Math 281X, Midterm problems. Total: 30 pts.

Due on March 19, 2018.

We work with the base number field $K = \mathbb{Q}$, so, there is only one complex embedding which we denote by σ. Of course $O_K = \mathbb{Z}$ and in our notation the base scheme will be $S = \operatorname{Spec}\mathbb{Z}$. This problem set concerns the "baby case" of Arakelov's intersection theory on $\mathscr{X} := \mathbb{P}^1_{\mathbb{Z}}$.

Consider any p.v.f $d\mu$ on $X_\sigma = \mathbb{P}^1_{\mathbb{C}}$; we'll later take a particular one. The definition of the Arakelov intersection pairing

$$(-.-) : \mathrm{Div}(\hat{\mathscr{X}}) \times \mathrm{Div}(\hat{\mathscr{X}}) \to \mathbb{R}$$

given in class, as well as its basic properties (well-defined, explicit, bilinear, symmetric, compatible with Arakelov linear equivalence) apply in this case, since our proofs at this point did not use semi-stability; we only used normal crossings and regularity, and both hold for $\mathscr{X} = \mathbb{P}^1_{\mathbb{Z}}$.

We fix the following notation: The Riemann surface $X_\sigma = \mathbb{P}^1_{\mathbb{C}}$ will be presented explicitly in homogeneous coordinates $[x_0 : x_1]$ to which we attach the open sets $U_0 = \{x_0 \neq 0\}$ and $U_1 = \{x_1 \neq 0\}$ with holomorphic charts $z = x_1/x_0$ and $w = x_0/x_1$ respectively.

(1) [5pts] On the open set U_0 with the chart z, write down explicitly the unique p.v.f. $d\mu$ that makes the Fubini-Study metric on $\mathscr{O}(1)$ admissible (and prove your formula is correct – in particular, it has to be positive at infinity). From now on we only use this $d\mu$.

(2) [10pts] Write down explicitly the Green function on $U_0 \times U_0 \subseteq \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ attached to $d\mu$, and prove your formula is correct. Please be careful about normalizations.

(3) [15pts] Consider the algebraic points $P = [1 : \sqrt{2}]$ and $Q = [1 : i]$ in $\mathbb{P}^1_{\mathbb{Q}}(\overline{\mathbb{Q}})$, where $i = \sqrt{-1}$. Compute $(D_P.D_Q)$ and approximate this real number to a couple of decimal places.