension L/K such that if B is the integral closure of A in L, then X_L admits a semi-stable integral model over* $\operatorname{Spec} B$.

Reference and comments. When $g = 1$ this is reduced to the well-known fact that elliptic curves acquire semi-stable reduction after finite base change.

For $g \geq 2$, it is an important theorem of Deligne and Mumford that such curves acquire *stable* reduction after finite base change (cf. Corollary 2.7 in [2]) —the definition of "stable" is slightly more restrictive than "semi-stable" as one requires that rational lines meet other components in at least three points. The rough idea of the argument is that stable reduction of integral models of curves can be linked to semi-stable reduction of the Neron model of their Jacobians (in the usual sense for abelian varieties). Then one concludes by a theorem of Grothendieck, which proves that Abelian varieties achieve semi-stable reduction upon finite base change. □

Thus, after finite base change, normal crossings and semi-stable models exist. Moreover, they are normal. Their desingularizations admit a simple description after a finite extension —with some additional work the finite extension is not necessary for desingularizations, but it is crucially necessary for the existence of semi-stable models. We describe desingularizations in the next paragraph.

3.3. **Regular semi-stable models.**

Theorem 3.3. *Let $\mathscr{X}/S$ be an arithmetic surface with smooth generic fibre X/K. Assume that $K = H^0(X, \mathscr{O}_X)$ and that $\mathscr{X}$ has normal crossings. There is a finite extension K_0/K such that for all finite extensions L/K_0 if B is the integral closure of A in L and $T = \operatorname{Spec} B$, then the following holds for the normal-crossings T-arithmetic surface $\mathscr{Y} = \mathscr{X}_T$:*

Let $s \in T$ be a closed point with a uniformizer $\varpi \in B$. The only singularities of $\mathscr{Y}$ above s (if any) are ordinary double points, at κ_s-rational points located at singularities of $\mathscr{Y}_s$, and they admit a completed-local description on $\mathscr{Y}$ of the form

$$\hat{\mathscr{O}}_{\mathscr{Y},y} \simeq \hat{B}[[X,Y]]/(XY - \varpi^n)$$

for some $n \geq 2$. Repeated blow up gives a minimal resolution of such a singularity, replacing it with a chain of $n-1$ copies of $\mathbb{P}^1_{\kappa_s}$, each with self-intersection -2, hence, non-exceptional. After resolving all singularities of $\mathscr{Y}$ in this way, we obtain a proper birational T-morphism $\mathscr{Y}' \to \mathscr{Y}$ with $\mathscr{Y}/T$ a regular normal crossings arithmetic surface.

All of this is valid if "normal crossings" is replaced by "semi-stable".

Idea/reference. This desingularization is standard and in the normal-crossings case can be found in detail in p.514-515 in [5] (note that in [5] the definition of "semi-stable" is our definition of "normal crossings" since the condition "rational lines meet other components in at least two points" is omitted). See also [2].

Suppose that we assume instead that $\mathscr{X}$ is semi-stable, then so is $\mathscr{Y}$ by the previous lemma. By the explicit description of the desingularization, the special fibre $\mathscr{Y}_s$ is obtained from $\mathscr{Y}'_s$ by contracting certain (-2)-curves, each with two intersection points with other components. Thus, $\mathscr{Y}'_s$ is semi-stable for all closed $s \in T$. □

References

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