Overview and recap of Dustin’s talk on quantization

February 15, 2010

As throughout the last semester, let us begin by fixing a smooth projective curve $X$ of genus $g > 1$ over a field $k$, and let $G$ be a reductive group. Our discussion started from the classical Hitchin map:

$$T^*\text{Bun}_G \rightarrow \text{Hitch}(X) = \text{Sect}(X, C \times_{\mathbb{G}_m} \omega_X).$$

The actors here are $\text{Bun}_G$ (the moduli stack of principal $G$–bundles over $X$), $C = g^*///G$ (the affine quotient of $g^*$ with respect to the adjoint action of $G$) and $\omega_X$ (the sheaf of regular differentials on $X$). Passing to the level of rings of functions, we get a map:

$$\mathcal{z}^{cl}(X) := \mathcal{O}(\text{Hitch}(X)) \xrightarrow{h^{cl}} \Gamma(T^*\text{Bun}_G, \mathcal{O}).$$

(1)

The connected components of $\text{Bun}_G$ are $\text{Bun}_G^\gamma$, indexed by elements $\gamma \in \pi_1(G)$. In Andrei’s Oct 22 lecture, we proved the following:

**Proposition 1** The map $h^{cl}$ becomes an isomorphism when we restrict it to any connected component $\text{Bun}_G^\gamma \subset \text{Bun}_G$.

**Proposition 2** The algebra $\Gamma(T^*\text{Bun}_G, \mathcal{O})$ has trivial Poisson bracket.

Our main focus last semester was to quantize the map $h^{cl}$, i.e. to prove the following theorem:

**Theorem 1** There exists a filtered commutative algebra $\mathfrak{z}(X)$ such that $\text{gr } \mathfrak{z}(X) \cong \mathfrak{z}^{cl}(X)$, and a map:

$$\mathfrak{z}(X) \xrightarrow{h} \Gamma(\text{Bun}_G, D'),$$

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such that the vertical maps in the following diagram are isomorphisms, and the following diagram commutes:

\[
\begin{array}{ccc}
\text{gr } \mathfrak{z}(X) & \overset{\text{gr } h}{\longrightarrow} & \text{gr } \Gamma(\text{Bun}_G, D') \\
\approx & & \downarrow \\
\mathfrak{z}^{cl}(X) & \overset{h^{cl}}{\longrightarrow} & \Gamma(T^*\text{Bun}_G, \mathcal{O})
\end{array}
\]

In the above, \( D' \) denotes the sheaf appropriately twisted differential operators on the stack \( \text{Bun}_G \).

Of course, one can restrict the above to any connexted component \( \text{Bun}^\gamma_G \):

\[
h^\gamma : \mathfrak{z}(X) \overset{h}{\longrightarrow} \Gamma(\text{Bun}_G, D'_\text{Bun}_G) \overset{\text{rest}}{\longrightarrow} \Gamma(\text{Bun}^\gamma_G, D'),
\]

and:

\[
\begin{array}{ccc}
\text{gr } h^\gamma : \text{gr } \mathfrak{z}(X) & \overset{\text{gr } h}{\longrightarrow} & \text{gr } \Gamma(\text{Bun}_G, D') \\
\approx & & \downarrow \\
\text{gr } h^{cl, \gamma} : \mathfrak{z}^{cl}(X) & \overset{h^{cl}}{\longrightarrow} & \Gamma(T^*\text{Bun}_G, \mathcal{O}) \overset{\text{rest}}{\longrightarrow} \Gamma(T^*\text{Bun}^\gamma_G, \mathcal{O})
\end{array}
\]

Then we have the following corollaries:

**Corollary 1** The morphism \( \text{gr } h^\gamma \) is an isomorphism.

**Corollary 2** The morphism \( h^\gamma \) is a filtered isomorphism (the quantization of Proposition 1).

**Corollary 3** The algebra \( \Gamma(\text{Bun}_G, D') \) is commutative (the quantization of Proposition 2).

**Corollary 4** The vertical morphism on the right in (2), while a priori just injective, is actually an isomorphism.

The Theorem was ultimately proved in Dustin’s Dec 3 talk, and today we will review both the construction of \( \mathfrak{z}(X) \) and the proof of the theorem. First, we will recall how we proved the “classical” Propositions 1 and 2, via
the local-to-global principle.

Take any closed point \( x \in X \), and consider the ind-scheme \( \text{Bun}_G^{\infty,x} \) of principal \( G \)-bundles on \( X \) with level structure at \( x \) (i.e. with a fixed trivialization on the formal neighborhood \( \text{Spec} \, O_x \)). The group ind-scheme \( G(K_x) \) acts on \( \text{Bun}_G^{\infty,x} \) by changing the trivialization.

Whenever we have an action of a group scheme \( H \) on a stack \( \mathcal{Y} \), this induces an “infinitesimal action” \( \mathfrak{h} = \text{Lie} \, H \to \text{Vect}(\mathcal{Y}) \). Taking the dual of this, we get a “moment map” \( T^* \mathcal{Y} \to \mathfrak{h}^* \). In our case, this construction provides a map:

\[
T^* \text{Bun}_G^{\infty,x} \to (\mathfrak{g} \otimes K_x)^* \cong \mathfrak{g}^* \otimes \omega_{K_x}. \tag{3}
\]

On the rings of functions, this corresponds to a map:

\[
\overline{\text{Sym}(\mathfrak{g} \otimes K_x)} \xrightarrow{h^\text{cl}_x} \Gamma(T^* \text{Bun}_G^{\infty,x}, O). \tag{4}
\]

Modding out by the \( G(O_x) \) action means forgetting the trivialization, and therefore \( \text{Bun}_G^{\infty,x}/G(O_x) = \text{Bun}_G \). This implies that the subscheme:

\[
T^* \text{Bun}_G \times_{\text{Bun}_G} \text{Bun}_G^{\infty,x} \hookrightarrow T^* \text{Bun}_G^{\infty,x}
\]

consists of cotangent vectors that are killed by the \( G(O_x) \)-action. Therefore, the restriction of (3) gives:

\[
T^* \text{Bun}_G \times_{\text{Bun}_G} \text{Bun}_G^{\infty,x} \to (\mathfrak{g} \otimes O_x)^{\perp} \cong (\mathfrak{g} \otimes K_x/O_x)^* \cong \mathfrak{g}^* \otimes \omega_{O_x}. \tag{5}
\]

Passing to rings of functions, we get:

\[
\overline{\text{Sym}(\mathfrak{g} \otimes K_x/O_x)} \xrightarrow{h^\text{cl}_x} \Gamma(T^* \text{Bun}_G \times_{\text{Bun}_G} \text{Bun}_G^{\infty,x}, O). \tag{6}
\]

Now we take \( G(O_x) \)-invariants in the above, which corresponds to the following map on spaces:

\[
T^* \text{Bun}_G \to (\mathfrak{g} \otimes K_x/O_x)^*/G(O_x) \to \text{Sect}(\text{Spec} \, O_x, C \times_{\mathbb{G}_m} \omega_{O_x}) =: \text{Hitch}_x. \tag{7}
\]
The second map was proved to be an isomorphism in the lectures. Then, the above gives rise to the following morphism on rings:

\[ z^{cl} := \text{Sym}(g \otimes K_x/O_x)^{G(O_x)} \xrightarrow{h^{cl}_x} \Gamma(T^*Bun_G, \mathcal{O}). \tag{8} \]

The map (7) is called the **local Hitchin map**. The natural inclusion Hitch\((X) \hookrightarrow \text{Hitch}_x\) has the property that the following composition is precisely the local Hitchin map:

\[ T^*Bun_G \longrightarrow \text{Hitch}(X) \hookrightarrow \text{Hitch}_x. \]

At the level of functions, we just reverse all arrows:

\[ h^{cl}_x : z^{cl}_x \to z^{cl}(X) \xrightarrow{h^{cl}_x} \Gamma(T^*Bun_G, \mathcal{O}). \tag{9} \]

As \(x\) varies, the local Hitchin maps can be “glued” together, by means of the \(D_X\)-scheme:

\[
\begin{array}{c}
\text{Hitch} \\ \downarrow \\
\text{Jets}(C \times_{G_m} \omega_X)
\end{array}
\]

The fiber of Hitch over \(x\) is just the local Hitch\(_x\), while the scheme of all horizontal sections HorSect\((X, \text{Hitch})\) coincides with the global Hitch\((X)\). We will write \(z^{cl} = \mathcal{O}(\text{Hitch})\), and then the compositions (9) patch up over all \(x\) to give a global morphism:

\[ h^{cl}_{gl} : z^{cl} \to z^{cl}(X) \otimes \mathcal{O}_X \xrightarrow{h^{cl}_x} \Gamma(T^*Bun_G, \mathcal{O}) \otimes \mathcal{O}_X. \tag{10} \]

The above composition merely reflects the properties of conformal blocks: recall that for a \(D_X\)-algebra \(B\), there exists an algebra \(H_\nabla(X, B)\) of **conformal blocks** and a horizontal morphism:

\[ \phi_B : B \to H_\nabla(X, B) \otimes \mathcal{O}_X, \]

which is universal in the following sense: any horizontal surjection \(B \to B \otimes \mathcal{O}_X\) factors through \(\phi_B\). In other words, the functor \(H_\nabla(X, \cdot)\) is left adjoint to the functor \(\cdot \otimes \mathcal{O}_X\). Last semester, we proved the following:
Lemma 1 The map $\mathfrak{z}^{cl} \to \mathfrak{z}^{cl}(X) \otimes \mathcal{O}_X$ of (10) is horizontal, and

$$H_{\nabla}(X, \mathfrak{z}^{cl}) = \mathfrak{z}^{cl}(X).$$

Therefore, (10) merely reflects the left-adjointness of the functor $H_{\nabla}$.

Let us present the general strategy for quantizing the above discussion (as in Sam’s third lecture), emphasizing the places where we run into trouble. Back up to the group $G(K_x)$ acting on $\text{Bun}_G^{\infty,x}$. From this, we get an infinitesimal action:

$$g \otimes \mathcal{K}_x \longrightarrow \Gamma(\text{Bun}_G^{\infty,x}, \text{Vect}) \hookrightarrow \Gamma(\text{Bun}_G^{\infty,x}, D_{\text{Bun}_G^{\infty,x}}),$$

where $D_{\text{Bun}_G^{\infty,x}}$ denotes the sheaf of differential operators on $\text{Bun}_G^{\infty,x}$. Since $D_{\text{Bun}_G^{\infty,x}}$ is a sheaf of algebras, we get a map:

$$U(\mathfrak{g} \otimes \mathcal{K}_x) \xrightarrow{\sim} \Gamma(\text{Bun}_G^{\infty,x}, D_{\text{Bun}_G^{\infty,x}}).$$

(11)

Modding out by the $G(\mathcal{O}_x)$ vector fields gives us a map:

$$\mathbb{V}_x := U(\mathfrak{g} \otimes \mathcal{K}_x) \boxtimes \mathbb{C} \xrightarrow{\sim} \Gamma(\text{Bun}_G^{\infty,x}, \pi^* D_{\text{Bun}_G^{\infty,x}}),$$

(12)

where $\pi : \text{Bun}_G^{\infty,x} \longrightarrow \text{Bun}_G$ is just the map that quotients out the $G(\mathcal{O}_x)$ action. Here, $\mathbb{V}_x$ denotes the vacuum module, defined by the property:

$$\text{Hom}_{\mathfrak{g} \otimes \mathcal{K}_x}(\mathbb{V}_x, M) \cong M^{G(\mathcal{O}_x)}.$$

Therefore, take $G(\mathcal{O}_x)$–invariants in (12):

$$\mathbb{V}_x^{G(\mathcal{O}_x)} \xrightarrow{h_x} \Gamma(\text{Bun}_G, D_{\text{Bun}_G}).$$

(13)

One would like this map to be the quantization of (8), but alas! It turns out that both the left and the right hand side of (13) are trivial: they are equal to $\mathbb{C}$. To get some non-trivial objects, we must twist both $\mathbb{V}_x$ and $D_{\text{Bun}_G}$, as in Dustin’s first talk. Let’s describe how this works.

Take the canonical line bundle $\mathcal{L}_{\det}$ of $\text{Bun}_G$, whose fiber over a principal $G$–bundle $P_G$ is canonically:
\[ \mathcal{L}_{\text{det}}|_{P_G} \cong \text{det}(R\Gamma(X, \mathfrak{g}_P)). \]

On the representation-theoretic side, take the central extension:

\[ 1 \to \mathbb{G}_m \to \hat{G}(\mathcal{K}_x) \to G(\mathcal{K}_x) \to 1. \]

The line bundle \( \pi^*\mathcal{L}_{\text{det}} \) on \( \text{Bun}_G^{\infty,x} \) is not \( G(\mathcal{K}_x) \)-equivariant, but it is \( \hat{G}(\mathcal{K}_x) \)-equivariant, where the central \( \mathbb{G}_m \) acts fiberwise by homotheties.

Taking Lie algebras, we obtain a map:

\[ \mathfrak{g} \otimes \hat{\mathcal{K}}_x \to \Gamma(\text{Bun}_G^{\infty,x}, D(\pi^*\mathcal{L}_{\text{det}}, \pi^*\mathcal{L}_{\text{det}})), \quad (14) \]

But this is not exactly what we need. In Sam’s talk, we showed how to define the sheaf \( D(\mathcal{L}_{\text{det}}^1, \mathcal{L}_{\text{det}}^1) \) for any complex number \( \lambda \). It is called the algebra of \textbf{twisted differential operators}. We will use \( \lambda = \frac{1}{2} \), so define:

\[ D_{\text{crit}}^{\text{Bun}_G} := D(\mathcal{L}_{\text{det}}^1, \mathcal{L}_{\text{det}}^1) \]

Together with this, we also define the Kac-Moody extension \( \mathfrak{g}^{\text{crit}} \) to be “half” of the extension \( \mathfrak{g} \otimes \hat{\mathcal{K}}_x \), i.e. constructed using \( \frac{1}{2} \) times the Killing form. As in (14), we obtain a map:

\[ \mathbb{U}(\mathfrak{g}^{\text{crit}}) \overset{\sim}{\longrightarrow} \Gamma(\text{Bun}_G^{\infty,x}, D(\pi^*\mathcal{L}_{\text{det}}^1, \pi^*\mathcal{L}_{\text{det}}^1)). \]

This is the correct twist of the map (11). Now it’s time to go through the usual story: mod out by the \( G(\mathcal{O}_x) \) directions:

\[ V^{\text{crit}}_x := \mathbb{U}(\mathfrak{g}^{\text{crit}}) \otimes_{\mathbb{U}(\mathfrak{g} \otimes \mathcal{O}_x \oplus \mathbb{C})} \mathbb{C} \overset{h_x}{\longrightarrow} \Gamma(\text{Bun}_G^{\infty,x}, \pi^* D_{\text{Bun}_G}^{\text{crit}}). \]

The critical twisted vacuum \( \mathfrak{g}^{\text{crit}} \)-module \( V^{\text{crit}}_x \) is defined by the property:

\[ \text{Hom}_{\mathfrak{g}^{\text{crit}}}(V^{\text{crit}}_x, M) \cong M^{G(\mathcal{O}_x)}. \]

Therefore taking \( G(\mathcal{O}_x) \)-invariants, we obtain:

\[ \mathfrak{z}_x := \text{End}_{\mathfrak{g}^{\text{crit}}}(V^{\text{crit}}_x) = (V^{\text{crit}}_x)^{G(\mathcal{O}_x)} \overset{h_x}{\longrightarrow} \Gamma(\text{Bun}_G, D_{\text{Bun}_G}^{\text{crit}}). \quad (15) \]

This is the correct quantization of the map (8). As in the classical case, these maps can be glued as \( x \) ranges over \( X \). Namely, there exists a \textbf{commutative}
$D_X$–algebra $\mathfrak{g}$ whose fiber over $x \in X$ is just $\mathfrak{g}_x$ defined above. Moreover, the morphisms (15) glue and give rise to a morphism:

$$\mathfrak{g} \xrightarrow{h_{\text{gl}}} \Gamma(\text{Bun}_G, D_{\text{Bun}_G}^{\text{crit}}) \otimes \mathcal{O}_X.$$  \hfill (16)

We claim (and will later argue) that this morphism is horizontal. Therefore, we are led to define:

$$\mathfrak{g}(X) = H^\nabla(X, \mathfrak{g}),$$

which is the correct quantization of the Poisson algebra $\mathfrak{g}^{\text{cl}}(X)$ of (1). From the left-adjointness of $H^\nabla$ and the horizontality of the map $h$, we deduce the existence of an algebra morphism:

$$\mathfrak{g}(X) \xrightarrow{h} \Gamma(\text{Bun}_G, D_{\text{Bun}_G}^{\text{crit}}),$$  \hfill (17)

which is the correct quantization of the map $h^{\text{cl}}$ from (1), as stated in Theorem 1. Now let us try to justify the claim we just made: why is the morphism $h_{\text{gl}}$ from (16) horizontal? This can be sketched in several sentences:

1. The assignment $x \mapsto V \otimes \mathcal{K}_x$ defines a crystal of l.l.c.v.s over $X$, for any finite-dimensional vector space $V$.

2. The assignment $x \mapsto \mathfrak{g} \otimes \mathcal{K}_x$ defines a crystal of Lie algebras of l.l.c.v.s over $X$.

3. The assignment $x \mapsto \mathcal{V}_x^{\text{crit}}$ defines a crystal of $\mathfrak{g} \otimes \mathcal{K}_x$ modules over $X$.

4. The assignment $x \mapsto \text{End}_{\mathfrak{g}^{\text{crit}}}(\mathcal{V}_x^{\text{crit}}) = \mathfrak{g}_x$ defines a crystal of associative algebras over $X$. In particular, Jacob’s talk on crystals implies the existence of the $D_X$–algebra $\mathfrak{g}$.

5. The assignment $x \mapsto \text{Bun}^\infty_{G,x}$ defines a crystal of schemes over $X$.

6. The assignment $x \mapsto G(\mathcal{K}_x)$ defines a crystal of group ind-schemes over $X$, and its action on $\text{Bun}^\infty_{G,x}$ is compatible with the crystal structure.

7. The assignment $x \mapsto \hat{G}_x^{\text{crit}}$ defines a crystal of group ind-schemes over $X$, and its action on $\pi_x^* \mathcal{L}_{\text{det}}$ is compatible with the crystal structure.
8. The maps $\tilde{h}_x, \widetilde{h}_x, h_x$ are compatible with the crystal structure. In other words, the morphism (16) is horizontal.

9. Finally, the filtration on the vacuum modules $\mathcal{V}^\text{crit}_x$ and the filtration on the algebras $\mathfrak{z}_x$ are compatible with the crystal structure. Therefore, we obtain a filtration on the $D_X$–algebra $\mathfrak{z}$ and on its algebra of conformal blocks $\mathfrak{z}(X)$.

The canonical injections $\operatorname{gr} \mathfrak{z}_x \hookrightarrow \mathfrak{z}_x^{cl}$ are also compatible with the crystal structure, so they induce an injection $\operatorname{gr} \mathfrak{z} \hookrightarrow \mathfrak{z}^{cl}$. It was proved by Feigin and Frenkel that this injection is actually an isomorphism:

$$\operatorname{gr} \mathfrak{z} \cong \mathfrak{z}^{cl} \Rightarrow H_{\nabla}(X, \operatorname{gr} \mathfrak{z}) \cong H_{\nabla}(X, \mathfrak{z}^{cl}).$$

Moreover, the canonical morphism $\mathfrak{z} \rightarrow H_{\nabla}(X, \mathfrak{z}) \otimes \mathcal{O}_X$ induces a surjection:

$$\operatorname{gr} \mathfrak{z} \twoheadrightarrow H_{\nabla}(X, \mathfrak{z}) \otimes \mathcal{O}_X = \operatorname{gr} \mathfrak{z}(X) \otimes \mathcal{O}_X.$$

By the left-adjointness of conformal blocks, this yields a surjection:

$$H_{\nabla}(X, \operatorname{gr} \mathfrak{z}) \twoheadrightarrow \operatorname{gr} \mathfrak{z}(X).$$

So let’s see where we stand: the map (17) induces the commutative diagram:

$$\begin{array}{ccc}
\operatorname{gr} \mathfrak{z}(X) & \xrightarrow{\operatorname{gr} h} & \operatorname{gr}(\text{Bun}_G, D^\text{crit}_{\text{Bun}_G}) \\
\downarrow^a & & \downarrow^b \\
H_{\nabla}(X, \operatorname{gr} \mathfrak{z}) & \cong & \Gamma(T^*\text{Bun}_G, \mathcal{O}) \\
\downarrow & & \uparrow^{h^{cl}(X)} \\
H_{\nabla}(X, \mathfrak{z}^{cl}) & & \mathfrak{z}^{cl}(X)
\end{array}$$

As we previously said, $a$ is surjective and $b$ is injective, while the map $h^{cl}(X)$ is an injection (it becomes an isomorphism only when we restrict to a connected component). Therefore, we deduce that $a$ must be injective, and thus an isomorphism. This proves Theorem 1.