VECTOR BUNDLES ON TREES OF SMOOTH RATIONAL CURVES

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ABSTRACT. Given a vector bundle $E$ on a tree of smooth rational curves $C$ and a vector bundle $E'$ on $\mathbb{P}^1$ of the same degree and rank, we show that $E$ deforms to $E'$ if such a deformation is permitted by the usual semicontinuity condition and $C \cong \mathbb{P}^1 \cup \mathbb{P}^1$. When $C$ is assumed to be a complex curve, we additionally show $E$ deforms to $E'$ if the semicontinuity condition permits and $E'$ has rank 2.

1. Introduction

Let $C/k$ be a connected curve of arithmetic genus 0 such that every irreducible component of $C$ is isomorphic to $\mathbb{P}^1$. In what follows, we will refer to $C$ as a tree of smooth rational curves, following the convention of [5]. Given a DVR $R$ over $k$, it is easy to construct flat projective families $\pi : \mathcal{C} \to \Delta$ where $\Delta = \text{Spec}(R)$ such that the special fiber of $\mathcal{C}$ is $C$ and the generic fiber is $\mathbb{P}^1_K$, where $K$ is the field of fractions of $R$. A natural question is under what circumstances a vector bundle on $C$ deforms likewise. More precisely, we have the following.

Question 1.1. Let $C$ be a tree of smooth rational curves, and let $E$ be a vector bundle on $C$. Let $E'$ be a vector bundle on $\mathbb{P}^1$. Under what circumstances does $E$ deform to $E'$? That is, when is there a flat family $\pi : \mathcal{C} \to \Delta$ where $\Delta$ is as above and a vector bundle $\mathcal{E}$ on $\mathcal{C}$ such that $\mathcal{C}_0 \cong C$ and $\mathcal{E}|_{\mathcal{C}_0} \cong E$, while $\mathcal{C}_\eta \cong \mathbb{P}^1_K$ and $\mathcal{E}|_{\mathcal{C}_\eta} \cong E$ for $\eta$ the generic point?

It is clear that $E$ and $E'$ must have the same degree and rank for $E$ to deform to $E'$. Beyond this, the best-known obstruction to a vector bundle $E$ deforming to $E'$ as above is a failure of the upper semicontinuity condition for cohomology of coherent sheaves. Every line bundle of degree $d$ on $C$ deforms to $\mathcal{O}(d)$ on $\mathbb{P}^1$, so the upper semicontinuity condition [4, Theorem 12.8] states that if $E$ deforms to $E'$, then the inequality

$$h^0(C, E \otimes L) \geq h^0(\mathbb{P}^1, E'(\deg L))$$

holds for all line bundles $L$ on $C$.

In the simplest case, when $C \cong \mathbb{P}^1$, the semicontinuity condition ([1]) is well-known to be sufficient to allow deformation.
Proposition 1.2 ([3], Theorem 14.7). Let $E$ and $E'$ be vector bundles on $\mathbb{P}^1$ of the same degree and rank. If we have
\[ h^0(\mathbb{P}^1, E'(d)) \leq h^0(\mathbb{P}^1, E(d)) \]
for all integers $d$, then $E$ deforms to $E'$ on $\mathbb{P}^1$.

Even outside of the case $C \cong \mathbb{P}^1$, the inequalities in [1] substantially refine our main questions.

Question 1.3. For any $C$, $E$, $E'$ as above, if (1) is satisfied for all line bundles $L$ on $C$, does $E$ deform to $E'$?

We will answer Question 1.3 affirmatively in a few key cases. The most straightforward case occurs when $E'$ is balanced.

Proposition 1.4. If $C$ is a tree of smooth rational curves, $E$ is a vector bundle on $C$, and $E'$ is a balanced vector bundle on $\mathbb{P}^1$ of the same degree and rank as $E$, then $E$ deforms to $E'$.

Proof. Let $L$ be a line bundle on $C$ such that $E \otimes L$ is globally generated. We show that $E \otimes L$ deforms to $E'((\deg L))$. Suppose $E$ has rank $r$ and $h^0(C, E \otimes L) = N$. Then, letting $X$ be the Grassmannian
\[ X = Gr(H^0(C, E \otimes L), N - r), \]
$E \otimes L$ gives a map $f : C \to X$ sending $p \in C$ to the space of global sections of $E \otimes L$ vanishing at $p$. And $TX$ is globally generated, so $f^*(TX)$ is also globally generated. Then by [5, Theorem II.7.6], $f$ occurs as the flat limit of a family $f_t : \mathbb{P}^1 \to X$ with $t \in T$ for $T$ some smooth curve. Letting $Q$ be the universal quotient bundle on $X$, we have that $E \otimes L \cong f^*(Q)$ deforms to the vector bundles $f_t^*(Q)$ on $\mathbb{P}^1$. Since every vector bundle on $\mathbb{P}^1$ deforms to a balanced bundle, we conclude that $E \otimes L$ deforms to $E' \otimes L$, so $E$ deforms to $E'$, as was to be shown. □

If $E$ has rank 2, the study of vector bundles on $C$ is closely related to the study of $\mathbb{P}^1$ bundles over $C$. Coskun [1] analyzed the deformations of these bundles, and using his analysis, we prove an affirmative answer to Question 1.3 in the rank 2 case.

Theorem 1.5. If $C$ is a complex tree of smooth rational curves, $E$ and $E'$ are as above, $E$ is rank 2, and (1) is satisfied for all line bundles $L$ on $C$, then $E$ deforms to $E'$.

Beyond rank 2, the geometry of vector bundles on trees of smooth rational curves becomes more complicated. The bulk of this article will be devoted to an analysis of higher rank vector bundles on the specific curve $C = \mathbb{P}^1 \vee \mathbb{P}^1$.

Theorem 1.6. Set $C = \mathbb{P}^1 \vee \mathbb{P}^1$, that is, $\mathbb{P}^1$ glued to $\mathbb{P}^1$ at a single node. If $E$ and $E'$ are vector bundles on $C$ and $\mathbb{P}^1$ respectively of the same rank and degree, and (1) is satisfied for all line bundles $L$ on $C$, then $E$ deforms to $E'$. 
To prove this result, we construct a sequence of degenerations starting with \( E' \) and ending with \( E \) explicitly. All of these degenerations are fairly simple on their own, and are listed in Section 3.1, but we believe they are likely to be sufficient to address cases beyond \( \mathbb{P}^1 \lor \mathbb{P}^1 \). We believe that for any \( C \) such that a vector bundle \( E \) on \( C \) automatically splits as the direct sum of line bundles, a method like the one we use to prove Theorem 1.6 could work. Those \( C \) are precisely chains of projective lines by \cite[Theorem 4.3]{6}, so we have the following.

**Conjecture 1.7.** Let \( C = \mathbb{P}^1 \lor \cdots \lor \mathbb{P}^1 \) be a finite chain of projective lines, and \( E \) a vector bundle on \( C \). If \( E' \) is a vector bundle on \( \mathbb{P}^1 \) allowed by semicontinuity to be a deformation of \( E \), then one can construct \( E \) as a degeneration of \( E' \) by iterating the degenerations listed in Section 3.1.

There are several open questions related to the deformations we study in this paper. Most obviously, it is unclear whether Question 1.3 ever has a negative answer. Even on a curve as simple as a projective line meeting three other disjoint projective lines in nodes, vector bundles need not split and can encode interesting geometric information. Even to answer Question 1.3 affirmatively in this case would likely require new techniques.

In addition, Deopurkar \cite{2} observes that even if a vector bundle \( E \) deforms to \( E' \), it may not be the case that \( E \) deforms to \( E' \) over a flat family \( C \to \Delta \) with \( C \) smooth. In general, it is unclear for what triples \( C, E, E' \) with \( E \) deforming to \( E' \) the family \( C \to \Delta \) exhibiting the deformation can have \( C \) smooth.

This paper is organized as follows. In Section 2 we prove Theorem 1.5. In section 3 we explain the general algorithm we use to deform a vector bundle on \( \mathbb{P}^1 \lor \mathbb{P}^1 \) to one on \( \mathbb{P}^1 \), and prove Theorem 1.6 conditional on several technical lemmas. In Section 4 we establish these lemmas.

## 2. The rank 2 case

Let \( C \) be an connected nodal complex tree of smooth rational curves, \( E \) a vector bundle of rank 2 on \( C \), and \( E' \) a vector bundle of identical degree and rank on \( \mathbb{P}^1 \). We will show that the condition (1) is sufficient to show \( E \) deforms to \( E' \), proving Theorem 1.5. To do so we use a closely related result of Coskun. Let \( S_{k,\ell} \subset \mathbb{P}^{k+\ell+1} \) denote the rational normal surface scroll given by fixing two rational normal curves \( C_k \subset \mathbb{P}^k \) and \( C_{\ell} \subset \mathbb{P}^{\ell} \) in disjoint subspaces of \( \mathbb{P}^{k+\ell+1} \), choosing an isomorphism \( C_k \to C_{\ell} \) and taking the union of all the lines between identified points in \( \mathbb{P}^{k+\ell+1} \). If \( k \leq \ell \), we call \( C_k \) the directrix of \( C \); if \( k < \ell \), \( C_k \) is the unique curve of degree \( k \) in \( S_{k,\ell} \) that is a section of the projection \( S_{k,\ell} \to C_k \). Coskun proves the following theorem about the flat limit of surface scrolls.

**Theorem 2.1.** \cite[Theorem 4.5]{1} Let \( S \) be a connected tree \( \bigcup_{1 \leq i \leq N} S_{k_i,\ell_i} \) of surface scrolls in \( \mathbb{P}^n \), where two scrolls intersect either transversally along a single fiber or not at all.
Let $C$ be a degree $d$ connected curve on $S$ whose restriction to each $S_{k,t_i}$ is a section class. Suppose $2d \leq \deg(S)$.

Then there is a one-parameter family of surface scrolls $S_{d,\deg(S)-d}$ with flat limit $S$ and with directrices specializing to $C$.

We will use this result to prove Theorem 2. To do so, we will use the following technical proposition, which essentially says that if $E$ is positive enough that all degree 0 twists of $E$ still have global sections, then we can construct a curve on $\mathbb{P}(E)$ associated to the global sections of twists of $E$.

**Proposition 2.2.** Let $C$ be a tree of smooth rational curves, and let $E$ be a vector bundle on $C$ such that $H^0(C, E(L)) \neq 0$ for all $L$ of degree 0. Then there is some $L$ of degree 0, a decomposition $C = \bigcup_{1 \leq i \leq k} C_i$ of $C$ into connected closed subcurves $C_i$ sharing no irreducible components of $C$ in common, and disjoint collections of points $B_i \subset C_i$ such that

$$\bigcup_{1 \leq i \leq k} B_i = \bigcup_{1 \leq i < j \leq k} C_i \cap C_j$$

and such that there exists a section $s_i$ of each $(E \otimes L)|_{C_i} \otimes \mathcal{I}_{B_i}$ that does not vanish identically on any irreducible component of $C_i$.

**Proof.** We proceed by induction on the number of components of $C$. The case $C = \mathbb{P}^1$ is trivial. Suppose the proposition is established for all $C$ with at most $n$ components, and let $C$ be a curve with $n+1$ components, and $E$ a vector bundle on $C$ such that all degree 0 twists of $E$ have global sections. The proposition does not depend on which degree 0 twist of $E$ we use, so without loss of generality assume $E$ is chosen to minimize $h^0(C, E)$ among twists of degree 0. Let $s$ be a section of $E$. If $s$ is not identically zero on any irreducible component of $C$, the proposition is valid with $k = 1$ and $C_1 = C$. Otherwise, let $C'$ be the union of the irreducible components of $C$ on which $s$ is not identically zero. Without loss of generality, we can assume $C'$ connected, as if $C'$ has at least 2 connected components $C'_1$ and $C'_2$, we can replace $s|_{C'_2}$ by the zero section and produce a global section of $E$ nonzero on $C'_1$ but zero on $C'_2$. So assume $C'$ is connected. So $C \setminus C'$ is a union of connected nodal subcurves $C_1, \ldots, C_k$ meeting $C$ in nodes $p_1, \ldots, p_k$ respectively.

We now construct a degree 0 line bundle $L$ on $C$ such that $h^0(C, E \otimes L) = h^0(C, E)$, there exists a section $s$ of $E \otimes L$ not vanishing on any irreducible component of $C'$, and $h^0(C_i, (E \otimes L)|_{C_i} \otimes L_i) > 0$ for all $i$ and degree 0 line bundles $L_i$ on $C_i$. We do this iteratively. Start with $L$ the trivial line bundle. Then iterate the following procedure.

1. If $H^0(C_i, (E \otimes L)|_{C_i}) = 0$, let $L'$ be the line bundle $L \otimes L''$ where $L''$ is a degree 0 line bundle satisfying $L''|_{C'} \cong O(-p_1)$, $L''|_{C_i} \cong O(p_i)$ and $L''$ is trivial on every other component. We then have a map

$$g : H^0(C, E \otimes L') \to H^0(C, E \otimes L)$$
given by multiplying \( s \in H^0(E \otimes L') \) by a nonzero section of \( L'^n \). \( g \) is injective (having kernel \( H^0(E \otimes L|_{C_i}) = 0 \)) and hence an isomorphism since \( h^0(E \otimes L) \) is minimal. So replacing \( L \) by \( L' \) produces a twist which still has minimal \( h^0 \) but has larger degree on \( C_i \). And \( g^{-1}(s) \) is a section of \( H^0(C, E \otimes L') \) vanishing on no components of \( C' \).

(2) If \( H^0(C_i, (E \otimes L)|_{C_i}) \neq 0 \) but for some degree 0 line bundle \( L_i \) on \( C_i \), \( H^0(C_i, (E \otimes L)|_{C_i} \otimes L_i) \neq 0 \), let \( L'' \) be the line bundle that is trivial except on \( C_i \) and has \( L''|_{C_i} = L_i \). Then setting \( L' = L \otimes L'' \), we have \( h^0(C, E \otimes L) = h^0(C, E \otimes L') \), and since \( (E \otimes L)|_{C'} \) was not affected by the twist, \( E \otimes L' \) still has a global section that does not vanish identically on any irreducible component of \( C' \).

(3) If neither of the steps above apply, we are done.

This process terminates, because each time we apply the first twist, we increase the degree of \( E \otimes L|_{C_i} \), and each time we apply the second twist, we then can immediately apply the first twist. Once the degree of \( (E \otimes L)|_{C_i} \) is sufficiently large, \( (E \otimes L)|_{C_i} \) automatically has global sections.

Now that we’ve constructed \( L \) satisfying the properties we desired, we have that \( (E \otimes L)|_{C_i} \) satisfies the hypotheses of the proposition on \( C_i \), and hence by the inductive hypothesis there exist connected subcurves \( C_{i,j} \) of each \( C_i \) and collections of points \( B_{i,j} \) such that the conclusion of the proposition is satisfied on each \( C_i \). Then setting \( B' = \{ p_1, \ldots, p_k \} \), we have that \( C \) decomposes into the union of connected curves \( C' \cup \bigcup_{i,j} C_{i,j} \) whose pairwise intersections are the union \( B' \cup \bigcup_{i,j} B_{i,j} \). To show the proposition is satisfied for these choices of subcurves and collections of points, we only need to check now that \( H^0(C', (E \otimes L)|_{C'}(-p_1 - \ldots - p_k)) \) has some section \( s \) that does not vanish on any component of \( C' \). Our original \( s \) is a section of \( (E \otimes L)|_{C'} \) that vanishes on \( p_1 + \ldots + p_k \), so such a section exists and the proposition is proved. \( \square \)

**Proof of Theorem 2**. Let \( C, E, \) and \( E' \) be as in the statement of Theorem 2. We show that the projective bundle \( \mathbb{P}(E) \) deforms to \( \mathbb{P}(E') \). Without loss of generality, we can assume \( E \) and \( E' \) are twisted by sufficiently negative line bundles that \( H^0(C', E|_{C'}) = 0 \) for all closed subcurves of \( C \) and \( H^0(\mathbb{P}^1, E) = 0 \). Under these assumptions, \( E \) embeds \( \mathbb{P}(E) \) embeds as a tree of scrolls in \( \mathbb{P}(H^0(E^*)^*) \), and \( \mathbb{P}(E') \) embeds in \( \mathbb{P}(H^0(E'^*)^*) \) as a scroll. Suppose \( E' = \mathcal{O}(-d) \oplus \mathcal{O}(-e) \), with \( 0 \leq d \leq e \). Then by hypothesis and by Proposition 2.2 there is some degree \( k \) twist of \( E \) such that there is a deformation of \( C \) as in the proposition. Choose sections \( s_i \) as in the proposition, and let \( \tilde{C} \) be the union of the section curves \( \tilde{C}_i \) in \( \mathbb{P}(E) \) corresponding to each \( s_i \) on \( C_i \) and the fibers in \( \mathbb{P}(E) \) over the points of \( \bigcup_{1 \leq i \leq k} B_i \). \( \tilde{C} \) then embeds in \( \mathbb{P}(H^0(E^*)^*) \) as a connected section curve on the image of \( \mathbb{P}(E) \) of degree \( d \). So by Theorem 2.1 there is a family of scrolls \( S_{d,e} \) in \( \mathbb{P}(H^0(E^*)^*) \) with flat limit the image of \( \mathbb{P}(E) \) such that their directrices degenerate to \( \tilde{C} \). Thus \( \mathbb{P}(E) \) deforms to \( \mathbb{P}(E') \), and so \( E \) deforms to \( E' \). \( \square \)
3. Deforming vector bundles on \( \mathbb{P}^1 \lor \mathbb{P}^1 \)

In this section we describe how to deform vector bundles on \( C = \mathbb{P}^1 \lor \mathbb{P}^1 \) to vector bundles on \( \mathbb{P}^1 \), with the goal of proving Theorem \ref{thm:1.6}. To fix notation, let \( C_1 \) and \( C_2 \) be the two components of \( C \), let \( p \) be the node at which they meet, and let \( \mathcal{O}(d_1, d_2) \) denote a line bundle on \( C \) of degree \( d_1 \) on \( C_1 \) and degree \( d_2 \) on \( C_2 \). Let \( E \) be a rank \( r \) degree \( e \) vector bundle on \( \mathbb{P}^1 \lor \mathbb{P}^1 \), and \( E' \) a vector bundle on \( \mathbb{P}^1 \) such that

\[
h^0(C, E \otimes L) \geq h^0(\mathbb{P}^1, E'(d))
\]

for all line bundles \( L \) of degree \( d \) on \( C \). In this section, we construct a sequence of degenerations of vector bundles starting at \( E' \) and ending at \( E \).

3.1. The basic degenerations. Constructing a degeneration from \( E' \) to \( E \) will require the four basic operations listed here:

1. Degenerating a vector bundle \( E' \) on \( \mathbb{P}^1 \) to a vector bundle \( E'' \) on \( \mathbb{P}^1 \) such that \( h^0(\mathbb{P}^1, E''(d)) \geq h^0(\mathbb{P}^1, E'(d)) \) for all \( d \).
2. Degenerating a line bundle \( \mathcal{O}(d) \) on \( \mathbb{P}^1 \) to a line bundle \( \mathcal{O}(b, d - b) \) on \( \mathbb{P}^1 \lor \mathbb{P}^1 \), and analogously degenerating direct sums of line bundles.
3. Given \( b_1 > b_2 \) and \( c_2 > c_1 \), degenerating the rank 2 bundle \( \mathcal{O}(b_1, c_1) \oplus \mathcal{O}(b_2, c_2) \) to \( \mathcal{O}(b_2, c_1) \oplus \mathcal{O}(b_1, c_2) \).
4. Given \( c_1 \geq c_2 \) and \( b_1 \geq b_2 \), degenerating the rank 2 bundle \( \mathcal{O}(b_1, c_1) \oplus \mathcal{O}(b_2, c_2) \) to \( \mathcal{O}(b_2, c_1 + 1) \oplus \mathcal{O}(b_1, c_2 - 1) \).

All of these degenerations are reasonably easy to construct. The first follows from Proposition \ref{prop:1.2}. The second is trivial. To construct the degeneration in (3), let \( f : \tilde{C} \to C \) be the normalization of \( C \), and \( \tilde{E} \) the vector bundle \( \mathcal{O}(b_1) \oplus \mathcal{O}(b_2) \uplus \mathcal{O}(c_1) \oplus \mathcal{O}(c_2) \) on \( \tilde{C} \). Then letting \( p_1 \) and \( p_2 \) be the two points in the preimage of \( p \) under \( f \), a vector bundle with pullback to \( \tilde{C} \) given by \( \tilde{E} \) is given by the data of an isomorphism \( i : \tilde{E} \otimes \mathcal{O}_{p_1} \to \tilde{E} \otimes \mathcal{O}_{p_2} \). If \( i \) maps the fiber at \( p_1 \) of the line subbundle \( \mathcal{O}(b_1) \) of \( \tilde{E}|_{\tilde{C}_1} \) into the fiber at \( p_2 \) of the line subbundle \( \mathcal{O}(c_2) \) of \( \tilde{E}|_{\tilde{C}_2} \), then \( i \) corresponds to the vector bundle \( \mathcal{O}(b_2, c_1) \oplus \mathcal{O}(b_1, c_2) \); otherwise, \( i \) corresponds to the vector bundle \( \mathcal{O}(b_1, c_1) \oplus \mathcal{O}(b_2, c_2) \). This condition on \( i \) is closed, so the desired degeneration exists.

The degeneration in (4) exists because under the hypotheses given, \( \mathcal{O}(b_1, c_1) \oplus \mathcal{O}(b_2, c_2) \) is a nontrivial extension of \( \mathcal{O}(b_2, c_1 + 1) \) by \( \mathcal{O}(b_1, c_2 - 1) \).

3.2. Proof of Theorem \ref{thm:1.6}. Our construction of a degeneration from \( E' \) to \( E \) will proceed through three steps. We first degenerate \( E' \) to the vector bundle \( E_{\text{max}} \) defined in the following lemma.

**Lemma 3.1.** With \( E \) as above, there is a unique vector bundle \( E'_{\text{max}} \) on \( \mathbb{P}^1 \) of degree \( e \) and rank \( r \) satisfying

\[
h^0(C, E \otimes L) \geq h^0(\mathbb{P}^1, E'_{\text{max}}(\deg(L)))
\]
for all \( L \) and such that any other \( E' \) satisfying (1) degenerates to \( E_{\text{max}} \).

We then degenerate \( E_{\text{max}} \) to a vector bundle \( F \) on \( C \) using the second construction in Section 3.1 producing a vector bundle \( F \) such that \( F|_{C_1} \cong E|_{C_1} \). The exact description of \( F \) is given in (2) below. We also verify the following two lemmas.

**Lemma 3.2.** If \( F \) is the vector bundle on \( C \) given by (2), then \( h^0(C, F \otimes L) \leq h^0(C, E \otimes L) \) for all line bundles \( L \) on \( C \).

**Lemma 3.3.** Let \( G \) and \( G' \) be vector bundles on \( C \) of the same rank and degree such that 
\[
h^0(C, G \otimes L) \geq h^0(C, G' \otimes L) \quad \text{and} \quad \text{such that } G|_{C_1} \cong G'|_{C_1}.
\]
Then there is a composition of degenerations of vector bundles of forms (3) and (4) above that degenerates \( G' \) to \( G \).

By Lemma 3.2 and the construction of \( F \), we have that \( F \) and \( E \) satisfy the hypotheses of Lemma 3.3, and so a sequence of (3) and (4) type degenerations converts \( F \) into \( E \). Thus we have the overall deformation from \( E \) to \( E' \) given by first deforming \( E \) to \( F \), then deforming \( F \) to \( E_{\text{max}} \), then deforming \( E_{\text{max}} \) to \( E \), and Theorem 1.6 is proved.

## 4. The three lemmas

In this section our ultimate goal is to prove the three lemmas of Section 3.2. To do so, we will study the broader geometry of vector bundles on \( C \cong \mathbb{P}^1 \vee \mathbb{P}^1 \). We adopt the same notational conventions as in Section 3. Let \( d_1, \ldots, d_r \) be chosen so \( E' \cong \mathcal{O}(d_1) \oplus \ldots \oplus \mathcal{O}(d_r) \), and let \( b_1, \ldots, b_r \) and \( c_1, \ldots, c_r \) be chosen so \( E|_{C_1} \cong \mathcal{O}(b_1) \oplus \ldots \oplus \mathcal{O}(b_r) \) and \( E|_{C_2} \cong \mathcal{O}(c_1) \oplus \ldots \oplus \mathcal{O}(c_r) \). We adopt the convention that \( d_1 \geq \ldots \geq d_r, b_1 \geq \ldots \geq b_r, \) and \( c_1 \geq \ldots \geq c_r \). Our first claim is that \( E \) splits.

**Proposition 4.1.** Let \( V \) be a vector bundle on a chain of rational curves \( C' \). Then \( V \) splits as a direct sum of line bundles.

**Proof.** Say \( V \) has rank \( r \). Then \( V \) determines a principal \( GL_r \) bundle \( G \) on \( C' \), which is rationally trivial in the sense of [6, Variation 4]. Thus, by [6, Theorem 4.3], \( G \) is an extension of a principal \( T \) bundle, where \( T \) is a maximal torus in \( GL_r \). This \( T \) bundle determines a splitting of \( V \) into line bundles. \( \square \)

**Remark.** Vector bundles also split into line bundles on arbitrarily long chains \( \mathbb{P}^1 \vee \ldots \vee \mathbb{P}^1 \), but do not typically split on other trees of smooth rational curves. We will not need to use this more general fact.

Given Proposition 4.1, we can write \( E = \bigoplus_{1 \leq i \leq r} \mathcal{O}(a(i), c_i) \), where \( (a(1), \ldots, a(r)) \) is some permutation of \( (1, \ldots, r) \). There can be multiple permutations that produce the same final vector bundle, so as a convention we choose the permutation that is smallest in the lexicographic order on \( r \)-tuples. That is, we adopt the convention that if \( c_i = c_{i+1} \), then \( a(i) < a(i+1) \), and if \( b_i = b_{i+1} \), then \( a^{-1}(i) < a^{-1}(i+1) \). We define two functions
\[
\delta(i) = \begin{cases} 
0 & \text{if } a(i) > i \\
1 & \text{if } a(i) \leq i 
\end{cases}
\]
and
\[ \epsilon(i) = \begin{cases} 
0 & \text{if } a^{-1}(i) \geq i \\
1 & \text{if } a^{-1}(i) < i 
\end{cases} \]
Denote by \( m_i \) the quantity
\[ \#\{1 \leq j \leq i | a_j \leq i\}. \]
It is an easy observation that we then have \( m_i = \sum_{1 \leq j \leq i} (\delta(j) + \epsilon(j)) \). We will also regularly use the following easy fact.

**Proposition 4.2.** For all \( k \geq i \), we have
\[ m_k - k + i \leq \#\{1 \leq j \leq i | a_j \leq k\} \leq m_i + k - i \]
and
\[ m_k - k + i \leq \#\{1 \leq j \leq k | a_j \leq i\} \leq m_i + k - i. \]

**Proof.** We show \( \#\{1 \leq j \leq i | a_j \leq k\} \leq m_i + k - i \); the other three inequalities follow identically. We have the containment
\[ \{1 \leq j \leq i | a_j \leq k\} \setminus \{1 \leq j \leq i | a_j \leq i\} \subseteq \{j | i < a_j \leq k\}. \]
The set on the right-hand side clearly has cardinality at most \( k - i \), and \( \#\{1 \leq j \leq i | a_j \leq i\} = m_i \), so \( \{1 \leq j \leq i | a_j \leq k\} \) has cardinality at most \( m_i + k - i \), as was to be shown. \( \square \)

**Remark.** \( m_i \) gives the dimension of the intersection at \( p \) of the highest-degree rank \( i \) subbundles of \( E|_{C_1} \) and \( E|_{C_2} \). In cases where there is not a unique such rank \( i \) bundle, \( m_i \) gives the dimension of the intersection when the subbundles are chosen to give the largest possible intersection. It is also helpful note that from the definition, \( m_i - m_{i-1} \) is either 0,1, or 2, and is given by
\[ m_i - m_{i-1} = \delta(i) + \epsilon(i). \]

We also define
\[ h^0_{\min}(E,d) := \min_b (h^0(E \otimes O(b,d-b))). \]
Thus we have by assumption that \( h^0(E'(d)) \leq h^0_{\min}(E,d) \) for all \( d \). Our first task is to verify Lemma 3.1, which we will accomplish by explicitly describing the vector bundle \( E'_{\text{max}} \) that appears.

**Lemma 4.3.** Let the vector bundle \( E'_{\text{max}} \) on \( \mathbb{P}^1 \) be given by the formula
\[ E'_{\text{max}} \cong \bigoplus_{1 \leq i \leq r} O(b_i + c_i - 1 + \delta(i) + \epsilon(i)). \]
Then \( h^0(\mathbb{P}^1, E'_{\text{max}}(d)) = h^0_{\min}(E,d) \) for all \( d \). In particular, \( E'_{\text{max}} \) is the unique vector bundle described in Lemma 3.1.
Proof. Set \( d'_i = b_i + c_i - 1 + \epsilon(i) + \delta(i) \). We note that \( d'_1 \geq \ldots \geq d'_r \) because we have \( b_1 \geq b_2 + \epsilon(2) \geq \ldots \geq b_r + \epsilon(r) \) and \( c_1 + \delta(1) \geq \ldots \geq c_r + \delta(r) \). Fix some integer \( d \).

We will show \( h^0(\mathbb{P}^1, E'_{\max}(d)) = h^0_{\min}(E, d) \). If we have \( d < -d'_1 \), \( E(-b_1, b_1 - d_1) \) has no sections. If we have \( -d'_r \leq d < -d'_1 \), choose integers \( b \) and \( c \) such that \( b + c = d \), \( -b_1 - \epsilon(i - 1) \leq b \leq -b_i - \epsilon(i) \) and \( -c_{i-1} - \delta(i - 1) < c \leq -c_i - \delta(i) \). Then

\[
h^0(C, E \otimes \mathcal{O}(b, c)) = \sum_{1 \leq j \leq i-1} (c_j - c + b_j - b) + m_{i-1}.
\]

Since

\[
h^0(\mathbb{P}^1, E(d)) = \sum_{1 \leq j \leq i-1} (b_j + c_j - d) + m_{i-1},
\]

by direct calculation, we have \( h^0(C, E \otimes \mathcal{O}(b, c)) = h^0(\mathbb{P}^1, E(d)) \). It remains to show that no degree 0 twist of \( E \otimes \mathcal{O}(b, c) \) has lower \( h^0 \). We note first that for \( e \geq 0 \), we have \( h^0(C, E \otimes \mathcal{O}(e + b, c)) \geq h^0(C, E \otimes \mathcal{O}(e + b, c)) + e(i - 1) \), and that this inequality is strict once \( b + e \) is at least \(-b_i - \epsilon(i) + 1\). And we have \( h^0(C, E \otimes \mathcal{O}(e + b, c - e)) \geq h^0(C, E \otimes \mathcal{O}(e + b, c)) - e(i - 1) - \gamma \), where \( \gamma \) is 1 if \( e + b \geq -b_i \) and \( a_i \geq i \), and is 0 otherwise. In particular, \( h^0(C, E \otimes \mathcal{O}(e + b, c - e)) \geq h^0(C, E \otimes \mathcal{O}(e + b, c)) - e(i - 1) - 1 \), strictly unless \( b + e \) is at least \(-b_i - \epsilon(i) + 1\). So combining the two inequalities gives \( h^0(C, E \otimes \mathcal{O}(e + b, c - e)) \geq h^0(C, E \otimes \mathcal{O}(b, c)) \), so \( h^0(C, E \otimes \mathcal{O}(b, c)) = h^0_{\min}(C, d) \), as was to be shown. \( \square \)

4.1. Degree configurations. For the rest of the proof, set \( c'_i = c_i - 1 + \delta(i) + \epsilon(i) \). We now specialize \( E'_{\max} \) to the vector bundle

\[
F = \bigoplus_{1 \leq i \leq r} \mathcal{O}(b_i, c'_i)
\]

on \( C \). We must check to make sure \( h^0(F \otimes L) \geq h^0(F \otimes L) \) for all line bundles \( L \) on \( C \). This is cumbersome to do directly, so we first come up with an equivalent condition that’s easier to check. We will express this condition in terms of a simple combinatorial object associated to any vector bundle on \( C \), which we call a degree configuration.

Definition 4.4. A **degree configuration** is an ordered list \( \tau = (d'_1, \ldots, d'_r) \), with the \( d_i \) and \( e_i \) integers, such that \( d_1 \geq \ldots \geq d_r \) and, also, if \( d_i = d_{i+1} \) for some \( i \), satisfies \( e_i \geq e_{i+1} \). Its **rank** is \( r \), and its **degree** is \( \sum_{1 \leq i \leq r} d_i \). The **stratification set** of \( \tau \) is the unordered multiset \( \{e_1, \ldots, e_r\} \). Two degree configurations are compatible if they have the same rank, degree, and stratification set.

Given a vector bundle \( E = \bigoplus_{1 \leq i \leq r} \mathcal{O}(e_i, d_i) \) on \( C \), there is a corresponding degree configuration \( (d'_1, \ldots, d'_r) \), assuming the \( d_i \) are suitably ordered. Under this correspondence, there is a 1-1 correspondence between isomorphism classes of vector bundles of given rank \( r \) and degree \( d \) on \( C \) whose restriction to \( C_1 \) is some particular bundle
\( \bigoplus_{1 \leq i \leq r} \mathcal{O}(e_i) \) and degree configurations of rank \( r \), degree \( d - \sum_{1 \leq i \leq r} e_i \), and stratification set \( \{ e_1, \ldots, e_r \} \).

We can define a partial order on degree configurations of a fixed rank, degree, and stratification set as follows. First, given \( d \in \mathbb{Z} \) and \( k \in \{1, \ldots, r\} \), define the modified \((k, d)\) partial sum \( S_{k,d}(\tau) \) of \( \tau = (d_1^e, \ldots, d_r^e) \) by
\[
S_{k,d}(\tau) = d_1 + \ldots + d_k + \# \{ i \leq k | e_i \geq d \}.
\]
Given degree configurations \( \tau \) and \( \tau' \) with the same degree and stratification set, we say \( \tau \leq \tau' \) and say \( \tau' \) majorizes \( \tau \) if for all integers \( d \) and all \( k \in \{1, \ldots, r\} \) the inequality
\[
(3) \quad S_{k,d}(\tau) \leq S_{k,d}(\tau')
\]
holds.

The usefulness of degree configurations to the main problem of this paper stems from the following fact.

**Proposition 4.5.** Given two vector bundles \( G \) and \( G' \) on \( \mathbb{P}^1 \cup \mathbb{P}^1 \) with compatible degree configurations \( \tau_G \) and \( \tau_{G'} \) respectively, we have that \( h^0(G \otimes \mathcal{O}(b,c)) \leq h^0(G' \otimes \mathcal{O}(b,c)) \) for all \( b, c \) if and only if \( \tau \leq \tau' \).

**Proof.** We show that, given \( b \in \mathbb{Z} \), \( h^0(G \otimes \mathcal{O}(b,c)) \leq h^0(G' \otimes \mathcal{O}(b,c)) \) holds for all \( c \) if and only if we have \( S_{b,k}(\tau_G) \leq S_{b,k}(\tau_{G'}) \) for all \( k \). Write \( \tau_G = (d_1^e, \ldots, d_r^e) \) and \( \tau_{G'} = (d'_1^e, \ldots, d'_r^e) \) Define numbers \( \overline{d}_i \) by
\[
\overline{d}_i = \begin{cases} 
d_i & \text{if } e_i < b \\
d_i + 1 & \text{if } e_i \geq b.
\end{cases}
\]
Analogously define numbers \( \overline{d}'_i \). Then an easy computation shows
\[
h^0(G \otimes \mathcal{O}(b,c)) = h^0(C_1, G|_{C_1}(b-1)) + h^0(C_2, \bigoplus_{1 \leq i \leq r} \mathcal{O}(\overline{d}_i + c - 1)).
\]
We have \( G|_{C_1} \cong G'|_{C_1} \) since \( G \) and \( G' \) have compatible degree configurations, so we conclude that \( h^0(G \otimes \mathcal{O}(b,c)) \leq h^0(G' \otimes \mathcal{O}(b,c)) \) holding for all \( c \) is equivalent to the inequality
\[
(4) \quad h^0(C_2, \bigoplus_{1 \leq i \leq r} \mathcal{O}(\overline{d}_i + c - 1)) \leq h^0(C_2, \bigoplus_{1 \leq i \leq r} \mathcal{O}(\overline{d}'_i + c - 1))
\]
holding for all \( c \). And by [3, Theorem 14.7], (4) holds for all \( c \) if and only if the inequality
\[
\sum_{1 \leq i \leq k} \overline{d}_i \leq \sum_{1 \leq i \leq k} \overline{d}'_i
\]
holds for all \( k \). But this last inequality is just \( S_{b,k}(\tau_G) \leq S_{b,k}(\tau_{G'}) \), so we have \( h^0(G \otimes \mathcal{O}(b,c)) \leq h^0(G' \otimes \mathcal{O}(b,c)) \) holds for all \( c \) if and only if \( S_{b,k}(\tau_G) \leq S_{b,k}(\tau_{G'}) \) for all \( k \). \( \square \)
We are now in a position to show that the vector bundle $F$ defined above satisfies $h^0(E \otimes L) \geq h^0(F \otimes L)$.

**Proof of Lemma 3.2.** By Proposition 4.5, to prove the lemma it will suffice to show $\tau_F \leq \tau_E$, where $\tau_F$ and $\tau_E$ are the degree configurations associated to $F$ and $E$ respectively. The degree configuration $\tau_F$ is a slightly permuted form of $(c_1^{b_1}, \ldots, c_k^{b_k})$; more precisely, it takes this shape except when the $i$ and $i+1$ terms satisfy $c_{i+1} = c_i - 1$ and $m_{i+1} = m_i + 2$, in which case they (and possibly a block of terms before and after) get switched. To verify that $\tau_F$ is majorized by $\tau_E$, for each pair $(k,d)$, we have one of two cases to check.

**Case 1:** The first $k$ terms of $\tau_F$ are $c_1^{b_1}, \ldots, c_k^{b_k}$ (in some order).

Set $j = \max \{i | b_i \geq d\}$. Then the partial sums $S_{k,d}$ are given by

$$S_{k,d}(\tau_F) = \left( \sum_{1 \leq i \leq k} c_i \right) - k + m_k + \min(j, k)$$

and

$$S_{k,d}(\tau_E) = \left( \sum_{1 \leq i \leq k} c_i \right) + \# \{i \leq k | b_i \geq d\}.$$ 

If $d \leq b_k$, we have that $S_{k,d}(\tau_F) = (\sum_{1 \leq i \leq k} c_i) + m_k$, and the $S_{k,d}(\tau_E)$ is bounded below by $(\sum_{1 \leq i \leq k} c_i) + m_k$. If $d > b_k$, the quantity $\# \{i \leq k | b_i \geq d\}$ is bounded below by $m_k - k + j$, so we still have $S_{k,d}(\tau_F) \leq S_{k,d}(\tau_E)$.

**Case 2:** Some terms in $(c_1^{b_1}, \ldots, c_k^{b_k})$ are rearranged out of the first $k$ terms of $\tau_F$.

The maximal value of $m_i - m_{i-1}$ is 2, so the only way a term $c_i^{b_i}$ with $i \leq k$ can switch out of the first $k$ terms is if $m_i - m_{i-1} = 0$ and there is some $j > k$ satisfying $c_j = c_i - 1$ and $m_j - m_{j-1} = 2$, in which case we have $c_j' = c_i' + 1$. In particular, there are constants $i_{\min} \leq k \leq i_{\max}$ and $j_{\min} \leq j_{\max}$ with $j_{\min} = i_{\max} + 1$ such that $c_i' = c_j' - 1$ for all $i_{\min} \leq i \leq i_{\max}$ and $j_{\min} \leq j \leq j_{\max}$ and hence such that the $i_{\min}$ to $j_{\max}$ terms of $\tau_F$ are

$$c_{i_{\min}}^{b_{i_{\min}}}, \ldots, c_{i_{\max}}^{b_{i_{\max}}}, c_{j_{\min}}^{b_{j_{\min}}}, \ldots, c_{j_{\max}}^{b_{j_{\max}}}.$$ 

Set $c = c_{i_{\min}}$, so we know $c_i = c$ for $i_{\min} \leq i \leq i_{\max}$ and $c_j = c + 1$ for $j_{\min} \leq j \leq j_{\max}$.

We also know $m_{i_{\min} - 1} = m_{i_{\max}}$, while $m_{j_{\max}} = m_{i_{\max}} + 2(j_{\max} - i_{\max})$.

We assume for simplicity that $k \leq i_{\min} + j_{\max} - j_{\min}$. Then $S_{k,d}(\tau_F)$ is given by

$$\left( \sum_{1 \leq i \leq k} c_i \right) + m_k - i_{\min} + 1 + \# \{i < i_{\min} | b_i \geq d\} + \# \{j_{\min} \leq j \leq k - i_{\min} + j_{\min} | b_j \geq d\},$$
and for $\tau_E$ the sum is

$$S_{k,d}(\tau_E) = \left( \sum_{1 \leq i \leq k} c_i \right) + \#\{i \leq k | b_i \geq d\}.$$

In this case, we have $m_k = m_{i_{\min}} \leq i_{\min} - 1$, and hence, if $d$ is sufficiently large that all set counts in the $(k, d)$ sums above are 0, then we have the simplified equation

$$S_{k,d}(\tau_F) = \left( \sum_{1 \leq i \leq k} c_i \right) + m_k - i_{\min} + 1.$$

This quantity is bounded above by the analogous sum $S_{k,d}(\tau_E) = \left( \sum_{1 \leq i \leq k} c_i \right)$.

Otherwise, let $\ell$ be the largest integer such that $b_{\ell} \geq d$. If $\ell < i_{\min}$, then the last set count in the original formula for $S_{k,d}(\tau_F)$ contributes 0, and we have

$$S_{k,d}(\tau_F) = \left( \sum_{1 \leq i \leq k} c_i \right) + m_{i_{\min} - 1} - i_{\min} + 1 + \ell,$$

while for $\tau_E$ we have

$$S_{k,d}(\tau_E) = \left( \sum_{1 \leq i \leq k} c_i \right) + \#\{i \leq k | a_i \leq \ell\}.$$

By Proposition 4.2, the quantity $\#\{i \leq k | a_i \leq \ell\}$ satisfies the inequality

$$\#\{i \leq k | a_i \leq \ell\} \geq m_{i_{\min} - 1} - i_{\min} + 1 + \ell,$$

verifying semicontinuity in this case. And if $\ell \geq i_{\min}$, we can assume $\ell \geq j_{\min}$, because otherwise we have

$$S_{k,d}(\tau_F) = S_{k,b_{\min} + 1}(\tau_F) \leq S_{k,b_{\min} + 1}(\tau_E) \leq S_{k,d}(\tau_E).$$

Then we have

$$S_{k,d}(\tau_F) = \left( \sum_{1 \leq i \leq k} c_i \right) + m_k + \min(\ell - j_{\min} + 1, k - i_{\min} + 1),$$

whereas using Proposition 4.2 we have an inequality

$$S_{k,d}(\tau_E) \geq \left( \sum_{1 \leq i \leq k} c_i \right) + \max(m_{\ell} - \ell + k, m_{k + j_{\min} - i_{\min}} - j_{\min} + i_{\min}).$$
Since \( m_\ell = m_k + 2(\ell - j_{\text{min}} + 1) \), this sum simplifies to
\[
\left( \sum_{1 \leq i \leq k} c_i \right) + m_k + \ell - 2j_{\text{min}} + 2 + k.
\]
Comparing these two expressions gives \( S_{k,d}(\tau_F) \leq S_{k,d}(\tau_E) \) in the case \( k \leq i_{\text{min}} + j_{\text{max}} - j_{\text{min}} \). The other case, where \( k > i_{\text{min}} + j_{\text{max}} - j_{\text{min}} \) follows by essentially the same argument.

4.2. Specialization of degree configurations. It now remains to prove Lemma 3.3. To do so, we first produce two operations on degree configurations analogous to the degenerations (3) and (4) of Section 3.1. We define two elementary specializations of degree configurations. Let \( \tau = (d_1^{e_1}, \ldots, d_r^{e_r}) \) be a degree configuration. The two elementary specializations are:

1. Given \( i < j \) such that \( e_i < e_j \), replace the two terms \( d_i^{e_i} \) and \( d_j^{e_j} \) with \( d_j^{e_i} \) and \( d_i^{e_j} \) respectively. Then rearrange the resulting configuration so the terms are in the order required for a degree configuration. We will describe this specialization as “switching \( e_i \) with \( e_j \).”
2. Given \( i < j \), replace the two terms \( d_i^{e_i} \) and \( d_j^{e_j} \) with \( (d_i + 1)^{e_j} \) and \( (d_j - 1)^{e_i} \), respectively. Then rearrange.

Both of these specializations are easily seen to produce a configuration \( \tau' \geq \tau \). As the following proposition indicates, they are all we need.

**Proposition 4.6.** Given compatible degree configurations \( \tau \) and \( \tau' \) with \( \tau \leq \tau' \), there is a finite sequence of elementary specializations that converts \( \tau \) into \( \tau' \).

This proposition quickly implies Lemma 3.3.

**Proof of Lemma 3.3.** If \( G \) and \( G' \) are as in the statement of the lemma, their degree configurations \( \tau_G \) and \( \tau_{G'} \) are compatible and we have \( \tau_{G'} \leq \tau_G \) by Proposition 4.5. Then, by Proposition 4.6 there is a sequence of elementary specializations converting \( \tau_{G'} \) into \( \tau_G \). But the elementary specializations are precisely those induced by the (3) and (4) type degenerations on vector bundles listed in Section 3.1, so there is a sequence of degenerations of type (3) and (4) from \( G' \) to \( G \), as was to be shown.

**Proof of Proposition 4.6.** Given \( \tau < \tau' \), we describe an elementary specialization \( \tau'' \) of \( \tau \) such that \( \tau < \tau'' \leq \tau' \). Because each such specialization increases some \( S_{k,d} \) while keeping every \( S_{k,d} \) at most that of \( \tau' \), and there are only finitely many distinct \((k,d)\) counts, iterating this process will then take \( \tau \) to \( \tau' \) in finitely many steps.

Say \( \tau = (d_1^{e_1}, \ldots, d_r^{e_r}) \) and \( \tau' = (d_1^{e'_1}, \ldots, d_r^{e'_r}) \). There are three cases, depending on the initial terms:

1. \( d_1 = d'_1 \) and \( e_1 = e'_1 \);
2. \( d_1 = d'_1 \) and \( e_1 \neq e'_1 \);
(3) \( d_1 \neq d'_1 \).

In case (1), the first term of both degree configurations can be ignored. So one can construct elementary specializations from the sub-configuration \((d'^e_2, \ldots, d'^e_r)\) to \((d'^e_2, \ldots, d'^e_r)\) using this process, and use them without change on the whole configuration.

In case (2), let \( i \) be minimal such that \( e_i = e'_1 \), and determine the \( j < i \) such that \((e_j, j)\) is largest in the lexicographic order on pairs of integers among pairs with \( e_j < e_i \). Then our specialization from \( \tau \) to \( \tau'' \) is to switch \( e_j \) with \( e_i \). It is always possible to do this operation because we must have \( e_1 < e'_1 \).

In case (3), we have \( d_1 < d'_1 \). Let \( \ell \) be the minimal integer such that \( d_1 + \ldots + d_\ell = d'_1 + \ldots + d'_\ell \), and let \( e_{\text{max}} = \max(\{e_1, \ldots, e_\ell\}) \) and \( e_{\text{min}} = \min(\{e_1, \ldots, e_\ell\}) \). We have three possible specializations in this case. If \( e_1 < e_{\text{max}} \), specialize \( \tau \) by finding the least \( i \) such that \( e_i = e_{\text{max}} \), and the \( j < i \) such that \((e_j, j)\) is maximal lexicographically, and switching \( e_i \) with \( e_j \) to produce the new degree configuration \( \tau'' \).

If \( e_1 = e_{\text{max}} \) but \( e_\ell > e_{\text{min}} \), specialize \( \tau \) by finding the largest \( i \) such that \( e_i = e_{\text{min}} \), and the \( j \in \{i + 1, \ldots, k\} \) such that \((e_j, j)\) is minimal lexicographically, and switching \( e_i \) with \( e_j \) to produce \( \tau'' \).

Finally, if both \( e_1 = e_{\text{max}} \) and \( e_\ell = e_{\text{min}} \), specialize by sending \( \tau \) to the configuration \( \tau'' = ((d_1 + 1)^{ e_\ell }, d'^e_2, \ldots, d'^e_{\ell-1}, (d_\ell - 1)^{ e_1 }, d'^e_{\ell+1}, \ldots) \).

It is clear that every \( \tau'' \) produced by the above algorithm satisfies \( \tau < \tau'' \). So to conclude Proposition 4.6 we now verify that the \( \tau'' \) given satisfy \( \tau'' \leq \tau' \). All of the elementary specializations involving a switch of \( e_i \) and \( e_j \) have essentially the same proof that \( \tau'' \leq \tau \), so we only present the proof corresponding to the case (2) switch.

Suppose for contradiction that the \( \tau'' \) we produce by the switch in case (2) satisfies \( S_{k,d}(\tau'') > S_{k,d}(\tau') \) for some \( k \) and \( d \). Then in particular, we must have \( S_{k,d}(\tau'') > S_{k,d}(\tau) \). For the switch to alter \( S_{k,d} \) we must have \( j \leq k < i \) and \( e_i \geq d > e_j \). But then we have \( d > e_n \) for all \( 1 \leq n \leq k \). So \( S_{k,d}(\tau'') = S_{k,d}(\tau) + 1 \), and \( S_{k,d}(\tau) = S_{k,d}(\tau) + 1 \) for all \( d' \geq d \). Hence \( \tau'' \) satisfies \( S_{k,d}(\tau'') \leq S_{k,d}(\tau') + 1 \) for all \( d' \). But since \( d'^e_i \) is the very first term of \( \tau' \), we have \( S_{k,d}(\tau') \geq S_{k,d}(\tau') + 1 \) for \( d' \geq e_i \). So we have \( S_{k,d}(\tau'') \leq S_{k,d}(\tau') \), a contradiction.

Now we check \( \tau'' \leq \tau' \) for the last listed specialization, where \( e_1 = e_{\text{max}} \) and \( e_\ell = e_{\text{min}} \). Suppose for contradiction that we have that some \( k, d \) such that \( S_{k,d}(\tau'') > S_{k,d}(\tau') \).

\footnote{This degree configuration may not be appropriately ordered. In particular, the term \((d_\ell - 1)^{ e_1 }\) may shift to a later point in the configuration.}
Let \( k \) be minimal such that this happens. First we suppose \( k \leq \ell - 1 \). If \( d > e_{\max} \) then
\[
S_{k,d}(t'') = d_1 + \ldots + d_k + 1 \leq S_{k,d}(t') \quad \text{by our definition of} \ \ell.
\]
If \( e_{\min} < e \leq e_{\max} \), we have
\[
S_{k,d}(t'') = S_{k,d}(t') \leq S_{k,d}(t').
\]
And if \( d \leq e_{\min} \) we first note that \( e_{\min} \) is also at least as small as any \( e'_i \) with \( 1 \leq i \leq \ell \), by comparing the terms of \( S_{\ell,e_{\min}}(t) \) and \( S_{\ell,e_{\min}}(t') \) and using \( S_{\ell,e_{\min}}(t) \leq S_{\ell,e_{\min}}(t') \), so we have \( S_{k,d}(t'') = d_1 + \ldots + d_k + k + 1 \), whereas \( S_{k,d}(t') = d'_1 + \ldots + d'_k + k \), so \( S_{k,d}(t'') \leq S_{k,d}(t') \).

Now we suppose \( k \geq \ell \) and \( S_{k,d}(t'') > S_{k,d}(t') \). This can only occur if the term \((d_\ell - 1)^{e_1}\) is not in position \( \ell \) of \( t'' \) (appropriately ordered). Say it moves to position \( \ell + n \), so \( t'' \), properly ordered, is given by
\[
(\ell, e_1, d'_2, \ldots, d'_{\ell - 1}, d'_{\ell + 1}, \ldots, d'_{\ell + n}, (d_\ell - 1)^{e_1}, \ldots).
\]
Since we have \( d_\ell \geq d_{\ell + s} \) for all \( s \geq 0 \), we immediately conclude \( d_\ell - 1 = d_{\ell + 1} = \cdots = d_{\ell + n} \).
Moreover, by the definition of \( \ell \), we must have \( d'_\ell = d_\ell - 1 \), so we analogously have \( d'_{\ell + s} = d_\ell - 1 \) for all \( 0 \leq s \leq n \). Then we have the formula
\[
S_{k,d}(t'') = S_{k,d}(t') + \begin{cases} 
1 & \text{if } e_1 < d \leq e_{k - \ell + 1} \text{ and } k < \ell + n \\
0 & \text{if otherwise.}
\end{cases}
\]
We know \( S_{k,d}(t') \leq S_{k,d}(t'' \), so we must have \( S_{k,d}(t') = S_{k,d}(t'') \) and \( S_{k,d}(t'') = S_{k,d}(t') + 1 \).
In particular, we have \( e_1 < d \leq e_{k - \ell + 1} \) and \( k < \ell + n \). With this knowledge, we have the formulas
\[
S_{k,d}(t'') = \left( \sum_{1 \leq i \leq \ell - 1} d_i \right) + 1 + (k - \ell + 1)(d_\ell - 1) + \# \{1 \leq s \leq k - \ell + 1 \mid e_{\ell + s} \geq d \}
\]
and
\[
S_{k,d}(t') = \left( \sum_{1 \leq i \leq \ell - 1} d_i \right) + 1 + (k - \ell + 1)(d_\ell - 1) + \# \{1 \leq s \leq k \mid e'_s \geq d \}.
\]
So, since \( S_{k,d}(t'') > S_{k,d}(t') \), we have
\[
\# \{1 \leq s \leq k - \ell + 1 \mid e_{\ell + s} \geq d \} > \# \{1 \leq s \leq k \mid e'_s \geq d \}.
\]
However, since \( k \) was chosen minimal, we must have \( e'_k < d \) and \( e_{k + 1} \geq d \). We have
\[
e'_k \geq e'_{k + 1} \geq \cdots \geq e'_{k + n} \quad \text{since the corresponding } d'_k, d'_{k + 1}, \ldots, d'_{k + n} \text{ terms are equal, so}
\]
\[
\# \{1 \leq s \leq \ell + n \mid e'_s \geq d \} = \# \{1 \leq s \leq k \mid e'_s \geq d \}.
\]
Because \( S_{\ell + n,d}(t) \leq S_{\ell + n,d}(t') \), we likewise have
\[
\# \{1 \leq s \leq \ell + n \mid e_{\ell + s} \geq d \} \leq \# \{1 \leq s \leq \ell + n \mid e'_s \geq d \}.
\]
And we trivially have the inequality
\[
\# \{1 \leq s \leq k \mid e_{\ell + s} \geq d \} \leq \# \{1 \leq s \leq n \mid e_{\ell + s} \geq d \}.
\]
Combining (6), (7), and (8), we obtain the inequality
\[
\#\{1 \leq s \leq k|e_{\ell+s} \geq d\} \leq \#\{1 \leq s \leq \ell + n|e'_s \geq d\}
\]
This contradicts (5), so we must have \( S_{k,d}(\tau'') \leq S_{k,d}(\tau') \). \( \square \)

References