RESOLUTION OF SINGULARITIES IN POSITIVE CHARACTERISTICS

HEISUKE HIRONAKA

TO WAKA

ABSTRACT. A proof of resolution of singularities in characteristic $p > 0$ and all dimensions.

CONTENTS

Date: March 23, 2017.

1. Introduction

In this paper, we prove the existence of an "embedded resolution of singularities", abbreviated ERS, for an arbitrary algebraic scheme (or algebraic variety) X of any dimension, over any base field $\mathbb{K}$ of any characteristic $p > 0$. The adjective *embedded* means that X is given as a closed subscheme of a smooth irreducible ambient scheme Z and an ERS of X is induced by a finite permissible sequence of successive blowups with smooth centers over Z. An ERS of $X \subset Z$ is a diagram:

$$\begin{array}{ccccccc} X_r \subset Z_r & \xrightarrow{\pi_{r-1}} & \cdots \xrightarrow{\pi_1} & X_1 \subset Z_1 & \xrightarrow{\pi_0} & X_0 \subset Z_0, \\ & & & \bigcup & & \bigcup \\ & & & D_1 & & D_0 \end{array}$$

where $X_0 = X \subset Z_0 = Z$. We require that for all $0 \leq j < r$, the map $\pi_j : Z_{j+1} \to Z_j$ is a *permissible* blowup with center $D_j \subset X_j$, that is, D_j is a closed irreducible smooth subscheme contained in the singular locus of X_j; X_{j+1} is the strict transform of X_j by π_j; and X_r has no singular points.

The main contribution of this paper is a new inductive approach to the ERS problems, local and global. The outline of our proof of ERS in characteristic $p > 0$ goes as follows.

(1) We give a powerful application of the finitely generated graded $\mathcal{O}_Z$-algebra $\wp$ of [23] and [27] together with its extension $\tilde{\wp}$ introduced in §5 of this paper.
(2) We introduce a useful new technique we call local leverage up and exponent down, or LLED, in §9, and define its globalization GLUED in §12 and §13.
(3) Using (1) and (2) we devise a new ambient reduction technique (which is a workable substitute in $p > 0$ of the "maximum contact" ambient reduction in characterisitic zero). See the formulation of induction on ambient dimension by AR-schemes of Def.(15.5) after §15.1 of §15.

Our proof of ERS is given in §16 (Theorem ∇ I) and §16.3 (Theorems ∇ II-III).

1.1. **Acknowledgments.** I received invaluable mathematical influence and guidance from numerous excellent mathematical teachers and colleagues in the course of completing this work. I have been indebted so much and for so long to many mathematical seniors, first and foremost to my teachers Oscar Zariski, Masayoshi Nagata, Alexander Grothendiek and Shreeram Abhyankar. I must also mention the valuable influence and stimulation I have received from my colleagues Michael Artin, David Mumford, Bernard Teissier, Le Dung Trang, S-Moh, Jose-Manuel Aroca, Mark Spivakowski, U.O. Villamayor, Herwig Hauser, Hiraku Kawanoue, June-E and many other algebraic geometers with strong interests in the topics surrounding singularities. Finally, I am grateful for the constant continuous encouragement, and multiple proof-reading and misprints-checking by Woo Yang Lee of Seoul National University and Tadao Oda of Tohoku University during the past decade of earnest but slow research toward the proof of resolution of singularities in positive characteristics and for all dimensions.

2. Preliminaries

In this section, we review some of the basic definitions used throughout this paper: the ideal exponent (§2.1), normal crossing data (§2.2), and infinitely near singularities (§2.3).

Throughout this paper an *ambient scheme* Z always means a smooth irreducible scheme of finite type defined over a perfect base field $\mathbb{K}$ of characteristic $p > 0$. In particular we may choose $\mathbb{K} = \mathbb{Z}/p\mathbb{Z}$.

Let $\mathcal{O}_Z$ denote the structure sheaf of Z. We may write $\mathcal{O}$ for $\mathcal{O}_Z$ and $\mathcal{O}_\xi$ for the local ring $\mathcal{O}_{Z,\xi}$ at a point $\xi \in Z$. Sometimes the notation R_ξ or R for $\mathcal{O}_{Z,\xi}$ and M_ξ or M for the maximal ideal $\max(\mathcal{O}_\xi)$ of $\mathcal{O}_\xi$ are used. We write κ_ξ or κ for the residue field R_ξ/M_ξ. The symbol ρ denotes the Frobenius endomorphism, sending each element to its p-th power. It gives the filtration

$$\mathcal{O} \supset \rho(\mathcal{O}) \supset \rho^2(\mathcal{O}) \supset \cdots$$

where each $\rho^i(\mathcal{O})$ is a locally free $\rho^j(\mathcal{O})$-module for $j > i$. It is frequently used for polynomial approximations and differentiations in $\mathcal{O}$. Denote by Y_{cl} the set of closed points of a scheme Y.

2.1. **Ideal exponents.** For an ideal exponent $E = (J, b)$ we write $\mathrm{Sing}(E)$ for the singular locus of E which is a closed reduced subscheme of Z consisting of those points ξ with $\mathrm{ord}_\xi J \geq b$. A blowup $\pi : Z' \to Z$ with center D is said to be *permissible* for E if D is a closed smooth irreducible subscheme of $\mathrm{Sing}(E)$.

Definition 2.1. The transform E' of E by π is $E' = (J', b)$ in Z' with $J\mathcal{O}_{Z'} = I(D)^b J'\mathcal{O}_{Z'}$ where $I(D)$ denotes the ideal of D in $\mathcal{O}_Z$.

Note that $I(D)\mathcal{O}_{Z'}$ is a locally everywhere nonzero principal ideal whose b-power divides $J\mathcal{O}_{Z'}$ thanks to the permissibility of π for (J, b).

Definition 2.2. *The following graded algebras will be used frequently in this paper.*

(1) $Bl^{posi}(Z) = \bigoplus_{i>0} Bl(Z,i)$ *and* $Bl^{nega}(Z) = \bigoplus_{j<0} Bl(Z,j)$ *where,* $Bl(Z,j) = \mathcal{O}$ *for all integers* j*;*
(2) $Bl(Z)_\circ = Bl^{posi}(Z) \bigoplus \mathcal{O}$*;*
(3) $Bl(Z) = Bl^{posi}(Z) \bigoplus \mathcal{O} \bigoplus Bl^{nega}(Z)$*; and*
(4) *for* $bl_\xi(Z,j) = (M_\xi)^j$ *and* $\xi \in Z_{cl}$*,*

$$\begin{aligned} bl_\xi(Z) &= \bigoplus_{j\geq 0} bl_\xi(Z,j) \\ &\subset \bigoplus_{j\in\mathbb{Z}} Bl(Z,j) = Bl(Z). \end{aligned}$$

It should be noted that $Bl(Z)$ has positive part as well as negative part in homogeneity, unlike $bl_\xi(Z)$. We have the natural epimorphism

(1) $\quad bl_\xi(Z) \ \rightarrow \ gr_\xi(Z) = gr_{\max(\mathcal{O}_\xi)}(\mathcal{O}_\xi) = \bigoplus_{j\geq 0}(M_\xi)^j/(M_\xi)^{j+1}$

whose kernel is $M_\xi bl_\xi(Z) = \max(\mathcal{O}_\xi) bl_\xi(Z)$.

2.2. **Normal crossing data.** We will be working with *normal crossing data*, *NC*-data for short, using the notation $\Gamma = (\Gamma_1, \cdots, \Gamma_s)$ where each Γ_i is a normal crossing smooth irreducible hypersurface in Z. (Some Γ_i may be empty and even the system may be empty to begin with.)

Definition 2.3. A blowup $\pi : Z' \rightarrow Z$ is said to be *permissible* for Γ if its center D has *normal crossing* with Γ. The transform Γ' of Γ by π is defined to be $\Gamma' = (\Gamma'_1, \cdots, \Gamma'_s, \Gamma'_{s+1})$ where Γ'_i is the strict transform of Γ_i by π for every $i \leq s$ and $\Gamma'_{s+1} = \pi^{-1}(D)$ is the *exceptional divisor* of π.

Note that the strict transform Y' of any $Y \subset D$ becomes empty but the information is kept in Γ' as the new exceptional divisor Γ'_{s+1}.

2.3. **Infinitely near singularities $\mathfrak{S}(E)$.** There are many different definitions for infinitely near singular points, one by pointed arcs and another by blowups. In this paper we choose the one given in [23], which we review here.

Definition 2.4. *An* LSB *(finite sequence of localized smooth blowups) over Z means any diagram as follows.*

$$\begin{array}{ccccccccc} Z_r & \stackrel{\pi_{r-1}}{\rightarrow} & U_{r-1} \subset Z_{r-1} & \stackrel{\pi_{r-2}}{\rightarrow} & \cdots \stackrel{\pi_1}{\rightarrow} & U_1 \subset Z_1 & \stackrel{\pi_0}{\rightarrow} & U_0 \subset Z_0 = Z \\ & & \bigcup & & & \bigcup & & \bigcup \\ & & D_{r-1} & & & D_1 & & D_0 \end{array}$$

where $U_i \subset Z_i$ is open, D_i is a smooth (or only regular) irreducible closed in U_i and the $\pi_i : Z_{i+1} \to U_i$ is the blow-up with center D_i.

Definition 2.5. We define the t-indexed disjoint union with arbitrary finite systems of indeterminates t, and write it symbolically as follows.

$$\mathfrak{S}(E) = \bigcup_t \big\{ \text{the LSBs over } Z[t] \text{ permissible for } E[t] = (J[t], b) \big\}$$

called the totality of the *infinitely near singularities* of E in Z.

Remark 2.6. Permissibility of LSB for E implies that the transform E_i of E in Z_i is well defined and $\mathrm{Sing}(E_i)$ is then equal to the union of the images of the centers of all the permissible sequences of blowups that can follows. As a matter of fact we lose nothing by replacing $Z_j \supset \mathrm{Sing}(E_i)$ by the germ of E_i in all of the steps. It is important to include those ambient extensions Z to $Z[t]$. By using the extensions the definition dramatically advances its significance in an essential manner.

Remark 2.7 (Notational warning (2))**.** In this paper we make the following change of notation from the original source [23]:

$$\text{for all } i,\ \wp(E, i) \text{ replaces } J_{\max}(E, i). \tag{2}$$

Consider an ideal exponent $E = (J, b)$, where J is the coherent ideal $J \neq (0)$ in $\mathcal{O}_Z$ and an integer b, positive to begin with. The $\wp(E) = \bigoplus_{j \geq 0} \wp(E, j)$ is a finitely generated graded algebra where each homogenous part $\wp(E, j)$ is a coherent ideal in $\mathcal{O}_Z$ for every degree $j \geq 0$. It was introduced in [23] (with the slight change in notation mentioned above).

We have been using the notation $\wp(E) = \bigoplus_{i \geq 0} J_{\max}(E, i)$ in relation to Th.(4.1) of §4, while referring to results in [23]. The symbol $\wp$ is convenient, since we will work with an extension of $\wp$ that includes the negative parts $\wp^{nega}(E, i), \forall i < 0$ (see §5). As to the characteristic nature of $\wp$, in particular for the elements of the negative parts $\wp^{nega}$, we will take special care to explian

(1) the rule of transformation by blowups permissible for E; and
(2) the meaning of multiplications and differentiations.

3. Three results for basic E-operations: Diff, AR, NE

We review three *key technical theorems* involving the operations Differentiation down Diff, Ambient resolution AR and numerical exponents NE originally proved in [22] and [24]. These play an important role in studying the characteristic algebra and its relations with the theory of infinitely near singularities (see §4).

In this section we do not assume that the base field $\mathbb{K}$ is perfect. However we assume that the ambient scheme Z is smooth over $\mathbb{K}$ and thus it remains so under any base field extension.

3.1. **Differentiation Theorem.** The following theorem was originally proved in [22] and [24], and is an important tool for the proofs that follow in later sections.

Theorem 3.1 (Differentiation theorem)**.** *For every $\mathcal{O}_Z$-submodule $\mathcal{D}$ of $\mathrm{Diff}_Z^{(i)}$, we have $\mathfrak{S}(\mathcal{D}J, b-i) \supset \mathfrak{S}(J,b)$.*

In what follows we review some known results on differential operators, especially in the case of a perfect base field of characteristic $p > 0$. There are substantial differences in properties of differentiation from the characteristic zero case. For example, in $p > 0$, anti-differentiation is not well-defined, and hence the Weierstrass preparation theorem can not be used in inductions on dimensions or on the number of variables. A crucial point for our proof of ERS is that differential operators can, with the help of Frobenius p-th powers, be quite useful in an inductive treatment of singularities, but for this we need dramatically different techniques as we explain.

3.1.1. *Definition of differential operators in $p > 0$.* Write $\mathcal{O}$ for $\mathcal{O}_Z$. We define the $\mathcal{O}$-module $\mathrm{Diff}_Z^{(m)}$ of differential operators of orders $\leq m$ in $\mathcal{O}$, which is identified with the $\mathcal{O}$-module of differentials $\mathfrak{D}^{(m)}$.

$$\mathfrak{D}_{\mathbb{K}}{}^{(m)} \;=\; \mathrm{Hom}_{\mathcal{O}}\big(\mathcal{D}_{\mathbb{K}}{}^{(m)}, \mathcal{O}\big),\ \forall m \geq 0 \tag{3}$$

where $\mathcal{D}_{\mathbb{K}}{}^{(m)} = \big(\mathcal{O} \otimes_{\mathbb{K}} \mathcal{O}\big)/\Delta_{\mathbb{K}}{}^{m+1}$ for the diagonal ideal $\Delta_{\mathbb{K}}$, in which the $\mathcal{D}_{\mathbb{K}}{}^{(m)}$ is a coherent $\mathcal{O}$-module by the action of $\mathcal{O}$ on the left factor of the tensor product. Every element $\partial \in \mathfrak{D}_{\mathbb{K}}{}^{(m)}$ determines the action on every $f \in \mathcal{O}$ in the following manner.

$$f \mapsto 1 \otimes f \mapsto \partial(1 \otimes f) \mapsto \partial(f) \in \mathcal{O} \tag{4}$$

where $\partial(1 \otimes f) \mapsto \partial(f)$ denotes the identity map which is the definition of $\partial(f)$. It is then better to say that ∂ is the map $f \mapsto \partial(f)$. With this $\mathbb{K}$-automorphic action in $\mathcal{O}$ induced by ∂ is called a differential operator of order $\leq m$. The $\mathcal{O}$-module of those differential operators is denoted by $\mathrm{Diff}_Z^{(m)}$. It is important to note that this induced $\mathbb{K}$-automorphic action in $\mathcal{O}$ uniquely determines the $\partial \in \mathfrak{D}_{\mathbb{K}}{}^{(m)}$. In fact suppose $\partial \neq 0$. Then there exists the minimum integer $\mu : 0 < \mu \leq m$ such that $\partial(\ker(\mathcal{D}_{\mathbb{K}}{}^{(\mu)} \to \mathcal{D}_{\mathbb{K}}{}^{(\mu-1)}) \neq (0)$. In fact the kernel is nonzero locally free because Z is smooth of dimension $n > 0$. Moreover the kernel is generated by the images of $1 \otimes f, f \in \mathcal{O}$. We thus conclude the differential operator by the ∂ cannot be zero. We thus make the natural identification $\mathrm{Diff}_Z^{(m)} = \mathfrak{D}_{\mathbb{K}}{}^{(m)}$ as $\mathcal{O}$-modules.

Definition 3.2. Let $\mathrm{Diff}_Z = \cup_{m \geq 0} \mathrm{Diff}_Z^{(m)}$ which is called the sheaf of the differential operators on Z. Its stalk at a point $\xi \in Z$ is denoted by $\mathrm{Diff}_{Z,\xi}$ or Diff_ξ for short. It should be noted that any $\partial \in \mathrm{Diff}_Z$ is $\rho^N(\mathcal{O}$-linear for all large enough N depending upon ∂.

We now define partial differentiation of $f \in \mathcal{O}$ using Taylor expansions.

Definition 3.3. Pick a point $\xi \in Z_{cl}$ and any regular system of parameters $x = (x_1, \cdots, x_n)$ of $\mathcal{O}_\xi$. Let $\hat{\mathcal{O}}_\xi$ be the $\max(\mathcal{O}_\xi)$-adic completion of $\mathcal{O}_\xi$. Then $\hat{\mathcal{O}}_\xi = K[[x]]$ where K is the algebraic closure of the base field $\mathbb{K}$ in $\hat{\mathcal{O}}_\xi$. Since $\mathbb{K}$ is assumed perfect we have K is finite separable algebraic extension of $\mathbb{K}$. Take an element $f \in \mathcal{O}$ and consider the image of $1 \otimes f \in (\mathcal{O} \otimes_{\mathbb{K}} \mathcal{O})$ under the natural map

$$(\mathcal{O} \otimes_{\mathbb{K}} \mathcal{O}) \to \big(\mathcal{O} \otimes_K \mathcal{O}\big) \tag{5}$$

Then $f \otimes 1$ has a *Taylor*-expansion in $K[[x,t]]$ by taking the image of $f \otimes 1$ under the composition

$$K[[x \otimes 1]] \otimes_K K[[1 \otimes x]] \quad \to \quad K[[x \otimes 1, 1 \otimes x]] \to K[[x,t]] \tag{6}$$

where $x \otimes 1 \ \mapsto x$ in the first map and $(1 \otimes x - x \otimes 1) \mapsto t$ in the second map.

The set $\{t_i = 1 \otimes x_i - x_i \otimes 1, 1 \leq i \leq n\}$ generates the diagonal ideals $\Delta_{\mathbb{K}}$ and Δ_K in each of the tensor products of Eq.(5). Then we have natural maps $\mathcal{D}_{\mathbb{K}}^{(m)} \to \mathcal{D}_K^{(m)}$, where

$$\begin{aligned} \mathcal{D}_{\mathbb{K}}^{(m)} &= \big(\mathcal{O} \otimes_{\mathbb{K}} \mathcal{O}\big)/\Delta_{\mathbb{K}}{}^{m+1} \\ \mathcal{D}_K^{(m)} &= \big(\mathcal{O} \otimes_K \mathcal{O}\big)/\Delta_K{}^{m+1} \to K[[x,t]]/(t)^{m+1} K[[x,t]] \\ &\sim \bigoplus_{\alpha \in \mathbb{Z}_0^n, |\alpha| \leq m} K[[x]] t^\alpha \end{aligned} \tag{7}$$

The *partial derivatives* of f are the coefficient of t^α in the image of f in the Taylor expansion. We use the notation $(\partial^\alpha f)(x)$ or $\partial^\alpha f(x)$ for these derivatives.

Note that we have the following fact.

Lemma 3.4. *Every element of the form* $h \otimes 1 - 1 \otimes h$ *with* $h \in K \supset \mathbb{K}$ *has zero class in the tensor product* $\mathcal{O} \otimes_{\mathbb{K}} \mathcal{O}/{\Delta_{\mathbb{K}}}^{m+1}$. *In other words we claim that the natural homomorphism* $\mathcal{D}^{(m)}_{\mathbb{K}[x]} \to \mathcal{D}^{(m)}_{K[x]}$ *of Eq.(7) is in fact an isomorphism for every* $m > 0$.

Proof. For every integer $\ell \gg m$ we have $g \in K$ such that $\rho^\ell(g) = h$. Hence $h \otimes 1 - 1 \otimes h$ is equal to $\rho^\ell\big(g \otimes 1 - 1 \otimes g\big)$ which is in ${\Delta_{\mathbb{K}}}^{m+1}$ with the diagonal ideal $\Delta_{\mathbb{K}}$ of $\mathcal{O} \otimes_{\mathbb{K}} \mathcal{O}$. Hence $h \otimes 1 - 1 \otimes h$ has zero image in $\mathcal{D}^{(m)}_{\mathbb{K}[x]} \mod \Delta^{m+1}_{\mathbb{K}}$. □

Remark 3.5. Recall Eq.(7) following after Eq.(6) and then note that

$$1 \otimes x^\beta = (1 \otimes x)^\beta \mapsto \big((x \otimes 1) + t\big)^\beta = \textstyle\sum_{\beta \in \gamma + \mathbb{Z}_0^n} \binom{\beta}{\gamma} x^{\beta - \gamma} t^\gamma$$

to which the action of $Hom_{\mathcal{O}_\xi}(\cdot, \mathcal{O}_\xi)$ apples only to the t^γ's individually arbitrarily. In particular the differential operator ∂^α sends $t^\alpha \mapsto 1$ and $t^\gamma \mapsto 0$ for all $\gamma \neq \alpha$. Locally at $\xi \in Z_{cl}$ there exists a free basis $\{\partial^a = \partial^a_x, a \in \mathbb{Z}^n_0\}$ of the $\mathcal{O}_\xi$-module $\mathrm{Diff}_{Z,\xi}$. They are well defined by the following equality.

$$\partial^\alpha x^\beta = \begin{cases} \binom{\beta}{\alpha} x^{\beta-\alpha} & \textit{if } \beta \in \alpha + \mathbb{Z}^n_0 \\ 0 & \textit{if otherwise} \end{cases} \tag{8}$$

Another way to have the same is by using a dummy variable t for x as follows.

$$(x + t)^\beta = \sum_{\beta : \alpha \in \alpha + \mathbb{Z}^n_0} (\partial^\alpha x^\beta) t^\alpha \tag{9}$$

3.1.2. *Cartier operations, extension and contraction.*

Lemma 3.6. *Pick* $e \geq 1$ *and let* $q = p^e$. *Then with respect to the given* x *we can extend every* $\sigma \in \mathrm{Diff}_{\rho^e(\mathcal{O}_\xi)/\mathbb{K}}$ *to* $\partial \in \mathrm{Diff}_{\mathcal{O}_\xi/\mathbb{K}}$ *such that the following condition is satisfied. Namely for all* $\alpha \in \mathbb{Z}^n_0$ *with* $0 \leq \alpha_i < q, \forall i$, *and for all* $h_\alpha \in \rho^e(\mathcal{O}_\xi)$, *we have*

$$\partial\Big(\sum_\alpha x^\alpha h_\alpha\Big) \;=\; \sum_\alpha x^\alpha \sigma(h_\alpha) \textit{ for all } h_\alpha \in \rho^e(\mathcal{O}_\xi). \tag{10}$$

In particular we have $\partial(x^\alpha) = 0$ *for even* α *as above.*

Proof. With the indeterminate copy t of x we examine the expansion in terms of t of $(x + t)^\alpha \times (h^q)(x + t)$ for each $h \in \mathcal{O}_\xi$. The only summands of its monomial terms divisible by any factor of the kind t^a, where $0 \leq a_i < q$ for all i, must have $a = (0)$. Hence $\partial(x^\alpha h^q)$ must be $x^\alpha \sigma(h^q)$. Moreover $\{t^a : 0 \leq a_i < q, \forall i\}$ are $\rho^e(\mathcal{O}_\xi)$-linearly independent. □

Definition 3.7. *Following Lem.(3.6)* ∂ *will be called the* x*-Cartier extension of* σ *from* $\mathrm{Diff}_{\rho^e(\mathcal{O}_\xi)/\mathbb{K}}$ *to* $\mathrm{Diff}_{\mathcal{O}_\xi/\mathbb{K}}$*.*

Remark 3.8. (Agreement on elementary differential operators.)
We agree that if x is chosen and fixed then every extension of elementary differential operators from $\mathrm{Diff}_{\rho^\ell(R_\xi)/\mathbb{K}}$ to $\mathrm{Diff}_{R_\xi/\mathbb{K}}$ will be always assumed to be x-Cartier in the sense of Def.(3.7).

Definition 3.9. *In contrast to the* x*-Cartier extension Def.(3.7), which depends on the choice of the coordinate system* x*, we have Cartier contraction from* $\mathrm{Diff}_{\mathcal{O}}$ *down into* $\mathrm{Diff}_{\rho^\ell(\mathcal{O})}$ *which is defined and denoted by* ρ^ℓ *as follows.*

$$\partial \in \mathrm{Diff}_{\mathcal{O}} \mapsto \rho^\ell(\partial) \in \mathrm{Diff}_{\rho^\ell(\mathcal{O})} \tag{11}$$

where $\rho^\ell(\partial)(f^{p^\ell}) = \partial(f)^{p^\ell}$ *for all* $f \in \mathcal{O}$*. This Cartier contraction is independent of local coordinate systems.*

3.2. **Ambient reductions.** Cutting down the ambient dimension by taking sections of the given singularity data is a useful technique for proving many theorems using induction on the dimension of the ambient scheme. Hence it is important to investigate AR-techniques especially for resolution of singularities.

Ambient reduction is explicitly performed by taking differentiations in the normal direction along the smooth subscheme $Y \subset Z$. The equivalent reduction is also done by taking all differentiations instead of only normal differentiations because $\wp$-algebras are invariant by any differentiations before and after AR in the manner of the Diff-theorem Th.(3.1). Below is an explicit definition of AR by differentiations of the given ideal of E and any choice of local coordinate system.

Theorem 3.10. *(Ambient Reduction Theorem) Given an ideal exponent* $E = (J, b)$ *in* Z *we define*

$$J^\sharp \;=\; \sum_{j=0}^{b-1} \Big(\mathrm{Diff}_Z^{(j)} J\Big)^{\frac{b^\sharp}{b-j}} \quad \textit{with} \;\; b^\sharp = b!.$$

For every smooth subscheme $W \subset Z$ *we let* $F = (J^\sharp \mathcal{O}_W, b^\sharp)$*. Then* F *is an* ambient reduction *of* E *from* Z *to* W *in the following sense: Pick any* t *and any* LSB *over* $Z[t]$ *such that all of its centers are in the strict transforms of* $W[t]$*. Then we have the* LSB $\in \mathfrak{S}(E)$ *if and only if the* LSB *induces the one to* W *which belongs to* $\mathfrak{S}(F)$*.*

3.2.1. *AR in terms of* $\mathfrak{S}$*.* For any smooth closed subscheme Y in an open subset $V \subset Z$ and for any ideal exponent F in V we have the ambient reduction $F(Y)$ of $F|_V$ from V to Y where $F|_V$ dnotes the restriction of F to V and $F(Y)$ is an ideal exponent in Y. See [24]. It is symbolically expressed as:

$$F \ \neg_Y \ F(Y) \tag{12}$$

which is characterized as follows.

Definition 3.11. Pick any finite system of variables v and any finite sequence of blowups LSB over $Z[v]$, say A, in the sense of Def.(2.4) of §2.3. Then A induces an LSB over $Y[v]$, say $\bar{A}$, which belongs to $\mathfrak{S}(F(Y)[v])$ if and only if the centers of blowups in A are all contained in the strict transforms of $Y[v]$ by the blowups and the A induces LSB $\bar{A}$ over $Y[v] \subset Z[v]$ which belongs to $\mathfrak{S}(F(Y)[v])$. Here recall the definition of the symbol $\mathfrak{S}$ by Defn.(2.5). For any X in Z we let $X[v] = X \times_{\mathbb{K}} Spec(\mathbb{K}[v])$. For more details of the general theory of infinitely near singularities we can review [24].

Let us then define an ideal exponent $F\{Y\}$ in Z associated with the given ideal exponent F in Z with reference to the given locally closed smooth subscheme $Y \subset Z$ as follows. $F\{Y\}$ is characterized by

$$\mathfrak{S}(F\{Y\}) = \mathfrak{S}(F) \cap \Big(\mathfrak{S}\big(I(Y), 1\big)\Big) \tag{13}$$

Now consider

$$F \to F\{Y\} \neg_Y \ F(Y)$$

where $I(Y)$ denotes the ideal of $Y \subset Z$ and the intersection is in the sense of infinitely near singularities $\mathfrak{S}$ as above. It then follows that the minimum system of generators of $I(Y)_\eta$ is extendable to a system of edge-generators of $\wp(F\{Y\})$ at every $\xi \in \mathrm{Sing}(F\{Y\})_{cl} \cap V_\xi$. It should be noted that in this case we have a stronger sense of ambient reduction, called AR-equivalence, which we denote as follows.

$$F\{Y\} \ \rightleftharpoons_Y \ F(Y) \tag{14}$$

This stronger AR, called *AR-equivalence* is characterized as follows.

Definition 3.12. Pick any finite system of variables v and consider any LSB over $Z[v]$, say A, in the sense of Def.(2.4) of §2.3. Then A belong to $\mathfrak{S}(F\{Y[v]\})$ if and only if A has all its centers contained in the strict transforms of $Y[v] \subset Z[v]$ and A induces in $Y[v]$ an LSB $\bar{A}$ over $Y[v]$ which moreover belongs to $\mathfrak{S}(F(Y)[v])$.

Remark 3.13. In general, wether Eq.(12) or Eq.(13), the definition of $F(Y)$ is expressed only up to $\mathfrak{S}(F(Y))$ but this is enough from its geometric view-point or in terms of its algebra $\wp(F(Y))$. However if F is expressed by a pair (J, b) we then have an explicit expression as $\Big(\sum_{0<j\le b} \big(\mathrm{Diff}^{(j)}(J)\big)^{b-j}\mathcal{O}_Y, \ b!\Big)$. See AR-theorem in [22] and [24].

Lemma 3.14. *Consider any open subset $V \subset Z$ and any smooth irreducible closed subscheme $Y \subset Z$. Then any ideal exponent (I, b) in Y has a unique natural extension (J, b) in V such that we have AR-equivalence $(J, b) \rightleftharpoons_Y (I, b)$ from V to Y. We will call (J, b) the natural AR-extention of (I, b) from Y to V.*

Proof. We have the natural epimorphism $\phi : \mathcal{O}_V \to \mathcal{O}_Y$. Let $J = \big(\phi^{-1}(I), I(Y, V))^b\big)\mathcal{O}_V$ and then The proof is reduced to the case of an affine scheme V by choosing finitely many affine open covering of V and defining the extension in the case of each member of the covering. Hence assume V is a closed subscheme of (J, b) $spec(\mathbb{K}[v]$ with a finite number of independent variables v. Then has the required property. □

3.3. **Numerical exponent theorem connecting geometric $\mathfrak{S}$ and algebraic $\wp$.** The characteristic algebra $\wp(E)$ of an ideal exponent $E = (J, b)$ was defined by geometric means of infinitely near singularities $\mathfrak{S}$ and then it was proved its finite generation by showing it is also defined algebraically by means of differetiations applied to the ideal J, [23]. Here by reproving NE-theorem of [23] we prove more directly prove the equivalence of geometric $\mathfrak{S}$ and algebraic $\wp$.

Theorem 3.15. *(Numerical Exponent Theorem) Let $E_i = (J_i, b_i), i = 1, 2$, be two ideal exponents in Z. If $\mathfrak{S}(E_1) \subset \mathfrak{S}(E_2)$ then $\mathrm{ord}_\xi(E_1) \le ord_\xi(E_2)$ for every $\xi \in Z$ provided any one of the two is ≥ 1, that is unless ξ is outside their singular loci.*

(Recall the definition $\mathrm{ord}_\xi(E) = ord_\xi(J)/b$ for $E = (J, b)$ in general.) The proof of the theorem will be given after some remarks and propositions which are stated and proven blow.

Remark 3.16. First of all if $\mathrm{ord}_\xi(E_1) < 1$ then the claimed inequality is obvious because the last provision said $\mathrm{ord}_\xi(E_2) \ge 1$. Secondly if $\mathrm{ord}_\xi(E_1) = 1$ then $\xi \in \mathrm{Sing}(E_1)$ and hence $\mathfrak{S}(E_1)$ contains the blowup with center $C(0) = \bar{\xi} \setminus \mathrm{Sing}(\bar{\xi})$ where $\bar{\xi}$ denotes the closure of ξ in Z. To be precise the $C(0)$ is a smooth irreducible subscheme of $Z(0) = Z \setminus \mathrm{Sing}(\bar{\xi})$ and ξ is the generic point of $C(0)$. Thus the blowup with center $C(0) \subset Z(0)$ is a member of $\mathfrak{S}(E_1)$ and hence the same of $\mathfrak{S}(E_2)$ by the assumption of the theorem, that is $\mathfrak{S}(E_1) \subset \mathfrak{S}(E_2)$. We thus conclude that it is enough to prove the theorem under the assumption that $\mathrm{ord}_\xi(E_1) \ge 2$ and $\mathrm{ord}_\xi(E_2) \ge 1$.

Remark 3.17. We then define and compare LSBs belonging to $\mathfrak{S}(E_i), i = 1, 2$, respectively. The LSBs of our particular interest will be constructed upon a common ambient scheme $W(0) = Z(0)[t] = Z(0) \times_{\mathbb{K}}$

$Spec(\mathbb{K}[t])$ where t is an indeterminate over $Z(0)$ of Rem.(3.16). Our construction will be done by connecting two different types of sequences of blowups, first one called type (I) followed by the one called type (II). See Rem.(3.18) for (I) and Rem.(3.21) for (II) explained later. We will examine the questions of permissibility for the ideal exponents $F_i(0) = E_i[t]|_{W(0)}$ which signify $(J_i\mathcal{O}_{W(0)}, b_i), i = 1, 2$, respectively. This $F_i(0)$ is defined to be $(J_i[t], b_i)|_{W(0)}$, too. We have a smooth irreducible closed subscheme $T(0) = C(0)[t] \subset W(0)$. The $T(0)$ contains a smooth irreducible closed subscheme $D(0)$ defined by $t = 0$.

Remark 3.18. (The type (I) sequence.)
The integer r is chosen to be arbitrary large.

$$W(r) \overset{\pi(r-1)}{\rightarrow} \begin{array}{c} W(r-1) \\ \cup \\ D(r-1) \end{array} \overset{\pi(r-2)}{\rightarrow \cdots} \overset{\pi(0)}{\rightarrow} \begin{array}{c} W(0) \\ \cup \\ D(0) \end{array}$$

where $D(j)$ is the center of blowup $\pi(j)$ for $j = 0, 1, \ldots$. They are chosen as follows. The $D(0) \subset T(0)$ has been chosen in Rem,(3.17). Note that the strict transform $T(1)$ of $T(0)$ is isomorphic to $T(0)$ because the center of blowup $D(0)$ is a smooth hypersurface in in $T(0)$. The next center $D(1)$ is the one in $T(1)$ which is isomorphic to $D(0)$ by $\pi(0)$. Incidentally $D(1) = T(1) \cap H(1)$ where $H(1)$ denotes the exceptional divisor in $W(1)$ for $\pi(0)$. The same is applied to all the subsequent blowups $\pi(j)$ with center $D(j) = T(j) \cap H(j), j = 1, 2, \cdots$, with strict transform $T(j)$ of $T(j-1)$ and the exceptional divisor $H(j)$ for $\pi(j-1)$. Note that the intersection is transversal and the center $D(j)$ is disjoint with all the strict transforms of the earlier $H(\iota), \iota < j$.

Remark 3.19. We claim the following equalities for $j = 0, 1, \cdots, r-1$, for $i = 1, 2$:

$$\text{(15)} \qquad \mathrm{ord}_{D(j)} F_i(j) \; = \; \mathrm{ord}_{D(0)} F_i(0) + j\Big(\mathrm{ord}_{D(0)} F_i(0) \; - \; 1\Big)$$

where ord_D denotes the order at the generic point of D in general. In fact the order of the ideal J_i of E_i is constant at every point in an open dense subset V of $\bar{\xi}$ and so is in $V \times Spec(\mathbb{K}[t])$. This constant order is same for the points of an open dense subset of the center $D(0)$ of the blowup $\pi(0)$. It follows that the pull-back of the ideal of J_i of E_i by $\pi(0)$ is equal to the product of the following two ideals; the strict transform of J_i and the $\mathrm{ord}_\xi(J_i)$-multiple of the ideal of the exceptional divisor $H(1)$. The order of this pull-back is equal to $\mathrm{ord}_\xi(J_i) + \big(ord_\xi(J_i) - b_i\big)$, where the first summand $\mathrm{ord}_\xi(J_i)$ is equal to $\mathrm{ord}_\eta(J_i)$ with the generic point η of an open dense cylindrical subset of $T(0)$. Hence it remains unchanged by the blowups in the sequence of type (I) of Rem.(3.18).

The second summand $\text{ord}_\xi(J_i) - b_i$ is by the rule of transformation for the ideal exponent (J_i, b_j). To be exact the pull-back of J_i by $\pi(0)$ is $\text{ord}_\xi(J_i)$-fold the ideal of the exceptional divisor $H(1)$ at the generic point $\eta(1)$ of $H(1)$ and hence its transform by the rule it is reduced to $\text{ord}_\xi(J_i) - b_i$-fold at $\eta(1)$ which is mapped down to the generic point ξ of $D(0)$. Along the center $D(0)$ the order of J_i is $\text{ord}_\xi(J_i)$ which is equal to the order of the same ideal at the generic point $\zeta(0)$ of $T(0)$. We thus conclude that the order of the transform $F(1)$ at the generic point $x(1)$ of $D(1)$ must be

$$(16) \qquad \begin{aligned} \text{ord}_{\xi(1)} F_i(1) &= b_i^{-1}\big(\text{ord}_\xi(J_i) + (\text{ord}_\xi(J_i) - b_i\big) \\ &= \text{ord}_{D(0)} F_i(0) + \Big(\text{ord}_{D(0)} F_i(0) - 1\Big) \end{aligned}$$

The situation for the blowup $\pi(1)$ with center $D(1)$ is totally similar to that of $p(0)$ with center $D(0)$ for only one exception that is the difference of the order at the generic point $\xi(1)$ of the center. This order change is from $\text{ord}_{D(0)} F(0)$ to $\text{ord}_{D(1)} F(1)$ jn the manner of Eq.(16). Knowing the successive blowups $\pi(j)$ are isomorphic in neighborhoods of the generic points of the transforms $T(j) \subset Z(j)$ of $T(0) \subset Z(0)$ and all the $T(\iota), \iota < j$, are disjoint with center $D(j)$, we obtain Eq,(15) for all j, In particular we have

$$(17) \quad \text{ord}_{D(r-1)} F_i(r-1) = \text{ord}_{D(0)} F_i(0) + (r-1)\Big(\text{ord}_{D(0)} F_i(0) - 1\Big)$$

Then we deduce the order of $F_i(r)$ at the generic point $\zeta(r)$ of the last exceptional divisor $H(r)$ and the following proposition is proven about type (I) sequence.

Proposition 3.20. *Let $F_i(r)$ be the final transform of $F_i(0) = E_i[t]$ by the sequence of blowups of type (I) defined by Rem.(3.18). Then we have*

$$(18) \quad \text{ord}_{H(r)}(F_i(r)) \;=\; \text{ord}_\xi(E_i) + r(ord_\xi(E_i) - 1) \text{ for } i = 1, 2.$$

Here $\text{ord}_{H(r)}(F_i(r))$ *denotesx the order of the ideal exponent $F_i(r)$ at the generic point of $H(r)$ which is mapped to the generic point of $D(r-1)$.*

Pick any large number r and take the end result of the sequence of blowups of type (I) of Rem.(3.18). Starting from the end result of the sequence of type (I) we begin to perform the new sequence of blowups, called type (II) as follows.

Remark 3.21. (The type (II) sequence.)
We begin with ideal exponent $F_i(r), i = 1, 2$, and smooth hypersurface

$H(r)$ in its ambient scheme $W(r)$. Let $W(r,0) = W(r)$, $F_i(r,0) = F_i(r), i = 1,2$, and $H(r,0) = H(r)$.

$$\begin{array}{ccccccc} & \pi(r,s-1) & & \pi(r,s-2) & \pi(r,0) & \\ W(r,s) & \to & W(r,s-1) & \to \cdots & \to & W(r,0) \\ & & \bigcup & & & \bigcup \\ & & H(r-1) & & & H(r,0) \end{array}$$

where all the blowups $\pi(r,j)$ are isomorphisms because the centers $H(r,j)$ are smooth hypersurfaces, $0 \le j \le s-1$. The $H(r,j)$ are all isomorphic by themselves. The first center $H(r,0)$ of blowup of type (II) sequence is the the last exceptional divisor created by the type (I) sequence.

Remark 3.22. Here it is important that the ideal exponents $F_i(r,0) = F_i(r), i = 1,2$, given in the ambient scheme $W(r,0) = W(r)$ undergo nontrivial changes by blowups $\pi(r,k)$ according to the rule of transformation of ideal exponent in spite of no changes for their geometric carriers by the blowups of type (II) . The rule of transformation dictates the ideal exponent $F_i(r,k) = (J_i(r,k), b_i)$ in $W(r,k)$ to be transformed into $F_i(r,k+1) = (J_i(r,k+1), b_i)$ in $W(r,k+1)$ by the rule of $J_i(r,k+1) = h(r,k+1)^{-b_j} J_i(r,k)\mathcal{O}_{W(r,k+1)})$ where $h(r,k+1)$ denotes the locally principal ideal defining the exceptional divisor $H(r,k+1)$ in $W(r,k+1)$. In other words the pull-back of the ideal will loose b_i-fold exceptional divisor. Thus we have

$$\operatorname{ord}_{\zeta(k+1)}(F_i(r,k+1)) = \operatorname{ord}_{\zeta(k)}(F_i(r,k)) - 1 \tag{19}$$

where $\zeta(j)$ denotes the generic point of $H(r,j), j = k, k+1$..

Remark 3.23. The equality of Eq.(19) is under the condition that the blowup $\pi(r,k)$ is permissible for $F_i(r,k)$. This condition ls equivalent to saying that the center $H(r,k) \subset \operatorname{Sing}(F_i(r,k))$, i.e, $\operatorname{ord}_{\zeta(k)}(F_i(r,k)) \ge 1$ because $H(r,k)$ is exactly the closure of its generic point $\zeta(k)$. In view of the equality of Eq.(19) we easily conclude by induction on the number e of the type (II) sequence of Rem.(3.21).

$$\operatorname{ord}_{\zeta(s)}(F_i(r,s)) = \operatorname{ord}_{\zeta(0)}(F_i(r,0)) - s, \quad \text{where } F_i(r,0) = F_i(r) \tag{20}$$

where the permissibility condition $\operatorname{ord}_{\zeta(s-1)}(F_i(r,s-1)) \ge 1$ is assumed for the blowups $\pi(r,s-1)$.

Definition 3.24. After a type(I) sequence of Rem.(3.18) for $F_i(0)$ was chosen, the connecting type(II) sequence of Rem.(3.21) for $F_i(r)$ has always the stopping time under the permissibility condition. In other words we have the maximum number $s_i(r)$ such that

$$\operatorname{ord}_{H_i(r,s-1)} F_i(r,s-1) \ge 1 \quad \text{and} \quad \operatorname{ord}_{H_i(r,s)} F_i(r,s) < 1, \quad \text{for } s = s_i(r).$$

This number $s_i(r)$ is the *stopping number* for E_i for the given number $r \gg 1$. Thanks to Rem.(3.16) we always have $s_i(r) > 0$, for $i = 1, 2$, and in particular $s_1(r) > 1$.

Remark 3.25. In the case of type (I) it should be noted that the permissibility conditions are valid for all the blowups of the type (I) sequence of Rem.(3.18) thanks to Eq.(17) by which $\mathrm{ord}_{D(j)}(F_i(j))$ is monotone non-decreasing or increasing with respect $j \geq 0$ starting from $\mathrm{ord}_\xi(E_i) \geq 1$ for both $i = 1, 2$ while $\mathrm{ord}_\xi(E_1) > 1$. Remember that the (i)+(II) combined sequence can be an LSB belonging to $\mathfrak{S}(E_i)$ with an arbitrary $r \gg 1$ so long as the number s of $R : (II) - 1$ is bounded by $s_i(r)$.

The following result is the summary of the remarks and propositions about the (I) and/(II) type sequences of blowups which are LSB $\in \mathfrak{S}(E_i)$ for $i = 1, 2$, respectively

Proposition 3.26. *Under the assumption of Th.(3.15) we have the following consequences.*

1. *since* $\mathfrak{S}(E_1) \subset \mathfrak{S}(E_2)$, *we have* $s_1(r) \leq s_2(r)$ *for all choices of* $r \gg 1$;
2. $\lim_{r \mapsto \infty}(s_1(r)/r) \leq \lim_{r \mapsto \infty}(s_2(r)/r)$;
3. $\mathrm{ord}_\xi(E_i) + r\big(\mathrm{ord}_\xi(E_i) - 1\big) \leq s_i(r)$ *for* $i = 1, 2$;
4. $s_i(r) \leq ord_\xi(E_i) + r\big(ord_\xi(E_i) - 1\big) + 1$ *for* $i = 1, 2$;
5. $\lim_{r \mapsto \infty}(s_i(r)/r) = \mathrm{ord}_\xi(E_i) - 1$ *for* $i = 1, 2$; *and*
6. $\mathrm{ord}_\xi(E_1) \leq \mathrm{ord}_\xi(E_2)$.

Proof. Item (1) is clear and (2) follows from (1); (3) is already proven by Eq.(17); (4) follows from Def.(3.24) together with Eq.(17) and Eq.(18); (5) follows from (3) and (4), dividing the sides of the resulting inequality by $r + 1$; and (6) follows from (1) and (5). $\square$

Th.(3.15) follows from Prop(3.26).

4. Characteristic algebra $\wp(E)$

The fundamental tool in our proofs of resolution theorems will be the characteristic algebra $\wp(E) = \bigoplus_{d \geq 0} \wp(E, d)$. A very important and useful property of the $\wp(E)$ is that it is not only finitely generated (thus producing useful edge data and edge invariants) but also it has a strong tie with the geometry of infinitely near singularity $\mathfrak{S}(E)$ of E. To be more explicit we have two different definitions of the same $\wp(E)$, the one geometric and the other algebraic.

The *geometric definition* of $\wp(E)$ is given by letting each $\wp(E,a), a > 0$, be the union of those deals I such that $\mathfrak{S}(I,a) \supset \mathfrak{S}(J,b)$. It follows that $\wp(E)$ is integrally closed in the graded algebra $Bl(Z)* = Bl^{posi}(Z) \bigoplus Bl(Z,0)$ (defined in Definition 2.2). This is due to the fact that all the blowups in $\mathfrak{S}$ have smooth centers and hence the graded algebra by the powers of its ideal is integrally closed.

The *algebraic definition* is as follows: $\wp(E)$ is the integral closure in $Bl(Z)*$ of the graded $\mathcal{O}$-subalgebra generated by the following ideals of specified homogeneity degrees:

$$(\mathrm{Diff}^{(j)} J)\mathcal{O} \subset Bl(Z, b-j) \ \textit{for all} \ 0 \leq j < b$$

The equivalence of the two definitions has been proven in [23] under the assumption that the base field $\mathbb{K}$ is perfect. The reader should refer to the proof of the main theorem of [23], in particular to the Lemmas 2.1-2.2 and the claim ($\flat$) in page 918 of the [23]. These are contained in the proof of the theorem of the finite presentation of $\wp(E)$ by the theory of infinitely near singularities in [23]. It should be noted that the two definitions are not equivalent in general when the base field $\mathbb{K}$ is not perfect.

Theorem 4.1. *The $\wp(E)$ of $E = (J,b)$ is the smallest graded $\mathcal{O}$-subalgebra of $Bl(Z)* = \bigoplus_{a\geq 0} Bl(Z,a)$ such that*

(1) $J \ \subset \ \wp(E,b)$
(2) $\mathrm{Diff}_Z^{(\mu)}\big(\wp(E,a)\big) \ \subset \ \wp(E, a-\mu)$ *for all* $0 \leq \mu < a$ *and*
(3) $\wp(E)$ *is integrally closed in* $Bl(Z)*$.

In connection with the second property above, we may use Differentiation theorem of [24] together with the following lemma.

Lemma 4.2. *For* $a = \sum_{i=0}^{b-1}(b-i)\alpha_i$ *with* $\alpha \in \mathbb{Z}_0^b$ *and for* $\mu < a$,

$$\mathrm{Diff}_Z^{(\mu)}\Big(\prod_{i=0}^{b-1}\big(\mathrm{Diff}_Z^{(i)} J\ \big)^{\alpha_i}\Big) \ \subset \sum_{\substack{\beta\in\mathbb{Z}_0^b \\ \sum_{i=0}^{b-1}\beta_i(b-i)=a-\mu}} \Big(\prod_{i=0}^{b-1}\big(\mathrm{Diff}_Z^{(i)} J\ \big)^{\beta_i}\Big)$$

Proof. see Remarks (2.1)-(2.2) of the paper [23]. □

Remark 4.3. We add the following supportive comment on the inclusion relation of Lem.(4.2) in connection with the second property above of Th.(4.1). There the integer μ could be as big as we may want so long as $0 \leq \mu < a$ with respect to the given a which may be big per se. There the result of differentiation on the left hand side of Lem.(4.2) can have some non-zero factor in summands of the left hand which may have differentiation of order $> b$ such as $i + \mu > b$. There we then replace the factor by the unit ideal on the right hand side of Lem.(4.2).

With the help of this lemma we can restate the Th.(4.1) as follows.

Theorem 4.4. *The $\wp(E)$ is the integral closure of the $\mathcal{O}_Z$-subalgebra*

$$\mathcal{O}_Z\big[\bigoplus_{0\le j<b}(\mathrm{Diff}_Z^{(j)})J\big] \subset Bl(Z)*$$

Theorem 4.5. *For every pair of integers $a > b \ge 0$ we have the $\wp(E, a)$ viewed as an ideal in $\mathcal{O}_Z$ is included in the ideal $\wp(E, b)$.*

Proof. Let t be an indeterminate over $\mathcal{O}_Z$ and consider the extension $E[t]$ in $Z[t]$. Then $\wp(E[t], a) = \wp(E, a)\mathcal{O}_{Z[t]}$ which contains the ideal $t\wp(E, a) \subset \wp(E, a)$. Hence by the elementary differentiation $\partial^{(a-b)} \in \mathrm{Diff}_{Z[t]/Z}$ with respect to t we obtain the ideal $\wp(E, a) = \partial(t^{a-b}\wp(E, a)) \subset \wp(E[t], a - b) = \wp(E, a - b)\mathcal{O}_{Z[t]}$. Hence by the ambient reduction modulo $t\mathcal{O}_{Z[t]}$ we obtain $\wp(E, a) \subset \wp(E, a - b)$. □

4.1. **Edge data of $\wp(E)$.** Recall $bl_\xi(Z) = \bigoplus_{d\ge0} bl_\xi(Z, d)$ at each $\xi \in \mathrm{Sing}(E)_{cl}$. Refer to Definition 2.2 of §2. We have $\wp(E, d) \subset bl_\xi(Z, d) = \max(\mathcal{O}_\xi)^d$ for every d. Recall $\wp(E)_\xi = \bigoplus_{d\ge0}\wp(E, d)_\xi$ of §4.

We define the edge algebra $\bar\wp(E)(\xi)$ of E at ξ as follows.

$$\bar\wp(E)(\xi) = \wp(E)/\big(\wp(E) \cap \max(\mathcal{O}_\xi)bl(Z)\big) \tag{21}$$

where the respective grades and homogeneity of their elements must be honored in $\wp(E)$ and in $bl(Z)$. The elements of $\mathcal{O}_{Z,\xi}$ as multipliers have all zero degree. Then note the following properties of $\bar\wp(E)(\xi)$.

(1) Pick any regular system of parameters z of $\mathcal{O}_\xi$ and let $\bar z$ denote the image of z in $bl(Z, 1)/\max(\mathcal{O}_\xi)bl(Z, 1)$. We then have the surjection $\mathcal{O}_\xi \to K[\bar z]$ where $K[\bar z]$ is the polynomial ring of independent variables $\bar z$ over the residue field K of $\mathcal{O}_\xi$.

(2) With respect to the choice of z with $\bar z$ we obtain the inclusions:

$$\bar\wp(E)(\xi) \subset K[\bar z] \subset bl(Z)_\xi/\max(\mathcal{O}_\xi)bl(Z)_\xi \tag{22}$$
$$\textit{and surjection } \mathrm{Diff}_{Z,\xi}^{(m)} \to \mathrm{Diff}_{K[\bar z]/K}^{(m)}, \forall m \ge 0$$

which is induced by the surjection $\mathcal{O}_{Z,\xi} \to K[\bar z]$. which denotes the algebra of the differential operators in $\bar z$ over K.

(3) If $g = f + h \in \wp(E, d)$ with $\mathrm{ord}_\xi(h) > d$ then f and g have the same image in $\bar\wp(E)(\xi)$ by the natural epimorphism.

(4) If $g \in \wp(E,d)$ is as above and $\partial \in \mathrm{Diff}^{(m)}, m < d$, then $\partial(g)$ and $\partial(f)$ have the same image in $\bar{\wp}(E)(\xi)$ for the image of $\partial(h)$ must have its order $> d - m$. The proof is done by splitting ∂ into a sum by Diff-orders and by rule of $\wp$-order changes.

(5) Let $\bar{f} \in \bar{\wp}(E,d)(\xi)$ and pick any f of its lifts in $\wp(E,d)$, i.e, $f \mapsto \bar{f}$ by the surjection $\wp(E,d)_\xi \to \bar{\wp}(E,d)(\xi)$. Then the image of $\partial(f)$ into $\bar{\wp}(E)(\xi)$ is independent of the choice of f out of a given $\bar{f}$ where $\partial \in \mathrm{Diff}^{(m)}_{Z,\xi}, 0 < m < d$, acting upon $\wp(E)$.

(6) The final step is to choose the minimum system of homogeneous generators for the canonical image G of $\bar{\wp}(E)(\xi)$ into $K[\bar{z}]$. $K[\bar{z}]$ has the grading by polynomial degrees in terms of the variables $\bar{z}$, It is induced by the same grading in terms of the basis $z \subset bl_\xi(Z,1)$ which is the original grading in $bl_\xi(Z)$.

(7) The minimum system of generators of the graded ideal $G \subset K[\bar{z}]$ is chosen starting from the lowest degree and by lexicographic ordering of the exponent of nonzero monomial terms. This is indeed the canonical procedure and terminate by a finite number of steps. The rule of the successive selection is its minimization of the result in each step by eliminating leading monomial term that could be composed of earlier selections as much as possible. Say we have done with the selection with $\bar{g} = (\bar{g}_1, \cdots, \bar{g}_r)$. Here it is possible to have $r = -1$ and $\bar{g}$ is empty.

(8) The final important point is that each $\bar{g}_j, 1 \leq j \leq r$, of the chosen generators of $G \subset K[\bar{z}]$ is an element of $\rho^{e_j}(K[\bar{z}])$ with an integer $e_j \geq 0$ where $\mathrm{ord}_\xi(\bar{g}_j) = p^{e_j}$ for every j. In fact we have a surjective homomorphism from $\wp(E)_\xi$ to the ideal G and $\wp(E)_\xi$ is Diff-closed by the nature of characteristic algebra so that G must be Diff-closed in view of Eq.(22).

Definition 4.6. We define the natural surjection ν and use it to clarify the notion of edge algebra and edge data of E at $\xi \in \mathrm{Sing}(E)_{cl}$.

$$\nu : bl_\xi(Z) \to bl_\xi(Z)/\max(\mathcal{O}_\xi)bl_\xi(Z) \tag{23}$$

This induces $\wp(E)_\xi \to \nu(\wp(E)_\xi) = \bar{\wp}(E)(\xi)$ where $\bar{\wp}(E)(\xi)$ is called the edge algebra of E at ξ.

Note that the target of ν is a polynomial ring of n variables over the residue field K of $\mathcal{O}_\xi$ where $n = dimZ$. Such a system of n variables is obtained by applying ν to any regular system of parameters z of $\mathcal{O}_\xi$ via the identification $bl_\xi(Z,1) = \max(\mathcal{O}_\xi)$ by the definition $K[\bar{z}]$ of Eq.(22). Moreover the image $\nu(\wp(E)_\xi)$ is naturally a graded K-algebra inherited from the grading in $\wp(E) \subset bl_\xi(Z)$ but the image is also an ideal in the polynomial ring $K[\bar{z}]$. We denote $\nu(\wp(E)_\xi)$ by $\bar{\wp}(E)(\xi)$.

A differential operator ∂ in a graded algebra is said to be homogeneous if it sends every homogeneous part to some homogeneous part. In the cases of our graded algebras, such as $Bl(Z)$, $bl_\xi(Z)$ and $\wp(E)$, every homogeneous differential operators has its own degree, say a and sends all elements of degree k to elements of degree $k-a$. Our next task is to choose a minimum system of generators of the ideal $\bar{\wp}(E)(\xi) \subset K[\bar{z}]$. Keep in mind that K is a perfect field for it is a finite algebraic extension of the perfect field $\mathbb{K}$. It then follows that every element of K is any high power of another member of K.

Remark 4.7. There exists a minimum system of homogeneous generators $(\bar{g}_1, \cdots, \bar{g}_n)$ for the edge algebra $\nu(\wp(E)_\xi)$ of the characteristic algebra $\wp(E)$ of E localized at $\xi \in \mathrm{Sing}(E)_{cl}$. The edge algebra is also a homogeneous ideal in $\nu(bl_\xi(Z)$ which can be identified with the polynomial ring of n independent variables over the residue field K of $\mathcal{O}_\xi$. These n variables can be obtained as follows.

(1) Choose a regular system of parameters $x = (x_1, \cdots, x_n)$ of $\mathcal{O}_\xi$;
(2) consider $x \subset \max(\mathcal{O}_\xi) = bl_\xi(Z, 1) \to \nu(bl_\xi(Z)) \supset \bar{x}$ where $\bar{x}$ is the image of x by the composition of those maps; and
(3) $\nu(bl_\xi(Z)) = K[\bar{x}]$, a polynomial ring of n variables $\bar{x}$.

Definition 4.8. Write each element $\bar{g} \in \bar{\wp}(E)(\xi), \neq 0$, in the form $\bar{g} = \sum_\alpha c_\alpha \bar{z}^\alpha$ with all $c_\alpha \in K$. We then have $c_\beta \neq 0$ such that there is no $c_\alpha \neq 0$ having $(|\beta|, \beta) \prec (|\alpha|, \alpha)$ where $||$ means the sum of components and $\prec$ means that the left side is lexicographically smaller. The monomial term $c_\beta \bar{z}^\beta$ with such β is called the head of $\bar{g}$ and $(|\beta|, \beta)$ is called the signature of $\bar{g}$. A system of elements $(\bar{g}_1, \cdots, \bar{g}_r)$ is said to be minimum with respect to $\bar{z}$ if we have $\bar{g}_j \prec \bar{g}_{j+1}$ for all j.

Definition 4.9. Choose any minimum system of homogeneous generators $(\bar{g}_1, \cdots, \bar{g}_r)$ of the ideal $\wp(E)(\xi) = \nu(\wp(E)_\xi)$ in the manner of Rem.(4.7). Then any system of homogenous lift back into $\wp(E)$ of such generators, say $(g_1, \cdots, g_r)$, is called the edge data of E at ξ. Note that we then have $\mathrm{ord}_\xi(g_j) = deg(\bar{g}_j)$ and

$$g_j = f_j + h_j \qquad f_j \in \rho^{e_j}\big(bl_\xi(Z, 1)\big) \tag{24}$$

and

$$\mathrm{ord}_\xi(h_j) > q_j = p^{e_j} = \mathrm{ord}_\xi(\bar{g}_j).$$

The edge generators of $\wp(E)_\xi$ of the form Eq.(24) will be called edge generator type. We will then choose the regular system of parameters

x of $\mathcal{O}_\xi$ as follows.

$$f_j = x_j{}^{q_j} \qquad \text{for all } j = 1, \cdots, r. \tag{25}$$

Remark 4.10. Given $\xi \in \mathrm{Sing}(E)_{cl}$ we let and we can choose a *minimum* system of generators of $\wp(E)(\xi)$ as $\kappa-\xi$-algebra which are induced by a subsystem of generators of the graded algebra$\wp(E)$ for a regular system of parameters x of $\mathcal{O}_\xi$. To be more explicit about role of special subsystems of $x = (y^\circ(\xi), \omega)$ we have the following technical terms which will be referred to for later usage. Here we choose $\{g_j, 1 \le j \le r\}$ of Eq.(24) and x of Eq.(25). However we make renaming as follows: Write $g_j(\xi)$ for $g_j, 1 \le j \le r$, and in particular $x_k = y^\circ_k(\xi) = g_k$ for those g_k with $\mathrm{ord}_\xi(g_k) = 1$ if any such exist. Indeed it could happen to have empty set of $\{g_j\}$, i.e, $r = t = 0$ if E is not maximum hiked, i.e, $E \ne \hat{E}$ in the sense of §6.1.

We define the edge generators and edge parameters as follows.

Definition 4.11. The *edge generators* are given by

$$g(\xi) = (y^\circ(\xi), g_{t+1}(\xi), \cdots, g_r(\xi)) = (g_1(\xi), \cdots, g_r(\xi)) \tag{26}$$

where, for $t = t(\xi)$ and $r = r(\xi)$, and $\mathrm{ord}_\xi(y^\circ_i(\xi)) = 1$ for all i,

$$\begin{array}{rcl} g_k(\xi) & = & y^\circ_k(\xi), \qquad \forall k \le t \\ y^\circ(\xi) & = & (y^\circ_1(\xi), \cdots, y^\circ_t(\xi)) \end{array}$$

and

$$1 < \mathrm{ord}_\xi(g^\circ_{t+1}(\xi)) \le \cdots \le \mathrm{ord}_\xi(g^\circ_r(\xi)).$$

The *edge parameters* are given by

$$y(\xi) = (y_1(\xi), \cdots, y_r(\xi)), \tag{27}$$

which is a part of a regular system of parameters $x = (y^\circ(\xi), \omega)$ of $\mathcal{O}_\xi$ where

$$\begin{array}{rcl} y_i(\xi) & = & y^\circ_i(\xi) \in \wp(E, 1), \qquad 1 \le i \le t \\ y_{t+k}(\xi) & = & \omega_k, \qquad 1 \le k \le r - t \\ g_j(\xi) & = & y_j(\xi)^{q_j(\xi)} + \epsilon_j(\xi) \in \wp(E, q_j(\xi)) \\ q_j(\xi) & = & p^{e_j(\xi)}, e_j(\xi) > 0 \qquad t < j \le r \\ \mathrm{ord}_\xi(\epsilon_j(\xi)) & > & ord_\xi(g_j(\xi)) = q_j(\xi) \qquad t < j \le r. \end{array}$$

The *edge exponents* are given by

$$q(\xi) = (1, \cdots, 1, q_{t+1}(\xi), \cdots, q_r(\xi)). \tag{28}$$

Remark 4.12. When the point ξ is specified we often omit (ξ) from those symbols for the sake of notational simplicity. Incidentally there are included the following special cases: $t = 0$ and/or $r = t$. When $r = t = 0$ we say that E is *edge-trivial* at ξ. Being edge-trivial does not mean that $\wp(E)$ is trivial.

Remark 4.13. In positive characteristic cases the study of edge data of $\wp(E)$ becomes far more important than the case of characteristic zero for the nature of singularity is dramatically different at a non-closed point compared to a closed point. Interesting but sometimes hard questions arise in the cases of points having imperfect residue fields.

4.2. **Cotangent module** $\cot(E)$**.** Recall

$$\wp(E) = \bigoplus_{d \geq 0} \wp(E, d)$$

((2) in §2.7). When we work on the graded algebra $\wp(E)$ for the study of the singularities of E in terms of local equations we need to choose various coordinate systems at each point $\xi \in \mathrm{Sing}(E)_{cl}$ with respect to the cotangent structure of E at ξ. It is represented by cotangent module denoted by $\cot(E)_\xi$ at ξ.

Definition 4.14. With an integer $\ell \gg e$ where $q = p^e, e \geq 0$, is the maximum p-power among the edge exponents of Eq.(28) of §4.1 and the Frobenius p-power is denoted by ρ. We then define the cotangent module $\cot(E, \ell)_\xi$ and $\cot(E)_\xi$ as follow.

$$\cot(E)_\xi \quad = \bigcup_{all\, \ell \geq 0} \cot(E, \ell)_\xi \tag{29}$$

where

$$\cot(E, \ell)_\xi \quad = \quad \sqrt[p^\ell]{\rho^\ell Bl(Z, 1)_\xi \cap \Big(\wp(E, p^\ell)_\xi + \max(\mathcal{O}_\xi) Bl(Z, p^\ell)_\xi\Big)}$$

in which all constituents should honor their homogeneity degrees in $\wp(E) \subset BL(Z)$ but on occasions of later applications we will need to ignore homogeneity degrees. Since $\wp(E)$ is finitely generated it is clear that some single $\ell \gg e$ will define the union of Eq,(29). We then consider the constituents and results simply as ideals in $\mathcal{O}_Z$.

Definition 4.15. As for the integer ℓ whose choice is significant in later applications but it should be noted here that the residue class $\cot(E)_\xi \mod \max(\mathcal{O}_\xi) Bl(Z, 1)$ is independent of ℓ for all $\ell \gg q$. This residue class, denoted by $\mathfrak{v}(E)_\xi$, is called the *cotangent vector space* of E at ξ, which may be viewed as being dual to the *tangent space* of E

in Z at ξ. A coordinate system x in an open subset V of Z is said to be *allowable* for E in V if it contains a system y which induces a basis of $\bar{c}(E)_\eta$ for every $\eta \in V_{cl}$. Symbolically we write $x_V \supset y$, or $x \supset y$ in short, for an *allowable* coordinate system for E in V.

Proposition 4.16. *The quantity of Eq.(29) is monotone increasing in trims of ℓ and hence its local ideal is independent of all $\ell \gg e$.*

Proof. Evident by the noetherian property of $\mathcal{O}_\xi$. We denote by $\cot(E)_\xi$ for $\lim_{\ell\to\infty} \cot(E,\ell)_\xi$. □

So far has been a strictly local definition of cotangent modules at a point $\xi \in \mathrm{Sing}(E)_{cl}$. However we have a globalized version. Firstly we set a preliminary notion.

Definition 4.17. Let $I = I(S)$ be the ideal of $S = \mathrm{Sing}(E) \subset Z$ and define the global coherent ideal $I^{(m)}(S)$ in $\mathcal{O}_Z$ such that for every open affine subset $V = Spec(A) \subset Z$ and for $f \in A$ we have

$$ord_\eta(f) \geq m, \forall \eta \in V \cap S_{cl} \;\Leftrightarrow\; f \in I^{(m)}(S) \tag{30}$$

This $I^{(m)}(S)$ is uniquely defined as a global coherent ideal in $\mathcal{O}_Z$. We call it the m-th symbolic power of the ideal of $\mathrm{Sing}(E)$.

Definition 4.18. We now define the global coherent cotangent module $\cot(E)$.

$$\cot(E,p^\ell) \;=\; \sqrt[p^\ell]{\rho^\ell(Bl(Z,1)) \cap \big(\wp(E,p^\ell) + I(\mathrm{Sing})Bl(Z,p^\ell)\big)} \tag{31}$$

Then $\cot(E) = \bigcup_{\ell\gg e} \cot(E,p^\ell)$ with $q = p^e$, which is independent of the choices of $\ell \gg e$ and unique for each $\xi \in \mathrm{Sing}(E)_{cl}$.

Here the Frobenus ρ maps the quantity under the symbol $\sqrt[q]{\ }$ into its component-wise p-th power into $\wp(E,pq) + I(S)Bl(Z,pq)$. Hence $\cot(E,\ell)$ is monotone increasing global coherent ideals in $\mathcal{O}_Z$.

Hence by the noetherian property we obtain a global coherent ideal

$$\cot(E) = \textstyle\bigcup_{all\, p^\ell, \ell\gg 1} \cot(E,\ell) \tag{32}$$

and hence $\cot(E,\ell) = \cot(E)$ for all $p^\ell, \ell \gg 1$.

This $\cot(E)$ is uniquely defined and called *cotangent* module of E which will be effectively used for the study of singularities in particular for the core-edge focused $\check{E}$ of E later discussed later in §6.2.

4.3. Edge Decomposition Theorem.

Theorem 4.19. *We obtain the following equivalence which holds within a sufficiently small neighborhood U of $\xi \in Z$.*

$$E \;\sim\; \Big(\bigcap_{i=1}^{r} E_i \Big) \bigcap R(E)_\xi \tag{33}$$

or more precisely

$$\mathfrak{S}(E) = \Big(\bigcap_{i=1}^{r} \mathfrak{S}(E_i) \Big) \bigcap \mathfrak{S}(R(E))$$

in the sense of the infinitely near singularity $\mathfrak{S}$ where $E_i = (g_i \mathcal{O}_U, q_i), \forall i$, and $R(E) = (I, c)$ with $\mathrm{ord}_\xi(I) > c$.

(See [22], [23], [24] and [25]). The $R(E)$ is less canonical than the other E_i but there is an explicit way to choose it as follows.

Remark 4.20. Let us define $\mathfrak{R}_\xi = \wp(E)_\xi \cap \max(\mathcal{O}_\xi) bl(Z)_\xi$ which is finitely generated as a graded $bl(Z)_\xi$-module and hence generated by finitely many $\wp(E, j) \cap \max(\mathcal{O}_\xi) bl(Z)_\xi$. Hence with the maximum $m > 0$ of the orders of such a finite system of generators we can choose the ideal I of $R(E)_\xi$ to be $\wp(E, m!) \cap \max(\mathcal{O}_\xi) bl_\xi(Z)$. It should be noted that $\mathfrak{R}_\xi$ is finitely generated as $bl(Z)_\xi$-module thanks to the finite generation of $\wp(E)$.

We define the numerical invariants of $\wp(E)$ at ξ as follows.

$$\mathrm{Inv}_\xi(E) \;=\; (n, n-r, q_1, q_2. \cdots, q_r) \tag{34}$$

with $n =$ dim Z.

4.3.1. *Notational warning (1).* When we talk about an ideal exponent, such as $E = (J, b)$ as above, we often think of the geometric interpretation: the singular locus $\mathrm{Sing}(E)$ as a set, or better the set of all infinitely near singularities $\mathfrak{S}(E)$, defined in [23].

Readers are cautioned about logical symbols, in particular when $\subset$ and $\cap$ are used when dealing with ideal exponents. For instance writing $(J_1, b_1) \subset (J_2, b_2)$ in short cannot signify anything but $\mathfrak{S}(J_1, b_1) \subset \mathfrak{S}(J_2, b_2)$ in the ordinary set-theoretical sense. Also writing $(J_1, b_1) \cap (J_2, b_2) = (J_3, b_3)$ in short cannot mean anything truly but $\mathfrak{S}(J_1, b_1) \cap \mathfrak{S}(J_2, b_2) = \mathfrak{S}(J_3, b_3)$. The intersection is better understood to signify $\big(j_1{}^{b_2} + J_2{}^{b_1}, b_1 b_2\big)$. We should note that even when $b_1 = b_2 = b$ the equality $(J_1, b) \cap (J_2, b) = (J_3, b_3)$ imples

$$\mathfrak{S}(J_1, b) \cap \mathfrak{S}(J_2, b)\big) = \mathfrak{S}(J_1 + J_2, b)$$

while the converse is not always true. Also $J_1 \subset J_2$ rigorously implies the reversed inclusion $\mathfrak{S}(J_1, b) \supset \mathfrak{S}(J_2, b)$ but the converse is not always true.

It should be kept in mind that geometric implications are often convenient to deal with, while algebraic ideal-theoretic considerations are more delicate. This is especially so in the context of the induction required in our proof of resolution of singularities given in this paper. Just for instance the ideal J of $E = (J, b)$ with $b > 1$ can remain to be significant by itself even after we gained $\mathrm{Sing}(E) = \emptyset$ during the process of inductive proofs. However it is always true that

$$\mathfrak{S}(E_1) = \mathfrak{S}(E_2) \ \Leftrightarrow\ \wp(E_1) = \wp(E_2) \tag{35}$$

Refer to [23] and [24]. It will be be seen that our $\mathfrak{S}$-geometry is always closely tied with the ideal exponents up to $\wp$-equivalence.

Remark 4.21. Unlike $\cup$ and $\subset$ with ideal exponents the rule of transforms of ideal exponents by permissible blowups are always explicit and precise as to the pair expression $E = (J, b)$. Refer to Def.(2.1). This fact of matter should be kept in mind. For instance the transform by a permissible blowup can have empty singular locus but the ideal of the transformed ideal exponent remains often significant in itself as it is obtained. The significance may become apparent and important especially in our inductive proofs of resolution of singularities.

5. The negative degree part of $\wp(E)$.

Recall the graded algebra $Bl(Z)$ of §2.1 of §2 and the characteristic algebra $\wp(E) \subset Bl(Z)$ of Th.(4.1) of §4. Note that $Bl(Z)$ has negative degree part $Bl^{nega}(Z)$ unlike $\wp(E) = \wp^{posi}(E) \bigoplus \wp(E, 0)$ where $\wp^{posi}(E) = \bigoplus_{b>0} \wp(E, b)$. We write the positive part $\wp^{posi}(E, b)$ of $\wp(E, b)$ where $0 < b \in \mathbb{Z}$. We then want to extend $\wp(E)$ by adding $\wp^{nega}(E)$ called the negative part of $\wp(E, b)$.

Definition 5.1. Recall $BL(Z)_\circ = \bigoplus_{j \geq 0} BL(Z, j)$ of Definition 2.2. We know that the graded $\mathcal{O}_Z$-algebra $\wp(E)$ is finitely presented. This means that $\wp(E)$ has a finite system of generators as a $BL(Z)_\circ$-module. Fix an integer $m > 0$ divisible by all the degrees of such a finitely selected system of generators of $\wp(E)$. For every integer $a \in \mathbb{Z}$ (positive, zero and negative) we will define $\mathcal{O}_Z$-submodule $\tilde{\wp}(E, -a) \subset Bl(Z, -a)$.

$$\tilde{\wp}(E, -a) = \textstyle\sum_{d \in \mathbb{Z}, dm \geq |a|} D(m, a, d) \tag{36}$$

$$D(m, a, d) = \mathrm{Diff}_Z^{(dm+a)} \wp^{posi}(E, dm)$$

When $a \geq 0$ we write $\wp^{nega}(E, -a)$ for $\tilde{\wp}(E, -a)$ to emphasize $-a \leq 0$. Write $\wp^{nega}(E)$ for $\bigoplus_{a>0} \wp^{nega}(E, -a)$.

In this definition the new notion is only for the case of $a \geq 0$. As a matter of fact for $a < 0$ we have the old parts as follows.

Lemma 5.2. *If $a < 0$ then $\tilde{\wp}(E,i) = \wp(E,i)$ with $i = -a$*

Proof. Pick any $d > 0$. Then

$$D(m,a,d) = \mathrm{Diff}_Z^{(dm-i)} \wp^{posi}(E,dm) \subset \wp(E,i)$$

by the differentiation property inside $\wp(E)$ thanks to Th.(4.2) of §3. Note that for d with $md = |a|$ we have $md + a = 0$ so that

$$D(m,a,d) = \mathcal{O}_{Z,\xi}\wp(E,-a) = \wp(E,i).$$

□

Lemma 5.3. *Consider the case of $a > 0$. The sum in the definition Eq.(36) is generated by finitely selected summands. For each integer $a > 0$ the $\wp^{nega}(E,-a)$ is a coherent $\rho^\ell(\mathcal{O}_Z)$-module for every sufficiently large positive integers ℓ.*

Proof. Strait forward by the noetherian $\mathcal{O}$-module of $Bl(Z,-a)$. The coherency is thanks to the Noetherean Zariski topology. □

Remark 5.4. It is very important to note that $\tilde{\wp}(E,0)$ is not always equal to $\wp(E,0)$ and it could be strictly smaller. The symbol $\tilde{\wp}(E,0)$ is to show its distinction from $\wp(E,0)$. We then define $\tilde{\wp}(E)$ to mean $\bigoplus_{i\in\mathbb{Z}} \tilde{\wp}(E,i)$.

Remark 5.5. In view of our main goal of this paper, that is the embedded resolution of singularities in $p > 0$, we often restrict our attention locally at a point $\xi \in \mathrm{Sing}(E)_{cl}$ with the following assumption.

$$\exists g \in \wp(E,m)_\xi, \qquad \mathrm{ord}_\xi(g) = m > 0 \tag{37}$$

where the m can be replaced by any of its positive multiple together with the corresponding power of g. Therefore we may assume that the same m can be chosen for both Eq.(37) and Def.(5.1). Incidentary we will later introduce the notion of maximum base-hiked $\hat{E}$ of E defined in the section §6. $\hat{E}$ will have an element g of Eq.(37) locally at every given point $\xi \in \mathrm{Sing}(\hat{E})$. Incidentally our proof of embedded resolution of singularities for E will be deduced from the same of all maximum base-hiked ideal exponents in the same ambient dimension. Refer to the later comment of Th.(16.4) of §16.1 which will be stated and proven later.

From now on throughout this section we will be assuming the existence of the element g in Eq.(37) at every $\xi \in \mathrm{Sing}(E)_{cl}$. With this assumption we have

Lemma 5.6. *The function $D(m,a,d)$ is monotone increasing with respect to d for any fixed pair (m,a) with $a>0$.*

Proof. We prove the statement for the case of $a>0$ locally at each $\xi \in \mathrm{Sing}(E)_{cl}$. Pick any pair of integers $d'>d$ and we then have the proof locally as follows.

$$
\begin{aligned}
D(m,a,d')_\xi &= \mathrm{Diff}^{(md'+a)}\wp^{posi}(E,md')_\xi \\
&\supset \\
&\mathrm{Diff}^{(md'+a)}\left(g^{d'-d}\wp^{posi}(E,md)_\xi\right) \\
&\supset \\
&\left(\mathrm{Diff}^{(md'-md)}g^{d'-d}\right)\left(\mathrm{Diff}^{md+a}\wp^{posi}(E,md)_\xi\right) \\
&= \mathcal{O}_{Z,\xi}\left(\mathrm{Diff}^{md+a}\wp^{posi}(E,md)\right)_\xi = D(m,a,d)_\xi
\end{aligned} \tag{38}
$$

where $\mathrm{Diff}^{(md'-md)}g^{d'-d} = \mathcal{O}_{Z,\xi}$ by the assumption on g. □

Remark 5.7. The $D(m,a,d)$ is monotone increasing with respect to d for $a>0$, and fixed (m,a). Hence we can redefine Eq.(36) for $a>0$ by $\cup$ instead of $\sum$ as follows.

$$
\begin{aligned}
&\tilde{\wp}(E,-a) = \wp^{nega}(E,-a) = \textstyle\bigcup_{d:dm>a} D(m,a,d) \\
&\quad\text{with } D(m,a,d) = \mathrm{Diff}_Z^{(dm+a)}\wp^{posi}(E,dm)
\end{aligned} \tag{39}
$$

The $D(m,a,d)$ are all coherent ideals in $Bl(Z,-a) \sim \mathcal{O}_Z$ and the noetherian property of $\mathcal{O}_Z$ implies that the union Eq.(39) is equal to:

$$
\wp^{nega}(E,-a) = D(m,a,d) = \mathrm{Diff}_Z^{(dm+a)}\wp^{posi}(E,dm) \tag{40}
$$

for every sufficiently large d with respect to each (a,m). We summarize the notation for the extended $\wp(E)$ with the tilde-symbols as follows.

$$
\tilde{\wp}(E) = \wp^{posi}(E) \bigoplus \wp(E,0) \bigoplus \wp^{nega}(E) \tag{41}
$$

where

$$
\begin{aligned}
\wp^{nega}(E) &= \bigoplus_{a>0} \wp^{nega}(E,-a); \\
\wp^{posi}(E) &= \bigoplus_{b>0} \wp(E,b); \qquad \text{and} \\
\wp(E,0) &\sim \mathcal{O}_Z
\end{aligned}
$$

Recall that $\wp(E) = \wp^{posi}(E) \bigoplus \wp(E,0)$. We write

$$
\tilde{\wp}(E) = \bigoplus_{i\in\mathbb{Z}} \tilde{\wp}(E,i).
$$

Note that $\wp^{nega}(E,-a)$, for $a>0$, is independent of the choice of m among those having the property described in Def.(5.1) for $D(m,a,d)$ because we can choose the common multiple for any two different m. Thanks to the monotone increase by this Remark it follows that the definition Eq.(36) is not affected at all when we replace m by any of its positive multiples.

Lemma 5.8. *For every pair of integers (i,j) we have*

$$\tilde{\wp}(E,i)\tilde{\wp}(E,j) \subset \tilde{\wp}(E,i+j)$$

Proof. Let $a=-i$ and $b=-j$ and the product $\tilde{\wp}(E,i)\tilde{\wp}(E,j)$ becomes

$$\begin{aligned} & D(m,a,d)D(m,b,d) \\ &= \mathrm{Diff}^{dm+a}\wp^{posi}(E,dm) \times \mathrm{Diff}^{dm+b}\wp^{posi}(E,dm) \\ &\subset \\ & \mathrm{Diff}^{2dm+a+b}\Big(\wp^{posi}(E,dm)\Big)^2 \\ &\subset \\ & \mathrm{Diff}^{2dm+a+b}\Big(\wp^{posi}(E,2dm)\Big) \\ &= \tilde{\wp}(E,-a-b) \end{aligned} \tag{42}$$

where $2dm \gg |a|+|b|$. □

Lemma 5.9. *We have the following inclusion relation among ideals:*

1. *For integers $0<a<b$ the ideal of $\wp^{nega}(E,-b)$ contains the ideal of $\wp^{nega}(E,-a)$.*
2. *The sum of the two ideals of (1) is contained in the ideal of $\wp^{nega}(E,-b)$.*
3. *For $a<0<b$ the ideal of $\wp^{nega}(E,-b)$ contains the ideal of $\wp^{nega}(E,-a)$.*
4. *The sum of the two ideals of (1) is contained in the ideal of $\wp^{nega}(E,-b)$ contains the ideal of $\wp^{posi}(E,-a)$.*

Proof. (1) is proven by $\mathrm{Diff}^{(\ell-a)} \supset \mathrm{Diff}^{(\ell-b)}$ for all $\ell \gg b$ and by the definition Eq.(39) of Rem.(5.7). (2) follows directly. (3) is proven in the same way as (1) by means of Diff's. □

Remark 5.10. It should be noted that, if $a+b$ is negative, the resulting $\tilde{\wp}(E,-a-b)$ is nothing but the old $\wp(E,a+b)$ by Lem.(5.2), while if $a+b$ is positive, then $\tilde{\wp}(E,-a-b)$ is a newly created member of the $\wp^{nega}(E)$. In other words, the product operation in the positive part of the extended algebra $\tilde{\wp}(E)$ of $\wp(E)$ is identical with the old $\wp^{posi}(E)$. The negative part of $\tilde{\wp}(E)$ is closed by product operation in itself.

Theorem 5.11. *Assume Eq.(37) at every point of* $\mathrm{Sing}(E)_{cl}$. *The* $\tilde{\wp}(E)$ *is a* $\mathcal{O}_Z$*-graded algebra whose positive part is identical to* $\wp^{posi}(E)$. *Moreover its homogeneous parts are all coherent* $\mathcal{O}_Z$*-module.*

Proof. By Lem.(5.6), $\wp^{nega}(E)$ is coherent. Lem.(5.2) is about

$$\wp^{posi}(E) \subset \tilde{\wp}(E)$$

and Lem.(5.8) for product inside $\tilde{\wp}(E)$. As for the equality of positive parts of $\tilde{\wp}(E)$ with $\wp(E)$ is by Lem.(5.2) with Lem.(5.8). □

6. Base-hikes and core-focusing of E

In our inductive strategy for resolution of singularity our initial question is to ask what would be the best definition of the *worst* singular points of an ideal exponent E. Our ultimate answer will become clear only after the theory of LLUED (local Leverage-up and Exponent-down) is fully developed in later sections. Here we begin with some preliminary notions and preparatory work. They are the following two steps preliminary modifications of the given $E = (J, b)$ with $J \neq (0)$.

6.1. **Maximum base-hike $\hat{E}$.** With $E = (J, b)$, $J \neq (0)$, let d be the $max(ord_\xi(J))$ for $\xi \in \mathrm{Sing}(E)_{cl}$ and define $\hat{E} = (J, d)$, called *maximum base-hike* of E. The $\wp(\hat{E})$ has non-trivial edge data as follows.

Theorem 6.1. *For every* $\xi \in \mathrm{Sing}(\hat{E})_{cl}$ *we have* $r = r(\xi) > 0$ *for* $\hat{E}$ *(not necessarily for* E*) in the expression of edge-data of §26 and edge decompositions by Th.(4.19). Moreover for every* $d > 0$ *and every* $\xi \in \mathrm{Sing}(\hat{E})_{cl}$ *we have* $d = ord_\xi(\wp(\hat{E}, d)$ *provided* $\wp(\hat{E}, d) \neq (0)$.

Proof. Note that $\hat{E}$ is equivalent to $(\prod_i \wp(\hat{E}, i), \sum i)$ where i ranges through all positive integers $i \leq b$ such that $\wp(\hat{E}, i) \neq (0)$. With the maximum $\mathrm{ord}_\xi(J) = d > 0$ in $\hat{E} = (J, d)$ we have a nonzero element in $\wp(J, d) \mod \max(\mathcal{O}_\xi)Bl_\xi(Z), \forall \xi$. Its differentiations produce nontrivial edge generators and edge data at ξ. □

Remark 6.2. Later in §(16.1)) we prove Th.(16.4), renamed *Theorem* Δ, which states that if we have the resolution for every maximum base-hiked ideal exponent in every ambient scheme having the same dimension as $n = \dim Z$, then we have the resolution of E itself. At this point we continue to develop the main concepts and key techniques needed to prove our main result: resolution of singularities in $p > 0$. Theorem Δ of §(16.1) is a kind of prelude to the *main theorems* ∇ *I-II-III* of §(16.2) and §(16.3). Some readers may wish to skip forward to Th.(16.4) and its proof in §(16.1) because it is independent and simpler to state in comparison with §(16.2) and §(16.3) which will command much heavier preparation.

6.2. **Core focusing $\check{E}$.** Let $\mathrm{Inv}_{\max}(\hat{E})$ denote the lexicographical maximum of $\mathrm{Inv}_\xi(\hat{E})$ for all $\xi \in \mathrm{Sing}(\hat{E})_{cl}$. We then have an ideal exponent $\check{E}$ naturally associated with $\hat{E}$ such that

$$\wp(\check{E}) \supset \wp(\hat{E}) \text{ and} \tag{43}$$
$$\mathrm{Sing}(\check{E})_{cl} = \{\xi \in \mathrm{Sing}(\hat{E})_{cl} \mid \mathrm{Inv}_\xi(\hat{E}) = \mathrm{Inv}_{\max}(\hat{E})\}$$

where $\wp(\check{E})$ is the maximum under these conditions. We can define the same $\check{E}$ as follows. With the maximum invariant

$$\mathrm{Inv}_{\max}(\hat{E}) = (n, n-r, q_1 \cdots, q_r)$$

we pick any $\check{E}$ such that

$$\mathfrak{S}(\check{E}) \;=\; \cap_{1\le i\le r}\, \mathfrak{S}\Big(\big(\wp(\hat{E}, q_i), q_i\big)\Big) \;\cap\; \mathfrak{S}(\hat{E}) \tag{44}$$

holds. The $\check{E}$ will be called the *core-edge focusing* of $\hat{E}$ (also that of E). We write and use the following symbol.

$$E \dashrightarrow \hat{E} \dashrightarrow \check{E} \tag{45}$$

for the composed process of core-edge focusing after maximum base-hike. From now on we will only be working with $\check{E}$ until we come to the last section of Part 1 in which we begin making use of blowups for the first time in order to prove Main Theorems I,II,III.

6.2.1. *Relation between $\wp^{nega}$ and $\wp^{posi}$.*

Theorem 6.3. *Assume that we have an element $g \in \wp(E,q)_\xi$ of Eq.(37) at a point $\xi \in \mathrm{Sing}(E)_{cl}$. We then assert that an element f with $0 \neq f \in \mathcal{O}_\xi$ belongs to $\wp^{nega}(E,-a)_\xi$ with $a \in \mathbb{Z}$ if and only if*

$$g^{p^\ell} f \in \wp(E, qp^\ell - a) \;\; for\ all\ \ell \gg 1. \tag{46}$$

Here the a could be either positive, zero or negative. Moreover if $h \in \wp^{nega}(E,-a)^{alg}$ then $hf \in \wp(E, d-a)$ for all $h \in \wp(E,d), d \ge a$

Proof. The only-if-part of the claim is immediate from the algebraic property (in particular the graded product rule) of $\tilde{E}$. In order to prove the if-part that is to prove $f \in \wp(E,-a)$ for the given $f \in Bl(Z,-a)$ it is better to generalise the question a little as follows. Namely pick any $h \in \wp(\hat{E},d), d \ge 0$, we assume the Eq.46 applied $hf \in Bl(Z, d-a)$ instead of f and prove $hf \in \wp(E, d-a)$ including the case in particular we have $d - a \ge 0$. This way we can assume that the addition of $f \in \wp(E,-a)^{alg}, a > 0$, to $\tilde{E}$ does not affect maintaining $\wp^{posi}(E)$ as it is. Now we assume $g^{p^\ell}hf \in \wp(E, qp^\ell + d - a), \forall \ell \gg 1$, and we want to prove $fh \in \wp(\hat{E}, d-a)$. We pick and fix a regular system of parameters x of $\mathcal{O}_\xi$ and recall the notations of elementary differential operators with respect to x in the sense of Def.(3) with Eq.(8). Pick any $\xi \in \mathrm{Sing}(E)_{cl}$ where we work locally. Let $K[[x]]$ be the local completion of $\mathcal{O}_\xi$ with a regular system of parameters. Let ∂^α be an elementary differential operator such that $|\alpha| = q$ and $\partial^\alpha g$ is a unit $u \in \mathcal{O}_\xi$. Consider the elementary differential operator $\partial(\ell) = \partial^{p^\ell \alpha}$. We then have $\partial(\ell)(\rho^\ell(X)) = \rho^\ell(\partial^\alpha X)$ for every $X \in \mathcal{O}$. Consider an elementally differential operator ∂^α such that $|\alpha| = q$ and $\partial^\alpha g$ is a unit in $K[[x]]$. For sufficiently large $\ell \gg ord_\xi(fh)$ we have

$$\partial^{(p^\ell\alpha)}\big(g^{p^\ell} fh\big) \equiv u^{p^\ell} fh \mod \max(\mathcal{O}_\xi)^{p^\ell} \tag{47}$$

In fact we have, for $(\partial^{(\alpha)})^{p^\ell}(g) = u^{p^\ell} fh$,

$$\partial^{(p^\ell\alpha)}(g^{p^\ell} fh) = \sum_{\beta \in \mathbb{Z}_0^n : \alpha \in \beta + \mathbb{Z}_0^n} \binom{\alpha}{\beta} (\partial^{(\beta)} g)^{p^\ell} \partial^{(p^\ell(\alpha-\beta))}(fh),$$

which is contained in $\max(\mathcal{O})^{p^\ell}$. We therefore have

$$\partial^{(p^\ell\alpha)}(g^{p^\ell} fh) = u^{p^\ell} fh + B \ with \ B \in \max(\mathcal{O})^{p^\ell}$$

Eq.(47) is now proven. On the other hand we have $g^{p^\ell} fh = (g^{p^\ell} f)h \in \wp(E, qp^{p^\ell} - a + d)$ by the assumption on (g, f) and by $h \in \wp(E, d)$. Hence $\partial^{(p^\ell\alpha)}(g^{p^\ell} fh) \in \wp(E, -a+d)$ Knowing that u is a unit we thus conclude that for all $\ell \gg 1$ we have

$$gh \in \wp(E, -a+d) + \max(\mathcal{O})^{p^\ell} \tag{48}$$

Now by taking the limit $\ell \to \infty$ we obtain $fh \in \wp(d-a)K[[s]]$ with the local completion $K[[s]]$ of $\mathcal{O}_\xi$. Since $K[[s]]$ is faithfully flat over $\mathcal{O}_\xi$ we conclude $fh \in \wp(d-a)_\xi$. □

Theorem 6.4. *Assume that $E = \hat{E}$, i.e, E is maximum base-hiked in the sense of §(6.1). We claim that the extended $\tilde{\wp}(\hat{E})$ is differentiation-closed, or diff-closed for short, in the sense that for every degree d, either positive or negative, we have $\partial(\wp(\hat{E}, d)) \subset \wp(\hat{E}, d-\mu)$ for every $\partial \in \mathrm{Diff}_Z^{(\mu)}$ with $\mu \geq 0$.*

Proof. By Th.(6.1) of §(6.1) we have some g at any given point $\xi \in \mathrm{Sing}(E)_{cl}$. Now if $f \in \wp(\hat{E}, -a)_\xi, a > 0$, then $g^{p^\ell} f \in \wp(\hat{E}, qp^\ell - a)_\xi$ for all $\ell \gg 1$. Then for any given $\partial \in \mathrm{Diff}_Z^{(\mu)}$ and for any $p^\ell \gg q\mu$ we have $\partial(g^{p^\ell} f) = g^{p^\ell}\partial(f)$ belongs $\wp(\hat{E}, qp^\ell - a - \mu)_\xi$. We then have

$\partial(f) \in \wp(\hat{E}, -a-\mu)_\xi$ thanks to Th.(6.3). This works for any a, positive or negative. Refer to the three key theorems of [24]. □

Theorem 6.5. *Let g be an element of $\wp(\hat{E}, q)_V$ with $\mathrm{ord}_\xi(g) = q \geq a > 0$ where $V \ni \xi$ is an open neighborhood to which g extends with the same property. We then assert that*

$$\wp^{nega}(\hat{E}, -a) \subset Ker(\sigma : \mathcal{O}_V \to \wp(\hat{E}, q)/\big(\wp(\hat{E}, q-a) \cap \wp(\hat{E}, q)\big) \tag{49}$$

where σ is the $\mathcal{O}_V$-homomorphism which maps every $f \in \mathcal{O}_V$ to the class of gf in the target quotient module.

Proof. In fact we have that if $f \in Ker$ then $gf \in \wp(\hat{E}, q-a)_V \cap \wp(\hat{E}, q)_V$ which is equivalent to $gf \in \wp(\hat{E}, q-a)_V$. Hence $g^{p^\ell} f = g^{p^\ell - 1} gf \in \wp(\hat{E}, q(p^\ell - 1) + q - a)$ for all $\ell \gg a$. We obtain $f \in \wp(\hat{E}, -a)$. □

Theorem 6.6. *Assume that $E = \check{E}$ in the sense of §6.2. Then $\wp^{nega}(E, -a)$ is coherent $\mathcal{O}$-module for all integers $a > 0$.*

Proof. Immediate by Th.(6.5). □

Theorem 6.7. *Consider a pair of ideal exponents E and F such that $\wp(E) \supset \wp(F)$ so that $\wp(\check{E}) \supset \wp(\check{F})$. Assume that there exists some $g \in \wp(F, q)_\xi$ of Eq.(37) at every point $\xi \in \mathrm{Sing}(F)_{cl}$, We then have $\wp^{nega}(E) \supset \wp^{nega}(F)$. For instance if $F = \hat{F}$ then the same is true.*

Proof. The proof is straight-forward by the definition of $\wp^{nega}$ in Eq.(36) of Def.(5.1). □

Definition 6.8. Notation $x \supset y$ will be called an allowable coordinate system for $\check{E}$ at a point $\xi \in \mathrm{Sing}(\check{E})_{cl}$ if x is a regular system of parameters of $\mathcal{O}_{Z,\xi}$ such that y is member of x (usually the first component) and is also a generator of $\cot(\check{E})_\xi$.

Remark 6.9. The LL-head module is generated by cotangent parameters y extendable to a regular system of parameters $x \supset y$ of $\mathcal{O}_\xi$. The LL-head module a $\mathfrak{L}(\check{E})$ is marked with an integer $\ell > 0$ called its depth.

Definition 6.10. Firstly we define the coherent ideal $L(\check{E})$

$$L(\check{E}) = \sum_{j>0} \{f \in \wp(\check{E}, j) : \mathrm{Diff}_Z^{(j)} f \subset \wp(\check{E}, j)\} \tag{50}$$

and the cotangent algebra of $\wp(\check{E})$

$$\cot \wp(\check{E}) = \wp(\check{E})/L(\check{E}). \tag{51}$$

It should be noted that the fibre of $\cot \wp(\check{E})$ at each $\xi \in \mathrm{Sing}(\check{E})_{cl}$ is the polynomial algebra $\kappa(\xi)[\bar{g}(\xi)_i, 1 \le i \le r(\xi)]$ where $\bar{g}(\xi)$ denotes the system of the residue classes of the edge generators of $\wp(\check{E})_\xi$. Refer to the module of LL-heads $\mathfrak{L}(\check{E})$ below.

6.3. **Coherent LL-heads module $\mathfrak{L}(\check{E})$.** Recall Eq.(31) of Def.(4.18) after Eq.(29) of §4.2 about $\cot(\check{E}, q)$ and $\cot(\check{E})$.

Theorem 6.11. *Pick any element $g = y^q + \epsilon \in \wp(\check{E}, q)_\xi$ with $q = p^e, e > 0$, where $\xi \in \mathrm{Sing}(\check{E})_{cl}$. Assume that $\mathrm{ord}_\xi(y) = 1$ and $\mathrm{ord}_\xi(\epsilon) > q$. Then $y \in \cot(\check{E})$. Conversely every element $y \in \cot(\check{E})$ with $\mathrm{ord}_\xi(y) = 1$ we find g and ϵ satisfying the assumption of Th.(6.11).*

Proof. By the equation Eq.(29) the assumption implies $y \in \cot(\check{E})$. The converse is also true because of the same equation. □

Definition 6.12. Let V be an open subset of Z and consider integers $\ell \gg 1$. Then an LL-head of order q of $\check{E}$ along $\mathrm{Sing}(\check{E}) \cap V$ means an element $g \in \wp(\check{E}, q)(V)$ having the following property.

$$(52) \qquad \begin{gathered} g = y^q + \epsilon \in \wp(\check{E}, q)(V) \quad \text{for } q = p^e, e > 0, \\ \mathrm{ord}_\eta(y) = 1 \textit{ and } \mathrm{ord}_\eta(\epsilon) > q \quad \text{for all } \eta \in \mathrm{Sing}(\check{E}) \cap V_{cl}. \end{gathered}$$

We choose $\ell \gg q$ so that the set of all LL-heads g as above form an $\rho^\ell(\mathcal{O})_V$-module.

We let $\mathfrak{L}(\check{E}, q, \ell)_V$ denote the $\rho^\ell(\mathcal{O})_V$-module generated by all LL-heads of order q of $\check{E}$ along $\mathrm{Sing}(\check{E})_V$. The ℓ is called the depth of $\mathfrak{L}(\check{E}, q, \ell)_V$. It should be noted that with every sufficiently large integer ℓ we have a global coherent $\rho^\ell(\mathcal{O})$-module

$$(53) \qquad \mathfrak{L}(\check{E}, \ell) = \bigoplus_{q(j)=p^j, j \in \mathbb{Z}_0} \mathfrak{L}(\check{E}, q, \ell)$$

Locally a typical example of $g \in \mathfrak{L}(\check{E})_\xi$ is any of the edge generators of $\wp(\check{E})$ at ξ. Refer to §4.1. We write $\bar{q}$ for the maximum $q_r(\xi)$ among the powers of p in the Eq.(28) of §4.1. Then the numbers ℓ of Def.(6.12) must have $\ell > \bar{q}$ at least and often bigger as we see later with reference to the notion of LL-chains in §9.8 later. Another notion (weaker in general) called LL-prehead is introduced in Eq.(54) below. Note that the core-edge focusing is essential for the LL-heads to be non-trivial.

$$(54) \qquad \begin{gathered} L(\check{E}, q)_\xi = \wp(\check{E}, q) \cap \max(\mathcal{O}_\xi) bl_\xi(Z) = \\ \{ f \in \wp(\check{E}, q)_\xi \mid \mathrm{Diff}_Z^{(k)} f \subset bl_\xi(Z, q - k + 1), \forall k < q \} \end{gathered}$$

We write $L(\check{E})_\xi = \bigoplus_{j>0} L(\check{E}, q(j))$ with $q(j) = p^j$ and $L(\check{E}, q(j))$ is a $\rho^\ell(\mathcal{O})$-module for every $\ell > j$. Its member is called LL-prehead of degree $q(j)$ of $\check{E}$.

Remark 6.13. Note that we have

$$\mathfrak{L}(\check{E}, q(j)) \subset L(\check{E}, q(j)) \text{ for } \forall j \tag{55}$$
$$g \in \mathfrak{L}(\check{E}, q(j)) \Leftrightarrow g \in L(\check{E}, q(j)) \text{ and } ord_\xi(g) = q(j)$$

and

$$g \in \mathfrak{L}(\check{E}, q(j)) \text{ and } f \in L(\check{E}, q(j)) \setminus \mathfrak{L}(\check{E}, q(j)) \Rightarrow g + f \in \mathfrak{L}(\check{E}, q(j)).$$

Theorem 6.14. *For every point $\xi \in \mathrm{Sing}(\check{E})$ and for every positive integer $\ell(0) > 0$ there exists an open affine neighborhood V of $\xi \in Z$ such that for every $\ell \geq \ell(0)$*

(1) we have a finite system $g* = (g_1, \cdots, g_r)$ with $g_i \in \mathcal{O}(V)$ for all i and $g*$ is a system of edge generators of $\wp(\check{E})$ at every point $\eta \in V_{cl} \cap \mathrm{Sing}(\check{E})$ in the sense of Eq.(26). (For this we need the assumption that $\check{E}$ is core-edge focused.)

(2) We have a finite system of homogeneous elements generating

$$I(\mathrm{Sing}(\check{E}))Bl(Z) \cap \wp^{posi}(\check{E}) \textit{ that is} \tag{56}$$
$$\cap_{\eta \in \mathrm{Sing}(\check{E})_{cl}} \max(\mathcal{O}_\eta) bl_\eta(Z) \wp(\check{E})$$

which is an ideal in the noetherian graded algebra $\wp(\check{E})$.

where $I(\mathrm{Sing}(\check{E}))$ denotes the ideal of $\mathrm{Sing}(\check{E}) \subset Z)$. It should be noted that $\mathrm{Sing}(\check{E})$ is viewed as a reduced subscheme of Z and $I(\mathrm{Sing}(\check{E}))$ is a radical ideal.

(3) The Eq.(56) is an ideal in the noetherian graded $\rho^\ell(\mathcal{O}(V))$-algebra and hence it is generated by finitely many homogeneous elements $\{f_j\}$ as $\rho^\ell(bl_\zeta(V))$-module.

(4) The finite generation of $I(\mathrm{Sing}(\check{E}))Bl(Z) \cap \wp^{posi}(\check{E})$ as above should be understood as $\rho^\ell(Bl(Z))$-module with every $\ell \geq \ell(0)$.

Thus for every $\zeta \in V_{cl} \cap \mathrm{Sing}(\check{E})$ and for every $g \in \mathfrak{L}(\check{E})_\zeta$ we write $g = g' + \sum_j B_j f_j$ with $g' \in \rho^\ell(\mathcal{O})(V)[g^*]$ and $B_j \in \rho^\ell(bl_\zeta(V))$.

Proof. Follows immediately from Eq.(52), Eq.(54) and Eq.(55) □

Definition 6.15. A paired system $(g*; f)$ of Th.(6.14) is called a set of *edge-inner generators* of $\mathfrak{L}(\check{E})_V$ in an open affine V and with depth ℓ.

7. Permissible transformations and effects on $\wp(E)$

Given $E = (J, b)$ furnished with $\wp(E)$ and $\mathfrak{S}(E)$ in the ambient scheme Z we pick a point $\xi \in \mathrm{Sing}(E)_{cl}$ and choose an edge decomposition of E locally at ξ by Th.(4.19). We then define and examine the transforms of the edge data of $\wp(E)$ by permissible blowups for E and elucidate the changes in the edge invariants $\mathrm{Inv}_\xi(E)$ of Eq.(34).

7.1. **Effects on edge data of $\wp(E)$.** Consider a blowup $\pi : Z' \longrightarrow Z$ with center $D \ni \xi$ permissible for E. Write $I = I(D, Z)$, the ideal of $D \subset Z$. Note that π is then automatically permissible for every $E_i = (g_i R_\xi, q_i), 1 \le i \le r$, of the local edge decomposition of Th.(4.19).

Theorem 7.1. *We assert that the edge invariants never increase in the sense of lexcographical ordering. To be precise we pick any point $\xi' \in \pi^{-1}(\xi) \cap \mathrm{Sing}(E')_{cl}$ with the transform E' of E by π so that $\xi' \in \pi^{-1}(\xi) \cap \left(\cap_{1\le i\le r}\mathrm{Sing}(E_i')\right)$. We then have $\mathrm{Inv}_{\xi'}(E') \le_{lex} \mathrm{Inv}_\xi(E)$. (Refer Eq.(34).)*

Theorem 7.2. *Let $I = I(D, Z)$, the ideal of D in $\mathcal{O}_Z$. Pick any ξ' of Th.(7.1) and any exceptional parameter $\mathfrak{z}$ for π at ξ', i.e, $\mathfrak{z} \in I(D, Z)_\xi$ and $\mathfrak{z}\mathcal{O}_{Z',\xi'} = I(\pi^{-1}(D), Z')_{\xi'}$. We then assert:*

1. *There exists $c_i \in \mathcal{O}_{Z,\xi}$ such that $\mathfrak{y}_i = y_i - c_i \in I_\xi$ and $y_i' = \mathfrak{z}^{-1}\mathfrak{y}_i \in M_{\xi'}$ for all i. Moreover $(y', \mathfrak{z})$ is extendable to a regular system of parameters of $\mathcal{O}_{Z',\xi'}$ where $y' = (y_1', \cdots, y_r')$.*
2. *If $\mathrm{Inv}_{\xi'}(E') = \mathrm{Inv}_\xi(E)$ we then have $\mathrm{ord}_{\xi'}(g_i') = q_i$ with $g_i' = \mathfrak{z}^{-q_i} g_i$ for all i and moreover*
3. *the pair (g', y') with $g' = (g_1', \cdots, g_r')$ is an edge data yielding an edge decomposition of E' locally at ξ' as follows.*

$$\mathfrak{S}(E') = \left(\bigcap_{i=1}^{r} \mathfrak{S}(E_i')\right) \bigcap \mathfrak{S}(F') \tag{57}$$

where $E_i' = (g_i'\mathcal{O}_{Z',\xi'}, q_i), 1 \le i \le r$, and F' is the transform of F by π at ξ'. Refer to Th.(4.19) of (4.3).

The lemmas below are stated and proven in such a manner that proofs of the theorems above are all therein included. For notational simplicity we write R_ξ or R for $\mathcal{O}_{Z,\xi}$ and M_ξ or M for $\max(\mathcal{O}_{Z,\xi})$.

Lemma 7.3. *The permissibility of π implies that the ideal I of the center contains $y_i - c_i$, denoted by $\mathfrak{y}_i$, with $c_i \in M_\xi^2$ for every i.*

Proof. The permissibility implies $g_i = y_i^{q_i} - \epsilon_i \in {I_\xi}^{q_i}$ and with $\mathrm{ord}_\xi(\epsilon_i) > q_i$. Hence $\epsilon_i \mod I_\xi$ is a q_i-th power in $\bar{R} = R_\xi / I_\xi$. It is also in

$\max(\bar{R})^{q_i+1}$ because $\epsilon_i \in M_\xi{}^{q_i+1}$. Therefore ϵ_i mod I_ξ must be in $\max(\bar{R})^{2q_i}$ from which there follows the existence of c_i. □

Lemma 7.4. $(\mathfrak{y}_1, \cdots, \mathfrak{y}_r, \mathfrak{z})$ *extends to a basis of the ideal I as well as to a regular system of parameters of R_ξ. We can choose those c_i of Lem.(7.3) so that $\mathfrak{z}^{-1}\mathfrak{y}_i \in M_{\xi'} = \max(\mathcal{O}_{Z',\xi'})$ for all i. For instance if $\kappa_\xi = \kappa_{\xi'}$ we can choose $a_i \in R_\xi$ which is congruent to $\mathfrak{z}^{-1}\mathfrak{y}_i$ modulo $M_{\xi'}$ and then replace c_i by $c_i + a_i\mathfrak{z}$. By choosing this instead of c_i of Lem.(7.3) we gain the claimed property of $\mathfrak{z}^{-1}\mathfrak{y}_i, 1 \le i \le r$.*

Proof. The first claim is immediate from Lem.(7.3). Let us write $g_i = \mathfrak{y}_i^{q_i} + \epsilon_i^*$. Then we must have ϵ_i^* in $I_\xi{}^{q_i} \cap M_\xi^{q_i+1} = M_\xi\, I_\xi^{q_i}$. Now pick an exceptional $\mathfrak{z} \in I_\xi$ for π at ξ'. Then $\mathfrak{z}^{-q_i}\epsilon_i^* \in M_{\xi'}$ and hence $\mathfrak{z}^{-q_i}\mathfrak{y}_i^{q_i} \in M_{\xi'}$ by the assumption $\xi' \in \mathrm{Sing}(E_i')_{cl}$. We conclude $\mathfrak{z}^{-1}\mathfrak{y}_i \in M_{\xi'}$ for all i. This proves Lem.(7.4) as well as the following Lem.(7.5). □

Lemma 7.5. *After replacing $\mathfrak{y}$ by Lem.(7.4) we obtain the new edge parameters $y' = (\mathfrak{z}^{-1}\mathfrak{y}_1, \cdots, \mathfrak{z}^{-1}\mathfrak{y}_r, \mathfrak{z})$ extendable to a regular system of parameters of $R_{\xi'} = \mathcal{O}_{Z',\xi'}$.*

Lemma 7.6. *If $\mathrm{Inv}_{\xi'}(E') \ge_{lex} \mathrm{Inv}_\xi(E)$ then $\mathrm{Inv}_{\xi'}(E') = \mathrm{Inv}_\xi(E)$ and the transform E_i' is $(g_i'\mathcal{O}_{Z',\xi'}, q_i)$ with $g_i' = \mathfrak{z}^{-q_i} g_i$ for every i. Also the transform E' of E by π has an edge decomposition as follows.*

$$\mathfrak{S}(E') \;=\; \Big(\bigcap_{i=1}^{r} \mathfrak{S}(E_i') \Big) \bigcap \mathfrak{S}(F') \tag{58}$$

within a neighborhood of ξ' in Z'. cf.(4.3)

Proof. The assumption $\xi' \in \mathrm{Sing}(E')_{cl}$ implies $\xi' \in \mathrm{Sing}(E_i')_{cl}$ and hence $\mathrm{ord}_{\xi'}(g_i') = ord_\xi(g_i)$ for every i. Hence the assumption on the edge invariants at ξ' implies that $g_i', 1 \le i \le r$, are enough to generate the edge data of $\wp(E')$ at ξ'. Thus Eq.(57) and the Inv equality. □

7.2. **The rule of transformation of $\wp^{nega}$.**

Remark 7.7. The rule of transformation for $\wp^{naga}(E)$ is deduced by the known rule for $\wp^{posi}(E)$ thanks to the product rule of Lem.(5.8) for $\tilde{\wp}(E)$. Geometrically speaking the transform of $\wp^{nega}(E, -a)$ is adding a-times exceptional divisor while the transform of $\wp^{nega}(E, +a)$ is deleting a-times exceptional divisor after taking the pull-back by blowup. To be precise algebraically we have the following theorem.

In the following theorem we are assuming Eq.(37) at every point in $\mathrm{Sing}(E)_{cl}$. Indeed this condition is always true if $E = \hat{E}$ and/or $E = \check{E}$. As for the effect of permissible transformation upon the Inv-invaiants we have a good reference with Lem.(7.3), Lem.(7.4), Lem.(7.5), Lem.(7.6) following after Th.(7.1).

Theorem 7.8. *For every finite sequence of local blowups $\pi : Z' \to Z$ with centers $D_i, 0 \le i \le r$, permissible for the successive transforms $E_i \in Z_i$ from $E = E_0 \in Z = Z_0$ up to $E' = E_r \in Z' = Z_r$ in the sense of Def.(2.4). We then claim that for every point $\xi' \in \pi^{-1}(\xi) \cap \mathrm{Sing}(E'_{cl})_{cl}$ with $\mathrm{Inv}_{\xi'}(E') = \mathrm{Inv}_{\xi}(E)$ in the sense of Eq.(34) we have*

$$\wp^{nega}(E', -a)_{\xi'} = \Big(\prod_i I(D_i, Z_i)^a\Big)\Big(\wp^{nega}(E, -a)\mathcal{O}_{Z'}\Big)_{\xi'} \tag{59}$$

where $I(D_i, Z_i)$ is the ideal of D_i in $\mathcal{O}_{Z_i}$. The integer a may be any but the important cases are for $a > 0$. The cases for $i = -a > 0$ are direct from the definition of $\wp^{posi}(E)$.

Proof. A proof of Eq.(59) can be done by strait application of product rule Lem.(5.8) and by the known rule of transformation for each homogeneous parts of $\wp^{posi}(E)$. □

Remark 7.9. Here is a more direct argument. We have $\wp^{nega}(E, -a) = \mathrm{Diff}^{(md+a)}\wp(E, md)$ for $d \gg a$ in the manner of Def.(5.1) supported by Lem.(5.6) and Lem.(5.8). We also have $(\rho^\ell(g))(\wp^{nega}(E, -a)) \subset \wp^{posi}(E, mp^\ell - a)$. The $I(D, Z)\mathcal{O}_{Z'} = I(D')$ is the the ideal of the exceptional divisor $D' = \pi^{-1}(D)$ in Z'. This ideal is locally principal. Choose a generator z of the localized $I(D')\mathcal{O}_{\xi'}$ Now by the permissible π we have the blown-up results at ξ' in all details as follows.

$$\begin{aligned}
(60) \qquad & \wp^{nega}(E, -a)\mathcal{O}_{\xi'} \ \ \textit{(pulback by } \pi\textit{)} \\
= \ & g^{-p^\ell}\wp^{posi}(E, mp^\ell - a)\mathcal{O}_{\xi'} \ \ \textit{(degree up by multiplier)} \\
= \ & z^{-mp^\ell} z^{mp^\ell - a}\wp^{posi}(E', mp^\ell - a)\mathcal{O}_{\xi'} \ \textit{(rule for 'posi' case)} \\
= \ & z^{-a}\wp^{posi}(E', mp^\ell - a)\mathcal{O}_{\xi'} \\
= \ & z^{-a}(g')^{p^\ell}\wp^{nega}(E', -a)_{\xi'} \ \textit{(degree down by division)} \\
= \ & z^{-a}\wp^{nega}(E', -a)_{\xi'} \ \textit{(unicity of } g' = z^{-m}g \textit{ at } \xi'\textit{)} \\
= \ & I(D, Z)^{-a}\big(\wp^{nega}(E', -a)\big)_{\xi'}
\end{aligned}$$

With this for all ξ' there follows Eq.(59).

We have the following technical but very useful lemma.

Lemma 7.10. *Let $h \in BL_\xi(Z, -k)$ with $k \in \mathbb{Z}$ and assume that we have the inclusion of Eq.(62) for all $\ell \gg 1$. Then we claim $h \in \tilde{\wp}(E, -k)$. Note that in this lemma the number k can be positive, negative or zero.*

Proof. Pick and fix a regular system of parameter x of $\mathcal{O}_{Z,\xi}$. We then have a differential operator $\partial \in \mathrm{Diff}^{(m)}$ such that $\partial(h) = 1$. In fact by the condition on g of Eq.(37) we can find an elementary differential operator δ in x of degree m such that $u = \delta g$ is a unit. Then let $\partial = u^{-1}\delta$. Then let σ be the differential operator in $\rho^\ell(\mathcal{O}_\xi)$ which is $\sigma = \rho^\ell \partial \rho^{-\ell}$. Then let $\Delta(\ell) \in \mathrm{Diff}^{(mp^\ell)}$ be the x-Cartier extension of σ in the sense of Def.(3.7) supported by Lem.(3.6). Now let $K[[x]]$ denote the local completion $\hat{\mathcal{O}}_\xi$ where K is the algebraic closure of $\mathbb{K}$ inside the completion. Let us write

$$h = \textstyle\sum_\beta c_\beta x^\beta \text{ and } h(\ell) = \sum_{\beta:\beta_i<\ell,\forall i} c_\beta x^\beta \qquad (61)$$
$$\text{where } c_\beta \in K, \forall \beta \in {\mathbb{Z}_0}^n$$

Recall that every differential operator in $\mathcal{O}_\xi$ has its unique extension to such in $K[[x]]$. By the x-Cartier property we have $\Delta(h) \in \rho^\ell(\max(K[[x]])$. Remember that $\Delta(\rho^\ell(g)) = 1$ while $\Delta(x^\beta) = 0$ every β with $0 \le \beta_j < p^\ell, \forall j$. Therefore $\Delta(g^\ell x^\beta) = x^\beta$ and $h - \Delta(g^\ell h) \in (x)^\ell K[[x]]$ for every $\ell \gg 1$ where $\Delta(g^\ell h) \in \wp^{nega}(E, -k)$. Hence $h \in \wp^{nega}(E, -k) + (x)^\ell K[[x]], \forall \ell \gg 1$ whch implies $h \in \wp^{nega}(E, -k)K[[x]]$. This implies $h \in \wp^{nega}(E, -k)$ because $K[[x]]$ is faithfully flat over the local ring $\mathcal{O}$. □

Theorem 7.11. *Under the assumption of Th.(7.8) we have the following facts.*

(1) $\tilde{\wp}(E)$ is diff-closed in the sense that for every integer $j \in \mathbb{Z}, \neq 0$, we have $\mathrm{Diff}_Z^{(c)}\tilde{\wp}(E, j) \subset \tilde{\wp}(E, j - c)$ for every integer $c > 0$

(2) $\tilde{\wp}(E)$ is integrally closed in the graded algebra $Bl(Z)$

Proof. As for (1) if $j > 0$ then $\tilde{\wp}(E, j) = \wp^{posi}(E, j)$ and hence the inclusion is either by the property of $\wp(E)$ with $c \le j$ or by the definition Eq.(36) of $\tilde{\wp}(E, j - c)$ with $c > j$. If $j < 0$ let $a = -j$ and $\tilde{\wp}(E, j) = \mathrm{Diff}^{(md+a)}\wp^{posi}(E, md)$ with $d \gg 1$ by Eq.(40). Hence $\mathrm{Diff}^{(c)}\tilde{\wp}(E, j)$ is $\mathrm{Diff}^{(c)}\mathrm{Diff}^{(md)+a}\wp^{posi}(E, md)$ which is contained in $\tilde{\wp}(E, -a - c)$ by definition. Here $-a - c = j - c$. As for (2) it is enough to examine the case of negative integration thanks to Th.(5.11). Pick any $h \in BL_\xi(Z, -k), k > 0$, which is integral over $\tilde{\wp}(E)$. In view of the mutual independence of homogeneous parts we must have an equation $h^\alpha + \sum_{0<\beta\le\alpha} f_\beta h^{\alpha-\beta} = 0$ with $f_\beta \in \wp^{nega}(E, \beta)$. Then by choosing $\ell \gg 1$ we examine the multiplication of the equation above by $(\rho^\ell(g))^\alpha$. Namely

$$(\rho^\ell(g))h)^\alpha + \sum_{0<\beta\le\alpha} (\rho^\ell(g))^\beta f_\beta)(\rho^\ell(g))h)^{\alpha-\beta} = 0$$

with g and m of the assumption Eq.(37). Note that we then have $(\rho^\ell(g))^\beta f_\beta \in \wp^{posi}(E, mp^\ell\beta)$ for all β so that $(\rho^\ell(g))h$ is integral over $wp^{posi}(E)$ and hence

$$(\rho^\ell(g))h \in \wp^{posi}(E, mp^\ell - k) \text{ with } g \textit{ of } Eq.(37) \tag{62}$$

thanks to the integral closed $\wp(E)$. Note that Eq.(62) is valid for every choice of $\ell \gg 1$. Now thanks to Lem.(7.10) we conclude $h \in \wp^{nega}(E, -k)$. Th.(7.11) is thus proven. □

7.3. **Rule of transformation for $\wp^{nega}$.** On the rule of permissible transformation on $\wp^{nega}(E)$ here is an important addition to Th.(7.8) again due to Eq.(83) and Eq.(84).

Theorem 7.12. *Let $\pi : Z' \to Z$ with center D be permissible for E and let E' be the transform of E by π. Pick a point $\xi' \in \mathrm{Sing}(E')_{cl}$ with $\xi = \pi(\xi') \in \mathrm{Sing}(E)$. Assume that we have a nonzero edge generator g' of E' at ξ' which is the transform of an edge generator g of E at ξ. Then locally at ξ' we have that $\wp^{nega}(E')$ contains the transform $\wp^{nega}(E)'$ of $\wp^{nega}(E)$ by π.*

Proof. The proof is straight forward by Th.(6.3). In fact for every element $f \in \wp(E, -a)$ with $a > 0$ we have $g^{p^\ell} f \in \wp(\hat{E}, qp^\ell - a)$. Then $g'^{p^\ell} f' \in \wp(E', qp^\ell - a)$ with the transform f' of f by π. □

7.4. **Coordinate translation in $p > 0$.**

Definition 7.13. Let V be an open set in Z and let $x = (x_1, \cdots, x_n)$ be a system of elements in $\mathcal{O}(V)$. We call x a coordinate system of V if for even point $\xi \in V_{cl} \cap \mathrm{Sing}(\check{E})$ we find a regular system of parameters $z = (z_1, \cdots z_n)$ such that the differentiation matrix of x with respect to z is invertible in the local ring $\mathcal{O}_\xi$.

We consider a regular system of parameters $x = (y, \omega)$ at $\xi \in \mathrm{Sing}(\check{E})_{cl}$ which is naturally extended to regular parameters on a sufficiently small affine neighborhood $V \ni \xi$ in Z. For each point $\eta \in V \cap \mathrm{Sing}(\check{E})_{cl}$ we then have the local completion $\hat{\mathcal{O}}_\eta = K(\eta)[[x - a]]$ with $a \in K(\eta)^n$ where $K(\eta)$ is the algebraic closure of the base field $\mathbb{K}$ inside $\hat{\mathcal{O}}_\eta$. This fact is due to the assumption that $\mathbb{K}$ is a perfect field and κ_η is separable algebraic over $\mathbb{K}$. It should be noted here that $(x - a)$ is a regular system of parameters in $K(\eta)[[x - a]]$ but not necessarily in $\mathcal{O}_\eta$. However all elementary differential operators in x make sense as differential operators in $K(\eta)[[x - a]]$ which coincide with the elementally differential operators with respect to $x - a$ in $K(\eta)[[x - a]]$. Now we want to obtain a regular system of parameters of $\mathcal{O}_\eta$ which are approximations of $x-a$ well suited to our work related to LLED-operations of later sections such as §9.

Lemma 7.14. *Given $\eta \in V_{cl}$ and a unit $f \in \mathcal{O}_\eta$ we have $f_N \in \mathcal{O}_\eta$ for every integer $N \gg 1$ such that $f \equiv \rho^N(f_N) \mod \max(\mathcal{O}_\eta)$. It follows that in a situation of the formal translation $x-a$ with $a \in K^n$ we can find $a_N \in {\mathcal{O}_\eta}^n$ such that $a - \rho^N(a_N) \in \max(K[[x-a]])^n$. We will use $x-\rho^N(a_N)$ in ${\mathcal{O}_\eta}^n$ as a useful approximation of $x-a$ in later sections.*

Proof. Given f we let $\kappa \in \mathbb{K}$ be the residue of f at η. Then For $k = \sqrt[p^N]{\kappa} \in \mathbb{K}$ and pick $f_N \in \mathcal{O}_\eta$ whose residue class at η is k. □

Corollary 7.15. *Under the assumption of Lem.(7.14) and given any finite number of differential operators $\partial_j, 1 \le j \le s$ in Diff_V we have $N \gg 1$ such that for every monomial x^α we have $\partial_j\big((x-\rho^N(a_N))^\alpha\big) = \partial_j(x^\alpha)$ for every j.*

The proof of this corollary is immediate from Lem.(7.14). It should be noted that $x-\rho^N(a_N)$ with $N \gg 1$ is a good rational substitute for the formal translation $x-a$ with a in the algebraic closure of the base field $\mathbb{K}$. It is useful in many of the consequent sections. In particular refer to those differentiations later in §9.5 which will be indispensable for the proofs of the LLUED theorems stated in Th.(9.2) and (9.3).

Definition 7.16. We call $x-\rho^N(a_N)$ a regular translation of x with depth N at η.

Theorem 7.17. *For any coherent $\rho^\ell(\mathcal{O})$-submodule W of $\wp^{nega}(\check{E}, -a)$ with $\ell \gg a$ and for a point $\xi \in \mathrm{Sing}(\check{E})_{cl}$, if $W(\xi) = (0)$ then there exists an open affine neighborhood V of $\xi \in Z$ such that $W(\eta) = (0)$ for every $\eta \in \mathrm{Sing}(\check{E}) \cap V_{cl}$.*

Proof. Note that we have a finite number of differentiations $\{\partial f\}$ which locally generate $\wp^{nega}(\check{E}, -a)_\xi$ as $\rho^\ell(\mathcal{O})$-module within a small enough affine open neighborhood V of $\xi^i nZ$. Then for each $\eta \in \mathrm{Sing}(\check{E}) \cap V_{cl}$ we apply the translation from ξ to η such as $x-\rho^N(a_N)$ for a chosen coordinate system x in V in the manner of Lem.(7.14). □

7.5. **Cleaning by edge equation $g = y^q + \epsilon$.** The self-cleaning of g or g-cleaning of ϵ is defined first formally in the local completion $\hat{\mathcal{O}}_\xi$ and then by its approximation in $\mathcal{O}$ modulo $\rho^N(\mathcal{O})$ with a sufficiently large integer $N \gg (1+e)n$ with $q = p^e$ and $n = dim\, Z$.

Choose a regular system of parameters $x = (y, \omega)$ of $\mathcal{O}_\xi$. It is important that x contains y of the given LL-head $g = y^q + \epsilon$. Let us write the local completion $\hat{\mathcal{O}}_\xi = K[[y, \omega]]$ of the unique algebraic extension K of the base field $\mathbb{K}$.

Then we have $\hat{\mathcal{O}}_\xi = \sum_{0\le j<q} y^j K[[g,\omega]]$ thanks to the assumption that $\mathrm{ord}_\xi(\epsilon) > q$ by the definition of $\mathfrak{L}_\xi$. Hence we can write

$$\epsilon = \sum_{k\ge 0} g^k \big(\sum_{0\le j<q} y^j \sigma(k,j)\big) \text{ with } \sigma(k,j) \in K[[\omega]], \forall (k,j). \tag{63}$$

Definition 7.18. Define the formal *g-cleaning* of ϵ to be

$$\epsilon^\sharp(\infty) \;=\; \sum_{0\le j<q} y^j \sigma(0,j), \tag{64}$$

not necessarily any element of $\mathcal{O}_\xi$. We rather look for its approximations by elements of $\mathcal{O}_\xi$ which help to prove global resolution of singularities (not merely the local or formal uniformizations). We thus pick an integer $N \gg (1+e)n$. We note that

$$\epsilon - \epsilon^\sharp(\infty) = \sum_{k\ge 1, 0\le j<q} g^k y^j \sigma(k,j) = gA(N) + gB(N) \tag{65}$$

where

$$\begin{aligned} A(N) \;&=\; \sum_{k\ge 1, 0\le j<q} y^j \sum_{kq+|\beta|\le p^N} g^{k-1}\omega^\beta \tau(k,j,\beta) \\ \sigma(k,j) \;&=\; \sum_\tau \omega^\tau \tau(k,j,\beta) \text{ with } \tau(k,j,\beta) \in K, \forall (k,j,\beta) \\ B(N) \;&=\; \sum_{k\ge 1, 0\le j<q} g^k y^j \sum_{kq+|\beta|> p^N} g^{k-1}\omega^\beta \tau(k,j,\beta). \end{aligned}$$

Note that $B(N) \in \rho^\ell(\max(K[[x]])$. Then for each $\tau \in K$ we pick $f \in \mathcal{O}_\xi$ which is $\equiv \sqrt[q]{\tau} \mod \max(\mathcal{O}_\xi)$, and define $\tau_N = \rho^N(f) \in \mathcal{O}_\xi$. We then define an approximation A_N of A as follows:

$$A_N = \textstyle\sum_{k\ge 1, 0\le j<q} y^j \sum_{kq+|\beta|\le p^N} g^{k-1}\omega^\beta \big(\tau(k,j,\beta)_N\big) \tag{66}$$
$$\text{and } A - A_N \in \rho^N(\max(K[[x]]))$$

We now define

$$\epsilon^\sharp(N) \;=\; \epsilon \;-\; gA_N \tag{67}$$

which will be called a *(rational) g-cleaning* of ϵ to the depth N.

We have

$$\epsilon \;\equiv\; \epsilon^\sharp(N) \text{ modulo}$$
$$\textstyle\sum_{k>N-e} g^k K[[x]] + \sum_{j\le q} y^j \sum_{kq+|\beta|\le p^N} g^{k-1}\omega^\beta \max(\rho^N(K[[x]]).$$

Lemma 7.19. *We have*

$$\epsilon^\sharp(\infty) \equiv \epsilon^\sharp(N)$$

where the equivalence is modulo

$$g\rho^{N-e}(\max(K[[x]])) + g\mathcal{O}_\xi \max(\rho^N(K[[y,\epsilon]]).$$

Proof. This is proved by direct calculus. □

Corollary 7.20. *Assume that $N \gg (1+q)n$. Pick any γ with $\gamma_i \in \mathbb{Z}_0, 1 \le i \le n$, and any $T = x^\alpha x^{p\beta}$ with $\alpha_i < p, \beta_i < p, \forall 1 \le i \le n$. If $Tx^{q\gamma}$ has nonzero coefficient in the power series $\epsilon^\sharp(N) \in K[[x]]$, then $\partial^{(q\gamma)} y^q = 0$ and $\partial^{(q\gamma)} g = \partial^{(q\gamma)}\epsilon$.*

Proof. γ must be either divisible by y^{qp} or by $y^q h$ with a non-unit nonzero monomial $h \in \sum_{i<q} y^i K[\omega]$. □

8. Standard unit-monomial expression

This section is somewhat independent from the rest of the paper, and is about a special method of choosing a local polynomial expression of an element of $\mathcal{O}_\xi$ which we will call $\rho\tau$-resentation which is defined in §8.2. The name $\rho\tau$ is chosen simply to pay special respect to its connection with the Frobenius filtration of $\mathcal{O}_\xi$ and to tangential property. In this work we simply state and prove the $\rho\tau$-resentation in a formulation suited to our program toward the resolution prolem. Then apply the $\rho\tau$-presentation to each element of the form $g = y^q + \epsilon \in \mathfrak{L}(\check{E})$ and obtain the *standard expression* of ϵ by which we deduce a key technical result. This is symbolical denoted by

$$h(\epsilon) \in \boxminus(g,q) \subset \wp^{nega}(\check{E})_\xi \ for \ \forall \ g = y^q + \epsilon \in \mathfrak{L}(\check{E})_\xi \tag{68}$$

where $\sqrt[p]{g - h(\epsilon)} \in \mathcal{O}_\xi$ for each $\xi \in \mathrm{Sing}(\check{E})_{cl}$, and $\boxminus(g,q)$ is defined in §9 (Prop.(9.1)). This should be in accord with Th.(9.3), stated and proven in this section.

8.1. **Finiteness of unit-monomial sum expression.** In this section we pick and fix any regular system of parameters x of $\mathcal{O}_\xi$ with $\xi \in Z_{cl}$. The following proposition will be used to assure a certain finiteness of unit-monomial sum expressiom in terms of x for every element of $\mathcal{O}_\xi$. We have the local completion $\hat{\mathcal{O}}_\xi = K[[x]]$. Note that $\partial(K) = (0)$ with every differential operator $\partial \in \mathrm{Diff}^{(m)*}_{Z,\xi}, m > 0$.

Proposition 8.1. *Pick any nonzero $h \in K[[x]]$. We claim the existence of a positive integer $\ell = \ell(h,x)$ having the following property.*

$$h = \sum_{a \in \mathbb{Z}_0^n, a_i < \ell, \forall i} A_a x^a \tag{69}$$

where every A_a is either zero or unit in $\rho^\ell(K[[x]])$ and $x^a \notin \rho^\ell(K[[x]])$.

The proposition will be proven after the Lem.(8.3) below.

Remark 8.2. The essence of the claim is that ℓ is finite but large enough to have the presentation of the form Eq.(69). In fact the same proposition is applicable to each $A_a \neq 0$. To be precise we write $A_a = A'_a + B_a$ with $A'_a \in \rho^{\ell+1}(K[[x])$ and $B_a = A_a - A'_a \in \rho^{\ell}(\max(\mathcal{O}_\xi))$. For instance A'_a can be chosen as follows. Let κ be the residue of A_a modulo $\max(\mathcal{O}_\xi)$ and pick an element $f \in \mathcal{O}_\xi$ whose residue is $\sqrt[m]{\kappa}$ with $m = p^{\ell+1}$. Then let $A'_a = \rho^{\ell+1}(f)$. Then write $B_a = \sum_i \rho^{\ell}(B_{a,i})$ with $B_{a,i} \in \max(\mathcal{O}_\xi)$. Apply Prop.(8.1) to each $B_{a,i}$ and have $B_{a,i} = \sum_{c\in\mathbb{Z}_0^n, c_i<\ell, \forall i} B_{a,i,c}x^c$ where every $B_{a,i,c}$ is either zero or unit in $\rho^{\delta}(K[[x]])$ with $\delta > 0$. Say $B_{a,i,c} = \rho^{\delta}(D_{a,i,c})$ Therefore we have

$$h = \textstyle\sum_{a\in\mathbb{Z}_0^n, a_i<\ell,\forall i}(A'_a + B_a)x^a \tag{70}$$

where

$$\begin{aligned} B_a x^a &= \sum_i \rho^{\ell}(B_{a,i})x^a \\ &= \sum_i \rho^{\ell}(\sum_c B_{a,i,c}x^c)x^a \\ &= \sum_{ic} \rho^{\ell+\delta}(D_{a,i,c})x^{p^{c\ell}+a}. \end{aligned}$$

It follows that Prop.(8.1) remains valid when we replace ℓ by $\ell+1$. We can repeat this process and conclude that ℓ of the Prop.(8.1) can be chosen arbitrary large.

Lemma 8.3. *Let X be any subset, possibly infinite, of $\mathbb{Z}_0^n$ and let $X_\circ$ be any finite subset of X such that*

$$(X_\circ)K[[x]] \;=\; (X)K[[x]] \tag{71}$$

Then for every $b \in X$ there exists $a \in X_\circ$ such that $b \in a + \mathbb{Z}_0^n$. An $X_\circ$ with Eq.(71) always exists by the noetherian property of $K[[x]]$.

Proof. By assumption $x^b = \sum_j f_j x^{a_j}$ with $a_j \in X_\circ$ and $f_j \in K[[x]]$. Let $f_j = \sum_k f_{jk}x^{e_{jk}}$ with $e_{jk} \in \mathbb{Z}_0^n$ and $f_{jk} \in K$. We must then have

$$x^b = \sum_{e_{jk}+a_j=b} f_{jk}x^{e_{jk}}x^{a_j}$$

by the independence of homogeneity. Hence $1 = \sum_{e_{jk}+a_j=b} f_{jk}$. There exists at least one $f_{jk} \neq 0$ and $b \in a_j + \mathbb{Z}_0^n$. □

The proof of Prop.(8.1) is as follows. Write

$$h = \sum_{a\in\mathbb{Z}_0^n, h_a\in K} h_a x^a$$

the power series expansion of $h \in \mathcal{O}_\xi \subset K[[x]]$. Let $X(h) = \{a \mid h_a \neq 0\} \subset \mathbb{Z}_0^n$. Thanks to Lem.(8.3) applied to $X = X(h)$ we have a finite subset $X_\circ \subset X(h)$ such that every $b \in X(h) \setminus X_\circ$ admits $a \in X_\circ$ with $b \in a + \mathbb{Z}_0^n$. We have a positive integer ℓ such that $X_\circ$ is contained in $Y = \{a \in \mathbb{Z}_0^n \mid a_i < \ell, \forall j\}$. Now we write $h = \sum_{b \in X(h)} B_b x^b$ where B_b is a unit in $K[[x]]$ for every b. We begin the power series expansion with $B_b \in K$.

We modify the expression step by step, possibly infinitely many times with convergence, in order to reach the claim of Prop.(8.1). Indded if we find any $b \notin Y$ with $B_b \neq 0$ then we have $b \in X(h)$ and $a \in X_\circ \subset Y$ such that $b = a + c$ with $(0) \neq c \in \mathbb{Z}_0^n$. Then clearly $B_a + B_b x^{b-a}$ is a unit in $\mathcal{O}_x i$. Replace B_a by $B_a + B_b x^{b-a}$ and eliminate the term $B_b x^b$. Note that $B_a + B_b x^{b-a}$ is a unit in $K[[x]]$. This simple procedure from B_a to $B_a + B_b x^{b-a}$ can be repeated one after another for every $b \in X(h) \setminus X_\circ$. The repetition can be infinite but it is convergent in $K[[x]]$. This proves Prop.(8.1).

Definition 8.4. Eq.(69) is called the *unit-monom* presentation of h with depth ℓ in terns of x. Each of the $A_a x^a$ of Eq.(69) with $A_a \neq 0$ will be called unit-monom summand of h with depth ℓ in terns of x.

Definition 8.5. We will say that $\ell = \ell(h)$ of Lem.(8.3) is unit-monom-big enough for h with respect to the regular system x of $\mathcal{O}_\xi$. The the monomial base $\{x^\theta, 0 \le \theta_i < p^\ell, \forall i\}$, say $\Sigma(\ell(h))$, will be called ℓ-big unit-monom set for h. What is important about ℓ and Σ in this work is that ℓ is finite but big enough and Σ is preferably little but many enough to generate h.

Theorem 8.6. *Pick any nonzero $h \in \mathcal{O}_\xi$ and any integer $\ell_\circ$. Then there exists an integer $\ell^\circ \ge \ell_\circ$ accompanied by a finite non-empty set $\boxdot(h, \ell, x) \subset \mathbb{Z}_0^n$ (or often $\boxdot(h, \ell)$ or $\boxdot(\ell, x)$ or even $\boxdot(\ell)$ in short) for each integer $\ell \ge \ell^\circ$ such that the following conditions are satisfied.*

(1) We have the following expression.

$$h = \sum_{\theta \in \boxdot(\ell)} u(\theta) x^\theta \tag{72}$$

where each θ is of the form $a + pb + qc$ with $a_i < p, b_j < p^{e-1}, c \in \mathbb{Z}_0^n$. We write $q = p^e$ with $e > 1$.

(2) $u(\theta)$ is a unit in $\rho^\ell(\mathcal{O}_\xi)$ for every $\theta \in \boxdot(\ell)$ while x^θ is not in $\rho^\ell(\mathcal{O}_\xi)$ for any $\theta \in \boxdot(\ell)$.

(3) We have $\partial^{(\theta)}(\rho^\ell(\mathcal{O}_\xi)) = (0)$ for every $\theta \in \boxdot(\ell)$.

(4) For every $\theta \in \boxdot(\ell)$ *we have the following formal expression in* $\hat{\mathcal{O}}_\xi = K[[x]]$ *with* $K \supset \mathbb{K}$.

$$u(\theta) = k_{(0)}(\theta) + \sum_{\chi \neq (0): \chi_j \geq \ell - \theta_j, \forall j} k_\chi(\theta) x^\chi \tag{73}$$

where $0 \neq k_{(0)}(\theta) \in K$ *and* $k_\chi(\theta) \in K, \forall \chi$.

Proof. Refer to Th.(8.6). Pick any integer $\ell \geq \ell_\circ$ and write

$$h = \sum_{c \in \mathbb{Z}_0^n, c_j < p^\ell, \forall j} h(c)^{p^\ell} x^c \text{ with } h(c) \in \mathcal{O}_\xi \tag{74}$$

where those $\{x^c\}$ form a free basis of $\mathcal{O}_\xi$ as $\rho^\ell(\mathcal{O}_\xi)$-module. Then we apply Prop.(8.1) to every one of those $h(c) \neq 0$ out of Eq.(74). We thus obtain a positive integer $m > 1$ such that each of those $h(c)$ is written as

$$h(c) = \sum_{k \in \mathbb{Z}_0^n, k_j < p^m, \forall j} x^k h(c, k)$$

where every $h(c, k)$ is either zero or unit in $K[[x]] = \hat{\mathcal{O}}_\xi$ and the set of k with $h(c, k) \neq 0$ is non-empty. Let $\boxdot(\ell)$ be the set of those $c + p^\ell k$ for all (c, k) with $h(c, k) \neq 0$. It is clearly a finite subset of $\mathbb{Z}_0^n$ and we obtain $h = \sum_{(c,k) \in \boxdot(\ell)} x^{c + p^\ell k} h(c, k)^{p^\ell}$. Writing θ for $c + p^\ell k \in \boxdot(\ell)$ and $u(\theta)$ for $h(c, k)^{p^\ell}$ we obtain an equality formally equal to Eq.(72) where the difference disappears as soon as we prove that $u(\theta) \in \mathcal{O}_\xi$ for every θ. Now let $\theta = c + p^\ell k \in \boxdot(\ell)$ be the lexicographically maximum so that $\partial^\theta(h) = h(c, k)^{p^\ell}$ which is clearly an element of $\mathcal{O}_\xi$. Repeat successively by the lexicographical induction. We then have Eq.(72) proven or all $\theta \in \boxdot(\ell)$. □

Theorem 8.7. *For each* $\ell \geq \ell^\circ$ *the* $\boxdot(h, \ell, x)$ *and the equality Eq.(72) of Th.(8.6) are uniquely determined by* (h, ℓ, x).

Proof. Suppose two presentations in the manner of Eq.(72), say

$$h = \sum_{\gamma \in \boxdot(\ell)} u(\gamma) x^\gamma = \sum_{\gamma' \in \boxdot'(\ell)} u'(\gamma') x^{\gamma'}$$

Let θ be the lexicographically maximum in $\boxdot(\ell) \cup \boxdot'(\ell)$. Then $u(\theta) = \partial^{(\theta)}(h) = u'(\theta)$. This means that θ must be in $\boxdot(\ell) \cap \boxdot'(\ell)$ and the unit factors are $u(\theta) = u'(\theta)$. □

Theorem 8.8. *For* $\ell^* > \ell \geq \ell^\circ$ *of Th.(8.6) the* $\boxdot(h, \ell, x)$ *is obtained from* $\boxdot(h, \ell^*, x)$ *by means of the equivalence relation among elements of* $\boxdot(h, \ell^*, x)$ *defined by* $x^\theta - x^{\theta'} \in \rho^\ell(\mathcal{O}_\xi)$.

Proof. The proof is straight-forward thanks to Th.(8.6). □

Definition 8.9. The expression of h by Eq.(72) of Th.(8.6) is called $\boxdot$-expression of depth ℓ in terms of x, or $\boxdot(\ell)$-expression in short. The sequence $\{\boxdot(h,\ell,x), \forall \ell \geq \ell^\circ\}$ of Th.(8.6) will be called $\rho\tau$-sequence of h with respect to x.

8.2. **Frobenius filtration with unit-monomials.** In terms of x a unit-monomial presentation is obtained firstly by using a free x-monomial basis A of $\mathcal{O}$ as $\rho(\mathcal{O})$-module, secondly a free x-monomial basis B of $\rho(\mathcal{O})$ as $\rho^e(\mathcal{O})$-module and finally a free x-monomial basis C of $\rho^e(\mathcal{O})$ as $\rho^\ell(\mathcal{O})$-module. The integer $\ell \gg e$ is chosen depending on the required property by the required purpose of a given element $X \in \mathcal{O}_{cl}$. Especially we require that $\rho\tau$-presentation is to be written as a sum of unit-monomials in terms of the A, B and C. Recall that we are given an LL-head $g = y^q + \epsilon$ of $\mathfrak{L}(\check{E})_\xi$ for $\xi \in \mathrm{Sing}(\check{E})_{cl}$ and we have particular interest not only in the number e of $q = p^e$ but also in the selection of $\ell \gg e$ for $\rho\tau$-presentation of X and our primary interest is in the case of $X = \epsilon$.

8.3. **Standard expressions for $\epsilon = g - y^q$.** A standard expression of ϵ is a special case of the unit-monomial presentation with Frobenius filtration of (8.2). This is based on the finiteness of the presentation by unit-monomial summation of (8.1). However when we speak of standard expression of ϵ we assume

(1) We are assuming $x \ni y$ where $y = g - \epsilon$ where $\mathrm{ord}_\xi(g) = q = p^e < \mathrm{ord}_\xi(\epsilon)$
(2) g is a member of $(L)(E)$ in the sense of Eq.(52).

Refer to §(8.1) following after §(8.1), in which technically important are Th.(8.6) based upon Prop.(8.1) proven by Lem.(8.3). Refer to Def.(8.5).

8.4. **Cleaned standard expressions for $\epsilon = g - y^q$.** As for the standard expression of ϵ at $\xi \in \mathrm{Sing}(\check{E})$ we write ϵ as

$$\epsilon = \sum_{(a,b,c)} x^a x^{pb} x^{qc} u(a,b,c)$$

with units or zero $u(a,b,c) \in \rho^\ell(\mathcal{O})$ by means of Th.(8.6) with sufficiently large ℓ, where the components are $0 \leq a_i < p$, $0 \leq b_j < p^{e-1}$ and $0 \leq c_k < \infty$. We choose to write the summation using the descending lexicographical ordering of (a,b,c) (cf. Eq.(76) in §9.3).

9. Local Leverage-up Exponent-down (LLUED)

The concept of Local Leverage-up Exponent-down (or LLUED) plays a power role in the inductive approach to resolution of singularities used in this paper. The LLUED theorems are based on the selection of a subset $\boxminus(\check{E}) \subset \wp^{nega}(\check{E})$ as in the following proposition.

Proposition 9.1. *For every $g = y^q + \epsilon \in \mathfrak{L}(\check{E}, q)_\xi$ of Eq.(52) of §(6.3), we have $h \in \boxminus(\check{E})$ such that*

(1) $g(1) = \sqrt[p]{g-h} \in \mathcal{O}_\xi$

(2) $\mathrm{ord}_\xi(g(1)) \geq p^{-1}q$ *and* $\mathrm{ord}_\xi(h) \geq ord_\xi(\epsilon)$

To define $\boxminus(g, q)$ and have $h \in \boxminus(g, q) \subset \boxminus(\check{E})$ we need what we call a standard expression of $\epsilon = g - y^q$ to which appropriate differentiation-product operations are applied. The products of those diff-product operations will fill up the set $\boxminus(g, q)$ containing an h producing the solution $g(1)$ from g as above. We do this in §9.3.

9.1. **Master keys for the existence of LLUED.** The following Th.(9.2) is the master key to the conceptual development and applications of LLUED technique in our proof of resolution of singularities in $p > 0$. In some cases the generalized version Th.(9.3) is more directly useful than Th.(9.2. Logically Th.(9.2) is only a special case of Th.(9.3) but the actual proof will be seen identical in their procedural steps. The proof of Th.(9.2) and Th.(9.3) will be given starting from the assumed existence of the standard expression of $\epsilon = g - y^q$ stated in §(9.3). cf. Eq.(76) of §(9.3) below. The existence of a standard expression of ϵ is proven in §(8.1) of §8.

Theorem 9.2 (Mater key)**.** *Given $g = y^q + \epsilon \in \mathfrak{L}(\check{E})_\xi$ of Eq.(52) locally at $\xi \in \mathrm{Sing}(\check{E})_{cl}$ we have $h \in \wp^{nega}(\check{E})_\xi$ such that $\mathrm{ord}_\xi(h) \geq ord_\xi(\epsilon)$ and $\sqrt[p]{g-h} \in \mathcal{O}_\xi$.*

It should be noted that in general the h may be a finite sum of different homogeneity degrees in $\wp^{nega}(E)_\xi$ but we are ignoring their degrees and taking the sum as elements of $\mathcal{O}_\xi$. However when we are applying blowups permissible for E the transforms of h are defined in accord with their respective degrees of homogeneity.

Th.(9.2) can be improved as follows.

Theorem 9.3 (Generic Key)**.** *For $g = y^q + \epsilon$ as above, let $D \ni \xi$ be the center of blowup permissible for (g, q). Then there exists $h \in \wp^{nega}(g, q)$ such that $\sqrt[p]{g-h} \in \mathcal{O}_\xi$ and $\mathrm{ord}_\xi(h) \geq ord_\xi(\epsilon)$ as well as $\mathrm{ord}_D(h) \geq q$.*

This generalization of Theorem 9.2 follows from Lem.(7.3) of §3.

9.2. **Introductory comments on local vs global IUED conceots.** Recall Def.(5.1) of the extension $\tilde{\wp}(E)$ of $\wp(E)$ by creating new homogeneous part of degree zero and the parts of negative degrees. This extension is very important in order to make our reasoning rigorous in our dealling of the notion of LUED in particular in connection with geometry of transformation of singular data by permissible blowups. As for the detailed account of necessary properties of $\tilde{\wp}(E)$ we must refer to Lem.(5.3), Rem.(5.4), Lem.(5.6), Lem.(5.8), Rem.(5.10), Th.(5.11), and Def.(5.1).

However in this section our study of LUED is strictly focused on its local questions in terms of selection of local coordinate system and use of explicit expression of differential operators. This locally focused study of singularities begins now and continues until we have established the local LUED-chain, or *LLUED-chain*, written in the form of Eq.(83) and Eq.(84) of §9.9, which describes LL-chains headed by any member of special generators of $\mathfrak{L}(\check{E}, q)$ with $q = p^e$. We then switch our focus to the study of global coherent LUED, or GLUED. The study will be focused on a coordinate free LUED, starting from §11 and leading to the end result in the form of complete global GLUED-chain diagram of Eq.(104) of Rem.(13.1) of §13. We then examine the exact relation between the two results, the local LUED on one hand and the global coherent GLUED on the other hand. This relationship will elucidate the local and global properties of cotangent structure of the singularity and imply effective application of the AR-technique. This will lead us to the proof of resolution of singularities by induction on ambient dimension. For the moment we will concentrate our work on the local LLUED-structure of singularity.

9.3. **Frontier symbols of a standard expression.** We choose a regular system of parameters $x \ni y$ of $\mathcal{O}_\xi$ with $q = p^e$ and with $\ell \gg e > 0$. We should refer to the finiteness of unit-monomial generators and on $\rho\tau$-presentations. An important conclusion is the existence of the standard expression of $\epsilon = g - y^q$ with respect to $x \ni y$. It is written as a finite sum of unit-monomials in $x \ni y$ in the manner of an $\rho\tau$-presentation of depth $\ell \gg q$. It is chosen to fit the Frobenius filtration $\mathcal{O}_\xi \supset \rho(\mathcal{O}_\xi) \supset \rho^e(\mathcal{O}_\xi) \supset \rho^\ell(\mathcal{O}_\xi)$. An additional important a priori imposition is that we first apply g-cleaning of ϵ before we begin working with the standard expression of ϵ. The cleaning process of §(7.5) requires once again to choose a big enough $\ell \gg q$. We thus may need to enlarge the depth number ℓ for the standard expression of ϵ. In fact the real purpose of choosing g-cleaned standard expression is the equality needed later:

$$H^\flat(g) = H^\flat(\epsilon) \text{ or, equivalently, } H^\flat(y^q) = 0 \tag{75}$$

for the $H^\flat$-operators defined by diff-products of §9.5 later. At any rate a priori g-cleaning of ϵ is done by Cor.(7.20) in §7.5. Here is the g-cleaned standard expression of ϵ to depth ℓ at $\xi \in \mathrm{Sing}(\check{E})$.

$$\epsilon = u_0 x^\alpha x^{p\beta} x^{q\gamma_0} + \sum_{\gamma_j <_{lex} \gamma_0} u_j x^\alpha x^{p\beta} x^{q\gamma_j} + \sum_{b <_{lex} \beta} u_{bc^\dagger} x^\alpha x^{pb} x^{qc^\dagger} + \sum_{a <_{lex} \alpha} u_{ab^\dagger c^\ddagger} x^a \lambda^{pb^\dagger} x^{qc^\ddagger} \tag{76}$$

where u_0 is a unit in $\rho^\ell(\mathcal{O}_\xi)$ while $u_c, u_{bc^\dagger}$ and $u_{ab^\dagger c^\ddagger}$ are either zero or units in $\rho^\ell(\mathcal{O}_\xi)$. Denote by the symbol $<_{lex}$ lexicographical strict inequality. Then we have

(1) $0 \le \alpha_i < p$, $0 \le \beta_i < p^{e-1}$ and $0 \le \gamma_i < \infty$ for all i.
(2) u_0 is a unit and $\alpha \ne (0)$ so that $\epsilon \notin \rho(\mathcal{O}_\xi)$.

We will then apply the diff-products, called $H^\flat$ operation to produce members of $\boxminus(g, q) \subset \wp^{nega}(\check{E})$, thus proving Prop.(9.1). We choose $\ell \gg q$ big enough so that all the differentiations ∂ used for $H^\flat$ shall annihilate $\rho^\ell(\mathcal{O}_\xi)$. Hence if $u \in \rho^\ell(\mathcal{O}_\xi)$ is any unit of Eq.(76) then we have $\partial(uX) = u\partial X$.

9.4. **Frontier symbolics.** Let us define and write

(1) $AB(\epsilon)$ for $x^{\alpha+p\beta}$ which is called the top frontier.
(2) $ABC_j(\epsilon)$ for $u_j x^\alpha x^{p\beta} x^{q\gamma_j}$ *with* $j = 0, 1, 2, \cdots, m$
(3) $m + 1$ is the length of those with $u_j \ne 0$. Refer to Eq.(76).

These are what we will begin working with as being important frontier factors of the standard expression of ϵ. We pay special attention to those unit-monomial terms of the standard expression that are divisible by the top frontier $AB(\epsilon)$ and arrange them in strict lexicographical descending order. We write as follows.

$$\begin{aligned} AB\gamma(\epsilon) &= \{x^{q\gamma_0}, x^{q\gamma_1}, \cdots, x^{q\gamma_m}\} \\ ABC_j(\epsilon) &= u_j AB(\epsilon) x^{q\gamma_j}, j \ge 0, \text{ for } u_j \ne 0 \end{aligned} \tag{77}$$

where

$$\gamma_0 >_{lex} \gamma_1 >_{lex} \cdots >_{lex} \gamma_m.$$

Remark 9.4. If we replace ℓ by $\ell' > \ell$ then the top frontier $AB(\epsilon)$ does not change while $AB\gamma(\epsilon)$ can change only adding tails.

9.5. **Differentiation-product $H^\flat$-operator.** Our work is now programmed to knock out $ABC_j(\epsilon), j = 0, 1, 2, \cdots$, one after another by subtracting their preferable substitutions which are obtained by differentiation-products or diff-products in short. The substitutions will be selected out of the summands of ϵ and the selected ones will be denoted by $H^\flat(\epsilon(j)), j = 0, 1, 2, \cdots$. Firstly we pick an integer $\ell \gg q$ large enough and apply Cor.(7.20), to replace ϵ by its g-cleaning with depth ℓ. The result is called $\epsilon(0) = \epsilon^\sharp(\ell)$.

9.5.1. *Cleaning a priori.*

Lemma 9.5. *We choose $\ell \gg q$ and in particular $\ell > |\alpha + p\beta + q\gamma_0|$ in such a way that Cor.(7.20) of Eq.(68) is applicable to $[\epsilon : x^{\alpha+p\beta}]$ with respect to y^q. If ℓ is large enough then the the property of applicability remains valid by any change ℓ to a larger $\ell' > \ell$.*

Proof. Straight by the nature of the claim of Cor.(7.20) of Eq.(68). □

9.6. **$H^\flat$-operators for LLUED.** The $H^\flat$-operator is defined by selecting a substation of $ABC_j(\epsilon)$. It is done by induction on $j = 0, 1, \cdots, m$. The symbol ∂^X denotes the elementary differential operator of multi-index X with respect to a fixed parameters x in the sense of Eq.(8). The diff-products are used to define $H^\flat$ case by case as follows.

Lemma 9.6. *We first note that there are three different cases of the type of $ABC_j(\epsilon)$. Namely, (I) $|q\gamma_0| \geq 2q$, (II) $|\alpha + p\beta| > q$ and (III) $0 < |\alpha + p\beta| \leq |q\gamma_0| = q$ so that $|\gamma_0| = 1$*

Proof. We have $|\alpha + p\beta + q\gamma_0| \geq ord_\xi(\epsilon(0)) \geq ord_\xi(\epsilon(0)) > q$. If $|q\gamma_0| < 2q$ and $|\alpha + p\beta| \leq q$ then the lower bound of $|q\gamma_0|$ is q. □

Definition 9.7. The symbol $[X : Y]$ is defined as follows. Take an example such as $[\epsilon : x^{\alpha+p\beta}]$. This means to consider the power series expansion of $X = \epsilon$ in $K[[x]]$ and make the replacement of each of the x-monomial terms of the expansion of ϵ as follows.

(1) Replace the term by its quotient if it is divisible by the monomial $Y = x^{\alpha+p\beta}$ while
(2) we replace the term by zero if it is not divisible by Y.

Remark 9.8. Cleaning is a unit multiplication. Hence differentiation after cleaning is a differentiation, too.

9.7. **Local $H^\flat$ case by case.** We choose $H^\flat$ in each of the three cases of Lem(9.6).

Remark 9.9. Case(I): $H^\flat(\epsilon(0)) = {u_0}^{-1}\partial^{(\alpha+p\beta)}(\epsilon(0)) \times \partial^{(q\gamma_0)}(\epsilon(0))$.
In this case we have

(1) $\mathrm{ord}_\xi(H^\flat(\epsilon(0))$, renamed by $d(A,I,0)$, is $\geq \mathrm{ord}_\xi(\epsilon(0)) - |\alpha+p\beta+q\gamma_0| + \mathrm{ord}_\xi(\epsilon(0)) \geq \mathrm{ord}_\xi(\epsilon(0))$
(2) $H^\flat(\epsilon(0)) \in \wp(\check{H}, -\delta(A,I,0))$ with $-\delta(A,I,0) = 2\mathrm{ord}_\xi(g) - |\alpha + p\beta + q\gamma_0|$ which is
$$2q - |\alpha+p\beta+q\gamma_0| = (q-|\alpha+p\beta|) + (q - |q\gamma_0| < -q$$

We should therefore infer that $\epsilon(0)$ is g-cleaned and hence $g(0) = y^q + \epsilon(0) \in \wp(\check{E}, q)$ and $\partial(\epsilon(0)) = \partial(g(0))$.

Remark 9.10. Case(II): $H^\flat(\epsilon(0)) = x^{\alpha+p\beta}\partial^{(\alpha+p\beta)}(\epsilon(0))$

(1) $\mathrm{ord}_\xi(H^\flat(\epsilon(0))$, which is denoted by $d(A,II,0)$, is equal to $\mathrm{ord}_\xi(\partial^{(\alpha+p\beta)}(\epsilon(0)) + ord_\xi(\epsilon(0))$ which is $\geq ord_\xi(\epsilon(0))$
(2) $H^\flat(\epsilon(0)) \in \wp(\check{E}, -\delta)$ with $-\delta = ord_\xi(g) - |\alpha+p\beta| < 0$.

Remark 9.11. Case(III) In this case we define preliminary operation $h^\flat$ which is repeated to make the definition of $H^\flat$ as follows.

(1) $h^\flat(X) = \partial^{(\alpha+p\beta)}(X)\partial^{(q\gamma_0)}(X)$ in general for any X and
(2) $h^\flat(\epsilon(0)) = \partial^{(\alpha+p\beta)}(\epsilon(0))\partial^{(q\gamma_0)}(\epsilon(0))$ in particular.
(3) $\mathrm{ord}_\xi(h^\flat(\epsilon(0))) \geq ord_\xi(\epsilon(0)) - |\alpha+p\beta+q\gamma_0| + ord_\xi(\epsilon(0)) \geq ord_\xi(\epsilon(0))$
(4) $h^\flat(\epsilon(0)) \in \wp(\check{E}, q - b(A,III,0))$ where $b(A,III,0) = |\alpha+p\beta+q\gamma_0| - q > 0$
(5) $(h^\flat)^k(\epsilon(0) \in \wp(\check{E}, q - kb(A,III,0)$ for $k = 1,2,\cdots$.
(6) We choose the first k which makes $q - kb(A,III,0) < 0$ and call it $\bar{k}$. We then define $-\delta(B,III,0) = q - \bar{k}b(A,III,0)$
(7) We now define $H^\flat(\epsilon(0)) = (h^\flat)^{\bar{k}}(\epsilon(0))$ with the $\bar{k}$.
(8) We have $\mathrm{ord}_\xi(H^\flat(\epsilon(0)) \geq ord_\xi(\epsilon)$ and at the same time $H^\flat(\epsilon(0)) \in \wp(\check{E}, -\delta(B,III,0))$.

Definition 9.12. We have thus defined the operator $H^\flat$ in all three cases of Lem.(9.6) of $\epsilon(0)$ with orders $\geq \mathrm{ord}_\xi(\epsilon)$, and homogeneity degrees < 0. We then define the iteration of $H^\flat$ as follows.

$$\epsilon(i+1) = \epsilon(i) - H^\flat(\epsilon(i)) \; for \; i = 0,1,\cdots,m \tag{78}$$

where m is the number defined by $\{ABC_j(\epsilon), 0 \leq i \leq m\}$ of §9.4 only. Here it may appear that $H^\flat$ has been defined only for $i=0$ so far with §(9.7). However this iteration makes perfect sense by explicit calculus in $Bl_\xi(Z)$ as well as in $\tilde{\wp}(\check{E})$. There are the new standard expressions of $\epsilon(i), i = 1, 2, \cdots, m+1$, in the manner of Eq.(76) of §(9.3). Namely we have its following form in each step as follows.

$$\begin{aligned} \epsilon(i+1) &= \sum_{m \geq j \geq i+1} w_j x^{\alpha} x^{p\beta} x^{q\gamma_j} \\ &+ \sum_{b <_{lex} \beta} w_{bc^\star} x^{\alpha} x^{pb} x^{qc^\star} \\ &+ \sum_{a <_{lex} \alpha} w_{ab^\star c^*} x^{a} \lambda^{pb^\star} x^{qc^*} \end{aligned} \tag{79}$$

where $w_j = {u_0}^{-(1+1)} u_j, j = i+1, \cdots, m$, and $w_{bc^\star}, w_{bc^\star}, w_{ab^\star c^*}$ are either zero or units $\rho^{\ell}(\mathcal{O}_{\xi})$.

Here the important point is that the top frontier AB remains unchanged while $AB\gamma$ in the sense of E q.(77) of §(9.4) undergoes losing the members indexed $j = 0, 1, \cdots, i$. Those members with $j > i+1$ change only their unit factors just as above to new unit factors. The rest summands change receiving the additions from the tails of $H^{\flat}(\epsilon(j)), j < i+1$, which are all lexicographically strictly smaller than the members of new $AB\gamma$. The expression Eq.(79) for this rest portion is is obtained by applying §(8.1) and Th.(8.6) if necessary. The end expression of $epsilon(i+1)$ will be exactly like Eq.(79). Incidentally we should remember that the task of Eq.(78) for $0 \leq i \leq m\}$ is to eliminate the summands of those terms divisible by the top frontier AB and $H^{\flat}$ operations for this end have no concern with the rest portion above, in particular whether this portion is g-cleaned or not. Thus Eq.(79) is justified. Moreover we have

$$\mathrm{ord}_{\xi}(\epsilon(m+1)) \geq \mathrm{ord}_{\xi}(\epsilon) \text{ and } \epsilon(m+1) \in \wp^{nega}(\check{E}) \cap \rho^{e-1}(\mathcal{O}_{\xi}) \tag{80}$$

Let us observe the change after k-times repetition of Eq.(78) in the standard expression of $\epsilon(k)$ obtained by applying $H^{\flat}$ to Eq.(76) as follows. All thanks to the differential calculus on the lexicographical arrangement of unit-monomials in the standard expression,

Remark 9.13. There we have only the following two possibilities:

(1) In the case of $m = 0$ the top frontier of Eq.(76) are eliminated and we must restart with a new frontier, lexicographically strictly smaller than before. Namely $AB(\epsilon(1)) <_{lex} AB(\epsilon(0))$
(2) We have the same top frontier as before while the $ABC(\epsilon(0))$ loose the first member. To be precise we have

$$AB\gamma(\epsilon(1)) = \{x^{q\gamma_1}, x^{q\gamma_2}, \cdots, x^{q\gamma_m}\}$$

in contrast with $AB\gamma(\epsilon(0))$ of Eq.(77).

(3) We then go on with Eq.(78) referring the effect of change to the new standard expression Eq,(79).

These claims are all justified by

(1) $\partial^{(\alpha+p\beta)}$ annihilates all those unit-monomial terms of the standard expression of $\epsilon(0)$ which are not divisible by $x^{\alpha+p\beta}$ thanks to $\ell \gg q$. (Pay attention to units of uni-monomials, too.)
(2) $\partial^{(\gamma_0)}$ annihilates all those unit-monomial terms which are divisible by $x^{\alpha+p\beta}$ but not divisible by x^{γ_k} in each step of $k = 0, 1, \cdots, m$.

We are then left to the case in which the top frontier terms, that is all the nontrivial multiples of $ABx^{\alpha+p\beta}$, are all eliminated.

Here is a simple but useful fact for our LLED strategy.

Lemma 9.14. *The top frontier AB drops many time in the strict lexicographical ordering but the number of times of AR-drops is finite.*

This is a special case of an abstract and general fact on strict lexicographical descent ordering which is used for Lem.(9.14) as well as indispensable for other important finiteness theorems claimed later.

Proposition 9.15. *Fix a positive integer N, and consider finite sequences Q of elements of*

$$\{C_i \in \mathbb{Z}^N \cup \{\infty\}, 1 \leq C_i \leq \infty\}$$

These are ordered by the lexicographical ordering $>_{\text{lex}}$, where $Q_1 >_{\text{lex}} Q_2$ if the strict lexicographic inequality holds after augmenting each of the Q_j after adding a tail extension consisting of only ∞ to the shorter sequence. Consider a decreasing sequence of such Q:

$$Q_1 >_{\text{lex}} Q_2 >_{\text{lex}} \cdots >_{\text{lex}} Q_j >_{\text{lex}} \cdots .$$

Then there exists a finite number s such that the sequence must stop with $Q_k = Q_s$ for all $k \geq s$.

Proof. Suppose we had an infinite sequence of strict drops:

$$Q(1) >_{\text{lex}} Q(2) >_{\text{lex}} \cdots >_{\text{lex}} Q(m) >_{\text{lex}} \cdots \quad (81)$$

We then claim contradiction. This will be verified by induction on N. If $N = 1$ then the finiteness is clear even after a finite number of ∞ had been added to its tail. Assume the finiteness was proven for the case of one less N. If we restrict our attention to the first components of the members of the given sequences then the first components cannot descent indefinitely. So the problem is reduced to the case of constant first component for all members of the sequence. Ignoring the first components the problem is then reduced to the case of $N-1$. Thus the finiteness of the strict lexicographical descending sequence if proven. □

Remark 9.16. We go back to the situation at the end of Rem.(9.13). Needless to say we must have a new standard expression for the new ϵ, that is $\epsilon(m+1)$ of Def.(9.12), though we use the same parameters $x = (y, \omega)$. Possibly we need renewed ℓ large enough for a priori cleaning in order to choose the renew standard expression. We also define and apply the new $H^\flat$ operators and thus obtain the iterated definition of Eq.(78) with renewed $ABC\gamma$ with new $i = 0, 1, 2, \cdots, m$. Starting from $\epsilon(0)$ which is g-cleaned

$$\epsilon(j+1) = \epsilon(j) - H_j^\flat(\epsilon(j))$$

where $j \geq 0$ with symbolic

$$H_j^\flat(X) = \partial_j^A X \times \partial_j^B(X)$$

where $\partial^A = \partial^{(\alpha+p\beta)}$ is independent of j and $\partial_j^B = \partial^{(q\gamma_j)}$, and $j = 0, 1, \cdots, m$, where m is given in Eq(77).

9.8. **LL-heads and LL-tails of LL-chains.** We are now going to introduce the notion of LL-chain headed by any $g = y^q + \epsilon \in \mathfrak{L}(\check{E}, q)$ with $q = p^e, e > 0$. we take a given g-cleaning of ϵ to the depth $\ell \gg q$ large enough to maintain the property that y^q is annihilated by all the differentiations of diff-products used for the $H^\flat$-operations. The y^q is hence annihilated by $H^\flat$ and hence $H^\flat(g) = H^\flat(\epsilon)$. This yields two important numbers: its order, $\mathrm{ord}_\xi(H^\flat(\epsilon)) \geq \mathrm{ord}_\xi(\epsilon)$, and its negative homogeneity degree $-\delta$, i.e, $H^\flat(g) \in \wp(\check{E}, -\delta)$.

9.9. **LL-chains headed by $\mathfrak{L}(\check{E}, q)$.** Now we begin with an LL-head $g \in \mathfrak{L}(\check{E}, q)_\xi$. Refer to Eq.(52) of §6.3. Apply LLUED-operations to $g(0) = g$ in the manner of Th.(9.3) and iterate LLUED-operation successively.

We write $g \in \mathfrak{L}(\check{E}, q)_\xi$ and assume that

$$g = g(0) = y^q + \epsilon(0) \tag{82}$$

where $\epsilon = \epsilon(0)$ is g-cleaned, $q = p^e, e > 0$, and $y \in \cot(\check{E})$, $\mathrm{ord}_\xi(y) = 1$ and $\mathrm{ord}_\xi(\epsilon) > q$.

Remark 9.17. We would wish that every step to the next of the following LL-chain the difference is always nothing but a differentiation-product (cleaning is swallowed by diff) successively contained in $\tilde{\wp}(\check{E})$ while keeping the cotangent vector unchained at all. This vector is y modulo $\max(\mathcal{O})^2$. We then define the LL-chain headed by $g = g(0)$:

$$g = g(0) = y^q + \epsilon(0) = g(1)^p + h(0) \tag{83}$$

where

(1) $\mathrm{ord}_\xi(g(0)) = q = p^e$ and $\mathrm{ord}_\xi(\epsilon(0)) = \lambda(0) > q$;
(2) $h(0) \in \boxminus(g(0), q)$, where $h(0)$ is a differentiation-product of ϵ as well as of $g(0)$, and is contained in $\big(\max(\mathcal{O}_\xi)^{q+1}\big) \cap \tilde{\wp}(\check{E})$, where $\tilde{\wp}(\check{E})$ was defined in Def.(5.1);
(3) $\mathrm{ord}_\xi(h(0)) \geq \lambda(0)$;
(4) $g(1) = y^{p^{-1}q} + \epsilon(1) = \sqrt[p]{g(0) - h(0)} \in bl_\xi(Z, p^{e-1})$;
(5) $\mathrm{ord}_\xi(g(1)) = p^{e-1}$;
(6) $\mathrm{ord}_\xi(\epsilon(1)) = \lambda(1) \geq \sqrt[p]{\lambda(0)} > ord_\xi(g(1)) = p^{e-1}$; and
(7) $\epsilon(1) = \epsilon(0) - h(0) \in \rho(\max(O_\xi))$

where we could let $y(1) = y(0) = y$ because the g-cleaning is a unit multiple on g and differentiation-product can be replaced so as to annihilate y^q with no change of g. (Note that if $g(0)$ is a p-powered then $h(0) = 0$ and $g(0) = g(1)^p$ and likewise for any of the later $g(i), i > 0$.) We express this process by $g(0) \dashrightarrow_\xi g(1)$. We omit ξ if understood. By iteration

$$g(0) \dashrightarrow_\xi g(1) \dashrightarrow_\xi \cdots \dashrightarrow_\xi g(j-1) \dashrightarrow_\xi g(j) \dashrightarrow_\xi \cdots \dashrightarrow_\xi g(e-1) \dashrightarrow_\xi g(e) \tag{84}$$

where

(1) $g(j) = y^{p^{e-j}} + \epsilon(j) = g(j+1)^p + h(j)$;
(2) $g(j+1) \in bl_\xi(Z, p^{e-j-1})$;
(3) $\mathrm{ord}_\xi(g(j+1)) = p^{e-j-1}$;
(4) $\mathrm{ord}_\xi(\epsilon(j)) = \lambda(j) \geq \sqrt[p]{\lambda(j-1)}$;
(5) $h(j) \in \boxminus(g(j), p^{e-j})$, $\mathrm{ord}_\xi(h(j)) \geq \lambda(j) > p^{e-j}$;
(6) $g(j) \in bl_\xi(Z, p^{e-j}), j = 1, 2, \cdots$;
(7) $g(e-1) = g(e)^p + h(e-1)$ with $\mathrm{ord}_\xi(g(e)) = 1$, $h(e) = 0$ and $g(e) = y(e)$; and
(8) $g(e) = y + \epsilon(e)$, $\mathrm{ord}_\xi(\epsilon(e)) > 1$.

At this point we may go back to the y that we began with and we could replace y by $y(e) = \sqrt[p]{g(e-1) - h(e-1)}$ so that $g(e) = y(e)$. Among the steps of the $LLUED$-chain Eq.(9.8) as above some $g(j-1)$ may happen to be p-powered by itself in which case we simply let $h(j) = 0$ and $g(j) = \sqrt[p]{g(j-1)}$.

Theorem 9.18. *The $y(e)$ of Eq.(83)+Eq.(84) has the following two properties.*

(1) $y(e) - y \in \max(\mathcal{O}_\xi)^2$ and $y(e) \in \tilde{\wp}(\check{E})$
(2) $y(e)\, mod \max(\mathcal{O}_\xi)^2$ is a nonzero member of $\mathfrak{v}(\check{E})_\xi$

Proof. Refer to Rem.(9.17) and check this remark at every step of Eq.(83)+Eq.(84). ☐

Corollary 9.19. *The cotangent vector space* $\mathfrak{v}(\check{E})_\xi$ *is generated by the images of tails of the LL-chains headed by the elements of* $\mathfrak{L}(\check{E})_\xi$*.*

Proof. Straight forward. ☐

Definition 9.20. The $g(e)$ (or $y(e)$) is called the *tail* of the LL-chain headed by g. Note that we have an open neighborhood $V \ni \xi$ such that $g(e)$ defines closed smooth irreducible hypersurface Y in V and that all the members and all their mutual relations extend themselves to every point $\eta \in V \cap \mathrm{Sing}(\check{E})_{cl}$. Note also that $Y \subset \mathrm{Sing}(\check{E})$ and that for every $\zeta \in V_{cl} \setminus \mathrm{Sing}(\check{E})$ we have some $Y \not\ni \zeta$. See Lem(10.1) of §10 below. Thus by iteration we have

$$(85) \qquad g(0) \dashrightarrow_\xi g(1) \dashrightarrow_\xi \cdots$$
$$\dashrightarrow_\xi g(j-1) \dashrightarrow_\xi g(j) \dashrightarrow_\xi \cdots \dashrightarrow_\xi g(e-1) \dashrightarrow_\xi g(e)$$

where

(1) $g(j) = y^{p^{e-j}} + \epsilon(j) = g(j+1)^p + h(j)$
(2) $g(j+1) \in bl_\xi(Z, p^{e-j-1})$
(3) $\mathrm{ord}_\xi(g(j+1)) = p^{e-j-1}$
(4) $ord_\xi(\epsilon(j)) = \lambda(j) \geq \sqrt[p]{\lambda(j-1)}$
(5) $h(j) \in \boxminus(g(j), p^{e-j})$ *and* $ord_\xi(h(j)) \geq \lambda(j) > p^{e-j}$
(6) $g(j) \in bl_\xi(Z, p^{e-j}), j = 1, 2, \cdots$
(7) and $g(e-1) = g(e)^p + h(e-1)$ with $\mathrm{ord}_\xi(g(e)) = 1$, $h(e) = 0$ and $g(e) = y(e)$
(8) $g(e) = y(e)$ with $\mathrm{ord}_\xi(g(e)) = 1$

Definition 9.21. We call the final $y(e)$ the tail of the LL-chain of Eq.(83) and Eq.(84). Note that g^{p^k} and g have the same tail of the LL-chains headed by them where the former is called p-powered prolongation. For any pair of LL-chains their $\rho^\ell(\mathcal{O})$-linear combination after their suitable prolongations is an LL-chain.

9.10. **LLED propagation from a point to a neighborhood.** Thanks to Rem.(9.23) supported by Th.(6.14) we have an affine open neighborhood V of any given point $\xi \in \mathrm{Sing}(\check{E})$ by which the following remarks are all justified and meaningful in all the following remarks, definitions and theorems.

Theorem 9.22. *Recall to Def.(6.15). We have a finite system of generators for all the tails of* local *LL-chains headed by* local *members of* $\mathfrak{L}(\check{E})$ *everywhere in some open affine neighborhood* V *of any given point* $\xi \in \mathrm{Sing}(\check{E})$.

Proof. Infer to Def.(6.15) following after Th.(6.14). □

Remark 9.23. Give any $\xi \in \mathrm{Sing}(\check{E})_{cl}$. Let us choose an open affine neighborhood $V \ni \xi$ of the given point $\xi \in \mathrm{Sing}(\check{E})_{cl}$ in accord with Th.(9.22). We recall the LL-chain headed by $g \in \mathfrak{L}(\check{E}, V)$ and write its tail $y(e)^q = g - H(g, x, V, e)$ with $q = p^e$ with respect to an allowable coordinate $x = (y, \omega)$ for $g(0)$. Refer to Def.6.8). We will often write $H(g, x)$ or $H(g, x, V)$ or $H(g, V)$ or $H(g, x)$ or even simply $H(g)$ for short of $H(g, x, V, e)$ when the omitted symbols are understood.

$$H(g) = \sum_{0 \le i < e} \rho^i(h(i)) \textit{ out of the LL-chain of } g \tag{86}$$

The finite sum is for those p^i-th power of explicit homogeneous elements $h(i)$ of $\wp^{nega}(\check{E})$ for each i while the $H(g)$ in the equality $y(e)^q = g - H(g)$ is understood as the sum taken by ignoring the degrees of the composed individuals. However we must go back to obey the original rules of transformation recorded in each individual member of the composition only when we need to apply any permissible blowups for $\check{E}$, which will become reality only in the course of proving the Main Theorems II-III in the last section of this paper. This methodology may appear too complicated but actually quite simple. Later in the proofs of the Main Theorems the matter of fact is that all we need are the inclusion $H(g) \in \wp^{nega}(\check{E})$ and the general principle on the rule of transformation of $\wp(\check{E})$. This principle is that by any blowup permissible for $\check{E}$ the pullback of every element of $\wp(\check{E}, d)$ with any positive d is divisible by the d-th power of the ideal of the exceptional divisor. When $d < 0$ the transform of every $f \in \wp(\check{E}, d)$ is determined by the rule of transforms of fg^{p^ℓ} and g^{p^ℓ} for $\ell \gg 1$ with any single $g \in \check{\mathfrak{E}}$.

We next define the number $\sigma(H(g))$ for $H(g)$ of Eq.(86) with $g \in \check{\mathfrak{E}}$.

$$\sigma(H(g)) = \sum_{0 \le i < e} s(i) \tag{87}$$

where $s(i) = min\{s \,|\, h(i) \in \wp^{nega}_{\le s}(V)\}$ with

$$\wp^{nega}_{\le s}(V) \;=\; \bigoplus_{0 < j \le s} \wp^{nega}(\check{E}, -j)(V) \tag{88}$$

Here the $h(i)$ are those out of the expression $H(g) = \sum_{0 \le i < e} \rho^i(h(i))$ in which $h(i)$ are defined by LL-chain headed by g. Refer to Eq.(86) and Eq.(83) and Eq.(84).

Remark 9.24. Then $H(g)$ of Eq.(86) has the following properties:

(1) $H(g) \in \wp^{nega}(\check{E})_V \cap I(\mathrm{Sing}(\check{E}))Bl(Z)$

(2) $\mathrm{ord}_\eta(H(g)) > q$ and $\mathrm{ord}_\eta(g - H(g)) = q$ for every $\eta \in V \cap \mathrm{Sing}(\check{E})_{cl}$
(3) $g - H(g) \in \rho^e(\mathcal{O})(V)$ so that $\sqrt[q]{g - H(g)} \in \mathcal{O}(V)$

where $I(\mathrm{Sing}(\check{E}))$ denotes the ideal defining the reduced subscheme $\mathrm{Sing}(\check{E}) \subset Z$. Note that those three properties of $H(g)$ are easily verified by its constructive definition by means of the LL-chain of Eq.(83), Eq.(84), Eq.(86) and $\boxminus(g, q) \subset \wp^{nega}(\check{E})$.

Remark 9.25. Once again we should note that $H(e)$ is a finite sum of terms with various homogeneity degree in $\wp^{nega}(\check{E})_V$. We maintain the rule of transformations of each of the homogeneous terms under the permissible blowups for $\check{E}$. Those degrees are ignored and the terms of $H(e)$ are summed up as if they are elements of $\mathcal{O}(V)$ and the sum is used in making the sense of the equational expression as $y(e)^q = g(0) - H(e)$. However whenever we come to talk about any permissible blowups we must go back to the original homogeneity summands and honor their inherited rule of transformations.

9.11. **Linear combinations of LL-chains.** Refer to the system of edge-inner generators in the sense of Def.(6.15) at each point $\zeta \in V_{cl} \cap \mathrm{Sing}(\check{E})$.

Definition 9.26. Take any pair of LL-heads $g = y^q + \epsilon$ and $g* = y*^{q*} + \epsilon*$ in $\mathfrak{L}(\check{E})_\xi$ where $q = p^e, e > 0$, and $q* = p*^{e*}, e* > 0$,. we then define an $\rho^\ell(\mathcal{O}_\xi)$-linear condination with $\ell \gg 1$ as follows. We first choose ℓ so large that all the differential operators used for defining the LL-chains headed by g and $g*$ annihilate $\rho^\ell(\mathcal{O}_x i)$. Say $e * -e \geq 0$. Then for A and $A*$ in $\rho^\ell(\mathcal{O}_\xi)$ we define

$$C(g, g*; A, A*) = A\rho^{e*-e}(LLch(g)) + A * (LLch(g*)) \tag{89}$$

where $LLch(X)$ denotes the LL-chain headed by X at ξ and the sum is then taken step by step of Eq.(83) and Eq.(84). Note that the combination result is an LL-chain of the adjusted linear combination

$$c(g, g*; A, A*) = A(\rho^{e*-e}(g) + A * (g*) \tag{90}$$

It should be noted that the linear combonation $C(g, g*; A, A*)$ of LL-chains are taken only to the components of the expressions of Eq.(83) and Eq.(84) but not every differentiation-produces used in the steps defining $H^\flat$-operators. This way let us gain the corresponding linear combination of the LL-tails.

Definition 9.27. For each point $\xi \in \mathrm{Sing}(\check{E})$ we define $\rho^\ell(\mathcal{O}_\xi)$-module $\mathfrak{t}(\check{E}; V, \ell)$ consisting of the tails of all LL-chains headed by LL-heads belonging to $\mathfrak{L}(\check{E})_\xi$ where $\ell \gg 1$. See the lemma below.

Lemma 9.28. *In the definition Def.(9.27) let us recall the finite system of edge-inner generators* $(g*, f)$ *of* $\mathfrak{L}(\check{E})_\xi$ *in the sense of Def.(6.15) following after Th.(6.14). Then for a fixed allowable regular system of parameters* x *containing the edge parameters* $y(\xi)_i, 1 \leq i \leq r(\xi)$ *we consider the set* W *of LL-chains of every members of* $(g*, f)$*. Then choose an integer* $\ell(0) > 1$ *such that all the differenciations used in defining the LL-chains in* W *annihilate* $\rho^{\ell(0)}(\mathcal{O}_\xi)$*. Then* $\mathfrak{t}(\check{E}; V, \ell)$ *is independent of all* $\ell \geq \ell(0)$*.*

Theorem 9.29. *we can find an affine open neighborhood* $V \ni \xi$ *and an integer* $\ell \gg 1$ *such that we obtain the following finitely generated* $\rho^\ell(\mathcal{O}_V)$*-module* $\mathfrak{t}(\check{E}; V, \ell)$ *consisting of all the tails of LL-chains headed by LL-heads belonging to* $\mathfrak{L}(\check{E})_V$*. It has the property that locally at every point* $\zeta \in V_{cl} \cap \mathrm{Sing}(\check{E})V_{cl}$ *we have that* $\mathfrak{t}(\check{E}; V, \ell)_\zeta$ *contains the LL-tails of all the LL-heads in* $\mathfrak{L}(\check{E})_\zeta$ *and it is moreover generated by these LL-tails.*

Proof. Infer to Def.(6.15) following after Th.(6.14) and the definition of the module of LL-preheads $L(\check{E}, q)_\xi$ defined by Eq.(54). □

It should be noted that locally at every $\zeta \in V_{cl}$ $\mathfrak{t}(\check{E}; V, \ell)_\zeta$ contains the tails of preheads, so to speak, which have orders > 1. It is necessary to include these prehead type ones to make $\mathfrak{t}(\check{E}; V, \ell)$ to be $\rho^\ell(\mathcal{O})$-module.

10. LL-chains modifications

Pick a $g = y^q + \epsilon \in \mathfrak{L}(\check{E}, q)_\xi$ with $q = p^e, e > 0$. We have the LL-chain headed by g in the manner of Eq.(83) and Eq.(84). Pick integers $N \gg \ell \gg q$ and an affine open neighborhood $V \ni \xi$ such that for every point $\eta \in \mathrm{Sing}(\check{E}) \cap V_{cl}$ there exist

1. $u(i, \eta) \in \rho^N(\mathcal{O}(V))$ and $u(i, \eta) \in \max(\mathcal{O}_\xi)$
2. with $y(i, \eta) = y(i) - u(i, \eta) \in \max(\mathcal{O}_\eta)$ for every $0 \leq i \leq e$.
3. $\rho^N(\mathcal{O}_\eta)$ is annihilated by all those differential operators used for the diff-products to define the $h(i) \in \boxminus(g, q)$ in the LL-chain Eq.(83) and Eq.(84). We assume $V \ni \xi$ small enough that all those differential operators extend throughout V.

Lemma 10.1. *We have the claim above and the modified LL-chain at* η *as follows.*

$$g(i) = y(i, \eta)^{q(i)} + \epsilon(i, \eta) \ \text{with}\ \epsilon(i, \eta) = \epsilon(i) + u(i, \eta)^{q(i)}$$
$$\text{for every } i \text{ of } 0 \leq i \leq e$$

which has the same head g *and the same* $h(i)$ *at* η *for every* i*. Finally the tail of this modified LL-chain vanishes at* η*.*

Proof. Pick a sufficiently small $V \ni \xi$ so that $g(i) = y(i)^{q(i)} + \epsilon(i)$ are all extendable to every point of $V \cap \mathrm{Sing}(\check{E})$. Then pick $N \gg \ell \gg q$ and $u(i,\eta) \in \mathcal{O}_\xi$ having all the properties lusted above. Such $u(i,\eta)$ can be found in the manner of the proof of Lem.(7.14) of (7.4). $\square$

Definition 10.2. We are given a coordinate system $x = (y, \omega)$ at the point $\xi \in \mathrm{Sing}(\check{E})_{cl}$ and a sufficiently small open affine neighborhood $V \ni \xi$ in accord with Lem.(10.1) such that there exist a finite system of generators $\{g_k \in \mathfrak{L}(\check{E})_V\}$ of $\mathfrak{L}(\check{E})_V$ viewed as $\rho^\ell(\mathcal{O}_V)$-module with $p^\ell \geq \max_k\{ord_\eta(g_k)\}$ where η ranges through $\mathrm{Sing}(\check{E}) \cap V_{cl}$. Note that $\check{E}$ is core-edge focused and hence $\mathrm{ord}_\eta(g_k)$ is independent of η. The existence of such $V \ni \xi$ is due to Th.(6.14) of §6.3.

10.1. **Enclosure of** $\mathrm{Sing}(\check{E})$ **by LL-tails.** We write $Y(g)$ for $g(e) = y(e)$ of Eq.(83) and Eq.(84). We call any $(F(g), V(g), Y(g))$ a *local triplet*, or simply *triplet*, of $\check{E}$ at $\xi \in \mathrm{Sing}(\check{E})$. We have the following theorem.

Theorem 10.3. *Write* $(V, Y.Y)$ *in short for* $(F(g), V(g), Y(g))$ *at* ξ *as above. The* $\mathrm{Sing}(\check{E}) \cap V$ *is contained in* Y *for every triplet* $(V, Y.Y)$. *At each point* $\xi \in \mathrm{Sing}(\check{E})$ *the germ of* $\mathrm{Sing}(\check{E})$ *at* ξ *coincides with the intersection of* $Y \cap \mathrm{Sing}(\check{E})$ *for all triplets* $(V, Y.Y)$ *with* $Y \ni \xi$. *Here we need only finitely many selected triplets* $(V, Y.Y)$ *thanks to Zariski topology.*

Proof. Direct from the combination of Th.(10.4) and Th.(10.5). $\square$

Theorem 10.4. *For every LL-chain of Eq.(83) and Eq.(84) at* ξ *we can find an open neighborhood* $V \ni \xi$ *such that the LL-chain extends uniquely to every point of* $V \cap \mathrm{Sing}(\check{E})_{cl}$ *and its scheme-tail defined by* Y *contains* $V \cap \mathrm{Sing}(\check{E})$.

Proof. The Lem.(10.1) applies to every $\eta \in V \cap \mathrm{Sing}(\check{E})$. $\square$

Pick any $\xi \in \mathrm{Sing}(\check{E})_{cl}$ and choose an open neighborhood $V \ni \xi$ such that a system of edge generators $\{(g(i) \in \wp(\check{E}, q(i)\}$ at ξ extends to such at every $\eta \in \mathrm{Sing}(\check{E})_{cl}$. This is possible because of the constancy of $\mathrm{Inv}_\eta(\check{E})$ throughout $\mathrm{Sing}(\check{E})_{cl}$. Pick any $\zeta \in V_{cl} \setminus \mathrm{Sing}(\check{E})$ and we then have at least one $g(i)$ such that $\mathrm{ord}_\zeta(g_i) = \delta < ord_\xi(g(i)) = q(i)$ thanks to the edge decomposition theorem Th.(T:edge-decomp) of (4.3). We then pick a differential operator $\partial \in \mathrm{Diff}^{(\delta)}_{Z,\xi}$ such that $\mathrm{ord}_\zeta(\partial(g(i))) = 1$. Let $v(i,\zeta)$

(1) We choose $N \gg \ell \gg q$ big enough that $\rho^N(\mathcal{O}_\xi)$ is annihilated by is annihilated by all those differential operators used for the diff-products to define the $h(i) \in \boxminus(g,q)$ in the LL-chain Eq.(83) and Eq.(84) at ξ.
(2) We choose $u(i,\zeta) = \rho^N(v(i,\zeta))$. Note that $u(i,\zeta)$ vanishes at ξ by $v(i,\zeta) \in \wp(\check{E}, q(i) - \delta)$ with δ defined above.

Theorem 10.5. *For every LL-chain of Eq.(83) and Eq.(84) at ξ we can find an open neighborhood $V \ni \xi$ such that for every $\zeta \in V_{cl} \setminus \mathrm{Sing}(\check{E})$ there exists a modification of the extended LL-chain of Th.(10.4) which has it tail not to vanish at ζ.*

Proof. Do the same as above except for a change $u(i,\eta)^{q(i)}$ to be in the ideal of $\mathrm{Sing}(\check{E})$ but not to vanish at ζ. This time we replace $y(i)$ by $y(i) - \rho^N(v(i,\zeta))$ and $\epsilon(i)$ by $\epsilon(i) + \rho^{e(i)+N}(v(i,\zeta))$ with $q(i) = p^{e(i)}$. □

Theorem 10.6. *Consider a smooth irreducible locally closed subscheme D which is contained in $\mathrm{Sing}(\check{E})$. Pick an LL-chain of $g \in \mathfrak{L}$ at $\xi \in D$ by Eq.(83) and Eq.(84). Then the LL-tail $g(e)$ of the LL-chain vanishes along D within a neighborhood of $\xi \in D$.*

Proof. Then by Lem.(7.3) of §7 of Part 1 we find $c(j) \in \max(\mathcal{O}_\xi)^2$ such that $\mathfrak{y}(j) = y(j) - c(j) \in I(D)$ where $I(D)$ denotes the ideal defining $D \subset Z$. Replace $y(j)$ by $\mathfrak{y}(j)$ and $g(j)$ by $\epsilon(j) + c(j)^{q(j)}$, say $= \epsilon(j)^+$, with $q(j) = p^{e-j}$ in Eq.(84) of §3. Then both of the $g(j)$ and $h(j)$ are unchanged and the LL-chain Eq.(84) is maintained as it is. Refer to the paragraph 3.6.1.2. on modification of ϵ. As to the change from $\epsilon(j)$ to $\epsilon(j)^+$ of $h(j)$ the permissibility condition $\mathrm{ord}_D(g(0)) \geq q(0)$ implies $\mathrm{ord}_D(g(0)) \geq q(0)$. Hence its tail $g(e) = y(e)$ is unchanged. □

11. Coordinate-free LL-chains and LL-tails

We will make use of $\wp^{nega}(\check{E}) = \bigoplus_{a>0} \wp(\check{E}, -a)$ in the way to build up LL-chains and LL-tails of the heads $g \in \mathfrak{L}(\check{E})$.

Recall the head type $g = y^q + \epsilon$ with $q = p^e, e > 0$, where $\mathrm{ord}_\eta(y) = 1$ and $\mathrm{ord}_\eta(\epsilon) > q$ for all η within an open affine $V \cap \mathrm{Sing}(\check{E})_{cl}$ in the manner of Eq.(52) of Def.(6.12).

Moreover we choose y to be a member of $\cot(\check{E})(V)$. We can have V such that for every $\ell \gg 1$ these types of g as above generate $\mathfrak{L}(V)$ as $\rho^\ell(\mathcal{O})$-module.

11.1. **Symbolic power and degree change.** We will need and use the following notation for the symbolic powers of the ideal $I(\mathrm{Sing}(\check{E}))$

of $\mathrm{Sing}(\check{E}) \subset Z$. We write Sing for $\mathrm{Sing}(\check{E})$ when $\check{E}$ is understood.

$$I^{(m)}(\mathrm{Sing}) = \big\{ f \in \mathcal{O}_Z : ord_\eta(f) \geq m, \forall \eta \in \mathrm{Sing}(\check{E})_{cl} \big\} \tag{91}$$

for each integer $m > 0$. Here the the $\forall \eta$ on f means the ranges η through the chosen domain V_{cl}.

The LL-chains and their tails depend upon the choice of coordinate system $x \ni y$ allowable to the given $\check{E}$. Refer to Def.(6.8) and Def.(9.20) Eq.(83), Eq.(84) and Eq.(85) of §9.8 for LL-chains and their tails for any LL-head $g = y^q + \epsilon$ chosen from the heads-module $\mathfrak{L}(\check{E}, q)$ of Def.(6.15) due to Th.(6.14) of §6.3.

The process called LL-chain from the given LL-head to its LL-tail shows a constructive LLUED process that is the essence of the master key theorems Th.(9.2) and Th.(9.3) of §9.1.

The head-module $\mathfrak{L}(\check{E}, \ell)$ has the global coherence as $\rho^\ell(\mathcal{O}_Z)$-module with any $\ell \gg 1$. However the definition of the LL-chains and LL-tails depend upon the choice of coordinate systems $x \ni y$ at $\xi \in \mathrm{Sing}(\check{E})_{cl}$. It is strictly local depending upon $x \ni y$.

In order to establish the proof of global resolution of singularities we must overcome the shortcoming of local nature of LL-chains closely depending upon the choice of coordinate system $x \ni y$. The task of this section is to define coordinate-free tails-module amenable to globalization. For this purpose we will use a little more abstract notions of LL-chans and LL-tails than the ones obtained by the explicit constructive LL-chains of §9.8.

11.2. **Abstract LL-chains with $\wp^{nega}$.** Review the edge data of $\wp(\check{E})$.

$$\big(1, \cdots, 1, q_{t+1}(\xi), \cdots, q_r(\xi)\big) \ \textit{defined by Eq.(26) of } §4.1. \tag{92}$$

In this section and from now on we use a little bit modified sequence of edge exponents. It should be noted that the edge exponent numbers for $\check{E}$ is constant along the $\mathrm{Sing}(\check{E})_{cl}$. Moreover we are only interested in those exponents greater than 1. Thus from now on we choose the following type of exponents sequence.

$$\big(p^e(= q_r), p^{e-1}, p^{e-2}, \cdots, p^1, p^0 = 1\big) \tag{93}$$

which has the length equal to $e + 1$ with $p^e = q_r(\xi)$ of Eq.(92).

The Diff_Z is the $\mathcal{O}_Z$-module of all differential operators in $\mathcal{O}_Z$ while $\mathrm{Diff}^*_Z \subset \mathrm{Diff}_Z$ denotes that of those vanishing on the base field $\mathbb{K}$. Recall that $\partial \in \mathrm{Diff}^*_Z \subset \mathrm{Diff}_Z$ if and only if $\partial(\mathbb{K}) = (0)$. Let $\mathrm{Diff}^*_{Z/Z(q)}$ denotes the submodule of Diff_Z consisting of those ∂ having $\partial\rho^e(\mathcal{O}_Z) = (0), q = p^e$. They are all global coherent $\mathcal{O}_Z$-modules.

With the integers of Eq.(93) and any sufficiently large integer $\ell \gg e$, we locally have a finite system of generators of $\mathfrak{L}(\check{E}, q)$ as $\rho^\ell(\mathcal{O}_Z)$-module. We can always choose a system of generators each of which is writen in a form $g = y^q + \epsilon$ where y is in $\cot(\check{E}, \ell)$ and has order 1 while ϵ has order $> q$. Recall $\cot(\check{E}, \ell)$ of Def.(4.14) of §4.2, and the heads module $\mathfrak{L}(\check{E}, q) \subset \wp(\check{E}, q)$ by Def.(6.12) and Th.(6.14) of §6.3.

Remark 11.1. We will be choosing a sufficiently large integer $\ell \gg e$. This ℓ does not effect the definition per se of various operators we are going to define, Λ, τ, θ and a few preliminary symbols. Important point is only that ℓ is big enough. In fact the bigger will be better for our purpose. How big in rigor will be made clear in the end. Roughly speaking just for this point we say that all the operators should be $\rho^\ell(\mathcal{O})$-linear and the created modules should become naturally $\rho^\ell(\mathcal{O})$-modules.

11.3. **Λ and τ operators.** We now define the following operation called $\tau\Lambda$ which is applied to any graded $\rho^\ell(\mathcal{O})$-submodule of $Bl(Z)$. Here we should recall that $Bl(Z) = Bl^{posi}(Z) \bigoplus Bl(Z, 0) \bigoplus Bl^{nega}(Z)$ of Definition 2.2 of §2. Recall $\mathfrak{L}(\check{E})$ of Eq.(55) of §6.3.

Remark 11.2. Here is a general agreement. A graded $Bl(Z)$-module X is said to be coherent if it is finitely generated as $\rho^k(Bl(Z))$-module with an integer $k \geq 0$ and its every homogeneous part is coheretnt as $\rho^k(\mathcal{O})$-module.

Definition 11.3. Let $\mathfrak{L}_e(0) = \mathfrak{L}(\check{E}, p^e)$ to begin with. The number e is the one chosen by *Eq.*(93). We will then define $\mathfrak{L}_{e-i}(j)$ for $j = 0, 1, 2, \cdots, \infty$ and for $i = 0, 1, 2, \cdots, e$ by double induction starting from $i = 0$ with $\mathfrak{L}_e(0)$ as above. Then for $i > 0$ we proceed on inductive definition, descending on i and ascending on j, as follows. For arbitrary graded $X \subset Bl(Z)$ we define Λ acting on X as follows.

$$\Lambda_{e-i} X = X + \left(\mathrm{Diff}^*_{Z/Z(p^i)} X\right)\left(\mathrm{Diff}_Z X\right) \tag{94}$$

Recall that $\mathrm{Diff}^* = \{\partial \in \mathrm{Diff} : \partial(\mathbb{K}) = (0)\}$.

Remark 11.4. If $X \subset \tilde{\wp}(\check{E})$ then $\left(\mathrm{Diff}^*_{Z/Z(p^i)} X\right)\left(\mathrm{Diff}_Z X\right)$ is contained in $\tilde{\wp}(\check{E})$ because this is a Diff-product of $X \subset \tilde{\wp}(\check{E})$. Adding X does not affect the same inclusion.

Next for arbitrary graded $Y \subset Bl(Z)$ we define τ as follows.

$$\tau_{e-i}(Y) = Y \bigcap \rho^i\Big(\, \rho^{e-i}(\cot(\check{E})) + I(\mathrm{Sing})^{(p^{e-i}+1)} Bl(Z)\,\Big) \tag{95}$$

Remark 11.5. If $Y \subset \tilde{\wp}(\check{E})$ then $\tau(Y) \subset \tilde{\wp}(\check{E})$ because the latter is a subst of Y.

Begin with the initial case $\mathfrak{L}_e(0) = \mathfrak{L}(\check{E}, p^e)$ by definition:

$$\text{(96)} \qquad If\ Y = \mathfrak{L}_e(0)\ then\ (\tau_e\Lambda_e)(X) = \tau(X) = X.$$

And then we will go up on j with each fixed i we define $\mathfrak{L}_{e-i}$ as follows.

$$\text{(97)} \qquad \mathfrak{L}_{e-i}(j+1) = \tau_{e-i}(\Lambda_{e-i}\mathfrak{L}_{e-i}(j))\ and$$
$$\mathfrak{L}_{e-i}(\infty) = \lim_{j\to\infty}\mathfrak{L}_{e-i}(j) = \bigcup_{j\geq 0}\mathfrak{L}_{e-i}(j)$$

Remark 11.6. Assume that $\mathfrak{L}_{e-i}(0)$ is a coherent $\rho^e(Bl(Z))$-module. Then $\mathfrak{L}_{e-i}(j)$ is also a coherent $\rho^e(Bl(Z))$-module for all $j > 0$ including $j = \infty$. In fact $\Lambda(\mathfrak{L}_{e-i}(0))$ is coherent $\rho^e(\mathcal{O})$-module because Diff_Z and $\mathrm{Diff}^*_{Z/Z(p^i)}$ are $\mathcal{O}$-modules and hence $\rho^e(\mathcal{O})$-modules. As for τ-operator we have that $\rho^e(\cot(\check{E}, \ell))$ is coherent $\rho^e(\mathcal{O})$-module but not so as $\mathcal{O}$-module in general. The $I(\mathrm{Sing})^{(p^{e-i}+1)}Bl(Z)$ is a coherent $Bl(Z)$-module. We conclude that $\mathfrak{L}_{e-i}(j)$ is a coherent $\rho^e(Bl(Z))$-module by induction on $j = 1, 2, \cdots$. Moreover $\mathfrak{L}_{e-i}(j)$ is monotone increasing in j by definition. Hence by the Noetherian property of $\mathcal{O}$ and by the compare Zariski topology we conclude that $\mathfrak{L}_{e-i}(\infty) = \mathfrak{L}_{e-i}(j)$ for all $j \gg e$ and hence $\mathfrak{L}_{e-i}(\infty)$ is coherent $Bl(Z)$-module, too.

Proposition 11.7. *Assume $\mathfrak{L}_{e-i}(0) \subset \tilde{\wp}(\check{E})$ then we have $\mathfrak{L}_{e-i}(j) \subset \tilde{\wp}(\check{E})$ for all $j \geq 0$ including $j = \infty$. Moreover for $j > 0$ we have that $\mathfrak{L}_{e-i}(j)$ is contained in $\rho^e(\cot(\check{E})) + I(\mathrm{Sing})^{(p^{e-i}+1)})Bl(Z)$. Refer to Prop.(4.16) for $\cot(\check{E})$ and recall $\cot(\check{E}) = \lim_{\ell\to\infty}\cot(\check{E}, \ell)$.*

Proof. $\tilde{\wp}(\check{E})$ is an algebra and diff-closed thanks to Lem.(5.8). The second claim is by the definition of τ. □

12. GLUED: the globalization of LLUED

We have obtained the sequence of $\mathfrak{L}_{e-i}(j), j = 0, 1, 2, \cdots, \infty$, for each i of $e > i \geq 0$ in the sense of Eq.(97) by means of Eq.(94) and Eq.(95) of §11.3 under the assumption that $\mathfrak{L}_{e-i}(0)$ had been defined for each $i, 0 \leq i \leq e - i$. The next important operator is to connect the sequences obtained above in order to complete the global coherent GLUED-chain diagram. Namely we want to define a new operator denoted by

$$\sigma_{e-i+1} : \mathfrak{L}_{e-i+1}(\infty) \searrow \mathfrak{L}_{e-i}(0)\ for\ each\ i : 1 \leq i \leq e$$

which will be called the step-down operators. Keep it in mind that the suffix $e - i + 1$ is important for the indication of source of the map especially when we use its inverse later.

12.0.1. *Symbols* $\| X \|$ *and* $[W]_i$. We use the following symbol for adding ideals by ignoring homogeneity degrees and then taking the sum of those ideals into $\mathcal{O}$. This result will be written simply $\| X \|$ which is an ideal in $\mathcal{O}$. Recall $Bl(Z) = Bl^{posi}(Z) \bigoplus Bl(Z,0) \bigoplus Bl^{nega}(Z)$.

$$\| X \| \subset Bl(Z,0) \quad for\ every\ X \subset Bl(Z) \tag{98}$$

which signifies the transform which . disregard the homogeneity degrees of the summands in the graded expression of X. Then make the sum of the degree-free summands in $\mathcal{O}_Z$. The result will be denoted by $\| X \|$. We introduce another symbol $[(\ )]_i$ for every integer $i \in \mathbb{Z}$ which signifies cutting out all those homogeneous parts whose degrees are less than i within ( ). To be precise it is defined as follows.

$$\begin{gathered} For\ every\ graded\ W = \textstyle\bigoplus_k W(k) \subset Bl(Z) \\ we\ define\ [W]_s = \textstyle\bigoplus_{k \geq s} W(k) \end{gathered} \tag{99}$$

where it should be noted that if W is contained in $\tilde{\wp}(\check{E})$ and $\rho^\ell(\mathcal{O})$-coherent then so is every $[W]_s$.

12.0.2. *Operator* θ *and its inverse.*

Remark 12.1. Recall the symbol $\| \ \|(k)$ of Eq.(98) in the sense of Def.(12.3) after Lem.(12.2) and the symbol $[\,]_s$ of Eq.(99). See §12.0.1. Using those symbols we define the operator θ by Def.(12.3) after Lem.(12.2) below.

Recall the $\| \ \|(k)$ of Eq.(98). First examine the case with $i = e$ of $q = p^e$ and then by descending induction on i of $p^{e-i}, 0 \leq i \leq e-1$.

Lemma 12.2. *Recall* $\tilde{\wp}(\check{E}) = \wp^{posi}(\check{E}) \bigoplus \tilde{\wp}(\check{E},0) \bigoplus \wp^{nega}(\check{E})$. *Note* $\tilde{\wp}(\check{E},0) \subset \wp(\check{E},0)$ *but not necessarily equal. By Prop.(11.7) we have*

$$\tilde{\wp}(\check{E}) \supset \mathfrak{L}_i(j)\ for\ \forall (i,j)\ and\ \tilde{\wp}(\check{E}) \supset \mathfrak{L}_i(\infty)\ for\ \forall i \tag{100}$$

Proof. Refer to Lem.(5.8), Lem.(5.9) and Th.(5.11). □

Definition 12.3. We have the natural contraction map as follows.

$$\theta_{e-i} : \mathfrak{L}_{e-i}(\infty) \rightarrowtail \|(\mathfrak{L}_{e-i}(\infty))\|_{p^{e-i}}\ for\ 0 \leq i \leq e \tag{101}$$

of which the target is contained in $Bl^{posi}(Z, p^{e-i})$.

Here it is important that we are assuming that $\mathfrak{L}_{e-i}(\infty)$ were already defined. Indeed the definition of $\mathfrak{L}_{e-i}(j)$ for all (i,j) and $\mathfrak{L}_{e-i}(\infty)$ for each i must be and will be defined by multi-induction on (i,j). There one important and decisive step is the way to select the definition of deriving $\mathfrak{L}_{e-i}$ out of $\mathfrak{L}_{e-i+1}$ which we call level step down of $\mathfrak{L}$. The rest of the operations are all natural and already defined. The key point is the Eq.(102) of ∇_{e-i+1} below which will be stated precisely after some preliminary remarks.

Remark 12.4. Note that for any ideal contained in the target ideal of θ we have that $\theta(\theta^{-1})$ acts as the identity because we are taking ideal-sum only in the definition of θ.

12.1. **Define the step down operator σ on $\mathfrak{L}$-levels.**

Definition 12.5. Here is the definition of the step down of $\mathfrak{L}_{e-i+1}$ from the level $e-i+1$ to $e-i$ for each $0 < i < e-1$ symbolically denoted:

$$\sigma_{e-i+1} : \mathfrak{L}_{e-i+1}(\infty) \searrow \mathfrak{L}_{e-i}(0) \tag{102}$$

where we assume $\mathfrak{L}_{e-i+1}(\infty) \subset \tilde{\wp}(\check{E})$ and define $\sigma_{e-i}(0)$ as follows.

$$\mathfrak{L}_{e-i}(0) = \big(\theta_{e-i+1}\big)^{-1}\Big(\big(\theta_{e-i+1}\big(\mathfrak{L}_{e-i+1}(\infty)\big)\bigcap \rho(\mathcal{O})\Big)$$

Here it is important to recall that the source of the map θ_{e-i} is $\mathfrak{L}_{e-i}(\infty)$ by Def.(12.3). Thus the inverse map $\theta_{e-i+1}\big)^{-1}$ defines $\sigma_{e-i}(0)$ as a submodule of $\mathfrak{L}_{e-i}(\infty) \subset \tilde{\wp}(\check{E})$.

Remark 12.6. Note here that some elements of $\mathfrak{L}_{e-i+1}(\infty) \subset \tilde{\wp}(\check{E})$ may be sent by θ_{e-i+1} into the outside of $\tilde{\wp}(\check{E})$. However the source of the map θ is contained in $\tilde{\wp}(\check{E})$. Hence $\theta^{-1}(C_{e-i+1})$ is contained in $\tilde{\wp}(\check{E})$. We thus conclude that $\mathfrak{L}_{e-i+1}(\infty) \subset \tilde{\wp}(\check{E}) \Rightarrow \mathfrak{L}_{e-i}(0) \subset \tilde{\wp}(\check{E})$. As a matter of fart in our situation we start with

$$\mathfrak{L}_e(0) = \mathfrak{L}(\check{E}) \subset \wp^{posi}(\check{E}, p^e) \subset \tilde{\wp}(\check{E}) \tag{103}$$

Therefore by the property of $\tau_{e-i}\Lambda_{e-i}$ and Δ_{e-i+1} we have proven the following theorem.

Theorem 12.7. *We have $\mathfrak{L}_{e-i}(j) \subset \tilde{\wp}(\check{E})$ for all (i, j) with $0 \le i \le e$ and with $0 \le j \le \infty$.*

13. The global coherent GLUED-chain diagram

Remark 13.1. The sequence of the global coherent LUED-chain (GLUED-chain in short) with the property of Th.(12.7) is obtained as follows.

$$
\begin{aligned}
S : glued\ chain\quad & \Big(\mathfrak{L}(\check{E}, p^e) = \Big)\, \mathfrak{L}_e(0) \rightarrowtail \mathfrak{L}_e(1) \rightarrowtail \mathfrak{L}_e(2) \rightarrowtail \mathfrak{L}_e(3) \cdots \rightarrowtail \\
& \mathfrak{L}_e(\infty) \searrow \mathfrak{L}_{e-1}(0) \rightarrowtail \mathfrak{L}_{e-1}(1) \rightarrowtail \mathfrak{L}_{e-1}(2) \rightarrowtail \\
& \cdots\cdots\cdots\cdots\cdots\cdots \rightarrow \\
& \mathfrak{L}_{e-i+1}(\infty) \searrow \mathfrak{L}_{e-i}(0) \rightarrowtail \cdots\cdots\cdots\cdots\cdots\cdots \\
& \cdots\cdots\cdots\cdots\cdots\cdots \searrow \\
& \mathfrak{L}_1(0) \rightarrowtail \cdots \rightarrowtail \mathfrak{L}_1(\infty) \searrow \mathfrak{L}_0(0) \rightarrowtail \cdots \rightarrowtail \mathfrak{L}_0(\infty) \\
& \subset \Big(\wp^{posi}(\check{E}, p^e) \cap \rho^e(Bl(Z,1))\Big).
\end{aligned} \tag{104}
$$

where all the $\rightarrowtail$ are the applications of $\tau\Lambda$ in the sense of Eq.(94) and Eq.(95), while the $\searrow$ are σ of Def.(12.5) defined above.

13.1. Define $\mathfrak{T}(\check{E})$ and $Cot(\check{E})$.

Recall Def.(12.3) of σ_{e-i} after Eq.(98) and examine the case of $i = e$ and consider $\sigma_0 : \mathfrak{L}_0(\infty) \searrow \mathfrak{L}_{-1}(0) = \sigma_0(\mathfrak{L}_0(\infty))$. Note that $\mathfrak{L}_0(\infty) = \tau(\mathfrak{L}_0(\infty)) \subset I(\mathrm{Sing})Bl(Z)$.

Definition 13.2. Define the global tails module $\mathfrak{T}(\check{E})$ as follows.

$$
\mathfrak{T}^\sharp(\check{E}) = \Big(\mathfrak{L}_0(\infty) \cap \wp^{posi}\Big) + \wp^{nega} \tag{105}
$$

where each of the two summands are contained in $\tilde{\wp}(\check{E})$
and $\mathfrak{T}^\flat(\check{E}) = \|\, \mathfrak{T}^\sharp(\check{E})\| \subset \mathcal{O}_Z$

and then we define the following $Cot(\check{E})$ which will be called the cotangent GLmodule of $\check{E}$

$$
Cot(\check{E}) \;=\; \sqrt[q]{\Big(\mathfrak{T}^\flat(\check{E}) \bigcap \rho^e(\mathcal{O})\Big)} \quad where\ q = p^e \tag{106}
$$

(1) In view of all those LL-chains needed to consider, we have a positive integer s such that $\mathfrak{T} \subset [\ \tilde{\wp}(\check{E})\ ]_{-s}$.
(2) $Cot(\check{E})$ is viewed as a $\rho^\ell(\mathcal{O})$-submodule of $\mathcal{O}$ with $\ell \gg \bar{\ell}$ in the sense of §14.3. Note that $\mathfrak{T}(\check{E})$ is a global coherent $\rho^\ell(\mathcal{O}_Z)$-module of $\tilde{\wp}(\check{E})$ with $\ell \gg e$ where ℓ is large enough integer in the sense explained later by Th.(14.8) of §14.3.

At this point we just say that all the differential operators in Diff_Z needed for the definition of the GLUED-chain diagram of Eq.(104) and those needed for local LL-chains, too.

14. Local LL-chains vs global GLUED-diagram

The $\mathfrak{T}^\sharp(\check{E})$ combined with $\mathfrak{T}^\flat(\check{E})$ exhibit the relation between the global coherent GLUED-chain diagram of Rem.(13.1) and all the local LLUED-chains of Eq.(83)+Eq.(84) of §9.9 at every point of $\mathrm{Sing}(\check{E})_{cl}$.

Theorem 14.1. *Consider every LL-chain headed by $g = y^e + \epsilon \in \mathfrak{L}(\check{E})$ in the manner of Eq.(82) of §9.9. Then we have that the LL-chain headed by $g = g(0)$ of Eq.(83)+Eq.(84) is related with the GLUED-chain diagram of Eq.(104). Choose a finite system of generators of $\mathfrak{L}(\check{E})$ with those g of Eq.(82). Here recall that $\mathfrak{L}(\check{E})$ is viewed as $\rho^\ell(\mathcal{O}_\xi)$-module with a chosen and fixed $\ell \gg e$. We then claim*

1. *Every $\mathfrak{L}_i(j)$ of Eq.(104) is contained in $\tilde{\wp}(\check{E})$ for all (i, j).*
2. *$\mathfrak{L}_i(j) \subset \rho^e(\cot(\check{E})) + \Big(\mathfrak{L}(\check{E}) \bigcap I(\mathrm{Sing})Bl(Z, p^e)\Big)$ for all (i, j) including $j = \infty$ where $\cot(\check{E}) = \lim_{\ell\to\infty} \cot(\check{E}, \ell))$*
3. *In particular $\mathfrak{L}_0(\infty)$ is congruent to $\rho^e(\cot(\check{E}))$ modulo addition of a certain subalgebra of $\wp^{nega}(\check{E})$.*
4. *$\mathfrak{T}(\check{E}) \subset \Big(\big(\cot(\check{E})\big) + I(\mathrm{Sing})^{(2)}Bl(Z)\Big) \cap \tilde{\wp}(\check{E})$*

Proof. Note that the sequence Eq.(104) start with $\mathfrak{L}(\check{E})$ which is contained in $\Big(\rho^e(\cot(\check{E})) + I(\mathrm{Sing})^{p^e+1}Bl(\check{e}, p^e + 1)\Big) \cap \big(\tilde{\wp}(\check{E})\big)$. The Λ of Eq.(94) is operations of differentiation-procucts which maps $\tilde{\wp}(\check{E})$ into itself because it is Diff-closed and algebra property. Moreover the factor τ of Eq,(95) of $\tau\Lambda$ denoted by $\rightarrowtail$ is acting as replacement by a subset of $\Big(\rho^e\big(\cot(\check{E}, \ell)\big) + \rho^i\big(I(\mathrm{Sing})^{(p^{e-i}+1)}\big)Bl(Z)\Big)$ for every $\mathfrak{L}_i(j)$. The same is true for ∇ denoted by $\searrow$ because of the souce of the operation. Thus (1) is proven. The (2) is again by the effect of τ_i for all $e \geq i \geq 0$. There the point is that τ-operator is decomposed into homogeneous operators. (3) follows from (2). $\square$

We now go back to local LL-chain at $\xi \in \mathrm{Sing}(\check{E})_{cl}$ which was denoted by $g(j) = y^{p^e} + \epsilon(j) = g(j + 1)^p + h(j)$ with $e \geq j \geq 0$ of Eq.(83)+Eq.(84) of §9.9 where $y \in \cot(\check{E}) = \lim_{\ell\to\infty} \cot(\check{E}, \ell)$ of Eq.(31) of Def.(4.18) in §4.2. We choose $g = g(0) = y^{p^e} + \epsilon(0) \in \mathfrak{L}(\check{E})$ such that $\mathrm{ord}_\xi(\epsilon(0)) > p^e$.

Theorem 14.2. *Pick any LL-chain headed by $g(0)$ at ξ as above. We then claim that the LL-chain headed by $g(0)$ is imbeded into the global GLUED-diagram of Eq.(104) of Rem.(13.1) up to appropriate p-powers* *of the components of the LL-chain in Eq.(83)+Eq.(84). To be precise:*

$$\rho^i(g(i)) \in \theta_{e-i}(\mathfrak{L}_{e-i}(\infty)) = \|\mathfrak{L}_{e-i}(\infty)\|_{e-i} \quad \text{for every } i : 0 \leq i \leq e \tag{107}$$

Recall the operator $\theta_{e-i} : \mathfrak{L}_{e-i}(\infty) \rightarrowtail \|(\mathfrak{L}_{e-i}(\infty))\|_{e-i}$ of Def.(12.3). In particular the LL-tail $g(e)$ headed by $g = g(0)$ of the LL-chain Eq.(84) is contained in the following ideal.

$$g \in \mathfrak{T}(\check{E}) = \theta_e\big(\mathfrak{L}_0(\infty)\big) \subset \rho^e\big(\operatorname{cot}(\check{E})\big) + I(\mathrm{Sing})^{(2)}\tilde{\wp}(\check{E})\Big) \tag{108}$$

in the sense of Def.(13.2).

Proof. Note that g-cleaning of g is a unit multiplication of g. Hence any Diff-procduct for cleaned g is equal to a Diff-product of the original g with the same pair of orders. In particular $H^\flat$ operators of §9.7 in (I)-(III) are all Diff-products contained in the corresponding members of GLUED-diagram of Rem.(13.1) Refer to Diff-product $H^\flat$ operators of §9.7 of §9.6. The condition of τ-operators and Δ-operators do not affect the containment in the same sense as above. □

The theorem Th.(14.2) show the relation between the local LL-chains Eq.(83)+Eq.(84) and the global GLUED-diagram Eq.(104) of Rem.(13.1). In particular we have the following important colorary.

Corollary 14.3. *We have the following existence theorem. At every $\xi \in \mathrm{Sing}(\check{E})_{cl}$ every element y of $\operatorname{cot}(\check{E})$ with $\operatorname{ord}_\xi(y) = 1$ is induced by an element of $\mathfrak{T}(\check{E})_\xi$ modulo $\max(\mathcal{O}_\xi)^2$. In other words we have*

$$\mathfrak{T}^\sharp(\check{E}) = \theta_1\big(\mathfrak{L}_0(\infty)\big) \equiv \operatorname{cot}(\check{E}) \quad \operatorname{mod}\ \Big(I(\mathrm{Sing}(\check{E}))Bl(Z,1) + \textstyle\bigoplus_{j\geq 2} Bl(Z,j)\Big) \tag{109}$$

Proof. The heading generators of $\mathfrak{L}(\check{E})$ include generators of $\operatorname{cot}\check{E}$. Moreover the Diff-prodicts included in those operators $\Lambda_{e-i}, 0 \leq i \leq e$, preserve the entire $\rho^e(\operatorname{cot}(\check{E}))\ \operatorname{mod}\ [Bl(Z)]_2$ by the nature of differentiations invalid in Λ in all steps of LLUED and GLUED. □

Theorem 14.4.

$$\mathfrak{L}_0(\infty) = \mathfrak{L}_0(\infty) \bigcap \rho^e(Bl(Z,1)) + \mathfrak{L}_0(\infty) \bigcap \rho^r\big(\wp^{nega}(\check{E})\big) \tag{110}$$

for every integer $r \gg e$.

Proof. Infer to Lem.(5.9) of §5. It is important to note that the □

14.1. **Rule of transformation of $\mathfrak{T}(\check{E})$.** We examine the effect of blowups permissible for $\check{E}$. In particular our interest is focused into its effect on $\mathfrak{T}(\check{E}) = \theta_0(\mathfrak{L}_0(\infty))$.

Lemma 14.5. *We have $\rho^e\big(Cot(\check{E})\big) \subset \mathfrak{L}_{e-i}(\infty)$ for all i.*

Proof. It is clear that $\Big(\rho^e\big(\cot(\check{E})\big) \subset \mathfrak{L}_{e-i}(0)$ implies $\rho^e\big(\cot(\check{E})\big) \subset \mathfrak{L}_{e-i}(\infty)$. Now assume this inclusion for $e-i>0$ and we want to prove $\rho^e\big(\cot(\check{E})\big) \subset \mathfrak{L}_{e-i-1}(0)$.

□

Theorem 14.6.

$$\rho^e\big(\mathfrak{T}(\check{E})\big) \subset \mathfrak{L}^\circ(\check{E}) + \wp^{nega}(\check{E}) \tag{111}$$

$$\text{where } \mathfrak{L}^\circ(\check{E}) = \big(\mathfrak{L}(\check{E})\big)\bigcap \rho^e\big(\mathcal{O}(\check{E})\big)$$

$$\mathfrak{S}\Big(\mathfrak{T}(\check{E}),\ 1\Big) \subset I(\mathrm{Sing}(\check{E}))Bl(Z)$$

where the symbol $\mathfrak{T}(\check{E})$ should be understood to be an ideal in $\mathcal{O}_Z$, so that $(\mathfrak{T}(\check{E}), 1)$ signifies an ideal exponent in Z. To be rigorous we should have written $\|\mathfrak{T}(\check{E})\|_0$ instead of $\mathfrak{T}(\check{E})$.

Proof. Recall that $\ell \gg \bar{\ell}$ of Th.(14.8) of §14.3 and that we are viewing $\mathfrak{L}(\check{E})$ and all $\mathfrak{L}_{e-i}(j)$ as $\rho^\ell(\mathcal{O}_Z)$-modules. The key is that the global GLUED-chain diagram of §13 is to begin with $\mathfrak{L}(\check{E})$ and then making a finite number of repetition of either adding a submodule of $\wp^{nega}(\check{E})$ or replacing by submodule. Thus the first line of the theorem is proven. The second line is implied by the first. In fact we have $\mathfrak{S}(\mathfrak{L}(\check{E}), q)$ contains $\mathfrak{S}(\check{E})$ which means every LSB permissible for $\check{E}$ is permissible for $(\mathfrak{L}(\check{E}), q)$. Moreover every LSB permissible for $\check{E}$ are always so for $wp^{nega}(\check{E})$ by the rule of transformation of the negative degree parts in general. See §5 and §7.3. □

Corollary 14.7. *For every* LSB $\in \mathfrak{S}(\check{E})$ *the transform $\check{E}'$ of $\check{E}$ by the* LSB *has the property that* $\mathrm{Sing}(\check{E}')$ *is contained in the transform of* $\mathrm{Sing}(\check{E})$*,*

Proof. Strait forward by the second inclusion of Th.(14.6). □

14.2. **Cotangent AR-heading.** In this section we examine the ambient reduction, AR in short, by means of $\tilde{\wp}$-extension of cotangent space. To be precise we first recall

$$\mathfrak{T}(\check{E}) = \theta_0\big(\mathfrak{L}_0(\infty)\big) \subset \rho^e\Big(\cot(\check{E})\big) + \Big(I(\mathrm{Sing}(\check{E}))^{(2)}\tilde{\wp}(\check{E})\Big) \tag{112}$$

$$\textit{and } \mathfrak{L}_0(\infty) \equiv \rho^e(\cot(\check{E})) \mod I(\mathrm{Sing}(\check{E}))\tilde{\wp}(\check{E})$$

in view of Eq,(105) of Def.(13.2) after the GLUED-chain diagram of Eq.(104). Th.(14.2) and its Cor.(14.3), in particular Eq.(109), of §14

14.3. **How large $\ell \gg e$ should be?** We have used such a supportive assumption $\ell \gg e$ in several occasions until now.

Theorem 14.8. *Here we summarize all the impositions on ℓ big enough for all our work and we prove the existence of single lower bound $\bar{\ell}$ to choose a single large enough $\ell \gg 0$ for the all the reasoning and claims accompanied by an ambiguous symbol $\ell \gg 0$ up to now.*

(1) *$\bar{\ell}$ is chosen to have $\mathrm{cot}(\check{E}, \ell) = \mathrm{cot}(\check{E})$ for all $\ell \geq \bar{\ell}$. Recall Def.(4.14) and Prop.(4.16) of §4.2.*
(2) *For each integer $a > 0$ we need only finitely many differential operators which we need to obtain generators of $[\wp^{nega}(\check{E})]_{-a}$ as $\mathcal{O}$-module. Hence we have a bound $\bar{\ell}$ such that those generating operators are all chosen in $\mathrm{Diff}_{Z/Z(p^e)}, \forall e \geq \bar{\ell}$. Refer to $D(m, a, d)$ of Def.(5.1) of §2.1 where $d \leq p^e$ are enough.*
(3) *$\bar{\ell}$ is such that $\rho^{\bar{\ell}}(\mathcal{O})$ are annihilated by the necessarily needed differential operators to make up the GLUED-chain diagram of Rem.(13.1). Refer to Rem.(11.6) applied to every process $\mathfrak{L}_i(0) \to \cdots \to \mathfrak{L}_i(\infty), 0 \leq i \leq e$. Note that the sequence is monotone increasing and $\mathcal{O}_Z$ is Noetherian over the compact scheme with Zariski topology.*
(4) *$\bar{\ell} \geq e$ is such that for every point $\xi \in \mathrm{Sing}(\check{E})$ we have an open affine neighborhood V_ξ of $\xi \in Z$ and a system of of elements $\{g_{\xi,j} \in \mathfrak{L}(\check{E})(V_\xi)\}$ with finitely many j such that*
   (a) *$g_{\xi,j} = {y_{\xi,j}}^e + \epsilon_{\xi,j}$ with $g_{\xi,j} \in \mathrm{cot}(\check{E})(V_\xi)$*
   (b) *$\mathrm{ord}_\eta(y_{\xi,j}) = 1$ and $\mathrm{ord}_\eta(\epsilon_{\xi,j}) > q = p^e$ and*
   (c) *$g_{\xi,j}$ is heading an LL-chain $C_{\xi,j}$ at every $\eta \in \mathrm{Sing}(\check{E})_{cl} \cap V_\xi$*
   (d) *only such differential operators belonging to $\mathrm{Diff}_{Z/Z(p^\ell)}(V_\xi)$*
   (e) *with a common integer $\ell_\xi \geq \bar{\ell}$ independent of η.*
   (f) *Then $\mathrm{Sing}(\check{E})$ is covered by a finite number of such V_ξ*
   (g) *and hence we can choose $\bar{\ell}$ that does the works for all ξ.*
(5) *Thus in conclusion there exists an integer $\bar{\ell} > e$ such that the whole chosen LL-chains and GLUED-chain diagram are created by using only finitely many differential operators which allow us to select a single $\bar{\ell}$ and any $\mathrm{Diff}_{Z/Z(\ell)}$ with $\ell \geq \bar{\ell}$.*

Proof. It is evident by definition in view of the finite generation of $\mathrm{Diff}_{Z/Z(k)}$ as $\rho^\ell(\mathcal{O}_Z)$-module for every positive integer k. □

Definition 14.9. From now on we will always assume $\ell \gg \bar{\ell}$ with $\bar{\ell}$ of the theorem Th.(14.8). The reason for writing $\ell \gg \bar{\ell}$ instead of $\ell \geq \bar{\ell}$ is that in later sections we may have to use differential operators, finitely many all the way to the end of this paper, which has not been mention explicitly before but fitting to the list of conditions of Th.(14.8). This is the reason for our conventional notation $\ell \gg \bar{\ell}$ in each of the later work.

14.3.1. *$\{g\}$ of $\mathfrak{L}$ and $\{h\}$ of $Cot(\check{E})$.*

(1) Choose a finite system of $\{g_k \in \mathfrak{L}(\check{E})\}$ such that each g_k is of the form.

$$g_k = \rho^e(y_k) + \epsilon_k \in \mathfrak{L}(\check{E})_\xi \quad \textit{which generate } \mathfrak{L}(\check{E})_\xi \qquad (113)$$
$$ord_\xi(y_i) = 1 \textit{ and and } ord_\xi(\epsilon) > p^e$$

(2) Chooce and fix a coordinate system $x = (y_1, y, \cdots, \omega)$ such that each (y_k, ω) is allowable.
(3) Then follow their LL-chains headed by the g_k with respect (y_k, ω) for every k. Callect their tails and call this set $\{y_k(e)\}$.
(4) Extend this system $\{y_k(e)\}$ to generate a full system of generators of $\mathfrak{T}^\sharp(\check{E})$ as $\rho^\ell(\mathcal{O})$. Call this generators $\{g_i^*\}$. However here the extension is done subject to the following condition.
(5) Each g_i^* is such that

$$g_i^* \ mod\Big(\max(\mathcal{O}_\xi)^{q+1} Bl(Z,1) + \wp^{nega}(\check{E})\Big)$$

is nonzero non-unit of the cotangent vector space $\rho^e(\mathfrak{v}(\check{E}))$ where $\mathfrak{v}(\check{E})$ is the cotangent vector space.
(6) Such $\{g_i^*\}$ is chosen and we let $\|g_i^*\| = \rho^e(h_i)$ for every i.

These selected $\{g_i^*\}$ and $\{h_i\}$ admit their local propagation to a neighborhood $V \ni \xi$ are faithfully fitted to the global coherency of the GLUED-chains diagram.

Definition 14.10. Each g^* is called LL-head type and the full system $\{g^*\}$ is called LL-head type generators of $\mathfrak{L}(\check{E})_V$. Each h_i is called CotLL type and the system $\{h_i\}$ is called CotLL type generators of $Cot(\check{E})_V$.

Theorem 14.11. *Recall $\mathfrak{T}^\flat(\check{E})$ of Def.(13.2) and $Cot(\check{E}) \subset \mathfrak{T}^\sharp(\check{E})$.*

$$\mathfrak{T}^\sharp(\check{E}) = {\sigma_0}^{-1}\rho^e\big(\mathfrak{T}^\sharp(\check{E})\big) \qquad (114)$$

There exist a finite set of generators $\{g_i\}$ for $\mathfrak{T}^\sharp(\check{E})$ and $\{\rho^e(y(e)_i)\}$ for $\mathfrak{T}^\flat(T)$ which are satisfying all the needed extension locally at $\xi \in$ $\mathrm{Sing}(\check{E})_{cl}$. We write Cot for $Cot\check{E}$, Sing for $\mathrm{Sing}(\check{E})$ and $I(\mathrm{Sing})$ for the ideal of $\mathrm{Sing}(\check{E}) \subset Z$ for notational simplicity. Recall the cotangent *vector space* $\mathfrak{v}(E)_\xi$ *of Def.(4.15.*

$$g_i = \rho^e(y_i) + \epsilon_i \in \mathfrak{L}(\check{E})_\xi \tag{115}$$
$$\mathfrak{v}(E)_\xi \text{ is generated by } \{y_i \mod I(\mathrm{Sing})Bl(Z)_\xi\}$$
$$g_i \in \left(\mathfrak{L}(\check{E}) + \wp^{nega}(\check{E})\right)$$
$$\mathrm{ord}_\xi(y_i) = 1 \text{ and } \mathrm{ord}_\xi(\epsilon) > p^e$$
$$\mathfrak{T}^\sharp(\check{E}) = \{\, g_i \,\}\rho^\ell(\mathcal{O})$$
$$\{\rho^e(y(0)_i)\} = \|\{\, g_i \,\}\| \text{ ignoring all homogeneity}$$
$$\mathfrak{T}^\flat(\check{E}) = \{\, \bar{g}_i \,\}\rho^\ell(\mathcal{O})$$

It should be noted that for each i we have $\|g_i\|$ *in* $\rho^e(\mathcal{O})$ *but not necessarily* $g_i \in \rho^e(Bl(Z))$ *in themselves.*

Proof. Staight forward. □

Remark 14.12. *Since* ℓ *is chosen to be* $\ell \gg \bar{\ell}$ *after Th.(14.8) and* $\|\{\, g_i \,\}\|$ *is only finitely many we can find an open neighborhood* V_ξ *of* $\xi \in Z$ *such that all the properties are held at every point* η *of* $\mathrm{Sing}(\check{E})_{cl} \cap V_\xi$ *thanks to the coherence of all* $\cot(\check{E})$, $\mathfrak{T}^\sharp(\check{E})$ *and* $\mathfrak{T}^\flat(\check{E})$*. Moreover we may add condition that* $\{Y_{ji} : g_i = 0\}$ *is a smooth irreducible closed hypersurfce in* V_ξ*.*

Remark 14.13. *Further more by the paracompact property of Zariski topology we can choose a finite set of open subsets* $\{V_j \subset Z\}$ *such that*

$$\mathrm{Sing}(\check{E}) \subset \cup_j V_j \tag{116}$$
$$\exists : \{g_i\}\, and\, \{\bar{g}_i\} \text{ in } V_j \text{ of } Eq.(115)$$
$$V_{ji} = \{\bar{g}_i = 0\} \subset V_j \forall (ji)$$
$$V_{ji} \text{ is smooth irreducible closed hypersurface in } V_j$$

14.4. **LLAR-type generators of $\mathfrak{T}$.** We want to choose a manner of selecting local generators $\{h_i\}$ of $\mathfrak{T}^\sharp(\check{E})_\xi$ which will be viewed as the $\rho^\ell(\mathcal{O})$-module where ℓ is an integer $\ell \gg \bar{\ell}$. Here we require that every h_i of the $\{h_i\}$ should be of the heading type in the sense of Eq.(115). Namely every h_i will be heading an LL-chain at ξ, Recall $\bar{\ell}$ is a large enough lower bound set by the conditions of Th.(14.8) of §14.3. Then in accord with $\{h_i\}$ we choose generators $\{z_i\}$ of $\mathfrak{T}^\flat(\check{E})_\xi$ by the equality $\rho^\ell(z_i) = \|h_i\|$. Recall that the symbol $\|(\,)\|$ means to ignore homogeneity degrees of the summands in the graded algebra $Bl(Z)_\xi$ and to have the result as an element in $\mathcal{O}_\xi$. These sets of generators can be and should be finite so that there natural extensions is possible to some open affine neighborhood V of $\xi \in Z$ in such a way that their property of generation of heading type as above should be maintained at every point $\eta \in \mathrm{Sing}(\check{E}) \cap V_{cl}$. of Eq.(115) below. It is possible thanks to the coherence of both $\mathfrak{T}^\sharp$ and $\mathfrak{T}^\flat$. Now the selection of $\{h_i\}$ and $\{\rho^e(z_i)\}$ are well done to suit to the AR-processing needed later and to have good relation with LL-chains headed by generators of $\mathfrak{L}(\check{E})$ and with the coherent GLUED-chain diagrams

We then consider $\mathfrak{L}_0(\infty)$ as $\rho^\ell(Bl(Z))$-module and $\mathfrak{T}^\sharp(\check{E})$ as $\rho^\ell(\mathcal{O}_Z)$-module. They are globally coherence and hence locally finitely generated. For each point $\xi \in \mathrm{Sing}(\check{E})_{cl}$ we can find an open affine neighborhood V_ξ of $\xi \in Z$ such that the restriction $\mathfrak{T}(\check{E})|_{V_\xi}$ is finitely generated as $\rho^\ell(\mathcal{O})_Z|_{V_\xi}$-module. There we choose a basis $h = (h_1, h_2, \cdots, h_N)$ of $\rho^\ell(\mathcal{O})|_{V_\xi}$-module $\mathfrak{T}(\check{E})|_{V_\xi}$. Here N is some positive integer bigger than the dimension of cotangent space. Here we can always choose V_ξ and h in such a way that $\mathrm{ord}_\eta(h_j) = 1$ for all j and for all point $\ell \in Z_{cl} \cap V_\xi$.

15. AR-schemes and AR-reductions

We introduce the notion of the ambient reduction scheme, AR-scheme in short, which is the central technique that is used to formulate our descending induction on the ambient dimension for the proof of ERS of given singular data in Z. Our approach to ERS in higher dimension is how to reduce the problems to those of lower dimensions. We thus come to the notion of AR-scheme, in particular *GLAR-scheme*, that is our way of cutting down the ambient dimension by combining two approaches, local and global. The local approach was the LLUED of §9 depending upon well chosen coordinate systems while the global approach used the coordinate free GLUED of §S: glued-chain . The local-global relation was shown in §14.

Definition 15.1. Here is the definition of AR-scheme denoted by $\mathfrak{Y}$ in general. It is defined by a finite family of triplets $\{(V_j, F_j, Y_{ji})\}$ in the ambient scheme Z which is characterized by the following conditions.

(1) F_j is an ideal exponent in the open subset $V_j \subset Z$ and Y_{ji} are smooth closed irreducible subschemes of V_j.
(2) For each j the triplet (V_j, F_j, Y_{ji}) has finitely many Y_{ji} and AR-reduction $F_j \rightharpoondown_{Y_{ji}} F_j(Y_{ji})$ for every j. The ideal exponent $F_j(Y_{ji})$ in Y_{ji} is uniquely determined by the (F_j, Y_{ji}).
(3) Let $m = dim(Y_{ji}) < n = dimZ$ and m is independent of (ji). This m is called the dimension of $\mathfrak{Y}$, denoted by $dim(\mathfrak{Y})$.
(4) $\mathfrak{S}(F_j|_{W_{jk}}) = \mathfrak{S}(F_k|_{W_{jk}})$ with $W_{jk} = V_j \cap V_k$ for every pair jk, i.e, the restricted pair have the same infinitely near singularities.
(5) Define $\mathrm{Sing}(\mathfrak{Y}) = \bigcup_{ji} \mathrm{Sing}(F_j(Y_{ji}))$. Then $\mathrm{Sing}(\mathfrak{Y})$ is a closed subscheme of Z, which is called the singular locus of $\mathfrak{Y}$.

(6) For each j let H_{ji} be the unique extension of F_{ji} to V_j having AR-equivalence $H_{ji} \rightleftharpoons_{Y_{ji}} F_{ji}$ Define H_j to be an ideal exponent with $\mathfrak{S}(H_j) = \bigcap_i \mathfrak{S}(H_{ji})$. We then have $\mathfrak{S}(H_j)|_{W_{jk}} = \mathfrak{S}(H_k)|_{W_{jk}}$ for every jk

As for the definition of ambient reductions we should refer to Eq,(12) and Eq.(14) of §3.2.1 of §3.2. Also refer to [23] and [24].

The conditions such as (4) or (7) produces a global ideal exponent by piecing together the given local ones. This is proven as follows.

Lemma 15.2. *Given a system of local ideal exponents H_i in open sets $V_i \subset Z$ and assume that $\mathfrak{S}(H_j)|_{W_{jk}} = \mathfrak{S}(H_k)|_{W_{jk}}$ for every pair jk where $W_{jk} = V_j \cap V_k$. Then there exists an ideal exponent H in the open set $V = \cup_i V_i$ such that $\mathfrak{S}(H_i) = \mathfrak{S}(H)|_{V_i}$ for every i.*

Proof. Thanks to Zariski topology in V we may assume the range of i is finite. We know that the equality of $\mathfrak{S}$ is equivalent to the corresponding equality of $\wp$ in general. Write $H_i = (J_i, b_i)$ with $b_i = b$ independent of i. We may also assume that J_i is the maximum choice of J_i of $H_i = (J_i, b)$ for each i with b fixed. The maximality is by integral closure of local nature. Thus we have $H_i|_{W_{jk}} = H_k|_{W_{jk}}$ for every ji. Thus the $\{H_i, \forall i\}$ piece together and produce a global H. □

Lemma 15.3. *Given a smooth closed subscheme $X \subset Z$ and an ideal exponent F in X, we can find an ideal exponent G in Z such that there exists AR-equivalence $G \rightleftharpoons_X F$.*

Proof. Let $I(X)$ be the ideal of $X \subset Z$. Write $F = (\bar{J}, b)$. Now let J be the inverse image of $\bar{J}$ by the natural epimorphism $\mathcal{O}_Z \to \mathcal{O}_X$. Then $G = (J + I(X)^b, b)$ is the answer for the lemma. In fact every permissible sequence LSB for $\mathfrak{S}(G)$ in the sense of of §2.3 must have all of its centers of blowups must be in the strict transforms of X because the ideal $I(X)^b$ is in G and X is smooth. □

15.1. **AR-schemes tied with GLUED-tails.** Once again we emphasize the chose of ℓ by $\ell \gg \bar{\ell}$ in the sense of Th.(14.8). Recall $Cot(\check{E})$ of Eq.(106) following after $\mathfrak{T}^\sharp(\check{E})$ and $\mathfrak{T}^\flat(\check{E})$ of Eq.(105) of Def.(13.2) of §13.1. Refer to the cotangentmodule $cot(E)_\xi$ of Prop.(4.16) and the cotangent vector space $\mathfrak{v}(E)_\xi$ of Def.(4.15).

We now choose a system of elements $\{h_{j\xi}\}$ with CotLL generators $\{h\}$ of $Cot(\check{E})$ in the sense of §14.3.1. Let $m = dim(\text{cot}(\mathfrak{v}(\check{E}))$ of Def.(4.15). Note that the rank m is constant at all points of $\text{Sing}(\check{E})_{cl}$ because $\check{E}$ is core focused in the sense of §6.2. Note that $Cot(\check{E})$ is finitely generated considered as $\rho^\ell(\mathcal{O})$-module with $\ell \gg \bar{\ell}$. It is here important that the system $\{h_{j\xi}\}$ must satisfy the conditions of the following remark.

Remark 15.4. *Onto $\{h_{j\xi}\}$ we impose the following conditions. Firstly we do it locally at $\xi \in \mathrm{Sing}(\check{E})_{cl}$ as follows.*

1. *The image of all $h_{j\xi}$ by the natural epimorohism from $Cot(\check{E})$ to $\bar{\mathfrak{v}}(E)_\xi$ is to generate the κ-vector space $\bar{\mathfrak{v}}(E)_\xi$ of Def.(4.15).*
2. *The system $\{h_{j\xi}\}$ is finite and its members generate the $\rho^\ell(\mathcal{O})$-module $Cot(\check{E})^m$.*
3. *We can choose a sufficiently small affine open neighborhood V_ξ of $\xi \in Z$ such that every $\{h_{j\xi}\}$ extends through V_ξ in such a way that all the preceding conditions are maintained bg $\{h_{j\xi}\}$ at every point η of $V_\xi \cap \mathrm{Sing}(\check{E})_{cl}$.*
4. *A smooth irreducible closed subscheme $Y_{j\xi}$ is defined by $h_{j\xi} = (0)$ in V_ξ for every $j\xi$.*

After we have chosen such a system $\{h_{j\xi}\}$ in the open subset V_ξ for every point $\xi \in \mathrm{Sing}(\check{E})_{cl}$ we realize that only finitely many of those V_ξ are enough to cover the entire $\mathrm{Sing}(\check{E})$. Having then selected such finitely many ξ we rename these as follows.
Write V_i instead of V_ξ, h_{ji} instead of $h_{j\xi}$ and Y_{ji} instead of $Y_{j\xi}$.

Definition 15.5. We now go on to define a special but most important kind of AR-scheme in Z, called CotAR-scheme of $\check{E}$. Let us denote it as $\mathfrak{Y}(E) = \{(F_j, V_j, Y_{ji})\}$ in the manner of $Def.$(15.1). The CotAR-scheme of $\check{E}$ will be made out of $Cot(\check{E})$ of $\check{E}$ after the $GLUED$-chain diagram of Eq.(104). Recall V_i, h_{ji} and Y_{ji} chosen by Rem.(15.4) of §15.1. Let $F_j = (\check{E})|_{V_i}$ for each i. The $\mathfrak{Y}(E) = \{(F_j, V_j, Y_{ji})\}$ is thus defined starting from the given dieal exponent E.

We symbolically express the whole process to define $\mathfrak{Y}(E)$ starting from the given ideal exponent E as follows.

$$E \Rightarrow \hat{E} \Rightarrow \check{E} \Rightarrow Cot(\check{E}) \Rrightarrow \mathfrak{Y}(E) = \{(F_j, V_j, Y_{ji})\} \quad \text{or just } \; Cot(\check{E}) \Rrightarrow \mathfrak{Y}(E) \; \text{ in short.} \tag{117}$$

The CotAR-scheme, i.e, AR-scheme by $Cot(\check{E})$, is defined below. It will be seen later that the notion is directly related with our inductive proof of the resolution of singularities in $p > 0$. In order to understand this direct relation we must refer to the later subsections of §16. Here we should note the following fact.

Theorem 15.6. *With $\mathfrak{Y}(E) = \{(F_j, V_j, Y_{ji})\}$ of Eq.(117), let us define ideal exponents G_j and $\{H_{ji}\}$ in V_j as follows.*

$$\text{(118)} \qquad Let\ F_j \rightharpoondown_{Y_{ji}} F_j(Y_{ji}) \ \ and\ define\ H_{ji} \rightleftharpoons_{Y_{ji}} F_j(Y_{ji})$$
$$Let\ \mathfrak{S}(G_j) = \bigcap_i \mathfrak{S}(H_{ji})\ \forall j$$

The $\mathfrak{S}$ is in the sense of infinitely near singularities. We then have an ideal exponent G in Z such that $G_j = G|_{V_j}$ for all j.

Proof. Note that the existence and uniqueness of H_{ji} are strait forward by Lem.(15.3) and by its definition by AR-equivalence. Next by the ambient reduction theorem of §3.2, applied to all of the $\{H_{ji}\}$, and in view of the way by which each H_{ji} is derived from $F_j(Y_{ji})$ we see the fullowing facts, Firstly it is clear that for $\eta \in Z_{cl} \cap V_j$ we have

$$\text{(119)} \qquad ord_\eta\big(Cot(\check{E})\big) \geq 1 \Longleftrightarrow \eta \in \cap_i Y_{ji}$$

It is then enough to prove the following equality.

$$\text{(120)} \qquad \mathfrak{S}(F_j) \cap \mathfrak{S}(Cot(\check{E}), 1) = \bigcap_i \mathfrak{S}(H_{ji}), \forall j$$

question on the $\mathfrak{S}$ equality which is easily proven by the definition of Eq.(118). We have the following equality in technical terms of $\mathfrak{S}$.

$$\text{(121)} \qquad \mathfrak{S}(F_j) \cap \mathfrak{S}(Cot(\check{E}), 1)\ of\ which$$
$$\mathfrak{S}(Cot(\check{E}), 1) = \cap_i \mathfrak{S}(h_{ji}, 1)$$
$$because\ the\ ideal\ Cot(\check{E})\ is\ generated\ by\ \{h_{ji}\mathcal{O}\}$$
$$Moreover\ \mathfrak{S}(F_j) \cap \mathfrak{S}(h_{ji}, 1) = \bigcap_i \mathfrak{S}(H_{ji})$$
$$by\ the\ definition\ of\ H_{ji}\ by\ h_{ji}.$$

The claimed existence of global G with equality $G_j = G|_{V_j}$ is now proven by Lem.(15.2). □

Definition 15.7. Rename E to $E(0)$ and then define $E(1)$ as follows. Let $\mathfrak{S}(E(1)) = \bigcap_i \mathfrak{S}(H_{ji})$ in the sense of Eq.(118) and Eq,(121). We thus have $\mathfrak{S}(E(1))$ equal to $\mathfrak{S}(F_j) \cap \mathfrak{S}(Cot(\check{E}(0)), 1)$ where $F_j = E(0)|_{V_j}$ and $Cot(\check{E}(0)) = \sqrt[q]{\mathfrak{T}^\flat(E(0)) \cap \rho^e(\mathcal{O}}$ with $q = p^e$. Refer to Eq.(106) of Def.(13.2). We then define the following chain of processing.

$$\text{(122)} \qquad \cdots \mathfrak{Y}(E(0)) \to E(1) \to \hat{E}(1) \to \check{E}(1)$$
$$\Rrightarrow \mathfrak{Y}(E(1)) \to E(2) \to \cdots .$$

in the manner of Eq.(117). We will then rename $\mathfrak{Y}(1)$ for $\mathfrak{Y}(E(1))$ and write the process from $\mathfrak{Y}(0)$ symbolically as $\mathfrak{Y}(0) \Rrightarrow \mathfrak{Y}(1)$.

15.2. **Iteration of ambient reduction of AR-scheme.** We have thus obtained the sequence of successive $\mathfrak{Y}$ as follows.

$$\mathfrak{Y}(0) \Rrightarrow \mathfrak{Y}(1) \Rrightarrow \cdots \Rrightarrow \mathfrak{Y}(s) \Rrightarrow \mathfrak{Y}(s+1) \cdots \tag{123}$$

Definition 15.8. Let us review the definition of $\mathfrak{Y}(s) \Rrightarrow \mathfrak{Y}(s+1)$. We are given an ideal exponent $E(s) = (J(s), b(s))$ with ideal $J(s) \neq (0)$ and integer $b(s) > 0$. We then proceed along the following steps.

(1) We have core-focused maximum hike $E(s) \to \hat{E}(s) \to \check{E}(s)$.
(2) Take the GRUED-chain of $\check{E}(s)$ and cotangent end $Cot(\check{E}(s))$. Pick an ℓ with $\ell \gg \bar{\ell}$ and view $Cot(\check{E}(s))$ as $\rho^{\ell}(\mathcal{O}_Z)$-module.
(3) Choose any open affine covering $\{V_j\}$ of $\mathrm{Sing}(\check{E}(s))$ in Z such that $Cot(\check{E}(s))(V_j)$ admits a finite system of cotangential generators $\{c(ji)\}$ for each j. Then let $Y(s)_{ji}$ be the smooth irreducible hypersurface defined by $c(ji) = 0$ in Z for each (ji).
(4) We then have the AR-scheme $\mathfrak{Y}(s) = \{(V(s)_j, F(s)_j, Y(s)_{ji})\}$ where $F(s)_j = \check{E}(s)|_{V(s)_j}$ and AR $F(s)_j \rightharpoondown_{Y(s)_{ji}} F(s)_j(Y(s)_{ji})$.

$$\begin{array}{c} Define\ dim(\mathfrak{Y}(s)) = n - rank(\mathfrak{v}(\check{E}(s)))\ and \\ \mathrm{Sing}(\mathfrak{Y}(s)) = \bigcap_{ji} \mathrm{Sing}\big(F(s)_j(Y(s)_{ji})\big) \end{array} \tag{124}$$

Recall the definition of the cotangent vector space $\mathfrak{v}(\check{E})_\xi$ in the sense of Eq.(29) of Def.(4.14) and Def.(4.15). Namely we have

$$\begin{aligned} \mathrm{rank}\big(\mathfrak{v}(\check{E})_\xi\big) &= \mathrm{rank}\big(\,\mathrm{cot}(\check{E})\big) \\ &= \mathrm{rank}\big(Cot(\check{E}(s))_\xi / \max(\mathcal{O}_\xi)^2\big) \end{aligned} \tag{125}$$

at each point $\xi \in \mathrm{Sing}(\check{E})_{cl}$, which is independent of ξ for $E = \check{E}$.

(1) We have a global ideal exponent $E(s+1)$ in Z characterized by $\mathfrak{S}(E(s+1))|_{V_j} = \bigcap_i \mathfrak{S}(F(s)_j(Y(s)_{ji}))$ for every j.
(2) Each $F(s)_j(Y(s)_{ji})$ in Y_{ji} is uniquely extended to G_{ji} in Z in such a way as AR-equivalence $G(s+1)_{ji} \rightleftharpoons_{Y(s)_{ji}} F(s)_j(Y(s)_{ji})$. We then have $\mathfrak{S}(E(s+1)) = \bigcap_{ji} \mathfrak{S}(G_{ji})$
(3) We take the core-edged maximum hike $E(s+1) \to \hat{E}(s+1) \to \check{E}(s+1)$ and GLUED-chain of $\check{E}(s+1)$ producing $Cot(\check{E}(s+1))$
(4) Finally $\mathcal{Y}(s+1) = \{(V(s+1)_{ja}, F(s+1)_{ja}, Y(s+1)_{jai})\}$ with $F(s+1)_{ja} = \check{E}(s+1)|_{V(s+1)_{ja}}$. The index a is added due to the possible need of refinement of the covering $\{V_j\}$ to have the cotangential generators $\{c(jai)\}$ of $Cot(\check{E}(s+1)|_{V(s+1)_{jai}}$ for every jai in the manner of $\{c(ji)\}$ of $Cot(\check{E}(s)|_{V(s)_{ji}}$ before.
(5) The $\mathfrak{Y}$-sequence is finite and get the terminal plat. Thefore the fact is that the integer $\ell \gg \bar{\ell}$ and covering of $\mathrm{Sing}(\mathfrak{Y})$ could have been chosen once for all.

We will thus us the same indexing (j, i) for the triplet family presentation of those $\mathfrak{Y}(s) = \{(V(s)_j, F(s)_j, Y(s)_{ji})\}$ for every $s = 0, 1, 2. \cdots$. This will help us for notational simplicity from now on.

We are assuming that E is not trivial, i.e, $E = (J, b)$ has $J \neq (0)$. The singular loci of the $\mathfrak{Y}(j), j \geq 0$, have the following properties.

Theorem 15.9. *The sequence Eq.(122) has a monotone decorating sequence of singular loci as follows. There exists an integer $t : 0 \leq t \leq n$, where*

$$n = dim(Z) > dim(\mathfrak{Y}(0)) > dim(\mathfrak{Y}(1)) > \cdots,$$

such that

$$\begin{aligned} \mathrm{Sing}(\check{E}) &= \mathrm{Sing}(\mathfrak{Y}(0)) \supsetneqq \mathrm{Sing}(\mathfrak{Y}(1)) \supsetneqq \cdots \supsetneqq \mathrm{Sing}(\mathfrak{Y}(t)) \\ & \cdots \supsetneqq \mathrm{Sing}(\mathfrak{Y}(t+1)) = \mathrm{Sing}(\mathfrak{Y}(t+2)) = \cdots \end{aligned}$$

and thus $\mathfrak{Y}(t+1)$ is smooth, i.e, for all i, $F(t+1)_j$ is the zero ideal.

Proof. Firstly at each point $\xi \in \mathrm{Sing}(\check{E})_{cl}$ we have for all ji by the definition of the CLUED-chain by the operations, $\tau\Lambda$, θ and σ. In particular we then note $\mathfrak{L}_j(0)(\infty)(\xi) = 0$. Thus by definition $\mathfrak{L}^\sharp(\check{E})(\xi) = 0$ and hence $\mathfrak{L}^\flat(\check{E})(\xi) = 0$ and hence $Cot(\check{E})(\xi) = 0$. It follows that $\mathfrak{Y}(0)(\xi) = 0$, We thus obtain $\mathrm{Sing}(\check{E}) \subset \mathrm{Sing}(\mathfrak{Y}(0))$. Next the reverse inclusion is by Th.(10.4) and Th.(10.5) of §10.1. As for the sequence of the dimensions of $\mathfrak{Y}$, recall the number m of Def.(15.1) and apply it to those Y_{ji} of $\mathfrak{Y}(E) = \{(F_j, V_j, Y_{ji})\}$ of Eq.(117). □

As for the processing $\mathfrak{Y}(s) \Rrightarrow \mathfrak{Y}(s+1)$ via $E(s)$ and $Cot(\check{E}(s))$ we should refer to Eq.(117) of Def.(15.5). Pick $\ell \gg \bar{\ell}$ and we consider the LL-head type generators $h_j i$ of $Cot(\check{E}(s))^c$ as $\rho^\ell(\mathcal{O}_{v_j}$-module

15.3. **Terminal $\mathfrak{Y}$ and terminal plat.** We have the monotone decreasing sequence of $\dim(\mathfrak{Y}(E(j))$ for $\mathrm{Sing}(\mathfrak{Y}(E(j))$ using Thm.(T:sing-termi). We next elucidate the special property of the $\mathfrak{Y}(t)$ and $\mathfrak{Y}(t+1)$ in particular.

Theorem 15.10. *Recall and examine the process of $\mathfrak{Y}(s) \Rrightarrow \mathfrak{Y}(s+1)$ for any $s \geq 0$. In other words we recall the processing:*

$$\{(V(s)_j, F(s)_j, Y(s)_{ji})\} \to \{(V(s+1)_j, F(s+1)_j, Y(s+1)_{ji})\}.$$

where $F(s)_j \rightharpoondown_{Y(s)_{ji}} F(s)_j(Y(s)_{ji})$ for every i and $\{(F(s)_j(Y(s)_{ji}))\}$. We claim that the unique integer $t \geq 0$ has the property as follows.

(1) $\mathrm{Sing}(\mathfrak{Y}(t-1)) \supsetneqq \mathrm{Sing}(\mathfrak{Y}(t) = \mathrm{Sing}(\mathfrak{Y}(t+k)), \forall k \geq 1.$
(2) $dim(\mathfrak{Y}(t)) > dim(\mathfrak{Y}(t+1)) = dim(\mathfrak{Y}(t+k)), \forall k \geq 2.$

(3) $\mathrm{Sing}(\mathfrak{Y}(t+1))$ *is a smooth closed non-empty subset of* Z*.*

Proof. This proof will be completed after two steps as follows.

Step(1)

We pick $\ell \gg \bar{\ell}$. Then consider the CotLL type generators $\{h_{ji}\}$ of $Cot(\check{E}(s))$ as $\rho^{\ell}(\mathcal{O}_{V(s)}$-module and $Y(s)_{ji}$ in the sense of §14.3.1. Recall Def.(15.7) after Eq.(118) and Eq,(121). Take $F(s)_j \rightharpoondown_{Y(s)_{ji}} F(s+1)_{ji}$. Extend $F(s+1)_{ji}$ to $H(s+1)$ in Z such that $H(s+1) \rightleftharpoons_{Y(s)_{ji}} F(s+1)_{ji}$. Then $\mathfrak{S}(E(s+1)) = \mathfrak{S}(E(s)) \cap_{ji} \mathfrak{S}(H(s+1))$. Thus $E(s+1)$ defined $\mathfrak{Y}(s+1) = \{(V(s+1)_j, F(s+1)_j, Y(s+1)_{ji})\}$. Now if $\mathrm{Sing}(\mathfrak{Y}(s)) \neq \mathrm{Sing}(\mathfrak{Y}(s+1))$ then $H(s+1)$ is not trivial, i.e, there exists $\xi \in Z_{cl} \cap \mathrm{Sing}(H(s+1))$ such that $H(s+1)_{\xi}$ is not trivial. This implies $dim(\mathfrak{Y}(s+1)) < dim(\mathfrak{Y}(s))$.

Step(2)

On the other hand if $\mathrm{Sing}(\mathfrak{Y}(s+1)) = \mathrm{Sing}(\mathfrak{Y}(s+1))$ we then claim the following fact, Pick any system $c = (c_1, c_2, \cdots, c_m)$ with $c_k \in Cot(\check{E}(s)), \forall k$, which induces a free base of cotangent vector space $\mathfrak{v}(E(s))_{\xi}$ in he sense of Def.(4.15) for any point $\xi \in \mathrm{Sing}(\check{(}E(s)))$ where $m = dim(\mathfrak{Y}(s))$. We then claim that

$$\mathfrak{S}((f)\mathcal{O}, 1) = \mathfrak{S}(\check{E}(s)) \cap \mathfrak{S}(Cot(\check{E})(s), 1)$$

which is $\mathfrak{S}(E(s+1))$. In fact pick a local coordinate $x = (f, \omega)$ of Z at any $\xi \in \mathrm{Sing}(\check{E}(s))$. Then for all $\ell \gg \bar{\ell}$ we must have not only that the ideal $J(s)$ of $\check{E}(s) = (J(s), b(s))$ have generators contained in $(f)^{b(z)}\rho^{\ell}(\mathcal{O}$ but also that $Cot(\check{E}(s))$ has generators in $(f)^{b(z)}\rho^{\ell}(\mathcal{O}$, too. In important point is that this is so for every $\ell \gg \bar{\ell}$. It follows that $E(s+1) = \hat{E}(s+1) = \check{E}(s+1)$ and $\mathfrak{Y}(s+1) = \mathfrak{Y}(s)$. Moreover it follows that $\mathrm{Sing}(\check{E}(s+1))$ is defined by $f = (0)$ and hence is a smooth irreducible subschema of Z. □

Definition 15.11. Define the succesive iteration all the way as follows.

$$\mathfrak{Y}(0) \Rrightarrow \mathfrak{Y}(1) \Rrightarrow \cdots \Rrightarrow \mathfrak{Y}(t) = \mathfrak{Y}(t+1) \tag{126}$$

This sequence terminates when every member $F(t;j)$ of $\mathfrak{Y}(t)$ has the $\wp(F(t)_j)$ generated by $\wp(F(t)_j, 1)$ so that every member $F(t+1)_j$ of $\mathfrak{Y}(t+1)$ has $F(t+1) = (\mathcal{O}, 1)$ with $\mathrm{Sing}(F(t+1)) = \emptyset$. We will say that $\mathfrak{Y}(m), m > t$, smooth. After that we have all smooth $\mathfrak{Y}(s+1) = \mathfrak{Y}(s+j), \forall j > 1$.

Definition 15.12. The $\mathrm{Sing}(\mathfrak{Y}(t))$ will be called the $\mathfrak{Y}$-terminal plat, or the terminal plat for short, of E and/or of $\check{E}$. Remember that it is smooth closed non-empty subscheme of Z. Moreover thanks to the monotone sequence in Th.(15.9) we have the terminal plat is included in $\mathrm{Sing}(\check{E})$ while this $\mathrm{Sing}(\check{E})$ is contained in $\mathrm{Sing}(E)$ with the original ideal exponent E. We will call $\nabla(E)$ or $\nabla(\check{E})$ for the terminal plat of $\mathfrak{Y}(E)(t)$.

16. PROOF OF RESOLUTION OF SINGULARITIES, $p > 0$

We are now ready to prove the ERS, embedded resolution of singularities in each characteristic $p > 0$ and for all ambient dimension $n = dimZ$, in the language of formalism called ideal exponents. Recall the introduction at the top of this paper that explained the logical connection of ideal exponents with ERS problems.

Remark 16.1. *It should be noted that so far no blowups has been performed uppon the given singular data in Z. The theorem Th.(15.10) is an important key to start inductive proof of ERS but it is not yet straight forward to the end of ERS. None the less the road to the end of ERS became much more clearly visible by the discovery of smooth terminal plat ∇ in the sense of Def.(15.12) after Th.(15.10).*

Remark 16.2. (1) Blowups with center the terminal plat ∇, or any closed smooth subscheme of ∇, are permissible for all members of the GLUED-chain diagram for every $E(j)$ in the sequence ideal exponents $E = E(0) \to \check{E}(0) \cdots \check{E}(t)$ of Eq.(117) and Eq.(122). Also the same is true for the CotAR sequence $\mathfrak{Y}(E(0)) \Rrightarrow \cdots \mathfrak{Y}(E(t))$ of Eq.(123).

(2) Blowups with center ∇ strictly improve the singularities of $\check{E}$ in terms of explicitly defined numerical invariants at every closed point corresponding to a point of ∇.

(3) The NC-data Γ must still be resolved, but this can be done by an algorismic procedure independent of characteristic $p \geq 0$.

(4) The gap between E and $\check{E}$ can be separately and independently overcome by permissibility condition of blowups that maintains the normal crossings of the transforms of Γ.

Remark 16.3. In the whole process starting from the given $E = (J, b)$ until establishing Th.(15.10) of the terminal plat $\nabla(\check{E})$, we may need some base field extensions for technical convenience in proofs of supporting lemmas. However we can eliminate base extension from the final result simply by taking into account all conjugates before applying any permissible blowups following after Th.(15.10).

16.1. **Theorem Δ and Γ-monomials.** Here we consider an arbitrary ideal exponent $E = (J, b)$ in the ambient scheme Z with $J \neq (0)$. Recall the definition of maximum base hike denoted by $\hat{E}$ for any E. Refer to §(6.1).

Theorem 16.4. *Assume that ERS had been proven for every maximum base-hiked ideal exponent accompanied by normal crossing data* Γ *in* Z *in any ambient scheme of dimension* $n = dimZ$*. We then assert that any ideal exponent* E *will admit a finite sequence of blowups* $\pi : Z' \to Z$ *permissible for* (E, Γ) *such that the transform* E' *of* E *by* π *has empty* $\mathrm{Sing}(E')$.

Proof. The proof is to follow the sequence of transforms of $E = (J, b)$ combined with $\hat{E} = (J, d)$ by repeated applications of the assumption of Th.(16.4), where the transforms of E will be defined in accord with usual succesive procedure while the transfoms of $\hat{E}$ will be changed by appropriate modification in each step of the course. Rename:

$$E = E(0) = (J(0), b(0)) \text{ and } \hat{E} = \mathcal{K}(0) = (K(0), d(0))$$

and $\Gamma(0)$ for Γ. We apply the assumption of Th.(16.4) to $(\mathcal{K}(0), \Gamma(0))$, where $\mathcal{K}(0)$ is the maximum base-hiked. We thus obtain a finite sequence of permissible blowups $\pi(0) : Z(1) \to Z(0) = Z$. Note that it is automatically permissible for $E(0)$. Thus we obtain the resolved $\mathcal{K}(0)' = (K(0)', d(0))$ with empty $\mathrm{Sing}(\mathcal{K}(0)')$. Remember that the ideal $K(0)'$ is maintained even after the resolution. We let $E(1)$ be the transform of $E(0)$ by $\pi(0)$ and $\mathcal{K}(0)' = (K(0)', d(0))$ be the transform of $\mathcal{K}(0)$. We then define $d(1)$ to be the maximum of $\mathrm{ord}_\eta(K(0)')$ for all $\eta \in \mathrm{Sing}(E(0))_{cl}$ provided that

(1) $\mathrm{Sing}(E(0))_{cl}$ is not empty. Note that if it is empty then the resolution of E is already achieved and we will be excluding this case.
(2) Therefore we must have $d(1) > 0$ and we will be assuming this.
(3) Yet we have $d(1) < d(0)$ because of the above resolution.

We then define and examine $\mathcal{K}(1) = (K(0)', d(1)) \cap E(1)$ which is maximum base hiked thanks to the choice of the number $d(1)$. We apply the assumption of Th.(16.4) to $\mathcal{K}(1)$ (and for the transform $\Gamma(1)$ of $\Gamma(0)$ as always) and we obtain a finite sequence of permissible blowups $\pi(2) : Z(2) \to Z(1)$. We thus obtain the resolved $\mathcal{K}(1)' = (K(1)', d(1))$ with empty $\mathrm{Sing}(\mathcal{K}(1)')$. We then define $d(2)$ to be the maximum of $\mathrm{ord}_\eta(K(1)')$ for all $\eta \in \mathrm{Sing}(E(1))_{cl}$ provided that

(1) $\mathrm{Sing}(E(1))_{cl}$ is not empty. Again if it is empty then the resolution of E is now achieved by $\pi(2)\pi(1)$. Thus we will be excluding this case.
(2) Then we must have $d(2) > 0$ and we would have $d(2) < d(1)$.

Unless the transform of E happens to be resolved we repeat the process all the way to get $d(k) = 0$. Clearly the repetition terminates after a finite number of times for we have $d(0) > d(1) > d(2) > \cdots$. Therefore in this repetition we should finally obtain that either E ends up to be resolved on the way or we finally reach the number $d(k) = 0$. Let us work on this last event in which case the end is that the final transform $K(k)$ of the ideal $K(0)$ becomes the unit ideal within $\text{Sing}(E(k))$. If $E(k)$ is not yet resolved then what remains in $E(k)$ must be nothing but the gap between the transforms of E and $hatE$. The gap is nothing but locally $\Gamma(k)$-monomial within in a sufficiently small neighborhood of $\text{Sing}(E(k)$. Thus the proof of Th.(16.4) is reduced to the special case in which J is Γ-monomial and the base number $b = 1$. A canonical process in this special case, given below, will complete the proof. □

16.1.1. Γ-*monomial cases.* The following lemma, the statement as well as its proof, does not depend on the characteristic of the base field $\mathbb{K}$. In fact its idea and application were effectively used in the characteristic zero case of [19], too. Also refer to [22].

Lemma 16.5. *We have a canonical resolution in the* Γ-*monomial case. This is the case in which we have* $E = (\prod_{1 \le i \le s} H(i)^a(i), b)$ *where* $H(i)$ *is the ideal of the hypersurface* $\Gamma(i), 1 \le i \le s$. *Let* $H = (H_1, \cdots, H_s)$ *and assume* $H_i \cap H_j \ne \emptyset, \forall i \ne j$.

Proof. We define the following numbers.

(1) Let s be the shortest length of those subsystems of H having non-empty intersection. Let S be the set of these subsystems with s and let $|S|$ be cardinality of S.
(2) Let $a(C)$ be the sum of the $a(i)$ for $H_i \in C \in S$
(3) and let A be the maximum of $a(C)$ for all $C \in S$.
(4) Assume $A \ge b$ let $T = \{C \in S, a(C) = A\}$ and pick any $C^\circ \in T$
(5) Let D denote the intersection of $H_j \in C^\circ$.

Then apply the blowup $\pi : Z' \to Z$ with center $D(C)$ with $C \in T$. The result is then the numerical ststem $(n-s, A, |T|)$ strictly decreases lexicographically by π. The proof of the last claim is as follows. The strict transforms of the $H_i \in C$ have no common point. Let $D' = \pi^{-1}(D)$ which is a new member of Γ' other than the strict transforms H'_i of the H_i's. Hence $s\prime \ge s$. Assume $s\prime = s$. The possible members of S', other than strict transforms from $T\backslash$, are some of the unions of the form $\{D', C^\circ(j)'\}$ where $C^\circ(j)'$ denotes the strict transform of C° deteted $H_j \in C^\circ$. Now $a(C^\circ(j)') = a(C^\circ(j)) + a(C^\circ) - b$. Since $C^\circ(j) \notin S$ we have $a(C^\circ(j)) < b$ we have $a(C^\circ(j)') < a(C^\circ(j))$. Hence if $A' = A$ we must have $|T'| < |T|$. This completes the proof of Lem(16.5). □

Note that the proof of Th.(16.4) is now complete.

16.2. **Theorem ∇ I: Domino-falls by ∇ blowups.** This subsection is devoted to the proof of the following theorem.

Theorem 16.6. *(Main Theorem I)*
Let D be any smooth closed irreducible subscheme of $\nabla(E)$ such that the blowup $\pi : Z' \to Z$ with center D is permissible. Let $D' = \nabla' \cap \pi^{-1}(D)$ where the ∇' is the strict transform of ∇ by π, i.e, the blowup of ∇ with center $D \subset \nabla$. We then have:

(1) π is permissible for every ideal exponent of $\mathfrak{Y}(i)$ for all i.
(2) Locally at every point $\xi' \in \pi^{-1}(D)_{cl} \setminus D'$ we have

$$\big(\mathrm{Inv}_{\xi'}(\mathcal{Y}'(0)), \mathrm{Inv}_{\xi'}(\mathcal{Y}'(1)), \cdots, \mathrm{Inv}_{\xi'}(\mathcal{Y}'(m'-1))\big) \tag{127}$$
$$<_{\mathrm{lex}}$$
$$\big(\mathrm{Inv}_{\xi}(\mathcal{Y}(0)), \mathrm{Inv}_{\xi}(\mathcal{Y}(1)), \cdots, \mathrm{Inv}_{\xi}(\mathcal{Y}(m-1))\big)$$

where the actual number m of stops before reaching the terminal may (or may not) drop at ξ', that is to say $m' \le m$. In any event the strict inequality holds true as is written if we extend the sequences by adding (0) repeatedly.

(3) For every closed point η' of the strict transform ∇' of ∇ by π we have the equality

$$\big(\mathrm{Inv}_{\xi'}(\mathcal{Y}'(0)), \mathrm{Inv}_{\xi'}(\mathcal{Y}'(1)), \cdots, \mathrm{Inv}_{\xi'}(\mathcal{Y}'(m-1))\big) \tag{128}$$
$$=$$
$$\big(\mathrm{Inv}_{\xi}(\mathcal{Y}(0)), \mathrm{Inv}_{\xi}(\mathcal{Y}(1)), \cdots, \mathrm{Inv}_{\xi}(\mathcal{Y}(m-1))\big)$$

(4) Consider the case in which $D = \nabla$ and hence $\nabla' = \emptyset$. Then for every $F(i;\alpha) \in \mathcal{Y}(i)$ with every $0 \le i \le m$ the $\wp^{posi}(F'(i;\alpha))$ is empty at every point $\zeta' \in \pi^{-1}(\nabla)_{cl}$ where $F'(i;\alpha)$ denotes the transform of $F(i;\alpha)$ by π.

Th.(16.6) will be proven after the sequence of lemmas stated and proven below. Here we have simple but useful techniques which are called *LL-DOMINO FALL*. The lemmas below are the invaluable keys to the proof of Th.(16.6) and for their applications.

Let us consider the following LL-chain at $\xi \in \mathrm{Sing}(\check{F}(0))_{cl}$ with its head $g(0) = y(0)^q + \epsilon \in \wp(\check{F}(0), q), q = p^e, ord_\xi(\epsilon) > q$:

$$g(0) = g(1)^p + h(0) \dashrightarrow g(1) = g(2)^p + h(1) \dashrightarrow \cdots \dashrightarrow \tag{129}$$
$$g(i) = g(i+1)^p + h(i) \dashrightarrow \cdots \dashrightarrow g(e)$$
$$\textit{also } g(i) = y(i)^{q(i)} + \epsilon(i) \textit{ with } ord_\xi(\epsilon(i)) > q(i) = p^{e-i}$$
$$\textit{ending with } ord_\xi(g(e)) = 1 \textit{ and } \nabla = \{g(e) = 0\}$$

We define ideal exponent $\check{F}(i) = (g(i)\mathcal{O}, q(i))$ with $q(i) = p^{e-i}, e \ge 1$, and with $h(i) \in \wp^{nega}(\check{E}), ord_\xi(h(i)) > q(i)$, while $g(e)$ defines a locally closed smooth hypersurface in a neighborhood $V(\xi)$ of ξ. Let $\pi : Z' \to Z$ a blowup with center $D \subset \nabla$ which is permissible for $\check{F}(0)$ and let $\xi \in D_{cl}$. We can then modify the chain without affecting the $\{\check{F}(i), 0 \le i \le e\}$ at all in such a way that $\mathrm{ord}_D(g(i)) \ge p^{e-i}$ for all i.

Lemma 16.7. *For this LL-chain, the blow-up π is permissible for every $\check{F}(i), i \ge 1$.*

Proof. Any cleaning of $\epsilon(i)$ by $g(i)$ does not change $\check{F}(i)$, Moreover by Th(7.1) and Th.(7.2) of §13 we can modify the parameters $y(i)$ so that $\mathrm{ord}_D(y(i)) \ge p^{e-i}, \forall i$, without changing any $g(i)$. By our choice of $h(i) \in \boxplus(g(i)) \subset \wp(\check{F}(i))$ of §12 we have $\mathrm{ord}_D(h(i)) \ge ord_D(\epsilon(i)) \ge p^{e-i}$. We conclude $\mathrm{ord}_D(g(i)) \ge p^{e-i}$. $\square$

Lemma 16.8. *Follow the notation of Eq.(129). Thanks to Lem.(16.7) the LL-chain Eq.(129) is transformed by π into:.*

$$\begin{aligned} g'(0) = g'(1)^p + h'(0) \dashrightarrow g'(1) = g'(2)^p + h'(1) \\ \dashrightarrow \cdots \dashrightarrow g'(i) = g'(i+1)^p + h'(i) \\ \dashrightarrow \cdots \dashrightarrow g'(e) \end{aligned} \tag{130}$$

Let ξ' be a point of $\pi^{-1}(\xi)_{cl}$ with $\xi \in D_{cl}$.

1. *Assume that $g'(e)$ does not vanish at ξ'. Then $g'(e-1) = g'(e)^p + h'(e-1)$ does not vanish at ξ' because $h'(e-1)$ vanishes there by the rule of transformation of $\wp^{nega}(\check{F}(1))$ by the permissible π. After the domino-fall for $i = e-1, e-2, \cdots$, by the same reason we obtain in particular $g'(0)$ does not vanish at ξ'. In other words the transform $\check{F}'(0)$ is empty locally at ξ'.*
2. *Consider the case of $\check{F}(0) = \check{E})$ created by $F \dashrightarrow \hat{F} \dashrightarrow \check{F}$. Then under the same assumption we must have $\mathrm{Inv}_{\xi'}(\hat{F}') <_{\mathrm{lex}} \mathrm{Inv}_{\xi}(\hat{F})$.*

Proof. (1) is just to repeat the same reasoning from $i = e$ to $i = 0$. (2) is by (1) in view of the definition of $\hat{E} \dashrightarrow \check{E}$. As to the inequality there we infer to theorems and lemmas of §13. $\square$

Corollary 16.9. *If ξ' is a closed point of $\pi^{-1}(\nabla) \setminus \nabla'$ with the strict transform ∇' of ∇ by π then we have*

$$\mathrm{Inv}_{\xi'}(\mathcal{Y}'(m-1) <_{\mathrm{lex}} \mathrm{Inv}_{\xi}(\mathcal{Y}(m-1)$$

Proof. Then $g(0)'$ does not vanish at ξ' because ξ' is not in the strict transform ∇' of ∇ by π while $\pi(\xi') = \xi \in D \subset \nabla$. $\square$

Lemma 16.10. *Consider the case of $\xi' \in \pi^{-1}(\xi) \cap \nabla'$ with $\xi \in D$. Then ξ' in the closure of $\nabla' \setminus \pi^{-1}(D)$ and we have*

$$\mathrm{Inv}_{\xi'}(\mathcal{Y}'(m-1) = \mathrm{Inv}_{\xi}(\mathcal{Y}(m-1) \tag{131}$$

Proof. If $D_\xi = \nabla_\xi$ then we must have ∇' has no common point with $\pi^{-1}(xi)$ by the definition of strict transform. Thus we must have $D_\xi \neq \nabla_\xi$ which means that D is not dense in ∇ within a sufficiently small neighborhood of ξ. The first claim of the lemma is proven. Now the equality of the kind of Eq.(131) is evident at any point ζ outside $\pi^{-1}(D)$ where π is isomorphic within a sufficiently small neighborhood of ζ. We know that $\mathrm{Inv}_\eta(\mathcal{Y}'(m-1))$ is upper semi-continuous in $\mathrm{Sing}(\mathcal{Y}'(m-1))_{cl}$ because the same is true for every ideal exponent belonging to $\mathfrak{Y}'(m-1)$. (Recall that every two members have the same *wp*-algebra at every common point.) There then follows the claimed equality thanks to the theorems and lemmas of §13. □

Lemma 16.11. *Let us consider the case of ξ' of Lem.(16.10). Then the transform Eq.(130) of Eq.(129) by $\pi : Z' \to Z$ of Lem.(16.7) is an LL-chain which is LL-headed by $g'(0)$.*

Proof. The only important point of claim is that $h'(i)$ of Eq.(130) belongs to $\wp^{nega}(\check{F}'(i))$. This is proven by the fact that $\wp^{nega}(\check{F}'(i))$ contains the transform of $\wp^{nega}(\check{F}(i))$ by π. □

The proof of Th.(16.6), Main Theorem I, is completed as follows. Thanks to Lem.(16.8) with Cor.(16.9) and repeated application of the case of Lem.(131) using Lem.(16.8) and Lem.(16.5), we can reduce the problem to the following case: the ideal exponents of $\mathfrak{Y}(m-1)$ are all resolved by a finite sequence of permissible blowups whose centers are nowhere dense in the respective strict transforms of the original ∇, and the permissibility condition to the NC-data (beginning with the given Γ) is faithfully observed at every step of the sequence of blowups. The end result so far will be the situation in which the NC-data has normal crossing with the ∇ which remains to be smooth all the way. This means that the blowup with center ∇ is permissible (Lem.(16.8) can be applied again) and the strict transform of *nabla* turns to be empty. This means that $\mathrm{Sing}(\check{F}(m-1))$ turns out to be be empty. Thus we reach the situation in which the resulting $\mathfrak{Y}'(m-1)$ has the strict inequality

$$\mathrm{Inv}_{\max}\mathcal{Y}'(m-1) <_{\mathrm{lex}} \mathrm{Inv}_{\max}\mathcal{Y}'(m-1)$$

unless more happily $\mathfrak{Y}'(m-1)$ turns to be empty, and $\mathfrak{Y}$-string becomes shorter. This proves Th.(16.6).

Remark 16.12. There may be questioned about any gap between the individual member $F(i)$ (such as in Lems.(16.7)∼(16.11), and the collective deal of $\mathfrak{Y}(m-1)$ with many ideal exponents. But there is actually no problem at all. In fact thanks to paracompact Zariski topology we can find a finite number of triplelets $(Y(ij), V(ij), F(ij))$ with $F(i) \leftrightarrows_{Y(ij)} F(ij)$ such that Sing($\mathcal{Y}$) is covered by the $V(ij)$'s. Then we have a simple way to create a single ideal exponent F such that

$$\mathfrak{S}(F)|_{V(ij)} \;=\; \mathfrak{S}(F(i))|_{V(ij)} \; for \; \forall (ij)$$

Then all we need is the claims in Lems.(16.7)∼(16.11) applied to this F. The results on F are only up to $\wp(F(i)$ (only up to integral closure of graded algebras) but they are clearly good enough for the Main Theorem.

16.3. **Theorems ∇ II-III: Inductive proof of resolution.** We know that for an ideal exponent $E = (J, b)$ with $J \neq (0)$ the maximum degree of the edge generators are bounded from above for all transforms of E by all possible sequences of the permissible blowups. With this fact in mind we apply Th.(16.6) (Main Theorem I) repeatedly starting from the sequence Eq.(126) of $\check{E}$ and replacing $\mathfrak{Y}$'s by their transforms in accord with Th.(16.6) with some deletions of those with Sing emptied. It should be noted that core-edge focusing is renewed in Eq.(126)-chain after each sequence of blowups by Th.(16.6) and eventually the repetition should sweep out the whole Sing. This is easily observed at the tail end of Eq.(126)-chain in particular where $\hat{\mathcal{Y}}$ will turn into empty (after empty ideal exponent members are deleted). There the chain is made shorter and shorter. In the end the the core-edge focusing after Th.(16.6) will sweep out the entire $\hat{E}$ and eventually $\hat{E}$ will turn to have the empty empty Sing.

Theorem 16.13. *(Main Theorem II)*
By applying Th.(16.6) to Eq.(126)-chain repeatedly but finitely many times, with all blowups of Z permissible for the given $E = (J, b)$ and for the given Γ, we obtain the resolution of singularities of $\hat{(}E)$.

The proof is already included in what has been done above.

Theorem 16.14. *(Main Theorem III)*
Given any ideal exponent E and any NC-data Γ in Z we have a resolution of singularities of E by a finite sequence of blowups of Z permissible for E and Γ. Here the resolution means to make the singular locus empty.

Proof. The proof is done by making use of Th.(16.4) together with Th.(16.13). □

16.4. $\Theta(\check{E})$ **headed by** $\sigma\mathfrak{L}(\check{E})$**.** Recall that with sufficiently large integer ℓ the members of the form $g = y^q + \epsilon \in \mathfrak{L}(\check{E}, q(j))$ generate the whole $\mathfrak{L}(\check{E})$. Refer to Eq.(6.3), Eq.(53), and Def.(6.12) of §6.3.

Recall the sequence of edge-exponents $(q_1, q_2, \cdots, q_r)$ among the edge generators of $\wp(\check{E})$ we should refer to Eq.(34) and Eq.(26). In what follows we will use the completed (and order reversed) sequence of exponents:

$$q(e) = q_r = p^e, q(e-1), \cdots, q(0) = 1 \text{ where } q(k) = p^k \tag{132}$$

We will then define $\Theta(\check{E}, q(k))$ and $\Sigma\Theta(\check{E}, q(k))$ by descending induction on $k : e \geq k \geq 0$, as follows. To begin, we let $\Theta(\check{E}, q(e)) = \mathfrak{L}(\check{E}, q_r)$ which is the $\rho^\ell(\mathcal{O}_Z)$-module generated LL-heads producing all LL-chains of $\wp^{posi}(\check{E})$. Refer to §6.3 where ℓ is sufficiently large integer. We sometimes write Sing below signifying $\mathrm{Sing}(\check{E})_{cl}$ for short in accord with Eq.(91).

Definition 16.15 (Induction Method)**.** For all k of $q(k)$ with $e \geq k \geq 0$, we proceed with descending induction:

Step 1.

$$\Theta(\check{E}, q(e)) = \mathfrak{L}(\check{E}, q(e))$$

Step 2.

$$(\sigma\Theta)(\check{E}, q(e)) = \Theta(\check{E}, q(e)) + \Sigma_{a>0}\big(\wp^{nega}(\check{E}, -a) \bigcap I^{(q(e)+1)}(\mathrm{Sing})\big),$$

where the symbol $\bigcap I^{(q(e)+1)}(\mathrm{Sing})$ with a graded module denotes application of the intersection to the ideal of every homogeneous part of the graded module.

Definition 16.16 (Induction for σ and Θ)**.** Take any k with $e \geq k \geq 0$,.

Step 1.

$$\Theta(\check{E}, q(e)) = \mathfrak{L}(\check{E}, q(e))$$

Step 2.

$$\begin{aligned} (\sigma\Theta)(\check{E}, q(e)) &= \Theta(\check{E}, q(e)) + \| \sum_{a>0} \big(\wp^{nega}(\check{E}, -a) \bigcap I^{(q(e)+1)}(\mathrm{Sing})\big) \|_{q(e)} \\ \Theta(\check{E}, q(e-1)) &= \sqrt[p]{\rho(Bl(Z, q(e-1)) \bigcap \big(\sigma\Theta(\check{E}, q(e))\big)} \end{aligned}$$

considered as $\rho^\ell(\mathcal{O})$-module with any $\ell \gg e$. We then proceed by descending induction on k:

Step 1.

$$(\sigma\Theta)(\check{E}, q(k)) = \Theta(\check{E}, q(k) + \sum_{a>0}(\wp^{nega}(\check{E}, -a) \bigcap I^{(q(k)+1)}(\mathrm{Sing})$$

Step 2.

$$\Theta(\check{E}, q(k-1)) = \sqrt[p]{\rho(Bl(Z, q(k-1)) \bigcap (\sigma\Theta)(\check{E}, q(k))}\,.$$

The next definition is the $\wp^{nega}(\check{E})$ version of Def.(16.16). In other words we take away the symbol $\| \;\|$ from Def.(16.16) and then apply Frobenius ρ which means taking the p-th power back after deleting two operations, $\sqrt[p]{}$ and $\| \;\|$. This elucidates the undergoing process behind Def.(16.16), and we regain the close tie of $\wp^{nega}(\check{E})$ with the inductive process.

The results of the new version will be denoted by $\Theta^*(\check{E}, q(k))$ and $(\sigma\Theta)^*(\check{E}, q(k))$. Thus the homogeneity degrees of components are revived in accord with the graded algebra $\tilde{\wp}(\check{E})$. We have

Step 1.

$$\Theta^*(\check{E}, q(e)) = \Theta(\check{E}, q(e)) = \mathfrak{L}(\check{E}, q(e))$$

Step 2.

$$(\sigma\Theta)^*(\check{E}, q(e)) = \Theta^*(\check{E}, q(e)) + \sum_{a>0} \big(\wp^{nega}(\check{E}, -a) \bigcap I^{(q(e)+1)}(\mathrm{Sing}) \big)$$

Step 3.

$$\Theta^*(\check{E}, q(e-1)) = \rho\big(\Theta(\check{E}, q(e-1)\big) = \rho(Bl(Z, q(e-1)) \bigcap (\sigma\Theta)^*(\check{E}, q(e))$$

(without $\| \;\|$ and $\sqrt[p]{\;}$). For general k we have

$$\begin{aligned} \Theta^*(\check{E}, q(k)) &= \rho^{e-k}\big(\Theta(\check{E}, q(k))\big) \\ (\sigma\Theta)^*(\check{E}, q(k)) &= \Theta^*(\check{E}, q(k)) + \sum_{a>0} (\wp^{nega}(\check{E}, -a) \bigcap I^{(q(k)+1)}(\mathrm{Sing})) \end{aligned}$$

and

$$\begin{aligned} \Theta^*(\check{E}, q(k-1)) &= \rho^{e-k+1}\big(\Theta(\check{E}, q(k-1))\big) \\ &= \rho^{e-k+1}\Big(\rho(Bl(Z, q(k-1)) \bigcap (\sigma\Theta)^*(\check{E}, q(k))\Big). \end{aligned}$$

17. Comments on the methodology of this paper

Our methodology of using leverage-up exponent-down seems to be more significant than just proving the resolution of singularities. Its featured property is that the leverage-techniques are all defined in the given ambient scheme Z independent of birational transformations which are chosen afterwords for actual desingularization.

In this work the base field $\mathbb{K}$ is always assumed to be a finite field or $\mathbb{Z}/p\mathbb{Z}$ because our resolution is for all dimension. When the $\mathbb{K}$ has transcendence degree d we can reformulate the resolution problem to the case of dimension $d + dimZ$.

Moreover for instance we can deduce the problem of equivariant resolution of singularities with a group of automorphisms of the given E in Z. We then let the group act on the collection $\{\boxminus(g, q)\}$ for all

$g \in \mathfrak{L}(\check{E})$. The quasi-compact Zariski topology is always helpful behind the actual formulations. Then the equivariant resolution of singularities is automatically obtained by the means of the global equivariant covering of leverage-ups data for the given singular data in Z

For instance in this paper we have no need to hesitate using any algebraic extensions of the base field $\mathbb{K}$ in any intermediary step of reasonings in this paper. It causes no trouble in the end to regain the resolution over the original base field. Simply take the Galois group of the base field extension which is always separable because our $\mathbb{K}$ is perfect. Then apply the Galois-equivariant covering of leverage-up packages.

References

[1] Abhyankar, S., *Local uniformization on algebraic surfaces over ground fields of chracteristic $p \neq 0$*, Annals of Mathematics, 63 (1956), pp.491-526.

[2] Abhyankar, S., *Resolution of singularities of embedded algebraic surfaces*, Pure and Applied Mathematics, 24 (1966), Academic Press, New York.

[3] ———., *Desingularization of plane curves*, Summer Institute on Algebraic Geometry, Arcata 1981, Proc. Symp. Pure Appl. Math.40, AMS.

[4] Aroca, Jose M., Hironaka, H., Vicente, Jose L., *Desingularization theorems*, vol 30 (1977), Mem. de Math., Institut Jorge Juan, Madrid

[5] Abramovich, Dan, de Jong, A. J., *Smoothness, semistability, and toroidal geometry*, J. Algebraic Geom. 6 (1997), no. 4 , 789-801

[6] Abramovich, Dan, Wang, Jianhua, *Equivariant resolution of singularities in characteristic 0*, Math. Res. Lett. 4(1997), no.2-3, 427-433

[7] Bierstone, Ed., Milman, P. D., *Uniformization of algebraic spaces*, J. Amer. Math. Soc. 2 (1989), no.4, 801-836

[8] Bierstone, Ed., Milman, P. D., *A simple constructive proof of canonical resolution of singularities, Effective methods in algebraic geometry*, (Castiglioncello, 1990). Progr. Math., vol. 94, Birkhauser Boston, Boston, MA, (1991), 11-30

[9] Bierstone, Ed., Milman, P. D., *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Inven. Math. 128 (1997), no.2, 207-302

[10] Bierstone, Ed., Milman, P. D., *Desingularization algorithm.I. Role of exceptional divisors*, Mosc. Math. J. 3 (2003), no.3, 751-805

[11] Cano, F.,*Reduction of the singularities of codeimension one singular folitions in dimension three*, Ann.of math, 160(2004), 907-1011

[12] Cossart, V., *Polyhedre caracteristique d'une singularite*, These d'Etat, Orsay 1987

[13] Cossart, V., Giraud, J., Orbanz, U., *Resolution of surface singularities*, Lecture Notes in Math. vol.1101, Springer 1984

[14] Cossart, V., Galindo C. et Piltant O. *Un example effectif de gradue non noetherien associe a une valuation divisorielle*, Ann. de l'Inst. Fourier, Grenoble, 50, 1(2000), 105-112

[15] Giraud, J., *Sur la theorie du contact maximal*, Math. Z. 137(1974), 286-310.

[16] ______, *Contact maximal en caracteristique positive*, Ann. Scient. E.N.S.8 (1975), 201-234
[17] Hauser, H. *Seventeen obstacles for resolution of singularities*, in Singularities, Oberwolfach, 1996, 289-313, Progr.Math., 162, Birkhauser, Bazel
[18] ______, *The Hironaka theorem on resolution of singularities (or: A proof we always wanted to understand)*,Bull.Amer.Math.Soc. (N.S.) 40 (2003), no.3, 323-403 (electronic).
[19] Hironaka, H., *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. of Math. 79(1964) pp.109-326.
[20] ______, *Gardening of infinitely near singularities*, Proc. Nordic Summer School in Math., Oslo (1970) pp.315-332.
[21] ______, *Introduction to the theory of infinitely near singular points*, Mem de Mat del Inst Jorge Juan, Madrid, 28 (1974)
[22] ______, *Idealistic exponents of singularity*, Algebraic Geometry, Johns Hopkins Univ. Press, Baltimore, Md. (1977) pp. 52-125 (J.J.Sylvester Symposium, Johns Hopkins Univ., 1976)
[23] ______, *Theory of infinitely near singular points*, J. Korean Math.Soc. 40, no.5, pp. 901-920 (Sept.2003)
[24] ______, *Three key theorems on infinitely near singularities*, Seminaires&Congres 10, Soc.Math.France, pp. 871-126 (2005)
[25] ______, *A program for resolution of singularities, in all characteristics and in all dimensions*, preprint for series of lectures in "Summer School on Resolution of Singularities" at Internatioanl Center for Theoretical Physics, Trieste, June 12-30, 2006
[26] J. de Jong, *Smoothness, semi-stability and alterations,* Publ.IHES 83 (1996), 51-93
[27] Kawanoue, Hiraku, *Toward Resolution of Singularities over a field of Positive Characteristic* Publ.RIMS, Kyoto Univ. 43 (2007), 819-909
[28] Kollar, Janos, *Resolution of singularities-Seattle lecture*, arXiv:math.AG/0508332 v1, 17 Aug 2005
[29] Moh, T.T. *On a stability theorem for local uniformization in characteristic p*, Journal of RIMS, 23 No.6,Nov.1987
[30] Moh, T.T. *On a Newton Polygon approach to the uniformization of singularities of characteristic p*, Algebraic Geometry and Singularities (eds. A. Campillo, L. Narvaez). Proc. Conf. on Singularities La Rabida. Birkhauser 1996
[31] Moh, T.T. *Desingularization of three-varieties in positive characteristic*, Progress in Mathematics 134, pp.49-93, (year?)
[32] Moh, T.T., Zhang,Y.T. *On a local uniformization theorem for 3-fold in characteristic* $p > 0$, Lecture Note (?)
[33] Youssin, Boris, *Newton Polyhedra without coordinates*, Mem. AMS 433(1990), 1-74, 75-99.
[34] Spivakovsky, Mark, *A solution to Hironaka's polyhedra game*, Arithmetic and Geometry, Papers dedicated to I. R. Shafarevich on the occasion of his sixtieth birthday, vol.II, Birkhauser,1983, pp.419-432
[35] Spivakovsky, Mark, *A counterexample to Hironaka's "hard polyhedra game*, Publ. RIMS Kyoto University 18,3 (1982), 1009-1012

[36] Spivakovsky, Mark, *A counterexample to the theorem of Beppo Levi in three dimensions*, Invent. Math 96 (1989), 181-183

[37] Spivakovsky, Mark, *Resolucion de singularidades y raices aproximadas de Tschirhausen*, Notes by Fernando Sanz, Seminarios tematicos - Instituto de Estudios con Iberoamerica y Portugal, Seminario Iberoamericano de Matematicas IV (1997), 3-17

[38] Zariski, O., *Local uniformization on algebraic varieties*, Ann. of Math., 41(1940) pp.852-896.

[39] ______, *The reduction of singularities of an algebraic three dimensional varieties*, Ann. of Math., 45 (1944) pp.472-542.

Japan Association for Mathematical Sciences, Coop Olympia 506, Jingumae 6-35-3, Shibuya-ku, Tokyo 150-0001, Japan