Notes on Beilinson’s “How to glue perverse sheaves”

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In this paper I provide something of a skeleton key to A.A. Beilinson’s “How to glue perverse sheaves” [1], which I found hard to understand as well as lacking a discussion of some of its more interesting consequences. In order to maintain readability, I will work in the sheaf-theoretic context of the classical topology on complex algebraic varieties; for the necessary modifications to étale sheaves, one should consult Beilinson’s paper: aside from the shift in definitions the only change is some Tate twists. For the $D$-modules case, read Sam Lichtenstein’s undergraduate thesis, [2].

1 Theoretical preliminaries

The topic at hand is perverse sheaves and nearby cycles; I give a summary of the definitions and necessary properties here.

1.0 Diagram chases

Especially in §3, we will need to do some pretty serious diagram chasing. Some people make a big deal about how you can’t chase diagrams in a general category because the objects aren’t sets, but in fact in an abelian category (or even a Quillen-exact category, if one wishes to adapt the results in §3 to the more general context in which Beilinson stated them in [1]) chasing elements is perfectly possible.

The important fact is that an object in an abelian category $A$ can be considered to be a sheaf on the canonical topology of $A$. This is, by definition, the largest Grothendieck topology in which all representable functors $\text{Hom}_A(-, x)$ are sheaves, and its open covers are precisely the strict universal epimorphisms. Such a map is, in a more general context, a map $f: u \to x$ such that the fibered product $u' = u \times_x u$ exists, the coequalizer $x' = \text{coker}(u' \to u)$ exists, the natural map $x' \to x$ is an isomorphism, and that all of this is also true when we make any base change along a map $g: y \to x$, for the induced map $f \times_x \text{id}: u \times_x y \to y$. In an abelian category, however, this is all equivalent merely to the statement that $f$ is a surjection.

Recall the definitions of the various constructions on sheaves:

1. Kernels of maps are taken sectionwise; i.e. for a map $f: F \to G$, $\ker(f)(U) = \ker(f(U)): F(U) \to G(U))$. Likewise, products and limits are taken sectionwise.

2. Cokernels are locally taken sectionwise: any section $s \in \text{coker}(f)(U)$ is, on some open cover $V$ of $U$, of the form $t$ for $t \in G(V)$. Likewise, images, coproducts, and colimits are taken locally.

In an abelian category, where all of these constructions exist by assumption, these descriptions are even prescriptive: if one forms the sheaves thus described, they are representable by the objects claimed. Therefore, the following common arguments in diagram chasing are valid:

1. A map $f: x \to y$ is surjective if and only if for every $s \in y$, there is some $t \in x$ such that $s = f(t)$. This is code for: for every “open set” $U$ and every $s \in y(U)$, there is a surjection $V \to U$ and a section $t \in y(V)$ such that $s|_V = f(t)$.

2. If $s \in y$, then $\overline{s} = 0 \in \text{coker}(f)$ if and only if $s \in \text{im}(f)$. This is code for: if $s \in y(U)$ and $\overline{s} = 0 \in \text{coker}(f)(U)$, then there is some surjection $V \to U$ and $t \in x(V)$ with $s|_V = f(t)$. 

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3. $f$ is zero if and only if for all $s \in x$, $f(x) = 0$. This is actually the definition of the zero map.

Any other arguments involving elements and some concept related to exactness can also be phrased in this way. Thus, a naïve diagram-chasing argument can be converted into a rigorous one simply by replacing statements like $s \in \mathcal{F}$ with correct ones $s \in \mathcal{F}(U)$ for some open set $U$, and passing to surjective covers when necessary. The process being mechanical, I will just give the naïve proof.

### 1.1 Derived category and functors

All the action takes place in the derived category; specifically, let $X$ be an algebraic variety and denote by $\mathcal{D}(X)$ its derived category of bounded complexes of sheaves with constructible cohomology. By definition, a map of complexes $f^\bullet: \mathcal{A}^\bullet \to \mathcal{L}^\bullet$ in $\mathcal{D}(X)$ is an isomorphism if its associated map on cohomology sheaves $H^i(f): H^i(\mathcal{A}^\bullet) \to H^i(\mathcal{L}^\bullet)$ is for all $i$. We have a notation for the index-shift: $K^{-1} = (K[1])^i$ (technically, the differential maps also change sign, but we will never need to think about this). The derived category $\mathcal{D}(X)$ is a “triangulated category”, which means merely that in it are a class of triples of complexes and maps

$$\mathcal{A}^\bullet \to \mathcal{B}^\bullet \to \mathcal{C}^\bullet \to \mathcal{A}^\bullet[1]$$

in which two consecutive arrows compose to zero, satisfying some axioms we won’t need, and with the property that the associated long sequence of cohomology sheaves

$$\ldots H^{-1}(\mathcal{C}^\bullet) \to H^0(\mathcal{A}^\bullet) \to H^0(\mathcal{B}^\bullet) \to H^0(\mathcal{C}^\bullet) \to H^1(\mathcal{A}^\bullet) \to \ldots$$

is exact (note that $H^0(\mathcal{A}[1]) = H^1(\mathcal{A})$); we say that the $H^i$ are “cohomological”. If $f: \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ is given, there always exists a (unique up to non-unique isomorphism) triangle whose third term $\mathcal{C}^\bullet = \text{Cone}(f)$ is the “cone” of $f$. A functor between two triangulated categories is “triangulated” if it sends triangles in one to triangles in the other.

In $\mathcal{D}(X)$ we also have some standard constructions of sheaf theory. For any two complexes there is the “total tensor product” $\mathcal{A}^\bullet \otimes \mathcal{B}^\bullet$ obtained by taking in degree $n$ the direct sum of all products $\mathcal{A}^i \otimes \mathcal{B}^j$ with $i + j = n$ (and some differentials that are irrelevant) and its derived functor $\mathcal{A}^\bullet \otimes \mathcal{B}^\bullet$, with $H^i(\mathcal{A}^\bullet \otimes \mathcal{B}^\bullet) = \text{Tor}^i(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$, which is a triangulated functor. We also have the functor $\mathcal{H}\text{om}(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$, where $\mathcal{H}\text{om}(\mathcal{A}^\bullet, \mathcal{B}^\bullet)(U) = \text{Hom}(\mathcal{A}^\bullet|_U, \mathcal{B}^\bullet|_U)$, and its derived functor $R\mathcal{H}\text{om}(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$, with $H^i R\mathcal{H}\text{om}(\mathcal{A}^\bullet, \mathcal{B}^\bullet) = \text{Ext}^i(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$, which is triangulated. Of course, these two have an adjunction:

$$R\mathcal{H}\text{om}(\mathcal{A}^\bullet \otimes \mathcal{B}^\bullet, \mathcal{C}^\bullet) \cong \mathcal{R}\mathcal{H}\text{om}(\mathcal{A}^\bullet, R\mathcal{H}\text{om}(\mathcal{B}^\bullet, \mathcal{C}^\bullet)).$$

For any Zariski-open subset $U \subset X$ with inclusion map $j$, there are triangulated functors $j_!, j_\ast, j^\ast: \mathcal{D}(U) \to \mathcal{D}(X)$ and $j^\ast = j_{\ast}^1: \mathcal{D}(X) \to \mathcal{D}(U)$; if $i$ is the inclusion of its complement $Z$, then there are likewise maps $i^\ast, i_\ast: \mathcal{D}(X) \to \mathcal{D}(Z)$ and $i_\ast = i_\ast^1: \mathcal{D}(Z) \to \mathcal{D}(X)$. (Again, technically $j_\ast$ and $j_\ast$ are derived functors and should be denoted $Lj_\ast$ and $Rj_\ast$, but we will never have occasion to use the plain versions so we elide this extra notation.) They satisfy a number of important relations, of which I will only use one here: there is a natural triangle for any complex $\mathcal{A}^\bullet$:

$$j_!(\mathcal{A}^\bullet) \to j_\ast(\mathcal{A}^\bullet) \to i_\ast i_\ast^1(\mathcal{A}^\bullet) \to$$

There is also a triangulated duality functor $D: \mathcal{D}(X) \to \mathcal{D}(X)^{\text{op}}$ which interchanges $!$ and $\ast$, in that $Dj_!(\mathcal{A}^\bullet) = j_!(D\mathcal{A}^\bullet)$, etc., and is an involution. In fact, if we set $D^\bullet = \mathcal{D}^\bullet$, then $\mathcal{D}(\mathcal{A}^\bullet) = R\mathcal{H}\text{om}(\mathcal{A}^\bullet, D^\bullet)$. The only thing we need to know about $D$ is that it exchanges $!$ and $\ast$; i.e. that $Dj_!F = j_!D^\ast F$, etc.

For any map $f: X \to Y$ of varieties, we have $f^!, f^\ast$ as well (also $f_!, f_\ast$, and none of them are equal), with the same relationships to $D$, and the useful identity

$$f^! \mathcal{R}\mathcal{H}\text{om}(\mathcal{F}, \mathcal{G}) = \mathcal{R}\mathcal{H}\text{om}(f^\ast \mathcal{F}, f^! \mathcal{G}).$$
1.2 Perverse sheaves

Within $D(X)$ there is an abelian category $M(X)$ of “perverse sheaves” which has nicer properties than the category of actual sheaves. It is specified by means of a “t-structure”, namely, a pair of full subcategories $D(X)^{\leq 0}$ and $D(X)^{\geq 0}$, also satisfying some conditions I won’t use, and such that

$$M(X) = \,^pD(X)^{\leq 0} \cap \,^pD(X)^{\geq 0}.$$  

There are truncation functors $\tau_{\leq a}: D(X) \to \,^pD(X)^{\leq a}$ and likewise for $\tau_{\geq a}$, fitting into a distinguished triangle for any complex $K^\bullet$:

$$\tau_{\leq a}K^\bullet \to K^\bullet \to \tau_{>a}K^\bullet \to$$

(where $\tau_{>a} = \tau_{>1} = [-1] \circ \tau_{\geq a} \circ [1]$). This triangle is unique with respect to the property that the first term is in $\,^pD(X)^{\leq a}$ and the third is in $\,^pD(X)^{\geq a}$. They have the obvious properties implied by the notation: $\tau_{\leq a} \tau_{\leq b} = \tau_{\leq \min(a,b)}$ if $a \leq b$, and likewise for $\tau_{>a}$. Furthermore, there are “perverse cohomology” functors $\,^pH^i: D(X) \to M(X)$, where of course $\,^pH^0(A^\bullet) = \,^pH^0(A^\bullet[i])$ and $\,^pH^0 = \tau_{\geq 0} \tau_{\leq 0} = \tau_{\leq 0} \tau_{\geq 0}$; these are cohomological just like the ordinary cohomology functors. The abelian category structure of $M(X)$ is more or less determined by the fact that if we have a map $f: F \to G$ of perverse sheaves (this is the notation I will be using), we will not think of perverse sheaves as complexes), then

$$\ker f = \,^pH^{-1} \operatorname{Cone}(f) \quad \quad \quad \coker f = \,^pH^0 \operatorname{Cone}(f).$$

If $X$ is connected of dimension $D$, then the category $M(X)$ is closed under a translate of the duality functor, $D[d]$, but not necessarily under the six functors defined for an open/closed pair of subvarieties. However, it is true that $j_*(F), \,^\dagger(F) \in \,^pD^{\geq 0}$ and $j_!(F), \,^\ast(F) \in \,^pD^{\leq 0}$, while $j^!(F), \,^\ast_!(F) \in M$; we say these functors are right, left, or just “t-exact”. Furthermore, when $j$ is an affine morphism (the primary example being when $Z$ is a Cartier divisor), both $j_!$ and $j_*$ are t-exact, and thus their restriction to $M(U)$ is exact with values in $M(X)$. There is also a “middle extension” functor $j_{\ast!}$, defined so that $j_{\ast!}(F)$ is the image of $\,^pH^0(j_*F)$ in $\,^pH^0(j_*F)$ along the natural map $j_! \to j_*$; it is the unique perverse sheaf such that $i^*j_{\ast!}F \in \,^pD^{\leq 0}$ and $i^!'j_{\ast!}F \in \,^pD^{\geq 0}$, but for me the most useful property is that when $j$ is an affine, open immersion, then we have a sequence of perverse sheaves

$$i^*j_{\ast!}F[-1] \hookrightarrow j_!F \to j_*F \hookrightarrow j_*F \to i^!'j_{\ast!}F[1];$$

i.e. $i^*j_{\ast!}F[-1] = \ker(j_!F \to j_*F)$ and $i^!'j_{\ast!}F[1] = \coker(j_!F \to j_*F)$ are both perverse sheaves.

Perverse sheaves have good category-theoretic properties: $M(X)$ is both artinian and noetherian, so every perverse sheaf has finite length.

Finally, we will use the sheaf-theoretic fact that if $\mathcal{L}$ is a locally constant sheaf on $X$, then $\mathcal{F} \otimes \mathcal{L}$ is perverse whenever $\mathcal{F}$ is. Note that since $\mathcal{L}$ is locally free, it is flat, and therefore $\mathcal{F} \otimes \mathcal{L} = \mathcal{F} \otimes \mathcal{L}$.

1.3 Nearby cycles

If we have a map $f: X \to \mathbb{A}^1$ such that $Z = f^{-1}(0)$, the “nearby cycles” functor $R\Psi_f: D(U) \to D(Z)$ is defined. Namely, let $u: G_m \to G_m$ be the universal cover of $G_m = \mathbb{A}^1 \setminus \{0\}$, let $v: \bar{U} = U \times_{G_m} G_m \to U$ be its pullback, forming a diagram

$$
\begin{array}{c}
Z \to i \to X \to j \to U \leftarrow v \to \bar{U} \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\{0\} \to \mathbb{A}^1 \leftarrow G_m \\
\end{array}
$$

and set

$$R\Psi_f(A^\bullet) = i^*Rj_*Ru_v^*A^\bullet: D(U) \to D(Z).$$
Since \( i^* \) and \( v^* \) are exact, indeed we have \( R\Psi_f = R(i^*j_*v_*v^*) \) is the right derived functor of a left-exact functor \( \Psi_f \) from sheaves on \( U \) to sheaves on \( Z \). The fundamental group \( \pi_1(G_m) \) acts on any \( j^*F \) via deck transformations of \( G_m \) and therefore acts on \( \Psi_f \) and \( R\Psi_f \). There is a natural map, obtained from \((u^*,v_*)\)-adjunction, from \( i^*F \to \Psi_f(j^*F) \), on whose image \( \pi_1(G_m) \) acts trivially. We set, by definition,

\[
i^*F \to \Psi_f(j^*F) \to \Phi_f(F) \to 0
\]

where \( \Phi_f(F) \) is the “vanishing cycles” sheaf, and for the generator \( t \in \pi_1(G_m) \) we get a map \( \text{Var}(t) : \Phi_f(F) \to \Psi_f(F) \) factoring \( 1-t \) on \( \Psi_f(F) \) through the quotient. Using some homological algebra tricks the above sequence induces a natural distinguished triangle

\[
i^*A^* \to R\Psi_f(j^*A^*) \to R\Phi_f(A^*) \to
\]

where \( R\Phi_f \) is (morally) the right derived functor of \( \Phi_f \).

If before taking the derived functor we single out the subsheaf \( \Psi^\text{un}_f(F) \) on which \( 1-t \) is locally nilpotent (in fact, by the constructibility theorem of SGA7 it is actually nilpotent), we get the functor of unipotent nearby cycles. The unit of \( \Psi^\text{un}_f(F) \) is nilpotent, hence zero. Thus \( \cokernel \) of the map \( j^* \) as promised.

Theorem 2.1. We start off with a nice result about nearby cycles.

**The unipotent nearby cycles functor**

We proceed to give a construction of \( \Psi^\text{un}_f(F) \). If before taking the derived functor we single out the subsheaf \( \Psi^\text{un}_f(F) \) on which \( 1-t \) is locally nilpotent (in fact, by the constructibility theorem of SGA7 it is actually nilpotent), we get the functor of unipotent nearby cycles. The unit of \( \Psi^\text{un}_f(F) \) is nilpotent, hence zero. Thus \( \cokernel \) of the map \( j^* \) as promised.

\[
\cdots \to H^{-3}(R\Psi^\text{un}_f A^*) \xrightarrow{1-t} H^{-3}(R\Psi^\text{un}_f A^*) \to H^{-2}(i^*j_*A^*) \to H^{-2}(R\Psi^\text{un}_f A^*) \xrightarrow{1-t} H^{-2}(R\Psi^\text{un}_f A^*) \to \\
\to H^{-1}(i^*j_*A^*) \to H^{-1}(R\Psi^\text{un}_f A^*) \xrightarrow{1-t} H^{-1}(R\Psi^\text{un}_f A^*) \to H^0(i^*j_*A^*) \to \\
\to 0 \to H^0(R\Psi^\text{un}_f A^*) \xrightarrow{1-t} H^0(R\Psi^\text{un}_f A^*) \to 0 \to \cdots
\]

For \( i \geq 1 \), the map \( H^i(R\Psi^\text{un}_f M) \to H^i(R\Psi^\text{un}_f M) \) is both given by a nilpotent operator and is injective, therefore zero. For \( i = 0 \) it is nilpotent and surjective, so also zero. It follows that \( R\Psi^\text{un}_f(A^*) \in pD(Z)^{\leq 0} \), as promised.

Now let \( M \in M(U) \) be a perverse sheaf. Then \( i^*j_*M \in pD(Z)[−1,0] \) since it consists of the kernel and cokernel of the map \( j_*M \to j_*M \). This means that for \( i \leq -2 \) in the above sequence, all the maps \( 1-t \) are injective and nilpotent, hence zero. Thus \( R\Psi^\text{un}_f(M) \in pD(Z)^{−1} \), as desired.

Now let \( L^a \) be the vector space of dimension \( a \geq 0 \) together with the action of a matrix \( J^a = [\delta_{ij} + \delta_{i,-j}] \), a unipotent Jordan block of dimension \( n \). Let \( L^a \) be the locally constant sheaf on \( G_m \) with underlying space \( L^a \) and monodromy action where a generator \( t \) of \( \pi_1(G_m) \) acts by \( J^a \); we consider it a complex in cohomological degree zero.

**Lemma 2.2.** Let \( A^* \in D(U) \); then \( D(A^* \otimes f^*L^a) \cong D(A^*) \otimes f^*L^a \).
Proof. First we prove a related fact: if we set $\hat{L}^a = \mathcal{H}om(L^a, \mathbb{C})$, then there is an isomorphism $\hat{L}^a \cong \mathcal{L}^a$. Indeed, $L^a$ has a given basis $e_1, \ldots, e_a$ associated to $J$, and we fix the basis $e_{a}', \ldots, e_{\Lambda}'$ of dual vectors for $\mathcal{L}^a$, sending $J$ to $J'$, and thus inducing the desired map of local systems.

In general, then, we compare:

$$\mathbb{D}(F) \otimes f^* \mathcal{L}^a = R \mathcal{H}om(F, \mathcal{D}^\bullet) \otimes f^* \mathcal{L}^a \quad \mathbb{D}(F \otimes f^* \mathcal{L}^a) = R \mathcal{H}om(F \otimes \mathcal{L}^a, \mathcal{D}^\bullet) = R \mathcal{H}om(F, \mathbb{D}\mathcal{L}^a).$$

There is a natural map from the former to the latter induced by $(\otimes, R \mathcal{H}om)$-adjunction:

$$R \mathcal{H}om(F, \mathcal{D}^\bullet) \to R \mathcal{H}om(f^* \mathcal{L}^a, R \mathcal{H}om(F, f^* \mathcal{L}^a)) = R \mathcal{H}om(F, R \mathcal{H}om(f^* \mathcal{L}^a, \mathbb{D}\mathcal{L}^a)).$$

Note that $\mathcal{D}^\bullet = f^! \mathcal{D}^\bullet$ (one on $X$, the other on $\mathbb{G}_m$) by definition and $\mathbb{D}f^* \mathcal{L}^a = f^! \mathbb{D}\mathcal{L}^a$. By the property of $f^!$, we have

$$R \mathcal{H}om(f^* \mathcal{L}^a, f^! \mathbb{D}\mathcal{L}^a) = f^! R \mathcal{H}om(\mathcal{L}^a, \mathbb{D}\mathcal{L}^a)$$

and therefore the desired map is the one induced on $R \mathcal{H}om(F, -)$ from the one obtained by applying $f^!$ to a certain map

$$\mathcal{D}^\bullet \to R \mathcal{H}om(\mathcal{L}^a, \mathbb{D}\mathcal{L}^a) = \mathbb{D}(\mathcal{L}^a \otimes \mathcal{L}^a).$$

This map, in turn, is obtained by first replacing the $\otimes$ with $\otimes$ (since $\mathcal{L}^a$ is locally free), replacing one copy of $\mathcal{L}^a$ with its dual, and applying $\mathbb{D}$ to the trace map

$$\mathcal{L}^a \otimes \hat{L}^a \to \mathbb{C}.$$  

The map $\mathbb{D}(F) \otimes f^* \mathcal{L}^a \to \mathbb{D}(F \otimes f^* \mathcal{L}^a)$ is an isomorphism locally since it would be an isomorphism if $\mathcal{L}^a$ were replaced with a constant sheaf. Thus, it is an isomorphism. \hspace{1cm} $\square$

For each $a, b \geq 0$ there is a natural exact sequence

$$0 \to \mathcal{L}^a \xrightarrow{g^{a,b}} \mathcal{L}^{a+b} \xrightarrow{g^{a+b,-b}} \mathcal{L}^b \to 0;$$

that is, for any $r \in \mathbb{Z}$, $g^{a,r}$ sends $\mathcal{L}^a$ to the first $r$ coordinates of $\mathcal{L}^{a+r}$ if $r \geq 0$, and to the quotient $\mathcal{L}^{a-(-r)}$ given by collapsing the first $-r$ coordinates if $-r \geq 0$ (that is, $r \leq 0$). This sequence respects the action of $J'$ on the terms and under the identification of Lemma 2.2, the $(a, b)$ sequence is dual to the $(b, a)$ sequence; that is,

$$\mathbb{D}g^{a,r} = g^{a+r,-r}.$$  

These have various formal identities:

1. When $r \geq 0$, we have $g^{a+r, -a} \circ g^{a,r} = 0$;
2. When $r \leq 0$ and $a + r \geq 0$, we have $g^{a+r, -r} \circ g^{a,r} = (1-t)^r$;
3. When $r$ and $s$ have the same sign, and $a + r \geq 0$, we have $g^{a,r+s} = g^{a+r,s} \circ g^{a,r}$.

Let $\mathcal{M} \in \mathcal{M}(U)$, so $\mathcal{M} \otimes f^* \mathcal{L}^a$ is also perverse and has an action of the monodromy via $J^a$ acting on the second factor, which we will call simply $t$. The maps id $\otimes g^{a,r}$ between these will be called $g_{\mathcal{M}}^{a,r}$, or just $g^{a,r}$ again if there is no need to mention $\mathcal{M}$, and we have

$$\mathbb{D}g_{\mathcal{M}}^{a,r} = g_{\mathcal{M}}^{a+r,-r}.$$  

Note that since the $\mathcal{L}^a$ are locally free, the $g_{\mathcal{M}}^{a,r}$ are all injective when $r \geq 0$ and surjective when $r \leq 0$. We can prove a strong uniformity lemma based on the monodromy action:

**Lemma 2.3.** Let $\alpha^a : j_h(\mathcal{M} \otimes f^* \mathcal{L}^a) \to j_* (\mathcal{M} \otimes f^* \mathcal{L}^a)$. There exists an integer $N$ such that $(1-t)^N$ annihilates both $\ker \alpha^a$ and $\coker \alpha^a$ for all $a$.  

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Proof. We consider again the triangle (1); if we could find such an $N$ annihilating $\Psi^m_f(M \otimes f^*L^a)$, then it would annihilate $i^*j_*(M \otimes f^*L^a)$ as well, and hence its perverse cohomologies $\text{ker} \alpha^a$ and $\text{coker} \alpha^a$. We know that $\Psi^m_f(M \otimes f^*L^a) \cong \Psi^m_f(M) \otimes L^a$ and that $1 - t$ is nilpotent on $\Psi^m_f(M)$, say $(1 - t)^N = 0$, so it suffices to show that this power kills the tensor product as well. Indeed, the short exact sequences (2) with $(a, b) = (1, a - 1)$: $0 \to L^1 \to L^a \to L^{a-1} \to 0$ induce short exact sequences of perverse sheaves respecting the monodromy action:

$$0 \to \Psi^m_f(M) \to \Psi^m_f(M) \otimes L^a \to \Psi^m_f(M) \otimes L^{a-1} \to 0.$$ 

Thus the claim is proved by induction. \hfill \square

Corollary 2.4. There exists a map $\beta^a : j_!(M \otimes f^*L^a) \to j_*(M \otimes f^*L^a)$ such that $\alpha^a \beta^a = \beta^a \alpha^a = (1 - t)^{2N}$.

Proof. Consider the diagram

$$
j_!(M \otimes L^a) \begin{array}{c} \alpha_1 \downarrow \beta_1 \downarrow \alpha_* \downarrow \beta_* \downarrow \end{array} j_* (M \otimes L^a)$$

where $\alpha = \alpha^{a}$, and set $\beta = \beta_1 \beta_*$. Note that $\beta_1$ and $\beta_*$ exist and $\beta_1 \alpha_1 = \alpha_1 \beta_* = (1 - t)^N$ because $(1 - t)^N$ annihilates $\text{ker} \alpha$ and $\text{coker} \alpha$, respectively.

We claim that $\alpha_1 \beta_1 = \beta_* \alpha_* = (1 - t)^N$ as well. Indeed, consider the latter and apply $\alpha_1$ to both sides: $\alpha_1 \beta_1 \alpha_* = (1 - t)^N \alpha_* = \alpha_1 (1 - t)^N$, since $\alpha_1$, and hence $\alpha_1$ and $\alpha_1$, respect the monodromy action. Since $\alpha_1$ is injective, $\beta_1 \alpha_* = (1 - t)^N$, as desired. Likewise for $\alpha_1 \beta_1$, applying it to $\alpha_1$ and using surjectivity.

Therefore,

$$\alpha^a \beta^a = \alpha_1 \alpha_1 \beta_* \beta_* = \alpha_1 (1 - t)^N \beta_* = (1 - t)^N \alpha_* \beta_* = (1 - t)^{2N}.$$ 

Likewise for $\beta^a \alpha^a$. \hfill \square

Let $r \in \mathbb{Z}$ with $a + r \geq 0$, and set $\alpha^{a,r} = j_!(g^{a,r}) \circ \alpha^a = \alpha^{a+r} \circ j_!(g^{a,r})$, so

$$\alpha^{a,r} : j_!(M \otimes f^*L^a) \to j_*(M \otimes f^*L^{a+r}).$$

Setting $\beta^{a,r} = j_!(g^{a+r,-r}) \circ \beta^{a+r} = \beta^a \circ g_*(g^{a+r,-r})$, we can generalize Corollary 2.4:

Corollary 2.5. We have:

1. When $r \geq 0$, $\alpha^{a,r} \circ \beta^{a,r} = (1 - t)^{2N+r}$;
2. When $r \leq 0$, $\beta^{a,r} \circ \alpha^{a,r} = (1 - t)^{2N+r}$;
3. For any $r$, $\beta^{a+2r,-r} \circ \alpha^{a,r} = (1 - t)^{2(N+r)} j_!(g^{a,2r})$ and $\alpha^{a+r,r} \circ \beta^{a+r,-r} = (1 - t)^{2(N+r)} j_!(g^{a,2r})$. \hfill \square

Proposition 2.6. For $a \gg 0$, the natural maps $j_!(g^{a,1})$ and $j_*(g^{a+1,1})$ respectively induce isomorphisms $\text{ker}(\alpha^{a,r}) \to \text{ker}(\alpha^{a+1,r})$ and $\text{coker}(\alpha^{a,r}) \to \text{coker}(\alpha^{a+1,r})$. When $r \geq 0$, the former are isomorphisms for all $a$, and when $r \leq 0$, the latter are. These statements also hold with $\alpha$ replaced by $\beta$ and $r$ negated.

Proof. The latter statement is obvious: when $r \geq 0$, $g^{a,r}$ and hence $j_!(g^{a,r})$ is injective, so $\alpha^{a,r}$ has the same kernel as $\alpha^a$; likewise for $r \leq 0$ and $j_!(g^{a,r})$ being surjective.
Suppose $r \leq 0$; then, applying Corollary 2.5, we have $\beta^{a,r} \circ \alpha^{a,r} = (1-t)^{2N+r}$, so $(1-t)^{2N+r}$ annihilates $\ker(\alpha^{a,r})$. Likewise, when $r \geq 0$, $\alpha^{a,r} \circ \beta^{a,r} = (1-t)^{2N+r}$, so $(1-t)^{2N+r}$ annihilates $\coker(\alpha^{a,r})$. Using the “natural maps” $j_!(g^{a,1})$ and $j_!(g^{a+r,1})$ we get a commutative square
\[
\begin{array}{ccc}
j_!(\mathcal{M} \otimes f^* \mathcal{L}^a) & \xrightarrow{\alpha^{a,r}} & j_!(\mathcal{M} \otimes f^* \mathcal{L}^{a+r}) \\
\downarrow & & \downarrow \\
j_!(\mathcal{M} \otimes f^* \mathcal{L}^{a+1}) & \xrightarrow{\alpha^{a+1,r}} & j_!(\mathcal{M} \otimes f^* \mathcal{L}^{a+r+1})
\end{array}
\]
showing that $j_!(g^{a,1})$ induces a map on kernels and $j_!(g^{a+r,1})$ on cokernels. Since the $j_!(g^{a,1})$ are injective, we get a long sequence of inclusions of kernels:
\[
\cdots \subset \ker \alpha^{a-1,r} \subset \ker \alpha^{a,r} \subset \ker \alpha^{a+1,r} \subset \cdots.
\]
Each kernel is annihilated by $(1-t)^{2N+r}$, whose kernel is (for $a \geq 2N+r$) the perverse sheaf $j_!(\mathcal{M} \otimes f^* \mathcal{L}^{2N+r})$. Since perverse sheaves have finite length, this chain must have a maximum, whence the claim for kernels. For cokernels, they are all quotients $j_!(\mathcal{M} \otimes f^* \mathcal{L}^{2N+r})$, and stabilized since their kernels are a similar increasing subsequence.

To obtain the results for $\beta$, just swap it with $\alpha$ in all the arguments and negate $r$; we have only used Corollary 2.5, which is symmetric under this transformation. \hfill \Box

Departing slightly from Beilinson’s notation, we denote these stable kernels $\ker \alpha^{\infty,r}$ for $r \geq 0$ and $\coker \alpha^{\infty,r}$ for $r \leq 0$; when $r = 0$ we drop it.

**Lemma 2.7.** Suppose that in Proposition 2.6 the stable cokernels $\coker(\alpha^{a,r})$, $\coker(\beta^{a,r})$ obtain for $a \geq L$. Then in this range the following natural maps are isomorphisms:

\[
\begin{align*}
\im(\alpha^{a+1,r})/\im(\alpha^{a,r}) & \cong j_*(\mathcal{M} \otimes f^* \mathcal{L}^{a+1+r})/j_*(\mathcal{M} \otimes f^* \mathcal{L}^{a+r}) \\
\im(\beta^{a+1,r})/\im(\beta^{a,r}) & \cong j_!(\mathcal{M} \otimes f^* \mathcal{L}^{a+1})/j_!(\mathcal{M} \otimes f^* \mathcal{L}^a).
\end{align*}
\]

In particular, $(\alpha^{a,r})^{-1}(\im(1-t)^n) \subset \im(1-t)^n$ and the same for $\beta$, for $n \leq a - L$.

**Proof.** This is of course merely a general algebraic fact: suppose we have objects $A_1 \subset A_2$ in an abelian category and subobjects $B_1 \subset A_1$ with $B_1 \subset B_2$. Then the following three conditions are equivalent:

1. The inclusion $B_2 \hookrightarrow A_2$ induces an inclusion $B_2/B_1 \hookrightarrow A_2/A_1$;
2. The inclusion $A_1 \hookrightarrow A_2$ induces an inclusion $A_1/B_1 \hookrightarrow A_2/B_2$;
3. The natural map $B_1 \rightarrow B_2 \times_{A_2} A_2$ is an isomorphism.

Furthermore, $B_2/B_1 \rightarrow A_2/A_1$ if and only if $A_1/B_1 \rightarrow A_2/B_2$. For the proof, assume by symmetry that the first map is injective (resp. surjective), and chase the following diagram:

\[
\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & B_1 & \rightarrow & A_1 & \rightarrow & A_1/B_1 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & B_2 & \rightarrow & A_2 & \rightarrow & A_2/B_2 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
B_1/B_1 & \rightarrow & A_2/A_1 & & & & & & 0
\end{array}
\]
Lemma 2.11. Suppose \( C \) is a triangulated category and let the core of \( C \) be the abelian category \( A \). We will assume that the objects of \( C \) are bounded above, meaning that \( C = \bigcup_{b \in \mathbb{Z}} C^{<b} \), and that \( C \) is nondegenerate, meaning that \( \bigcap_{b \in \mathbb{Z}} C^{\leq b} = 0 \). These assumptions hold for the perverse t-structure.

**Proposition 2.8.** For \( r \geq 0 \), there is an isomorphism \( \text{coker} \alpha_{\infty,r} \cong \ker \alpha_{\infty,-r} \).

**Proof.** Starting with the map \( \beta^a \) of Corollary 2.4, and for \( a \geq M \), let \( \gamma^{a,-r} = (1 - t)^{a-2N} \beta^{a+r} \), so

\[
\gamma^{a} : j_{i}(M \otimes f^* L^{a+r}) \to j_{i}(M \otimes f^* L^{a+r}).
\]

By definition of \( \alpha^{a,r} \), we have \( \beta^a \circ \alpha^{a,r} = (1-t)^{2N} j_{i}(g^{a,r}) \), so \( \gamma^{a,r} \circ \alpha^{a,r} = j_{i}(g^{a,r}) \circ (1-t)^a = 0 \) since \( (1-t)^a \) annihilates \( L^a \). Likewise, \( \alpha^{a+r,-r} \circ \beta^a = (1-t)^{2N} j_{i}(g^{a+r,-r}) \), so \( \alpha^{a+r,-r} \circ \gamma^{a,-r} = 0 \) for the same reason. So, in fact

\[
\gamma^{a,r} : \text{coker} \alpha^{a,r} \to \ker \alpha^{a+r,-r}.
\]

The kernel of \( \gamma^{a,r} \) is by definition \( (\beta^{a+r})^{-1} \text{im}(1-t)^{2N} + \text{mod} \alpha^{a,r} \), but since \( \alpha^{a,r} \circ \beta^{a,r} = (1-t)^{2N+r} \), to show that \( \gamma^{a,r} \) is injective it suffices to show that \( (\beta^{a+r})^{-1} \text{im}(1-t)^{2N} + \text{mod} \alpha^{a,r} \), which follows from Lemma 2.7 when \( a \gg 0 \). For surjectivity, using the fact that \( (1-t)^{2N} \alpha^{a+r,-r} \), we have \( \alpha^{a+r,-r} \circ \beta^a = (1-t)^{2N} j_{i}(g^{a+r,-r}) \), or since \( \beta^{a+r} \circ \alpha^{a+r} = (1-t)^{2N}, \) that \( (\alpha^{a,r})^{-1} \text{im}(1-t)^{2N} + \text{mod} \alpha^{a,r} \), which is again true for \( a \gg 0 \) by Lemma 2.7.

Because they are equal, we will give a single name to the stable kernel and cokernel: \( \Pi_f^M(M) = \text{ker}(\alpha^{\infty,r}) = \text{coker}(\alpha^{\infty,-r}) \). The definitions of \( \alpha^{a} \) and \( \beta^{a,r} \) easily imply that \( D\Pi_f^M(M) \cong \Pi_f^M(DM) \).

**Theorem 2.9.** There is an isomorphism \( R\Psi_f^M(M)[-1] \cong \Pi_f^M(M), \) so that \( R\Psi_f^M \) commutes with \( D \).

**Proof.** This is really just a more specific version of the argument in Lemma 2.3. Consider again the triangle (1) with \( A^* = M \otimes f^* D^{a} \). Since we know that \( R\Psi_f^M(M)[-1] \) is a perverse sheaf and that \( i_! j_{*}(M \otimes f^* L^{a}) \in \mathcal{P} \mathcal{D}_{[-1,0]}(X) \), it suffices to show that \( \text{ker}(1-t) \cong R\Psi_f^M(M)[-1] \). Since

\[
R\Psi_f^M(M \otimes f^* L^a)[-1] \cong R\Psi_f^M(M)[-1] \otimes L^a \cong \bigoplus_{i=1}^a R\Psi_f^M(M)[-1]_{(a)},
\]

where the \( i \)th coordinate of \( t \) is \( t_{(i)} + t_{(i+1)} \). Thus, if \( c_i \) is the \( i \)th coordinate (function) of \( \text{ker}(1-t) \), we have

\[
c_{i+1} = (1 - t_{(i)}^{-1}) c_i \text{, or by induction, } c_i = (1 - t_{(i+1)}^{-1}) c_{i+1} \text{ and } (1 - t_{(i+1)}^{-1}) c_{i+1} = 0.
\]

Since \( 1 - t \) is nilpotent on \( R\Psi_f^M(M), \) for a sufficiently large, this formula for \( c_i \) in fact defines \( \text{ker}(1-t) \) to be the isomorphic image of \( R\Psi_f^M(M)[-1] \), as desired.

Note that by rearranging some definitions (and doing the completely analogous proof for the cokernel), we could have given Theorem 2.9 right after Lemma 2.2 and proven the \( r = 0 \) cases of Proposition 2.6 and Proposition 2.8 as a side effect. However, we need the \( r = 1 \) case for the gluing constructions. Note also the corollary:

**Corollary 2.10.** Suppose \( (1-t)^N \) annihilates \( R\Psi_f^M(M) \). Then we have (and conversely, of course)

\[
R\Psi_f^M(M)[-1] = i_* j_{*}(M \otimes f^* L^N)[-1] = i_! j_{*}(M \otimes f^* L^N)[1].
\]

**Nearby cycles and the perverse t-structure**

One thing Beilinson does not at all address in his paper is the interaction of \( R\Psi_f^M \) with the entire perverse t-structure. However, this an easy consequence of two general lemmas. To this end, let \( T \) be a triangulated functor on a t-category \( C \) and let the core of \( C \) be the abelian category \( A \). Then we have that the objects of \( C \) are bounded above, meaning that \( C = \bigcup_{b \in \mathbb{Z}} C^{<b} \), and that \( C \) is nondegenerate, meaning that \( \bigcap_{b \in \mathbb{Z}} C^{\leq b} = 0 \). These assumptions hold for the perverse t-structure.

**Lemma 2.11.** Suppose \( T \) is right t-exact and that \( TA \subset A \); then \( T \) is t-exact.
Proof. We will show that $T$ commutes with all truncations. Suppose we have an object $x \in C^{\leq b}$, so that there is a distinguished triangle

$$\tau_{< b}x \rightarrow x \rightarrow \tau_{\geq b}x \rightarrow$$

where by definition, $\tau_{\geq b}x = H^b(x)[-b] \in A[-b]$. By hypothesis on $T$, we have $T(x) \in C^{\leq b}$, $T(\tau_{< b}x) \in C^{< b}$, and $T(H^b(x)[-b]) \in A[-b] \subset C^{\geq b}$. Since $T$ is triangulated, there is a triangle

$$T(\tau_{< b}x) \rightarrow T(x) \rightarrow T(H^b(x)[-b]) \rightarrow$$

and therefore, by uniqueness of the truncation triangle, it must be that $T(\tau_{< b} x) = \tau_{< b} T(x)$. This is under the hypothesis that $x \in C^{\leq b}$; since then $\tau_{< b} x \in C^{\leq b-1}$ and since $\tau_{< b-1} \tau_{< b} = \tau_{< b-1}$, we can apply truncations-by-one repeatedly and conclude that for all $n$, we have $\tau_{\leq n} T(x) = T(\tau_{\leq n} x)$.

Now suppose we have any $x$, and for any $n$ form the distinguished triangle

$$\tau_{< n} x \rightarrow x \rightarrow \tau_{\geq n} x \rightarrow$$

to which we apply $T$. Since $T(\tau_{< n} x) = \tau_{< n} T(x)$, the cone of the resulting triangle

$$\tau_{< n} T(x) \rightarrow T(x) \rightarrow T(\tau_{\geq n} x) \rightarrow$$

must be isomorphic to $\tau_{\geq n} T(x)$, by uniqueness of cones and the truncation triangle for $T(x)$. Thus, $\tau_{\geq n} T(x) = T(\tau_{\geq n} x)$. Since then $T$ commutes with all truncations, it is a fortiori $t$-exact.

Lemma 2.12. Suppose that $T$ is $t$-exact and that $T$ vanishes on $A$; then $T = 0$.

Proof. $T$ commutes with cohomology, so for any $x$ we have $H^n(T x) = TH^n(x) = 0$. Thus, by nondegeneracy, we must have $T(x) = 0$.

In the first lemma, take $T = i_* R\Psi_f^{tn}[-1]$; by Theorem 2.1, it satisfies the hypothesis of Lemma 2.11, and therefore is $t$-exact. By definition, the Verdier duality functor $D$ is (contravariantly) $t$-exact for the perverse $t$-structure, so taking $T = D \circ i_* R\Psi_f^{tn}[-1] - i_* R\Psi_f^{tn}[-1] \circ D$ in Lemma 2.12 we get $T = 0$ on all of $D(X)$ by Theorem 2.9. That is:

Theorem 2.13. The unipotent nearby cycles $R\Psi_f^{tn}$ commute with Verdier duality.

3 Vanishing cycles and gluing

Since $\Pi_f^0 \cong R\Psi_f^{tn}[-1]$, we will refer to it simply as $\Psi_f(M)$, and $\Pi_f^1$ as $\Xi_f$, what Beilinson calls the "maximal extension functor".

Proposition 3.1. There are two natural exact sequences exchanged by duality and $M \leftrightarrow D^!M$:

$$0 \rightarrow j!(M) \xrightarrow{\alpha_-} \Xi_f(M) \xrightarrow{\beta_-} \Psi_f(M) \rightarrow 0$$

$$0 \rightarrow \Psi_f(M) \xrightarrow{\beta_+} \Xi_f(M) \xrightarrow{\alpha_+} j_!(M) \rightarrow 0,$$

where $\alpha_+ \circ \alpha_- = \alpha$ and $\beta_- \circ \beta_+ = 1 - t$.

Proof. Since $\alpha^{a-r} = \alpha^{a+r} \circ j_!(g^{a,r})$ we have $\text{im}(\alpha^{a-r}) \subset \text{im}(\alpha^{a+r})$ and thus we obtain a natural surjection

$$\text{coker}(\alpha^{a-r}) \rightarrow \text{coker}(\alpha^{a+r}).$$

The kernel of this map is $\text{im}(\alpha^{a-r}) / \text{im}(\alpha^{a})$ for $a \gg 0$; this receives a natural surjection, via $\alpha^{a+r}$, from $\text{coker}(1-t)^r$ inside $j_!(M \otimes f^*L^{a+r})$, which is injective by Lemma 2.7. In particular, for $r = 1$ we get the first short exact sequence.

Likewise, since $\alpha^{a+r} = j_!(g^{a+r,-r}) \circ \alpha^{a+r}$, we have $\text{ker}(\alpha^{a+r}) \subset \text{ker}(\alpha^{a+r-r})$, where the quotient is $\text{ker}(\alpha^{a+r-r}) / \text{ker}(\alpha^{a+r})$ for $a \gg 0$; it maps, via $\alpha^{a+r}$, injectively to $\text{ker}(g^{a+r-r}) = \text{im}(1-t)^a$, and surjectively by Lemma 2.7. For $r = 1$ we get the second short exact sequence.
The duality claim follows from the fact that $\mathbb{D}(\alpha_{\mathcal{M}}^{a,r}) = \alpha_{\mathcal{D},\mathcal{M}}^{a+r,-r}$ and that $\mathbb{D}$ is contravariant and exact. Since $j^*\Psi_f = 0$ and $j^!(\alpha^a) = \text{id}$, we conclude that $j^*(\alpha_+ \circ \alpha_-) = \text{id}$ as well; by adjunction, $\alpha_+ \circ \alpha_- = \alpha$.

Observe that the constructions above take place in $j_!(\mathcal{M} \otimes F^aC^{a+1})$. Recall the isomorphism $\gamma^{a,r} = (1-t)^{a-2N}\beta^{a+r}$ of Proposition 2.8; what we must really prove is that

$$\beta_- \circ (\gamma^{a,1})^{-1} \circ \beta_+ = (1-t)(\gamma^{a+1,0})^{-1}.$$  

First we show that $\beta_+ \circ \gamma^{a+1,0} \circ \beta_- = (1-t)\gamma^{a,1}$, which is immediately obvious from the definition of $\gamma^{a,r}$ (note that the symbolic formula given above is not the isomorphism of $\operatorname{coker}(\alpha_{\mathcal{M}}^{a,r})$ with $\ker(\alpha_{\mathcal{D},\mathcal{M}}^{a+r,-r})$, but that this isomorphism is canonically induced by it. This induction is precisely the contribution of the $\beta_+$.

Therefore, multiplying on the left by $\beta_- \circ (\gamma^{a,1})^{-1}$, we get:

$$\beta_- \circ (\gamma^{a,1})^{-1} \circ \beta_+ \circ \gamma^{a+1,0} \circ \beta_- = \beta_- \circ (1-t) = (1-t) \circ \beta_-.$$  

Since $\beta_-$ is surjective, we may remove it from the right, and we are done after multiplying by $(\gamma^{a+1,0})^{-1}$. \qed

The remainder of the paper is simply what Beilinson calls “linear algebra”. Let $\mathcal{M} = j^*\mathcal{F}$ for a perverse sheaf $\mathcal{F} \in \mathcal{M}(X)$. From the maps in these two exact sequences we can form a complex:

$$j_!j^*(\mathcal{F}) \xrightarrow{(\alpha_- \circ \gamma_-)} \Xi j_!j^*(\mathcal{F}) \oplus \mathcal{F} \xrightarrow{(\alpha_+ \circ - \gamma_+)} j_!j^*(\mathcal{F}),$$  

where $\gamma_-: j_!j^*(\mathcal{F}) \to \mathcal{F}$ and $\gamma_+: \mathcal{M} \to j_!j^*(\mathcal{F})$ are defined by the left- and right-adjunctions ($j_!, j^!$) and $(j^*, j_*)$ and the property that $j^*(\gamma_-) = j^*(\gamma_+) = \text{id}$.

**Proposition 3.2.** The complex (3) is in fact a complex; let $\Phi_f(\mathcal{F})$ be its cohomology sheaf. Then $\Phi_f$ is an exact functor whose values are supported on $Z$ and there are maps $u, v$ such that $v \circ u = 1 - t$ as in the following diagram:

$$\Psi_f j^*(\mathcal{F}) \xrightarrow{u} \Phi_f(\mathcal{F}) \xrightarrow{v} \Psi_f j^*(\mathcal{F}).$$

**Proof:** That (3) is a complex amounts to showing that $\gamma_+ \circ \gamma_- = \alpha = \alpha_+ \circ \alpha_-$, which is true by definition of the $\gamma_{\pm}$ and adjunction. To show that it is exact, suppose we have $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$, so that we get a short exact sequence of complexes

$$0 \to C^*(\mathcal{F}_1) \to C^*(\mathcal{F}_2) \to C^*(\mathcal{F}_3) \to 0,$$

where by $C^*(\mathcal{F})$ we have denoted the complex (3) padded with zeroes on both sides. Note that since $\alpha_-$ is injective and $\alpha_+$ surjective, $C^*(\mathcal{F})$ fails to be exact only at the middle term. Therefore we have a long exact sequence of cohomology sheaves:

$$\cdots \to (0 = H^{-1}C^*(\mathcal{F}_3)) \to \Phi_f(\mathcal{F}_1) \to \Phi_f(\mathcal{F}_2) \to \Phi_f(\mathcal{F}_3) \to (0 = H^1C^*(\mathcal{F}_1)) \to \cdots$$

which shows that $\Phi_f$ is functorial and an exact functor.

If we apply $j^*$ to (3), it becomes simply

$$j^*\mathcal{F} \xrightarrow{[\text{id}, \text{id}]} j^*\mathcal{F} \oplus j^*\mathcal{F} \xrightarrow{[\text{id}, -\text{id}]} j^*\mathcal{F}$$

which is actually exact, so $j^*\Phi_f(\mathcal{F}) = 0$; i.e. $\Phi_f(\mathcal{F})$ is supported on $Z$. Finally, to define $u$ and $v$, let $\text{pr}: \Xi j_!j^*(\mathcal{F}) \oplus \mathcal{F} \to \Xi j_!j^*(\mathcal{F})$, and set $u = (\beta_+, 0)$ in coordinates, and $v = \beta_- \circ \text{pr}$. Then $v \circ u = \beta_- \circ \beta_+ = 1 - t$ by Proposition 3.1. \qed

Define a vanishing cycles gluing data for $f$ to be a quadruple $(\mathcal{F}_U, \mathcal{F}_Z, u, v)$ modeled on Proposition 3.2; for any $\mathcal{F} \in \mathcal{M}(X)$, we have $F_f(\mathcal{F}) = (j^*\mathcal{F}, \Phi_f(\mathcal{F}, u, v))$ as there. Let $\mathcal{M}_f(U, Z)$ be the category of such data, and $F_f: \mathcal{M}(X) \to \mathcal{M}_f(U, Z)$ is a functor. Conversely, given a vanishing cycles data, we can form the complex

$$\Psi_f(\mathcal{F}_U) \xrightarrow{(\beta_+, u)} \Xi_f(\mathcal{F}_U) \oplus \mathcal{F}_Z \xrightarrow{(\beta_-, v)} \Psi_f(\mathcal{F}_U)$$

since $v \circ u = 1 - t = \beta_- \circ \beta_+$, and let $G_f(\mathcal{F}_U, \mathcal{F}_Y, u, v)$ be its cohomology sheaf.

We could prove Theorem 3.6 directly, but some of the formalism in Beilinson’s paper is too elegant to cut. Therefore, we make two further definitions (which are a little different from the terminology he uses):
Definition 3.3. Let a \textit{diad} be a complex of perverse sheaves on \(X\) of the form
\[
D^\bullet = \left( F_L \xrightarrow{L = (a_L, b_L)} A \oplus B \xrightarrow{R = (a_R, b_R)} F_R \right)
\]
in which \(a_L\) is injective and \(a_R\) is surjective (so it is exact on the ends). Let the category of diads be denoted \(\mathbf{M}_2(X)\).

Let a \textit{triad} be a short exact sequence of the form
\[
S = \left( 0 \rightarrow F_- \xrightarrow{(c_-, d^1_-, d^2_-)} A \oplus B^1 \oplus B^2 \xrightarrow{(c_+, d^1_+, d^2_+)} F_+ \rightarrow 0 \right)
\]
in which both \((c_-, d^-) : F_- \rightarrow A \oplus B^2\) are injections and \((c_+, d^+) : A \oplus B^1 \rightarrow F_+\) are surjections. Let the category of triads be denoted \(\mathbf{M}_3(X)\); it has a natural \textit{reflection functor} \(r : \mathbf{M}_3(X) \rightarrow \mathbf{M}_3(X)\) which invokes the natural symmetry \(1 \leftrightarrow 2\), and is an involution.

We can define a map \(T : \mathbf{M}_2(X) \rightarrow \mathbf{M}_3(X)\) by setting
\[
T(D) = \left( \ker(R) \xrightarrow{(\iota_A, \iota_B, h)} A \oplus B \oplus H(D^\bullet) \xrightarrow{(\pi_A, \pi_B, \pi_h)} \coker(L) \right),
\]
where:
1. \(\iota = (\iota_A, \iota_B) : \ker(R) \rightarrow A \oplus B\) is the natural inclusion,
2. \(\pi = (\pi_A, \pi_B) : A \oplus B \rightarrow \coker(L)\) is the natural projection,
3. \(h : \ker(R) \rightarrow H(D^\bullet)\) is the natural projection onto \(H(D^\bullet) = \ker(R)/F_L\),
4. \(k : H(D^\bullet) \rightarrow \coker(L)\) is the natural inclusion of \(H(D^\bullet) = \ker(\coker(L) \rightarrow F_R)\).

These four maps are related: we have \(\pi \iota = kh\). Define the inverse \(T^{-1}\) by the formula
\[
T^{-1}(S) = \left( \ker(d^2_-) \xrightarrow{(c_-, d^1_-)} A \oplus B^1 \rightarrow \coker(c_-, d^1_-) \right).
\]

Lemma 3.4. The functors \(T, T^{-1}\) are mutually inverse equivalences of \(\mathbf{M}_2(X)\) with \(\mathbf{M}_3(X)\).

Proof. First we have to show that \(T(D^\bullet) \in \mathbf{M}_3(X)\) at all. The conditions that \((\iota_A, \iota_B) = \iota\) be injective and \((\pi_A, \pi_B) = \pi\) be surjective are true by definition. To show that \((\iota_A, h)\) is injective we do a diagram chase: if \(x \in \ker(R)\) and \(\iota_A(x) = h(x) = 0\), then \(x \in \text{im}(L)\); \(x = (a_L(y), b_L(y))\) with \(y \in F_L\), and \(a_L(y) = \iota_A(x) = 0\). But \(a_L\) is injective, so \(y = 0\) and hence \(x = 0\). Likewise, we show that \((\pi_A, -k)\) is surjective: suppose that \(x \in \coker(L)\), and pick a lift \((y_1, y_2) \in A \oplus B\), with \(\pi_A(y_1) + \pi_B(y_2) = x\). Since \(a_R\) is surjective, there is some \(z \in A\) such that \(a_R(z) = b_R(y_2)\), and therefore \((R)(-z, y_2) = 0\): \((-z, y_2) \in \ker(R)\). By definition, \(kh(-z, y_2) = \pi(-z, y_2) = \pi_B(y_2) - \pi_A(z)\), so \(x = \pi_A(y_1 + z) - kh(z, -y_2)\), as desired.

Now we have to check exactness in the middle. First of all, \(T(D^\bullet)\) is a complex, since \(\pi \circ \iota = k \circ h\). We prove exactness: suppose \((x, y) \in \ker(\pi, -k)\), so \(\pi(x) = k(y) \in \coker(L)\), with \(x \in A \oplus B\) and \(y \in H(D^\bullet)\). Pick \(z \in \ker(R)\) such that \(y = h(z)\), so \(\pi(x) = k(y) = \pi_L(z)\); thus, \(x = \iota(z) \in \ker(\pi) = \text{im}(L) \subset \ker(R)\) and thus \(x \in \ker(R)\), or more precisely, there is \(w \in \ker(R)\) such that \(x = \iota(w)\). Also, \(k(y) = \pi(x) = \pi_L(w) = kh(w)\), so since \(k\) is injective, \(y = h(w)\); thus, \((x, y) = (\iota, h)(w)\), as desired.

Likewise, we must check that \(T^{-1}(S) \in \mathbf{M}_2(X)\). That it is a complex is obvious, so we must show that \(c_-\) is injective on \(\ker(d^2_-)\) and that \(A\) alone surjects onto \(\coker(c_-, d^1_-)\). The former follows from the fact that \((c_-, d^2_-)\) is assumed to be injective. For the latter, let \((x_1, x_2) \in A \oplus B^1\); since \((c_+, d^+)\) is surjective, there is some \((y_1, y_2) \in A \oplus B^2\) such that \(c_+(y_1) + d^1_+(y_2) = d^1_+(x_2)\). That is, \((y_1, -x_2, y_2) \in \ker(c_+, d^1_+, d^2_+)\), and hence in \(\text{im}(c_-, d^1_-, d^2_-)\). In particular, \((y_1, -x_2) \in \text{im}(c_-, d^1_-)\), so \((x_1, x_2)\) and \((x_1 + y_1, 0)\) have the same image in \(\coker(c_-, d^1_-)\), as desired.

It is easy to check that \(T^{-1} \circ T \cong \text{id}\): its \(F_L\) is \(\ker(h) = \text{im}(L) = F_L\); its \(A\) and \(B\) are indeed \(A\) and \(B\), and its \(F_R\) is \(\coker(\iota) = F_R\); one checks quickly that the maps are right as well. Conversely, \(T \circ T^{-1} \cong \text{id}\):
its $\mathcal{F}_-$ is $\ker(A \oplus B^1 \to \operatorname{coker}(c_-, d^1_-)) = \mathcal{F}_-$ since $(c_-, d^1_-)$ is an injection; its $A$ and $B^1$ are obviously the original $A$ and $B^1$; we will deal at greater length with $B^2$ and $\mathcal{F}_+$. 

For $B^2$, we must show that $\mathcal{F}_- / \ker(d_2^2) = B^2$, or in other words, that $d_2^2$ is surjective. This is another diagram chase: if $x \in B^2$, then since $(c_+, d^1_+)$ is surjective there is some $(y_1, y_2) \in A \oplus B^1$ such that $c_+(y_1) + d^1_+(y_2) = d^2_+(x)$. Therefore $(y_1, y_2, -x) \in \ker(c_+, d^1_+, d^2_+)$, so in particular $-x = d^1_+(-z)$, or $x = d^1_+(z)$, for some $z \in \mathcal{F}_-$, as desired.

For $\mathcal{F}_+$, we must show that the smaller sequence

$$0 \rightarrow \ker(d_2^2) \xrightarrow{(c_-, d^1_-)} A \oplus B^1 \xrightarrow{(c_+, d^1_+)} \mathcal{F}_+ \rightarrow 0$$

is still exact. The hypotheses on $S$ already show that it is exact at the ends, so we deal only with the middle. First of all, it is a complex, since $c_+ c_- + d^1_+ d^2_+ = -d^2_+ d^2_+ = 0$ on $\ker(d_2^2)$. Now we do still another diagram chase: if $x \in \ker(c_+, d^1_+)$, then $(x, 0) \in \ker(c_+, d^1_+, d^2_+)$, so there is some $y \in \mathcal{F}_-$ such that $(-y, d^{-1}_-(y)) = x$ and $d^1_2(y) = 0$; i.e. $y \in \ker(d_2^2)$, as desired.

\begin{corollary}
Corollary 3.5. The reflection functor on a diad $T^{-1}rT(D^*)$ is the complex

$$r(D^*) = \left(\ker(a_R) \xrightarrow{(a'_L, b'_L)} A \oplus H(D^*) \xrightarrow{(a''_L, b''_L)} \operatorname{coker}(a_L)\right),$$

where $a'_L$ is the natural inclusion and $a''_L$ the natural projection, $b'_L = h \circ (a'_L, 0)$, and $b''_L$ factorizes $k$ through $\operatorname{coker}(a_L) \subset \operatorname{coker}(L)$.
\end{corollary}

\begin{proof}
The $d^2_2$ of $rT(D^*)$ is $\iota_B$, whose kernel (in $\ker(R)$) we show is $\ker(a_R)$. Indeed, the map $(\text{id}, 0)$ exhibits $\ker(a_R) \subset \ker(\iota_B) \cap \ker(R)$; conversely, if $(x_1, x_2) \in \ker(\iota_B) \cap \ker(R)$, then $x_2 = 0$ and $a_R(x_1) = 0$, so $x_1 \in \ker(a_R)$.

Likewise, the $d^1_2$ of $rT(D^*)$ is $h$, so we must show that $\operatorname{coker}((\iota_A, h)) = \operatorname{coker}(a_L)$. The map $(\text{id}, 0)$ from $A$ to $A \oplus H(D^*)$ induces a map $A \rightarrow \operatorname{coker}(\iota_A, h)$; to show that it descends to $\operatorname{coker}(a_L)$, we must show that for $x \in F_L$, $(a_L(x), 0) = ((\iota_A)(y), h(y))$ for some $y \in \ker(R)$. In particular, $y \in \operatorname{im}(L)$ since $h(y) = 0$, so we can take $y = (L)(x) = (a_L(x), b_L(x))$. Thus we have a map $\operatorname{coker}(a_L) \rightarrow \operatorname{coker}(\iota_A, h)$, which this argument also shows is injective. It is also surjective: if $(x_1, x_2) \in A \oplus H(D^*)$, then there is some $y \in \ker(R)$ with $h(y) = x_2$, so $(x_1 - \iota(y), 0)$ has the same image in $\operatorname{coker}(\iota_A, h)$.

The injectivity argument just given shows that $\operatorname{coker}(a_L) \subset \operatorname{coker}(L)$, so the description of $b''_L$ makes sense. Indeed, $k$ does factor through $\operatorname{coker}(a_L)$, as shown by the surjectivity argument above, and in fact the computation there shows that it coincides with the map $H(D^*) \rightarrow \operatorname{coker}(a_L)$. Similar considerations prove the identities of the other maps.
\end{proof}

\begin{theorem}
The gluing category $M_f(U, Z)$ is abelian, and $F_f : M(X) \rightarrow M_f(U, Z)$ and $G_f : M_f(U, Z) \rightarrow M(X)$ are mutually inverse exact functors, so $M_f(U, Z)$ is equivalent to $M(X)$.
\end{theorem}

\begin{proof}
That $M_f(U, Z)$ is abelian amounts to proving that taking coordinatewise kernels and cokernels works. That is, if we have $(\mathcal{F}_U, \mathcal{F}_Z, u, v)$ and $(\mathcal{F}'_U, \mathcal{F}'_Z, u', v')$ with maps $a_U : \mathcal{F}_U \rightarrow \mathcal{F}'_U$, $a_Z : \mathcal{F}_Z \rightarrow \mathcal{F}'_Z$ and such that the following diagram commutes:

$$\begin{array}{ccc}
\Psi_f(\mathcal{F}_U) & \xrightarrow{u} & \mathcal{F}_Z & \xrightarrow{v} & \Psi_f(\mathcal{F}_U) \\
\Psi_f(a_U) \downarrow & & \downarrow a_Z & & \downarrow \Psi_f(a_U) \\
\Psi_f(\mathcal{F}'_U) & \xrightarrow{u'} & \mathcal{F}'_Z & \xrightarrow{v'} & \Psi_f(\mathcal{F}'_U)
\end{array}$$

then $(\ker a_U, \ker a_Z, \bar{u}, \bar{v})$ is a kernel for $(a_U, a_V)$, where $\bar{u}$ and $\bar{v}$ are induced maps; likewise for the cokernel; and we must show that $(a_U, a_V)$ is an isomorphism if and only if the kernel and cokernel vanish. The maps $\bar{u}$ and $\bar{v}$ are constructed from the natural sequence of kernels (or cokernels) in the above diagram, and the exactness of $\Psi_f$, and once they exist it is obvious from the definition of morphisms in $M_f(U, Z)$ that the
desired gluing data is a kernel (resp. cokernel). Since $M(U)$ and $M(Z)$ are abelian and kernels and cokernels are taken coordinatewise, the last claim follows.

That $F_f$ is exact follows from the form taken by kernels and cokernels, and the exactness of $\Phi_f$; that $G_f$ is functorial and exact is exactly the same proof as that $\Phi_f$ is. All that remains is to prove that they are mutually inverse. We do this by interpreting both $M(X)$ and $M_f(U,Z)$ as diads in the form given, respectively, by diagrams (3) and (4). Then Corollary 3.5 shows that these two types are interchanged by the reflection functor, with the correspondence explicitly realizing the two functors $F_f$ and $G_f$ previously given.

References
