Irreducible cuspidal representations

with prescribed local behavior

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Abstract

Let G be a simple algebraic group defined over the global field k. In this paper, we use the simple trace formula to determine the sum of the multiplicities of the irreducible representations in the cuspidal spectrum of G, with specified local behavior at a finite set of places of k and unramified elsewhere. This sum is expressed as the product of the values of modified Artin L-functions at negative integers.

Let k be a global field, with ring of adèles $\mathbb{A}$. Let G be a simple (= almost absolutely simple) algebraic group over k. Then $G(k)$ is a discrete subgroup of $G(\mathbb{A})$, and the quotient $G(k)\backslash G(\mathbb{A})$ has finite Haar measure. The unitary representation of $G(\mathbb{A})$ on $L^2(G(k)\backslash G(\mathbb{A}))$ has a discrete spectrum L^2_d, which contains the cuspidal spectrum L^2_0. Both decompose as a Hilbert direct sum of irreducible unitary representations π with finite multiplicities $m(\pi)$ [DKV]. Furthermore, every irreducible representation π of $G(\mathbb{A})$ is a restricted tensor product $\hat{\otimes}\pi_v$ of local representations π_v of $G(k_v)$, with π_v unramified for almost all finite places v of k. In this paper, we will use the simple trace formula to calculate the sum of multiplicities $m(\pi)$, for cuspidal π with specified local behavior at a finite set of places and unramified elsewhere.

In the number field case, we will assume that k is totally real and that the Archimedean components π_v all lie in the discrete series. The prescribed components at finite places will be either Steinberg representations (for v in S) or simple supercuspidal representations [G-R] (for v in T). The main term in our formula for the sum of multiplicities is given by a product of the values $L_{S,T}(V, 1-d)$ of modified Artin L-functions at negative integers, taken over the degrees d of the invariant polynomials for the Weyl group of G. This product calculates the volume of the quotient $G(k)\backslash G(\mathbb{A})$ with respect to a specific Haar measure; by our hypotheses on the sets S and T only the identity conjugacy class in $G(k)$ has a non-zero orbital integral for the relevant test function in the trace formula.

Our results are new even in the case of classical modular forms, where $k = \mathbb{Q}$ and $G = PGL(2)$. Using the fact that Steinberg representations are the unique representations of conductor p and simple supercuspidal represen-taions are the unique representations of conductor q^3, we obtain the following translation of our formula into classical terminology. Let S and T denote disjoint, finite, non-empty sets of rational primes, and let $N = \prod_S p . \prod_T q^3$. Then the dimension of the space of new forms of weight $2k$ for the group $\Gamma_0(N)$ is given by the formula

$$\frac{(2k-1)}{12} \prod_S (p-1) \prod_T (q^2-1)(q-1) \tag{1}$$

provided that either $\mathrm{Card}(T) \geqslant 2$ or $T = \{q\}$ with $q > 3$. Here, the appear-ance of Artin L-functions is in the value $\zeta(-1) = -1/12$. Some formulae of this type were known for square-free level N; the surprising feature is that the addition of the set T actually simplifies matters. Indeed, the condition $\mathrm{Card}(T) \geqslant 2$ or $T = \{q\}$ with $q > 3$ forces the orbital integrals over non-trivial torsion conjugacy classes to vanish.

In the function field case, our results are conditional on an extension to local fields of characteristic $p > 0$ of a result of Kottwitz on the vanishing of the orbital integrals of the Euler-Poincaré function, on classes which are not elliptic. Let $k = F(X)$ be the field of functions on a curve X of genus g over the finite field F of order q. The final formula for the sum of multiplicities is similar, given by a product of Artin L functions at negative integers, but the Artin L-functions $L_{S,T}(V, s)$ are now polynomials in q^{-s}, of degree equal to $\dim(V) \cdot (2g - 2 + \deg(S) + \deg(T))$. When the curve X has genus 0, and the sets S and T each contain a single place of degree 1, we obtain a surprising multiplicity one result in the cuspidal spectrum, which has many interesting ramifications [F-G]. We also use the stable trace formula to obtain a formula for the multiplicities when the set T is empty and the group G is defined over the finite field F, as a finite sum of orbital integrals over the stable elliptic torsion classes of $G(k)$. We then identify these stable classes with the semi-simple conjugacy classes in the group $G(F) = G(q)$, and transform the sum of orbital integrals into an expression involving the representation theory of the finite Lie group $G(q)$.

This paper is organized as follows. We begin by reviewing the basic rationality and integrality results for Artin L-functions at negative integers, as well as the appearance of these values in the computation of adelic volumes. We then introduce the specific local representations that we will consider, and construct a global test function which detects them in the trace formula. After using the trace formula to obtain a formula for the sum of multiplicities, we discuss the number field and function field cases separately. We then unwind the stable trace formula in the function field case, translating our results into character theory. We end with some specific examples.

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1. Rationality and integrality for Artin L-functions

In this section, we review some basic results of Weil, Siegel, Pierrette Cassou-Noguès, Barsky, and Deligne-Ribet on the values of Artin L-functions at negative integers.

Let k be a global field, and let V be a representation of $\mathrm{Gal}(k^s/k) = \Gamma$ on a finite-dimensional, *rational* vector space. We assume this representation is continuous, where $GL(V)$ is given the discrete topology, so factors through a finite quotient $\mathrm{Gal}(K/k)$ of Γ.

For each place v of k, we define the local factor $L_v(V,s)$ as follows (cf. [T]). If v is complex

$$L_v(V,s) = ((2\pi)^{-s}\Gamma(s))^{\dim V}.$$

If v is real, and $V = V^+ \oplus V^-$ under the action of the complex conjugation F_v in Γ,

$$L_v(V,s) = (\pi^{-s/2}\Gamma(s/2))^{\dim V^+}\left(\pi^{-(\frac{s+1}{2})}\Gamma((s+1)/2)\right)^{\dim V^-}.$$

Finally, if v is non-Archimedean, with inertia group I_v, geometric Frobenius F_v, and residue field of order q_v,

$$L_v(V,s) = \det\left(1 - F_v q_v^{-s} | V^{I_v}\right)^{-1}.$$

The last definition agrees with that of Artin, who used the arithmetic Frobenius $\sigma_v = F_v^{-1}$, because the representation V is defined over $\mathbb{Q}$, so is orthogonal.

The global L-function of V is defined by the infinite product, taken over all the places v of k

$$L(V, s) = \prod_v L_v(V, s).$$

This product converges in the half plane $\mathrm{Re}(s) > 1$. It has a meromorphic continuation to the entire complex plane, and satisfies the functional equation

$$L(V, 1-s) = A^{(1/2)-s} L(V, s).$$

In the functional equation, $A \geqslant 1$ is an integer that measures the ramification in k and V. When k is a number field

$$A = |D|^{\dim V} \cdot \mathbb{N}(\mathcal{F}(V))$$

where D is the discriminant of k and $\mathcal{F}(V) = \prod P_v^{f_v(V)}$ is the conductor ideal of V. When $k = F(X)$ is a function field,

$$A = q^{(2g-2)\dim V + f(V)},$$

where q is the order of F, g is the genus of X, and $f(V) = \sum_v f_v(V) \cdot \deg v$.

The function $L(V, s)$ has a pole of order $\dim V^\Gamma$ at $s = 0$ and $s = 1$. When k is a function field, it is regular elsewhere, by results of Weil. This is conjectured (by Artin) to be true in the number field case, and known to be true when the representation V is monomial.

We say V is ramified at a finite place v of k if the inertia group $I_v \subset \Gamma$ acts non trivially on V, or equivalently if the local conductor $f_v(V) \geqslant 1$. Let S be a finite, non-empty set of places of k, which contains all Archimedean places and all finite places where V is ramified. Define the modified L-function

$$\begin{aligned} L_S(V,s) &= L(V,s)\prod_{v\in S} L_v(V,s)^{-1} \\ &= \prod_{v\notin S}\det(1-F_v q_v^{-s}|V)^{-1} \quad \text{for} \quad \mathrm{Re}(s)>1. \end{aligned}$$

This function is regular at $s=0$, and has a pole of order $\dim V^\Gamma$ at $s=1$. It is regular at all negative integers, and the main result on rationality is due to Weil (cf. [W]) in the function field case, and to Siegel and Brauer in the number field case (cf. [S]).

Theorem 1.1 *For all $d \geqslant 1$, the value $L_S(V,1-d)$ is a rational number.*

Now let T be a non-empty set of places of k, which is disjoint from S. The places $v\in T$ are all finite, and V is unramified at v. We define

$$L_{S,T}(V,s) = L_S(V,s)\cdot\prod_{v\in T}\det(1-F_v q_v^{1-s}|V).$$

The function $L_{S,T}(V,s)$ is regular at $s=0$ and $s=1$, and takes rational values at negative integers. In fact, we will show that these special values are usually integral.

When $k=\mathbb{Q}$ and V is the trivial representation we have

$$L(V,s) = \pi^{-s/2}\Gamma(s/2)\prod_p (1-p^{-s})^{-1}$$

When $S=\{\infty\}$, we have

$$\begin{aligned} L_S(V,s) &= \prod_p (1-p^{-s})^{-1} \\ &= 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots . \end{aligned}$$

This is the Riemann zeta function $\zeta(s)$. When $S = \{\infty\}$ and $T = \{2\}$, we have

$$\begin{aligned} L_{S,T}(V,s) &= \zeta(s) \cdot (1 - 2^{1-s}) \\ &= 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots . \end{aligned}$$

This alternating sum was studied by Euler, who showed that its value at $s = 1 - d$ was an integer divided by 2^d. We will present a generalization of his result below.

We first consider the function field case. Here $k = F(X)$, with X a curve of genus g and F a finite field of order q. Then every v is finite, and $q_v = q^{\deg(v)}$. Hence

$$L_{S,T}(V,s) = L(V,s) \cdot \prod_{v \in S} \det(1 - F_v q_v^{-s})|V^{I_v}) \prod_{v \in T} \det(1 - F_v q_v^{1-s}|V)$$

is a series $1 + \sum_{n \geqslant 1} a_n q^{-ns}$ in q^{-s}, with integral coefficients a_n. For example

$$a_1 = \sum_{X(F)} \mathrm{Tr}(F_x|V^{I_x}) - \sum_{\substack{v \in S \\ \deg\ v=1}} \mathrm{Tr}(F_v|V^{I_v}) - q \sum_{\substack{v \in T \\ \deg\ v=1}} \mathrm{Tr}(F_v|V).$$

Let $\deg(S) = \sum_{v \in S} \deg v$ and $\deg(T) = \sum_{v \in T} \deg v$. Then we have the following result, which is due to Weil [W].

Theorem 1.2 *The function $L_{S,T}(V,s)$ is a polynomial in q^{-s}, with integral coefficients and constant coefficient 1. It has degree*

$$\dim V.[2g - 2 + \deg(S) + \deg(T)] + \mathrm{sw}(V)$$

where $\mathrm{sw}(V) = \sum \mathrm{sw}_v(V) \deg\ v$ is the global Swan conductor.

For all $d \geqslant 1$, the value $L_{S,T}(V, 1-d)$ is an integer. This value is non-zero once $d \geqslant 2$.

We note that $L_{S,T}(V, s)$ has degree $\geqslant 1$ in q^{-s}, except in the case where $g = 0$, $S = \{v\}$ and $T = \{w\}$ with deg v = deg $w = 1$, and $\mathrm{sw}(V) = 0$. In this case $X \simeq \mathbb{P}^1$. Moreover, V is unramified outside of S and is tamely ramified at S. Since the tame geometric fundamental group of $\mathbb{A}^1$ is trivial, V must be everywhere unramified. In the exceptional case, we have $L_{S,T}(V, s) = 1$.

We now consider the number field case. First, there are some parity issues to address.

Lemma 1.3 *Let k be a number field, and assume that $d \geqslant 2$. Then the value $L_S(V, 1-d)$ is equal to zero unless k is totally real, and for every real place v of k, $F_v = (-1)^d$ on V. In this case, $L_S(V, 1-d) \neq 0$.*

Indeed, $L(V, s)$ is regular and non-zero at $s = d$. The same holds at $s = 1-d$, by the functional equation. But the local factor $\prod_{v \in S} L_v(V, s)$ has a pole of order

$$m(V, 1-d) = \sum_{v \text{ complex}} \dim V + \sum_{v \text{ real}} \dim V^{F_v = (-1)^{d-1}}$$

at $s = 1-d$, so $L_S(V, s)$ must vanish to order $m(V, 1-d)$ there.

At $d = 1$ the situation is a bit more complicated. The order of $L_S(V, s)$

at $s = 1 - d = 0$ is equal to

$$m(V, 0) = \sum_S \dim V^{\Gamma_v} - \dim V^{\Gamma}.$$

If $V^{\Gamma} = 0$, the condition in the Lemma is not enough to ensure that $L_S(V, 0) \neq 0$. One also needs the local conditions $V^{\Gamma_v} = 0$, for all $v \in S$.

Here is the desired generalization of Euler's result on integrality, which is due to Cassou-Noguès [C], Barsky [B], and Deligne-Ribet [D-R].

Theorem 1.4 *Assume that the representation V is monomial.*

1) *If $T = \{v\}$ consists of a single finite place of residual characteristic p, then $L_{S,T}(V, 1-d)$ lies in the subring $\mathbb{Z}[1/p]$ of $\mathbb{Q}$.*

2) *If T contains two places of different residual characteristics, then $L_{S,T}(V, 1-d)$ is an integer. If $d \geqslant 2$, this integer is divisible by $2^{n.\dim V}$, where n is the degree of k over $\mathbb{Q}$.*

2. Adèlic volumes

Let k be a global field, and let G be a reductive group over k. We assume that the connected center of G is anisotropic over k, so the discrete subgroup $G(k)$ has finite covolume in $G(\mathbb{A})$. Our aim in this section is to compute the volume

$$\int_{G(k)\backslash G(A)} \mu$$

for certain product Haar measures $\mu = \Pi\mu_v$ on $G(\mathbb{A})$.

Let $\hat{G}$ be the complex dual group of G, with pinned action of $\Gamma = \mathrm{Gal}(k^s/k)$. Let $Z(\hat{G})$ be the center of $\hat{G}$. Our hypothesis on the connected center of G is equivalent to the statement that the group $Z(\hat{G})^\Gamma$ is finite. Let $M = \oplus_{d\geqslant 1} V_d(1-d)$ be the Artin-Tate motive attached to G over k, as defined in [G].

Let S be a finite, non-empty set of places of k, which contains all Archimedean places, as well as all finite places where G has bad reduction. Choose a reductive model $\underline{G}$ for G over the ring of S-integers $\mathcal{O}_S$ of k, which has good reduction at all primes $P \subset \mathcal{O}_S$. By the results in Chapter 1, the value

$$L_S(M) = \prod_{d\geqslant 1} L_S(V_d, 1-d)$$

is a rational number, as each V_d is a rational vector space with an action of Γ. By our hypothesis on the center of G, $L_S(M^\vee(1))$ is finite.

We define a Haar measure μ_S on $G(\mathbb{A})$ as follows. For $v \notin S$, let μ_v be the measure on $G(k_v)$ that satisfies

$$\int_{\underline{G}(\mathcal{O}_v)} \mu_v = 1.$$

For $v \in S$, let μ_v^{EP} be Serre's Euler-Poincaré measure on $G(k_v)$, which is non-zero if and only if G contains a maximal, anisotropic torus T over k_v. Let

$$\mu_S = \prod_{v \in S} \mu_v^{EP} \times \prod_{v \notin S} \mu_v.$$

Then the following conditions are all equivalent [G]:

1) μ_S is a non-zero measure on $G(\mathbb{A})$.

2) G contains a maximal torus T, which is anisotropic over k_v for all $v \in S$.

3) $L_S(M) \neq 0$ in $\mathbb{Q}$.

We assume them in what follows, and give a formula for the volume of $G(k)\backslash G(\mathbb{A})$ under μ_S in terms of the special value $L_S(M)$.

If $v \in S$ is finite, the measure μ_v^{EP} has sign $(-1)^{\mathrm{rank}(G/k_v)}$. We define the local factor

$$c_v(G) = \# H^1(k_v, G) = \# Z(\hat{G})^{\Gamma_v}.$$

If $v \in S$ is Archimedean, then by condition 2), v must be a real place of k. Therefore the number field k is totally real. Let T_v be a compact maximal torus in $G(k_v)$ and let K_v be a maximal compact subgroup containing T_v. Then μ_v^{EP} has sign $(-1)^{(1/2)\dim(G(k_v)/K_v)}$, and we define the local factor

$$c_v(G) = 2^{\dim(T_v)} / \#(W : W_c),$$

where $W_c = N_{K_v}(T_v)/T_v$ and $W = N_{G(\mathbb{C})}(T_{\mathbb{C}})/T_{\mathbb{C}}$. The following is the main global result of [G].

Theorem 2.1 *We have the integral formula*

$$\int_{G(k)\backslash G(\mathbb{A})} \mu_S = \tau(G) \cdot L_S(M) / \prod_{v \in S} c_v(G),$$

where $\tau(G)$ is the Tamagawa number of G.

We note that Kottwitz has shown that $\tau(G) = 1$ when G is simply-connected and k is a number field. From this, one obtains the general formula in the number field case [K]:

$$\tau(G) = \# Z(\hat{G})^{\Gamma} / \# \ker^1(k, Z(\hat{G})).$$

When G is absolutely almost simple, $\ker^1 = 1$, so

$$\tau(G) = \# Z(\hat{G})^{\Gamma}.$$

We now consider a modification of μ_S to obtain integral volumes. We recall that for $v \notin S$, H_v is a pro-p-Sylow subgroup of the profinite group $\underline{G}(\mathcal{O}_v)$, where p is the residual characteristic of v. Fix a finite set T, which is non-empty and disjoint from S. At each place $v \in T$, let μ_v^* be the Haar measure on $G(k_v)$ with

$$\int_{H_v} \mu_v^* = 1.$$

Since H_v is the inverse image of $U(q_v)$, the unipotent radical in a Borel, under the map

$$\underline{G}(\mathcal{O}_v) \to \underline{G}(q_v)$$

we have

$$\begin{aligned} \int_{\underline{G}(\mathcal{O}_v)} \mu_v^* = (\underline{G}(\mathcal{O}_v) : H_v) &= (\underline{G}(q_v) : U(q_v)) \\ &= (-1)^{\ell_v} \det(1 - F_v q_v | M) \end{aligned}$$

with ℓ_v the rank of G over k_v. Hence, if we define a modified global measure by

$$\mu_{S,T} = \prod_{v \in S} \mu_v^{EP} \times \prod_{v \in T} \mu_v^* \times \prod_{v \notin S \cup T} \mu_v$$

we have

$$\mu_{S,T} = \prod_T (-1)^{\ell_v} \det(1 - F_v q_v | M) \cdot \mu_S$$

as measures on $G(\mathbb{A})$.

Theorem 2.2 *Let G be a reductive group over k, whose connected center is anisotropic. Let S be a finite, non-empty set of places of k, which contains all Archimedean places and all finite places where G has bad reduction. Let T be a finite set of places of k which is disjoint from S, and define the product measure $\mu_{S,T}$ on $G(\mathbb{A})$ as above. Then we have the integral formula*

$$\int_{G(k)\backslash G(\mathbb{A})} \mu_{S,T} = \tau(G) L_{S,T}(M) / \prod_{v \in S} c_v(G) \prod_{v \in T} (-1)^{\ell_v}.$$

3. Local representations and their test measures

In this section, we describe some families of irreducible complex representations π_v of the locally compact groups $G(k_v)$, where G is a simple group over the local field k_v. For real groups, these are the representations in the discrete series. For groups over non-Archimedean fields, these representations are of three distinct types - unramified representations, unramified twists of the Steinberg representation, and simple supercuspidal representations. In each case, we will also describe local measures φ_v on $G(k_v)$, which are smooth and compactly supported and detect these representations in the trace formula.

First assume that $k_v = \mathbb{R}$ and that G contains a maximal isotropic torus T_v over k_v. Let K_v be a maximal compact subgroup containing T_v, and let $d_v = (1/2)\dim(G(k_v)/K_v)$. Let π_v be a representation in the discrete series of $G(k_v)$. Then π_v has a regular Harish-Chandra parameter λ in $X^{\cdot}(T_v) + \rho$, well-defined up to the action of W_c. Let V be the finite-dimensional, irreducible, complex representation of $G(k_v)$ with the dual central and infinitesimal characters of π_v.

For the $(W : W^c)$-discrete series representations π'_v with the same central character, and whose Harish-Chandra parameters λ' are the W-conjugates of λ, we have

$$\begin{aligned} H^i(G(k_v), \pi'_v \otimes V) &= 0 \qquad i \neq d_v \\ \dim H^{d_v}(G(k_v), \pi'_v \otimes V) &= 1. \end{aligned}$$

In particular, the Euler characteristic of the cohomology of π'_v with coefficients in V is equal to $(-1)^{d_v}$.

If the real group $G(k_v)$ is compact, or if the highest weight of V is regular, these are the *only* irreducible unitary representations of $G(k_v)$ having non-zero cohomology with coefficients in V. In any case, the test measure φ_v on $G(k_v)$ constructed by Clozel and Delorme, which satisfies [G-P]

$$\mathrm{Tr}(\varphi_v|\pi_v) = \chi(H^*(G(k_v), \pi_v \otimes V)),$$

has trace equal to $(-1)^{d_v}$ on these $(W : W^c)$ discrete series. When $G(k_v)$ is compact, we have the formula

$$\varphi_v = \mathrm{Tr}(g|V) \cdot dg,$$

where dg is the Haar measure with volume 1 on $G(k_v)$.

Next assume that k_v is non-Archimedean, with ring of integers $\mathcal{O}_v$, and let G be a simple group over k_v. Let St_v denote the Steinberg representation in the discrete series of $G(k_v)$, and let V denote a finite-dimensional, irreducible complex representation of $G(k_v)$ which is trivial on an Iwahori subgroup $I_v \subset G(k_v)$. Then $V = \chi$ has dimension 1, and the twisted representation $\pi_v = St_v \otimes \chi^{-1}$ of $G(k_v)$ is also irreducible, with a unique line fixed by I_v. This representation is unitary, and satisfies

$$\begin{aligned} H^i(G(k_v), \pi_v \otimes \chi) &= 0 \qquad i \neq \ell_v, \\ \dim H^{\ell_v}(G(k_v), \pi_v \otimes \chi) &= 1. \end{aligned}$$

In particular, the Euler characteristic of the cohomology of π_v with coefficients in $V = \chi$ is equal to $(-1)^{\ell_v}$.

If π_v is an irreducible, unitary representation of $G(k_v)$ having non-zero cohomology with coefficients in $V = \chi$, then Casselman [Ca] has shown that either $\pi_v \simeq St_v \otimes \chi^{-1}$ or $\pi_v \simeq \chi^{-1}$. Let φ_{EP} be the Euler-Poincaré measure on $G(k_v)$ constructed by Kottwitz [K]

$$\varphi_{EP} = \sum_{\sigma \in \mathcal{F}} \frac{\text{sign on } G_\sigma}{\int_{G_\sigma(k_v)} dg} \cdot (-1)^{\dim \sigma} \cdot dg.$$

Here sign is the sign of the permutation representation of the stabilizer G_σ of a facet σ on the vertices of that facet, as a function on G_σ with values in ± 1. We define $\varphi_v = \chi \cdot \varphi_{EP}$ as a compactly supported measure on $G(k_v)$. Then for all irreducible, unitary representations π_v of $G(k_v)$ we have,

$$\begin{aligned} \mathrm{Tr}(\varphi_v|\pi_v) &= \chi(H^*(G(k_v), \pi_v \otimes \chi)) \\ &= \begin{cases} 1 & \text{if} \quad \pi_v = \chi_v^{-1} \\ (-1)^{\ell_v} & \text{if} \quad \pi_v = St_v \otimes \chi_v^{-1} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Finally, we consider the case when the simple group G is unramified over k_v, and fix a hyperspecial maximal compact subgroup $\underline{G}(\mathcal{O}_v)$. Let $\varphi_v = \mathrm{char}(\underline{G}(\mathcal{O}_v))dg_v$, where dg_v is the unique Haar measure with volume 1 on $\underline{G}(\mathcal{O}_v)$. If π_v is irreducible, we have

$$\mathrm{Tr}(\varphi_v|\pi_v) = \dim \pi_v^{\underline{G}(\mathcal{O}_v)} = \begin{cases} 1 & \text{if} \quad \pi_v \text{ is unramified} \\ 0 & \text{otherwise} \end{cases}$$

Let $H_v \subset \underline{G}(\mathcal{O}_v)$ be a pro-p-Sylow subgroup and let $V = \chi$ be an affine generic character of H_v, in the sense of [G-R]. Let $\varphi_v = (\chi \;\text{ on }\; H_v) \cdot dh_v$, where dh_v is the unique Haar measure giving H_v volume 1. Then

$$Tr(\varphi_v|\pi_v) = \dim \mathrm{Hom}_{H_v}(\chi^{-1}, \pi_v).$$

The induced representation

$$R = \mathrm{Ind}_{H_v}(\chi^{-1})$$

is multiplicity-free, of finite length. Hence, for all irreducible representations π_v of $G(k_v)$,

$$\mathrm{Tr}(\varphi_v|\pi_v) = \begin{cases} 1 & \text{if } \pi_v \text{ is a summand of } R \\ 0 & \text{otherwise} \end{cases}$$

Finally, the irreducible summands of R are all supercuspidal representations so φ_v is a sum of supercuspidal matrix coefficients [G-R].

Since the groups H_v and $\underline{G}(\mathcal{O}_v)$ are compact, we have

$$\begin{aligned} \dim \mathrm{Hom}_{\underline{G}(\mathcal{O}_v)}(1, \pi_v) &= \chi(H^*(\underline{G}(\mathcal{O}_v), \pi_v)) \\ \dim \mathrm{Hom}_{H_v}(1, \pi_v \otimes \chi_v) &= \chi(H^*(H_v, \pi_v \otimes \chi_v)), \end{aligned}$$

as there is no higher cohomology.

4. The global test functions $\varphi_{S,T,V}$

For the rest of this paper, we assume that G is a simple group, defined over the global field k with ring of adèles $\mathbb{A}$. We fix sets S and T of places, as in section 2: S is non-empty, and contains all of the Archimedean places of k as well as the finite places where G has bad reduction, and T is disjoint from S (possibly empty). Since G is unramified outside S, we may choose a model $\underline{G}$ over the S-integers of k with good reduction at all primes. We will assume that G contains a maximal torus over k, which is anisotropic (compact) over k_v, for all v in S. In particular, when k is a number field, it is totally real.

At each place v in S we fix a finite-dimensional irreducible complex representation V_v of $G(k_v)$. When v is finite, we assume that V_v is trivial on an Iwahori subgroup, so V_v has dimension 1. Let φ_v be the test measure on $G(k_v)$ for the Euler characteristic of $G(k_v)$-cohomology with coefficients in V_v which was constructed in section 3.

At each place v in T, with residual characteristic p, we fix an affine generic character $V_v = \chi_v$ of the pro-p-Sylow subgroup H_v of $\underline{G}(\mathcal{O}_v)$. Let φ_v be the test measure on $G(k_v)$ for the Euler characteristic of H_v-cohomology with coefficients in V_v.

At each v not in $S \cup T$, let φ_v be the test measure on $G(k_v)$ for the Euler characteristic of $\underline{G}(\mathcal{O}_v)$-cohomology.

We define the locally compact open subgroup

$$G_{S,T} = \prod_{v \in S} G(k_v) \times \prod_{v \in T} H_v \times \prod_{v \notin S \cup T} \underline{G}(\mathcal{O}_v)$$

of $G(\mathbb{A})$, and the finite-dimensional, irreducible complex representation

$$V = \bigotimes_{v \in S \cup T} V_v \quad \text{of} \quad G_{S,T}.$$

The compactly supported measure $\varphi = \varphi_{S,T,V} = \Pi \varphi_v$ on $G(\mathbb{A})$ has the property that, for any irreducible representation π of $G(\mathbb{A})$:

$$\mathrm{Tr}(\varphi|\pi) = \chi(H^*(G_{S,T}, \pi \otimes V)).$$

The value $\varphi(1)$ at the identity is a Haar measure on $G(\mathbb{A})$, and we have the integral formula

Proposition 4.1

$$\int_{G(k)\backslash G(\mathbb{A})} \varphi(1) = \tau(G) \cdot \dim V \cdot L_{S,T}(M_G) / \prod_{v \in S} c_v(G) \prod_{v \in T} (-1)^{\ell_v}.$$

Indeed

$$\varphi(1) = \prod_{v \in S} \dim V_v \cdot \mu_{EP} \times \prod_{v \in T} \mu_v^* \times \prod_{v \notin S \cup T} \mu_v = \dim V \cdot \mu_{S,T}$$

and we have evaluated the adèlic integral of $\mu_{S,T}$ in Theorem 2.2.

5. The trace formula

In this section, we apply the work of Arthur [A] and Kottwitz [K] on the trace formula to the compactly supported measure $\varphi = \varphi_{S,T,V}$ on $G(\mathbb{A})$

Let γ be a semi-simple element in $G(k)$. We will use the algebraic definition of semi-simple in this paper: γ is diagonalizable in any representation of G over the algebraic closure of k. Then the centralizer G_γ is a reductive subgroup over k. Let I_γ denote the connected component of G_γ. If I_γ contains a maximal torus which is anisotropic over k_v, for all places v in S, we say that the element γ (or its conjugacy class) is S-elliptic.

Proposition 5.1. *Assume that k is a number field, and that $\#S + \#T \geqslant 2$. Then $\varphi = \varphi_{S,T,V}$ has a trace on the discrete spectrum $L^2_d(G(k)\backslash G(\mathbb{A}))$. This trace is given by a sum of orbital integrals*

$$\mathrm{Tr}(\varphi | L^2_d) = \sum_{\{\gamma\}} O_\gamma(\varphi)$$

where the sum is taken over the semi-simple, S-elliptic, torsion conjugacy classes of $\{\gamma\}$ in $G(k)$, and only finitely many terms in the sum are non-zero.

If T contains a place of residual characteristic p, then $O_\gamma(\varphi) = 0$ unless the torsion class γ has p-power order in $G(k)$. If T contains two places of different residue characteristics, then $O_\gamma(\varphi) = 0$ for all $\gamma \neq 1$ in $G(k)$. In this case

$$\mathrm{Tr}(\varphi | L^2_d) = O_1(\varphi) = \int_{G(k)\backslash G(\mathbb{A})} \varphi(1).$$

Proof. The fact that φ has a trace, and that the trace is a sum of orbital integrals over the semi-simple, S-elliptic classes is a consequence of the work of Arthur and Kottwitz. This is discussed in more detail in [G-P]. We note that the orbital integrals in question

$$O_\gamma(\varphi) = \int_{G_\gamma(k)\backslash G(\mathbb{A})} \varphi(g^{-1}\gamma g)$$

are defined using counting measure on the discrete subgroup $G_\gamma(k)$ to obtain a Haar measure on the quotient. Using the inclusions

$$I_\gamma(k) \subset G_\gamma(k)$$

$$I_\gamma(k) \subset I_\gamma(\mathbb{A}) \subset G(\mathbb{A}),$$

we can write the orbital integral as a product of three terms

$$O_\gamma(\varphi) = (G_\gamma(k) : I_\gamma(k))^{-1} \cdot \tau(I_\gamma) \int_{I_\gamma(\mathbb{A})\backslash G(\mathbb{A})} \varphi(g^{-1}\gamma g)/dg_\gamma,$$

where dg_γ is Tamagawa measure on $I_\gamma(\mathbb{A})$, and

$$\tau(I_\gamma) = \int_{I_\gamma(k)\backslash I_\gamma(\mathbb{A})} dg_\gamma$$

is the Tamagawa number. Finally, the integral of $\varphi(g^{-1}\gamma g)$ over $I_\gamma(\mathbb{A})\backslash G(\mathbb{A})$ can be written as a product of local orbital integrals over the places v of k (once dg_γ is written as a product of local measures), and almost all of the terms in this product are equal to 1.

It remains to show that $O_\gamma(\varphi) = 0$ unless γ is a torsion class, with the corresponding refinements coming from places in T. These properties depend crucially on the test measure $\varphi = \varphi_{S,T,V}$. For finite v, the support of the local measure φ_v is contained in a finite union of open, compact subgroups. The local orbital integrals of φ_v at the real places v of k are zero, unless γ_v is both semi-simple and elliptic (cf. [G-P]). Hence $O_\gamma(\varphi) = 0$ unless γ lies in a compact subgroup $K = \Pi K_v$ of $G(\mathbb{A})$. The intersection of K with the discrete group $G(k)$ is finite, so γ is torsion. At any place v of T, with residual characteristic p, the support of φ_v is a pro p-group. Hence $O_\gamma(\varphi) = 0$ unless γ has p-power order. Finally, the only torsion class which has p-power order and p'-power order is $\gamma = 1$, and by definition

$$O_1(\varphi) = \int_{G(k)\backslash G(\mathbb{A})} \varphi(1).$$

For v in T, the local measure φ_v is the sum of matrix coefficients of the simple super-cuspidal representations which appear in $\mathrm{Ind}(\chi_v^{-1})$. Hence, if T is non-empty, the endomorphism of $L^2(G(k)\backslash G(A))$ given by φ has image contained in the cuspidal spectrum L_0^2 (cf. [BKV]). In particular, if T is non-empty we have

$$\mathrm{Tr}(\varphi|L_d^2) = \mathrm{Tr}(\varphi|L_0^2) \qquad \textbf{(5.2)}$$

and Proposition 5.1 gives a formula for the trace of φ on the subspace of cusp forms.

In the function field case, we conjecture that a formula similar to Proposition 5.1 is true. However, since the fields k and k_v are not perfect, we will need to assume that $p = \mathrm{char}(k)$ does not divide the order of the fundamental group of G. Then any unipotent element γ in $G(k)$ is contained in the unipotent radical of a parabolic subgroup [Gi]. Otherwise, there can be elliptic, unipotent elements, like the involution

$$\gamma = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}$$

in $PGL_2(k)$, where $\mathrm{char}(k) = 2$ and a is non-trivial in $k^*/{k^*}^2$.

Conjecture 5.3. *Assume that k is a function field of characteristic p, that p does not divide $\#Z(\hat{G})$, and that $\#S + \#T \geqslant 2$.*

Then $\varphi = \varphi_{S,T,V}$ has a trace on the discrete spectrum $L^2_d(G(k)\backslash G(\mathbb{A}))$. This trace is given by a sum of orbital integrals

$$\mathrm{Tr}(\varphi | L^2_d) = \sum_{\{\gamma\}} O_\gamma(\varphi)$$

where the sum is taken over the semi-simple, S-elliptic torsion conjugacy classes $\{\gamma\}$ in $G(k)$, and only finitely many terms in the sum are non-zero.

This conjecture is true for $G = PGL_n$ (assuming that p doe not divide n) by work of Laumon [L]. Dac has shown that the conjecture would follow from a purely local statement on the orbital integrals of the Euler-Poincaré function φ_{EP} over k_v (generalizing work of Kottwitz [K, §2] in characteristic zero).

If conjecture 5.3 is true and the set T is non-empty, then

$$\mathrm{Tr}(\varphi | L^2_d) = \mathrm{Tr}(\varphi | L^2_0) = \int_{G(k)\backslash G(\mathbb{A})} \varphi(1). \qquad \textbf{(5.4)}$$

Indeed, the semi-simple torsion classes in $G(k)$ have order prime to p. But for all places v in T, the support of the measure φ_v is a pro p-group. So

$O_\gamma(\varphi) = 0$ unless $\gamma = 1$. Since φ_v is also super-cuspidal, the endomorphism φ of L^2 has image in L^2_0, so $\mathrm{Tr}(\varphi|L^2_d) = \mathrm{Tr}(\varphi|L^2_0)$

6. Totally real fields — the identity term

In this section, we assume that k is totally real, and that T contains two places of different residue characteristics, or one place whose residue characteristic p is not a torsion prime for G over k. Let $\varphi = \varphi_{S,T,V}$.

By Proposition 5.1, we have the identity

$$\mathrm{Tr}(\varphi|L_0^2) = \int_{G(k)\backslash G(\mathbb{A})} \varphi(1).$$

By Proposition 4.1, we have the integral formula

$$\int_{G(k)\backslash G(\mathbb{A})} \varphi(1) = \tau(G) \cdot \mathrm{dim}V \cdot L_{S,T}(M)/\prod_{v\in S} c_v(G) \cdot \prod_{v\in T}(-1)^{\ell_v}.$$

On the other hand, by the results in §3, the trace of φ computes the Euler characteristic of $G_{S,T}$ with coefficients in V. Hence we obtain

Theorem 6.1 *If k is totally real and T contains two places of different residue characteristics, then*

$$\chi(H^*(G_{S,T}, L_0^2 \otimes V)) = \tau(G) \cdot \mathrm{dim}V \cdot L_{S,T}(M)/\prod_{v\in S} c_v(G) \prod_{v\in T}(-1)^{\ell_v}.$$

We can make this identity more explicit when G is simply connected. Then $\tau(G) = 1$ by results of Weil, Langlands, Lai, and Kottwitz, and $c_v(G) = \#H^1(k_v, G) = 1$ for finite v, by results of Kneser. Furthermore, if $W_c \subset W$ denote the compact Weyl group and the complex Weyl group of an anisotropic torus in $G(k \otimes \mathbb{R})$, and $n\ell$ its dimension, we have

$$\prod_{v|\infty} c_v(G) = 2^{nl}/(W : W_c).$$

Hence

$$\chi(H^*(G_{S,T}, L_0^2 \otimes V)) = \dim V \cdot (W : W^c) \cdot \frac{1}{2^{n\ell}} L_{S,T}(M) \cdot \prod_{v \in T} (-1)^{\ell_v}.$$

We note that ℓ is the absolute rank of G and n is the degree of k over $\mathbb{Q}$.

If the subgroup $\prod_{v \in S} G(k_v)$ is *not* compact, then the product $G(k).G_{S,T}$ is equal to $G(\mathbb{A})$, by the strong approximation theorem for simply-connected groups. Let

$$\Gamma_{S,T} = G(k) \cap G_{S,T},$$

which is a torsion-free S-arithmetic subgroup of $\prod_{v \in S} G(k_v)$, which acts on the representation V. We therefore obtain a formula for the Euler characteristic of $\Gamma_{S,T}$ with coefficients in V:

$$\chi(H^*(\Gamma_{S,T}, V)) = \dim V \cdot (W : W^c) \cdot \frac{1}{2^{n\ell}} L_{S,T}(M) \cdot \prod_{v \in T} (-1)^{\ell_n} \qquad \textbf{(6.2)}$$

in terms of the values of Artin L-functions at negative integers.

We can also use Theorem 5.1 to count the number of irreducible representations π of $G(\mathbb{A})$ which occur in the space L_0^2 of cusp forms and have prescribed local behavior.

In such a count, we will always weight π with its multiplicity $m(\pi)$. First, we address the question of which irreducible representations have $\mathrm{Tr}(\varphi|\pi) = \prod \mathrm{Tr}(\varphi_v, \pi_v) \neq 0$. If the local components π_v of π satisfy the following conditions, which we denote by (*)

for v in S_∞, $\quad \pi_v$ is a discrete series representation of $G(k_v)$, and $\pi_v \otimes V_v$

	has trivial infinitesimal and central character
for v in S_f,	$\pi_v \otimes \chi_v$ is isomorphic to the Steinberg representation of $G(k_v)$
for v in T,	$\pi_v \otimes \chi_v$ has a non-zero vector fixed by H_v
for $v \neq S \cup T$,	π_v has a non-zero vector fixed by $\underline{G}(O_v)$

Then we have $\mathrm{Tr}(\varphi|\pi) = \prod_{v \in S_\infty} (-1)^{d_v} \prod_{v \in S_f} (-1)^{\ell_v}$. Note that the sign of this trace is independent of the representation π, whose local components satisfy (*). Let

$$\epsilon_G(S,T) = \prod_{v \in S_\infty} (-1)^{d_v} \prod_{v \in S_f} (-1)^{\ell_v} \cdot \prod_{v \in T} (-1)^{\ell_v}. \qquad \textbf{(6.3)}$$

We expect that there are no other irreducible representations π which occur in L_0^2 and have $\mathrm{Tr}(\varphi|\pi) \neq 0$. In this direction, we have the following

Theorem 6.4. *Assume that either $G(k \otimes \mathbb{R})$ is compact, or that the representation $\bigotimes_{v \in S_\infty} V_v$ of $G(k \otimes \mathbb{R})$ has a regular highest weight. Let $\pi = \otimes \pi_v$ be an irreducible representation of $G(\mathbb{A})$ which occurs in L_0^2, and has Tr$(\varphi|\pi) \neq 0$.*

Then the local components π_v of π satisfy (*) If T contains two places of different residue characteristics, we have the formula

$$\sum_{(*)} m(\pi) = \epsilon_G(S,T) \cdot \tau(G) \cdot \dim V \cdot (W : W^c) \cdot \frac{1}{2^{\ell n}} L_{S,T}(M) / \prod_{v \in S} c_v(G).$$

Proof. Our hypothesis at infinity implies that any unitary representation of $G(k \otimes \mathbb{R})$ with non-zero cohomology with coefficients in V_∞ is in the discrete series (with the correct central and infinitesimal characters). It remains to check that for every place v in S_f where $G(k_v)$ is not compact, $\pi_v \otimes \chi_v$ is isomorphic to the Steinberg representation. Otherwise, by Casselman's result [C], $\pi_v \otimes \chi_v$ has dimension 1 and π_v is trivial on a subgroup of finite index in $G(k_v)$. This implies, by the strong approximation theorem, that π is finite-dimensional. Such representations cannot occur in the cuspidal spectrum of G.

When S_f is non-empty, we expect Theorem 6.4 to be true without any hypothesis on $\underset{v \in S_\infty}{\otimes} V_v$. Even more should be true — we should be able to count the irreducible cuspidal representations π whose local components satisfy the more restrictive conditions, which we denote by (**):

for v in S_∞, π_v is a *fixed* discrete series representation of $G(k_v)$, such that

$\pi_v \otimes V_v$ has trivial central and infinitesimal character

for v in S_f, $\pi_v \otimes \chi_v$ is isomorphic to the Steinberg representation of $G(k_v)$

for v in T, $\pi_v \otimes \chi_v$ has a non-zero vector fixed by H_v, and $Z(k_v)$ acts by

the central character α_v

for $v \neq S \cup T$, π_v has a non-zero vector fixed by $\underline{G}(O_v)$.

For $v \in S$, let α_v be the central character of π_v. This is trivial for $v \in S_f$, and is equal to the inverse of the central character of V_v, for $v \in S_\infty$.

A representation $\pi = \otimes \pi_v$ with local components satisfying (**) has character

$$\prod_{v \in S_\infty} \alpha_v \prod_{v \in T} \alpha_v$$

on $\prod_{v \in S_\infty} Z(k_v) \times \prod_{v \in T} Z(k_v)$. For this representation to be automorphic, the restriction of this character to the diagonally embedded subgroup $Z(k)$

must be trivial. Assume that this is the case, that S_f is non-empty and that T contains two places of different residue characteristics. Then we conjecture that

$$\sum_{(**)} m(\pi) = \epsilon_G(S,T)\cdot \dim V \cdot \frac{\#Z(\hat{G})^{\Gamma}}{\prod\limits_{v\in S_f} \#Z(\hat{G})^{\Gamma_v}} \cdot \frac{\#Z(k)}{\prod\limits_{v\in T} \#Z(q_v)} \cdot \frac{1}{2^{\ell n}} L_{S,T}(M). \quad \textbf{(6.5)}$$

Here we have used the identities:

$$\begin{aligned} \tau(G) &= \#Z(\hat{G})^{\Gamma} \qquad \text{(for simple } G\text{)} \\ c_v(G) &= \#Z(\hat{G})^{\Gamma_v} \qquad \text{(for } V \text{ in } S_f\text{)}. \end{aligned}$$

7. Function fields — the identity term

In this section, we will assume that k is a function field, and that both sets S and T of places of k are non-empty. We want to examine the implications of Conjecture 5.3 on the cuspidal trace of $\varphi = \varphi_{S,T,V}$. We assume further that $p = \mathrm{char}(k)$ does not divide the order of $Z(\hat{G})$.

Arguing as in the previous section, using the results in §3 and §4, we find that

$$\mathrm{Tr}(\varphi|L_0^2) = \int_{G(k)\backslash G(\mathbb{A})} \varphi(1) \tag{7.1}$$

implies the identity

$$\chi(H^*(G_{S,T}, L_0^2 \otimes V)) = \tau(G) \cdot L_{S,T}(M) / \prod_{v \in S} c_v(G) \prod_{v \in T} (-1)^{\ell_v}. \tag{7.2}$$

We note that $\dim(V) = 1$ in this case.

We can use (7.2) to count the irreducible cuspidal representations $\pi = \hat{\otimes}\pi_v$ with prescribed local behavior (*). The argument is the same as in §6, and we obtain

Theorem 7.3. *Assume that S and T are not empty, that $p = char(k)$ does not divide $\#Z(\hat{G})$, and that Conjecture 5.3 is true. Then*

$$\sum_{(*)} m(\pi) = \epsilon_G(S,T) \cdot \tau(G) \cdot L_{S,T}(M) / \prod_{v \in S} \#Z(\hat{G})^{\Gamma_v}.$$

If we impose the stricter conditions (**), then there are no cuspidal π, unless the character $\prod_{v \in T} \alpha_v$ of $\prod_{v \in T} Z(k_v)$ is trivial when restricted to the diagonally embedded subgroup $Z(k)$. If this restriction is trivial, we obtain

$$\sum_{(**)} m(\pi) = \epsilon_G(S,T) \cdot \frac{\tau(G)}{\prod_{v\in S} \#Z(\hat{G})^{\Gamma_v}} \cdot \frac{\#Z(k)}{\prod_{v\in T} \#Z(q_v)} \cdot L_{S,T}(M). \qquad \textbf{(7.4)}$$

We expect that $\tau(G) = \#Z(\hat{G})^{\Gamma}$, as in characteristic zero.

8. Function fields — the stable trace formula

We continue to assume that k is a function field, with constant field the finite field F. We now examine the consequence of Conjecture 4.2 when the set S has $\#S \geqslant 2$ and the set T is empty.

We simplify the discussion in this case, which will involve stable conjugacy in G, by assuming that G is simply-connected. We will also assume that G defined over F, so is quasi-split, and everywhere unramified over k.

At all places $v \in S$, let φ_v be an Euler-Poincaré function defined by Kottwitz on $G(k_v)$. At places v not in S, let φ_v be the measure determined by the hyperspecial compact subgroup $G(O_v)$. Let $\varphi = \varphi_S = \prod \varphi_v$ on $G(\mathbb{A})$. Since $Z(\hat{G}) = 1$, Conjecture 4.2 implies that

$$\mathrm{Tr}(\varphi|L^2_d) = \sum_{\{\gamma\}} O_\gamma(\varphi) = \sum_{\{\gamma\}} \tau(G_\gamma) O_\gamma(\varphi|dg_\gamma). \tag{8.1}$$

Here the sum is taken over the semi-simple, S-elliptic, torsion conjugacy classes $\{\gamma\}$ in $G(k)$, and G_γ denotes the subgroup centralizing γ. Since G is simply-connected, G_γ is both connected and reductive. We let dg_γ denote Tamagawa measure on $G_\gamma(\mathbb{A})$ and

$$\tau(G_\gamma) = \int_{G_\gamma(k)\backslash G_\gamma(\mathbb{A})} dg_\gamma$$

$$O_\gamma(\varphi|dg_\gamma) = \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} \varphi(g^{-1}\gamma g)/dg_\gamma.$$

By the argument in §7, the left-hand side of (8.1) is equal to

$$\chi(H^*(G_S, L_d)) = 1 + \epsilon_G(S) \sum_{(*)} m(\pi). \tag{8.2}$$

The additional 1 on the right-side of (8.2) comes from the trivial representation of $G(\mathbb{A})$, which appears in L^2_d but not in L^2_0, and has $H^0(G_S, \mathbb{C}) = \mathbb{C}$.

Following Kottwitz, we will rewrite the right-hand side of (8.1) as a finite sum of stable orbital integrals. Recall that two semi-simple elements of $G(k)$ are stably conjugate if they are conjugate in $G(\overline{k})$, or equivalently, if their traces on the fundamental representations of G are equal. These traces are elements of the splitting field $k' = F'k$, and are stable under the action of $\Gamma = \mathrm{Gal}(k'/k) = \mathrm{Gal}(F'/F)$ (of [G]).

Proposition 8.3. *Every stable torsion class of order prime to p in $G(k)$ is represented by a semi-simple element in $G(F)$, which is uniquely determined up to conjugacy.*

Every stable torsion class of order prime to p in $G(k_v)$ is represented by a semi-simple element in $G(F_v)$, which is uniquely determined up to conjugacy. The intersection of the stable class S_γ of such a γ in $G(F_v)$ with $G(O_v)$ forms a single $G(O_v)$ conjugacy class. The group G_γ is unramified, and $S \cap G(O_v) \simeq G_\gamma(O_v)\backslash G(O_v)$.

Proof. Since γ has finite order, its eigenvalues on the fundamental representations of G are roots of unity in $\overline{F}$. Hence the traces lie in the field $\overline{F} \cap k' = F'$ (as X is geometrically connected). They are stable under Γ, so correspond to a stable class in $G(F)$. Over finite fields, stable conjugacy is equivalent to conjugacy, by Lang's theorem on the vanishing of $H^1(F, G_\gamma)$.

Hence we have a bijection:

$$\left\{ \begin{array}{c} \text{semi-simple conjugacy} \\ \text{classes in } G(F) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{semi-simple stable torsion} \\ \text{classes in } G(k) \end{array} \right\}$$

The proof of the result for $G(k_v)$ is similar, and the $G(O_v)$ action on $S_\gamma \cap G(O_v)$ follows from [K2, Proposition 7.1].

Proposition 8.3 implies that the motive $M(G_\gamma)$ of the centralizer of a stable torsion class of order prime to p in $G(k)$ can be defined over F, so is everywhere unramified over k. Its Artin L-function over k

$$L(M_{\gamma,S}) = \frac{\det(1 - Fr.q^{-s}|M_\gamma \otimes H^1(X))}{\det(1 - Fr.q^{-s}|M_\gamma)\det(1 - Fr.q^{1-s}|M_\gamma)} \tag{8.4}$$

is a rational function of degree $(2g-2) \cdot rank(M)$ in q^{-s}.

The crucial local result, on which Conjecture 5.3 depends, is that the orbital integrals of φ_{EP} behave as in characteristic 0. That is, we expect

$$\int_{G_\gamma(k_v)\backslash G(k_v)} \varphi_{EP}(g^{-1}\gamma g)/dg_\gamma \cdot dg_\gamma = \begin{cases} \text{Euler-Poincaré measure on } G_\gamma(k_v), \\ \qquad \text{when } \gamma \text{ is semi-simple} \\ 0 \qquad \text{otherwise.} \end{cases} \tag{8.5}$$

Assuming this, we can rewrite the right-hand side of (8.1) as a sum over the semi-simple stable torsion classes σ in $G(k)$ which are elliptic for all $v \in S$:

$$\sum_{\{\gamma\}} \tau(G_\gamma) O_\gamma(\varphi|dg_\gamma) = \sum_\sigma \sum_{\{\gamma\}} e(\gamma) O_\gamma(\varphi|dg_\gamma). \tag{8.6}$$

The inner sum is taken over the $G(\mathbb{A})$–conjugacy classes $\{\gamma\}$ which are stably conjugate to σ. The sign $e(\gamma) = \prod e_v(\gamma) = \pm 1$ is determined by the adèlic centralizer $G(\mathbb{A})_\gamma$, and dg_γ is an analog of Tamagawa measure on this centralizer (which may not be defined over k. For details, see [K] and [G-P].

We will henceforth use Proposition 8.3 to identify the stable classes σ with those semi-simple conjugacy classes in $G(F)$, which are elliptic in $G(F_v)$, for all $v \in S$.

To proceed, we need to compare Tamagawa measure dg_γ with the product measure

$$d\mu_{S,\gamma} = \prod_{v \in S} \mu_{EP,\gamma} \times \prod_{v \neq S} dg_{v,\gamma}$$

on $G(\mathbb{A})_\gamma$. The main formula of [G] gives their ratio in terms of the motive $M_\gamma = M(G_\gamma)$:

$$d\mu_{S,\gamma}/dg_\gamma = L_S(M_\gamma)/\prod_v c_v(G_\gamma) \cdot e(\gamma_v). \tag{8.7}$$

Here $c_v(G_\gamma) = \#H^1(k_v, G_\gamma)$ is the number of $G(k_v)$ conjugacy classes in the stable class of γ over k_v, and $e(\gamma_v) = \pm 1$ is the Kottwitz sign of G_{γ_v}. The motive M_σ depends only on the stable class σ containing γ.

Writing the terms in the sum

$$\sum_{\{\gamma\}} e(\gamma) O_\gamma(\varphi/d\mu_{S,\gamma})$$

as a product of local integrals using (8.7), and arguing exactly as in [G-P], we find that the total contribution of the stable class σ to the trace is equal to the rational number $L_S(M_\sigma)$. This is also true when σ is not elliptic at the places $v \in S$, as these $L_S(M_\sigma) = 0$.

Hence we have obtained the following formula (conditional on the vanishing of the non-elliptic local orbital integrals of φ_{EP}):

$$1 + \epsilon_G(S) \sum_{(*)} m(\pi) = \sum_{\{\sigma\}} L_S(M_\sigma). \tag{8.8}$$

The sum on the right of (8.8) is taken over the semi-simple conjugacy classes $\{\sigma\}$ in the finite group $G(F)$. The sum on the left of (8.8) is taken over the irreducible, cuspidal representations π of $G(\mathbb{A})$, which are Steinberg at S and unramified away from S. We recall that $\#S \geqslant 2$ and that $\epsilon_G(S) = \prod_{v \in S}(-1)^{\ell_v}$.

9. Character theory

In this section, we show how the sum

$$\sum_{\{\sigma\}} L_S(M_\sigma) \tag{9.1}$$

of rational numbers, which appears on the right side of formula (8.8), can be reinterpreted using the character theory of the finite group $G(F)$. We recall that the sum is over the semi-simple conjugacy classes $\{\sigma\}$ in the group $G(F)$. We have the explicit formula

$$L_S(M_\sigma) = \frac{\det(1 - Fr|H^1(X) \otimes M_\sigma)}{\det(1 - Fr.q|M_\sigma)} \cdot \frac{\prod_{v \in S} \det(1 - Fr^{deg_v}|M_\sigma)}{\det(1 - Fr|M_\sigma).}$$

We begin with a computation of the term $\det(1 - Fr.q|M_\sigma)$, which appears in the denominator

Proposition 9.3. *Let U_σ be the unipotent radical of a Borel subgroup of G_σ Then*

$$\det(1 - Fr.q|M_\sigma) = (-1)^{\ell_\sigma} \cdot \#G_\sigma(F)/q^{\dim U_\sigma}.$$

Proof. By definition

$$\det(1 - Fr.q|M_\sigma) = \prod_{i=1}^{\ell} (1 - \zeta_i q^{d_i}),$$

where the d_i are the degrees of G_σ and the ζ_i are roots of unity determined by the Γ-action on M_σ.

On the other hand, by Steinberg's formula (cf. [G, (3.1)])

$$\begin{aligned}\#G_\sigma(F) &= q^{\dim G_\sigma} \det(1 - Fr|M_\sigma^v(1)) \\ &= q^{\dim G_\sigma} \prod_{i=1}^{\ell}(1 - \zeta_i q^{-d_i}) \\ &= q^{\dim U_\sigma} \prod_{i=1}^{\ell}(q^{d_i} - \zeta_i)\end{aligned}$$

as $\sum d_i = \dim\, G_\sigma - \dim\, U_\sigma$. Since $\prod_{i=1}^{\ell}(-\zeta_i) = (-1)^{\ell_\sigma}$, where ℓ_σ is the rank of G_σ over F, this gives the desired result.

There are $\#G(F)/\#G_\sigma(F)$ elements g in the conjugacy class of $\{\sigma\}$. We may therefore rewrite the sum (9.1) as a sum over semi-simple elements g

$$\sum_{\{\sigma\}} L_S(M_\sigma) = \frac{1}{\#G(F)} \sum_g (-1)^{\ell_g} q^{\dim U_g} \cdot h_S(g) \tag{9.4}$$

where

$$h_S(g) = \det(1 - Fr|H^1(X) \otimes M_g) \frac{\prod_{v \in S} \det(1 - Fr^{deg_v}|M_g)}{\det(1 - Fr|M_g)} \tag{9.5}$$

is now an integral valued function, which depends only on the conjugacy class of the semi-simple element g.

The function $\chi_{St}(g) = (-1)^{\ell_g} q^{\dim U_g}$ is the character of the virtual Steinberg representation $(-1)^\ell St = \epsilon(G)St$ of $G(F)$. This character vanishes on all elements g' in $G(F)$ which are not semi-simple. Hence we have established the formula:

$$\sum_{\{\sigma\}} L_S(M_\sigma) = \frac{\epsilon(G)}{\#G(F)} \sum_{g \text{ in } G(F)} \chi_{St}(g) h_S(g) \tag{9.6}$$

where h_S is any class function in $G(F)$ which takes the values (9.5) at semi-simple elements. This gives our final formula for the multiplicities in the discrete spectrum:

$$1 + \epsilon_G(S) \sum_{(*)} m(\pi) = \epsilon(G) \langle \chi_{St}, h_S \rangle_{G(F)}. \tag{9.7}$$

We end with a discussion of the case where the curve X has genus 0 over F, and S contains a place ∞ of degree 1. Then, for semi-simple classes g,

$$h_S(g) = \prod_{\text{finite } v \text{ in } S} \det(1 - Fr^{deg_v} | M_g). \tag{9.8}$$

The following representation theoretic interpretation of h_S is due to Mark Reeder.

For simplicity, we will assume that G is split over F, and let W be the Weyl group (in the general case, one needs to work with the group $W \rtimes \mathrm{Gal}(F'/F)$ where F' is the splitting field). For each $w \in W$, we let $T_w \subset G$ be the corresponding maximal torus in G over F, and let $R(T_w, 1)$ be the virtual unipotent character of $G(F)$ defined by Deligne and Lusztig (cf. [Car, chapter 12]). They also define a map Φ from class functions C on W to class functions on $G(F)$:

$$\Phi(C) = \frac{1}{\#W} \sum_{w \text{ in } W} C(w) R(T_w, 1). \tag{9.9}$$

If C is the character of the regular representation of W, then $\Phi(C) = R(T_1, 1)$ is the character of the permutation representation of $G(F)$ on $G(F)/B(F)$. In general, Deligne and Lusztig prove the formula

$$\langle C, C' \rangle_W = \langle \Phi(C), \Phi(C') \rangle_{G(F)}.$$

Let R be the character of the reflection representation of W on $X(T)$, and let $\bigwedge^i R$ be the character of its ith exterior power. When G is simply-laced, $\Phi(R)$ is the character of the reflection representation $R(F)$ of $G(F)$, which has dimension $\sum_{i=1}^{\ell} q^{d_i-1}$. W-T Gan proved the formula [G, (3.2)].

$$\mathrm{Tr}(Fr|M_g) = \mathrm{Tr}(g|R(F))$$

in this case. In general, $\bigwedge^\ell R = \epsilon$ is the sign character of W, and

$$\Phi(\epsilon) = \frac{1}{\#W} \sum_w \epsilon(w) R(T_w, 1) = \chi_{St} \tag{9.11}$$

is the character of the Steinberg representation of $G(F)$

Reeder has generalized Gan's formula, and shown that

$$\mathrm{Tr}\left(Fr| \bigwedge^i M_g\right) = \Phi\left(\bigwedge^i R\right)(g)$$

for all $i \geqslant 0$, and all semi-simple g in $G(F)$. Hence the class function $g \mapsto \det(1 - Fr|M_g)$ on semi-simple elements is given by the Deligne-Lusztig function $\sum_{i=0}^{\ell}(-1)^i \Phi(\bigwedge^i R)$ on $G(F)$.

More generally, for $n \geqslant 1$ the class function $g \mapsto \det(1 - Fr^n|M_g)$ is given by restricting the coresponding Deligne-Lusztig function from $G(F_n)$ to $G(F)$. Taking a product of these functions, we get an interpretation of h_S in (9.8).

We apply this to the case when $S = \{\infty, 0\}$ consists of two places of degree 1 on the projective line X. Then $\epsilon_S(G) = +1$, $\epsilon(G) = (-1)^\ell$ and

$$h_S = \sum_{i=0}^{\ell} (-1)^i \Phi\left(\bigwedge^i R\right).$$

By (9.7), we have

$$
\begin{aligned}
1+\sum_{(*)} m(\pi) &= (-1)^{\ell}\langle St, h_S\rangle_{G(F)} \\
&= (-1)^{\ell}\langle \Phi\left(\bigwedge^{\ell} R\right), \sum_{i=0}^{\ell}(-1)^{i}\Phi\left(\bigwedge^{i} R\right)\rangle_{G(F)} \\
&= \sum_{i=0}^{\ell}(-1)^{i+\ell}\langle \Phi\left(\bigwedge^{\ell} R\right), \Phi\left(\bigwedge^{i} r\right)\rangle_{G(F)} \\
&= \sum_{i=0}^{\ell}(-1)^{i+\ell}\langle \bigwedge^{\ell} R, \bigwedge^{i} R\rangle_{W}.
\end{aligned}
$$

Since G is simple, the representations $\bigwedge^i R$ of W are all irreducible and distinct. Hence

$$1+\sum_{(*)} m(\pi) = 1$$

in this case. Since $m(\pi) \geqslant 0$ for all irreducible representation π of $G(\mathbb{A})$, this shows that there are no cuspidal representations of G over $k = F(t)$ which satisfy $(*)$: π_v is Steinberg for $v = 0, \infty$ and π_v is unramified elsewhere.

In general, the contribution of the central classes in (8.8) gives the estimate

$$\sum_{(*)} m(\pi) \approx \#Z(q)\cdot q^N$$

with

$$N = (g-1)\dim(G) + \deg(S)\dim(G/B). \tag{9.12}$$

It seems likely that the sum $\sum_{*} m(\pi)$ is positive in all cases, except when

$$g = 0 \quad \deg(S) = 2 \quad \text{(considered above)}$$

$$g = 0, \quad G = SL_2, \quad \deg(S) = 3, \quad q \text{ even.}$$

When

$$g = 0, \quad G = SL_2, \quad \deg(S) = 3, \quad q \text{ odd}$$

we find $\sum_{(*)} m(\pi) = 1$.

10. Examples

We first give some examples when $k = \mathbb{Q}$. Assume that $G = PGL_2$ and that $\pi = \otimes \pi_v$ has the form (**):

for $v \in S_f$, $\quad \pi_v \otimes \chi_v \simeq St_v$, with χ_v an unramified character of the quotient $k_v^*/k_v^{*^2}$ of $PGL_2(k_v)$

for $v \in S_\infty$ $\quad \pi_v$ is the discrete series of weight $2k$, with infinitesimal character dual to the irreducible representation V_v of dimension $(2k-1)$

for $v \in T$ $\quad \dim\, (\pi_v \otimes \chi_v)^{H_v}$ has dimension 1

for $v \notin S \cup T$ $\quad \dim\, \pi_v^{G(\mathcal{O}_v)}$ has dimension 1.

Since $M_G = \mathbb{Q}(-1)$, Theorem 6.3 gives the formula

$$\begin{aligned}\sum_\pi m(\pi) &= (-1)\frac{2}{\prod\limits_{v\in S_f}(-2)}\frac{1}{\prod\limits_{T}(-1)}(2k-1)\cdot\frac{1}{2}\zeta_{S,T}(-1)\\ &= \left(\frac{1}{2}\right)^{\#S_f}(2k-1)\frac{1}{12}\prod_{S_f}(p-1)\prod_T(q^2-1).\end{aligned}$$

The multiplicities $m(\pi)$ are known to be 0 or 1 in this case. The local representations π_v of the form $St \otimes \chi_v^{-1}$ give rise to newforms of level p. The local representations π_v where H_v acts by χ_v^{-1} on a line give rise to newforms of level q^3. Since there are $(q-1)$ choices for the orbit of affine generic character χ_v at each place $v \in T$, and $2^{\#S_f}$ choices for the unramified character χ_v at the places $v \in S$, we obtain the formula given in the introduction

$$\dim S_{2k}^{\text{new}}\left(\prod_{S_f} p \prod_T q^3\right) = \frac{2k-1}{12}\prod_{S_f}(p-1)\prod_T(q^2-1)(q-1).$$

We note that the local representation π_q at primes $q \in T$ is defined over the totally real subfield $E_q \subset \mathbb{Q}(\mu_q)$. In particular, the field generated by the Fourier coefficients of such a new form must contain the composition of the fields E_q, for all q in T (cf. [Sa]).

Next let G be the anisotropic form of the exceptional group G_2 over $\mathbb{Q}$, which is split over $\mathbb{Q}_p$ for all primes p and is compact over $\mathbb{R}$. Here $M_G = \mathbb{Q}(-1) + \mathbb{Q}(-5)$. We consider representations $\pi = \otimes \pi_v$ satisfing (**):

$\pi_v \simeq St_v$ for $v \in S_f$

$\pi_v \simeq V$, for $v \in S_\infty$

$(\pi_v \otimes \chi_v)^{H_v}$ has dimension 1, for $v \in T$

$\pi_v^{G(\mathcal{O}_v)}$ has dimension 1, for $v \notin S \cup T$.

Then

$$\begin{aligned}\sum_{\substack{\pi \text{ satisfies} \\ (**)}} m(\pi) &= \dim V \cdot \frac{1}{2}\zeta_{S,T}(-1)\frac{1}{2}\zeta_{S,T}(-5) \\ &= \frac{\dim V}{2^6.3^3.7}\prod_{S_f}(p-1)(p^5-1)\prod_{T}(q^2-1)(q^6-1).\end{aligned}$$

Again, there are $(q-1)$ orbits of χ_v, for each place $v \in T$.

Finally, we give some examples when $k = F(t)$ is a function field of genus 0. Assume that G is defined over F and that p does not divide the order of $Z(\hat{G})$. If $S = \{0\}$ and $T = \{\infty\}$, then $\zeta_{S,T}(s) = 1$. Hence, by Theorem 7.1 we find

$$\sum m(\pi) = 1,$$

where the sum is taken over all cuspidal representations π for $G(\mathbb{A})$ which are unramified outside 0 and ∞, are a fixed unramified twist of the Steinberg representation at 0, and are a simple supercuspidal component of $\mathrm{Ind}_{H_v}(\chi_v^{-1})$ at ∞. Since $m(\pi) \geqslant 0$, this implies that there is a *unique* automorphic representation with this local behavior. It is defined over the field of definition of π_∞, which is a subfield of $\mathbb{Q}(\mu_p)$.

Let $\hat{G}$ be the split, pinned dual group of G over $\mathbb{Q}$, let 2ρ be the sum of the positive co-roots of $\hat{G}$, let $e = 2\rho(-1)$ be the central element of $\hat{G}$ with $e^2 = 1$, and let $\hat{G}_1 = \hat{G} \times \mathbb{G}_m/(e, -1)$. For example, when $G = PGL_2$, we have $\hat{G} = SL_2$, $e = -I$, and $\hat{G}_1 = GL_2$. This construction of $\hat{G}_1$ was suggested by Deligne, to preserve rationality in the (conjectural) Langlands correspondence. We have a homomorphism $\nu : \hat{G}_1 \to \mathbb{G}_m$, with kernel equal to $\hat{G}$.

Associated to the automorphic representation π of $G(\mathbb{A})$, we expect that there is a compatible system of continuous homomorphims

$$\varphi_\ell : \mathrm{Gal}(k^s/k) \to \hat{G}_1(\mathbb{Q}_\ell(\mu_p)) \rtimes \Gamma,$$

for all primes ℓ not equal to p. The homomorphisms φ_ℓ should be unramified outside 0 and ∞, and the semi-simple conjugacy classes $\varphi_\ell(Frob_v)$ should all be rational over $\mathbb{Q}(\mu_p)$. At 0, φ_ℓ should be tamely ramified and should map tame inertia to a subgroup isomorphic to $\mathbb{Z}_\ell$, which is topologically generated by a principal unipotent element. At ∞, φ_ℓ should be wildly ramified, but trivial on the $(1/h)^+$ subgroup in the upper ramification filtration, where h is the Coxeter number of G. We can normalize φ_ℓ by insisting that the composite with ν is a fixed power of the cyclotomic character, which will be even if $e = 1$ and odd if e has order 2.

If $\rho : \hat{G}_1 \rtimes \Gamma \to GL(V)$ is any representation of the L-group over $\mathbb{Q}$, then $\rho \circ \varphi_\ell$ defines a lisse ℓ-adic sheaf $\mathcal{F}$ on $\mathbb{G}_m$ over F, with $\text{rank}(\mathcal{F}) = \dim(V)$. Katz has constructed the ℓ-adic sheaf $\mathcal{F}$ in the special cases where a principal SL_2 in $\hat{G}$ acts irreducibly on V; it is a challenge to construct $\mathcal{F}$ in general. For a discussion of what is known, and of the analogous problem in geometric Langlands theory, see [F-G].

We discuss this automorphic representation in more detail when $G = PGL_2$, so $\hat{G}_1 = GL_2$. Let V be the two-dimensional representation of $\hat{G}_1$. It follows from Katz's identification of the ℓ-adic sheaf that the traces of Frobenii at the unramified places v of k on $\mathcal{F}$ are classical Kloosterman sums. Such places v correspond to elements a in $\bar{F}^*$, up to Galois conjugacy, and the finite extension $F_v = F(a)$ is the residue field at v. The choice of an orbit of affine generic characters for H_∞ corresponds to the choice of non-trivial additive character $\psi : F^+ \to \mathbb{Q}(\mu_p)^*$, and

$$\begin{aligned}\text{Tr}(\text{Frob}_v|\mathcal{F}) &= -\sum_{\substack{x,y \in F_v \\ xy=a}} \psi(\text{Tr}(x+y)) \\ &= -\sum_{x \in F_v^*} \psi(\text{Tr}(x + a/x)).\end{aligned}$$

As a modular form on PGL_2 over $k = F(t)$, π has level $(0) + 3(\infty)$, as a divisor on $\mathbb{P}^1$. Apply the automorphism $\sigma(t) = t^{-1}$ of k, to obtain a form $\pi' = \pi^\sigma$ of level $(\infty) + 3(0)$. If we insist that π'_∞ is isomorphic to the

Steinberg representation (and not its unramified quadratic twist), then the modular forms π' appear in the study of the Drinfeld modular curve $X_0(t^3)$, which is hyperelliptic of genus $q-1$ over $k = F(t)$. Specializing even further to the case when $q = 2$, the curve $X_0(t^3)$ has genus 1, and is isomorphic to the elliptic curve E with equation

$$y^2 + txy = x^3 + t^2x$$

$$\Delta = t^8 \qquad j = t^4$$

over $F_2(t)$. It has degenerate fibres of type I_1^* at $t = 0$ and I_4 at $t = \infty$. Both component groups are cyclic of order 4, as is the Mordell-Weil group $E(k)$, which is generated by the point $P = (t, 0)$. The global L-function $L(E/k, s) = 1$, as it is a polynomial of degree 0 in q^{-s}. At the unramified primes v, which correspond to maximal ideals of the ring $F_2[t, t^{-1}]$, we have

$$\#E(F_v) = 1 + q_v + \sum_{\substack{x,y \in F_v \\ xy=a}} (-1)^{\mathrm{Tr}(x+y)}$$

where a is the image of t in the residue field F_v.

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