MATH 233B, FLATNESS AND SMOOTHNESS.

The discussion of smooth morphisms is one place were Hartshorne doesn't do a very good job. Here's a summary of this week's material. I'll also insert some (optional) exercises that I recommend that you have do in order to better understand the material.

1. A refresher on flatness

1.1. Let A be a commutative ring. Recall that an A-module M is flat if and only if the functor $M \underset{A}{\otimes} -$ is exact.

Exercise 1.1.1. (1) *Show that a module is flat if and only if its localization at any prime is flat.*
(2) *Show that M is flat if and only if* $\mathrm{Tor}_1(M, A/I) = 0$ *for any ideal $I \subset A$.*

Exercise 1.1.2. *Assume that A is Noetherian.*
(1) *Show that a module M is flat if and only if* $\mathrm{Tor}_1(M, A/\mathfrak{p}) = 0$ *for any prime $\mathfrak{p}$.*
(2) *Show that a module M is flat if and only if* $\mathrm{Tor}_i(M, k_{\mathfrak{p}}) = 0$ *for any prime $\mathfrak{p}$ and $i \in \mathbb{N}$.*
(3) *Assume in addition that M is f.g. over A. Show that in* (2) *it's enough to check only the maximal ideals and* Tor_1.
(4) *Show that the f.g. assumption above is necessary.*

1.2. Flat morphisms.

Definition 1.2.1. *We say that a morphism $f : X \to Y$ is flat if, locally on X, the structure sheaf $\mathcal{O}_X$ is flat as an $\mathcal{O}_Y$-module.*

Definition 1.2.2. *We say that a morphism $f : X \to Y$ is flat at x, if the local ring $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$.*

Easy exercise 1.2.3. *A morphism is flat if and only if it is flat at any point.*

More generally, we can define the notion of Y-flatness (resp., Y-flatness at a point x), a.k.a. flatness with respect to f (resp., flatness with respect to f at a point x) for $\mathcal{F} \in \mathrm{QCoh}(X)$.

Definition 1.2.4. *A map is faithfully flat if it is flat and surjective.*

Exercise 1.2.5.
(1) *Show that a map f if faithfully flat if and only if the functor f^* is exact and conservative.*
(2) *Deduce that the notion of faithful flatness is stable under base change.*
(3) *Show that a flat map is open.*

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1.3. Here are some basic facts about flatness in the Noetherian situation:

Exercise 1.3.1. *Assume that Y is Noetherian, and $\mathcal{F} \in \mathrm{QCoh}(X)$.*

(1) *Show that $\mathcal{F}$ is Y-flat if and only if, $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{F}, k_y)$ vanishes (as a q.c. sheaf on X) for all $y \in Y$ and $i \in \mathbb{N}$.*

(2) *Give a (stupid) counterexample of how the above fails if we only check the closed points.*

(3*) *Assume that both X and Y of finite type over a field, and assume that $\mathcal{F}$ is coherent on X. Show that in this case, it's enough to check only the closed points and Tor_1.*

Here's an important generic flatness theorem:

Proposition 1.3.2. *Let $f : X \to Y$ be a morphism of finite type with Y integral and Noetherian. Let $\mathcal{F}$ be a coherent sheaf of X. Then there exists a non-empty open subscheme $\overset{\circ}{Y} \subset Y$, such that $\mathcal{F}|_{f^{-1}(\overset{\circ}{Y})}$ is Y-flat.*

For the proof, google "generic flatness fantechi" and go to the first link (Gothendieck's FGA is explained).

Here is a relationship between the notions of flatness and flatness at a point:

Exercise 1.3.3. *Let $f : X \to Y$ be a morphism of finite type with Y Noetherian. Let $\mathcal{F}$ be a coherent sheaf on X. Let $y \in Y$ be a point such that for every $x \in X$ with $f(x) = y$, the sheaf $\mathcal{F}$ is Y-flat at x. Show that y admits a Zariski neighborhood over which $\mathcal{F}$ is flat.*

(This is a non-trivial exercise, and the proof that I know uses generic flatness.)

1.4. Here is a basic fact about the interaction between the notions of flatness and dimension.

If X is a Noetherian scheme and $x \in X$ is a point, we denote by $\dim(X)_x$ the "dimension of X at x", i.e., the dimension of the Noetherian local ring $\mathcal{O}_{X,x}$.

Proposition 1.4.1. *Let $f : X \to Y$ be a morphism between Noetherian schemes. Let $x \in X$ be a point and $y = f(x)$ its image in Y. Let X_y be the fiber of X over y, i.e., $X \underset{Y}{\times} k_y$.*

(1) *We have the inequality: $\dim(X_y)_x + \dim(Y)_y \geq \dim(X)_x$.*

(2) *If f is flat at x, then the above inequality is an equality.*

(3) *If Y is regular at y and X is CM at x, then the assertion of (2) is "if and only if".*

1.5. Let $X \to Y$ be a faithfully flat map with both X and Y locally Noetherian. Let's see how some favorable properties of X imply those for Y.

Proposition 1.5.1. *If X is (i) reduced, (ii) integral, (iii) regular, (iv) R_n, (v) S_n, (vi) locally factorial, then the same is true for Y.*

Proof. The proof of (i) and (ii) is easy, while (iii) was done in class. To prove (iv) let's localize Y at a point y of height $m \leq n$. Consider the scheme X_y, and let x be one of its generic points. By Proposition 1.4.1, $\dim(X)_x = \dim(Y)_y$, so X is regular at x by assumption. Hence, Y is regular at y by (iv).

Point (v) was done in class, but let's repeat it nonetheless. The assertion is local, so we can assume that both X and Y are affine. Property S_n means that for $\mathcal{F} \in \mathrm{Coh}(Y)$ with $\mathrm{codim}(supp_Y(\mathcal{F})) \geq n$, we have $\mathrm{Ext}^k(\mathcal{F}, \mathcal{O}_Y) = 0$ for $k \leq n-1$. By faithful flatness, it's enough to show that $f^*(\mathrm{Ext}^k_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y)) = 0$, while the latter, by flatness and the fact that $\mathcal{F}$ is coherent, is isomorphic to $\mathrm{Ext}^k_{\mathcal{O}_X}(f^*(\mathcal{F}), \mathcal{O}_X)$. By Proposition 1.4.1, $\mathrm{codim}(supp_X(f^*(\mathcal{F}))) = \mathrm{codim}(supp_Y(\mathcal{F}))$, so the latter group vanishes since X was S_n.

Let's prove (vi). Let $j : \overset{\circ}{Y} \hookrightarrow Y$ be an open subset whose complement is of codim ≥ 2. Let $\mathcal{L}$ be a line bundle over $\overset{\circ}{Y}$. We need to show that it admits an extension to a line bundle on the entire Y. We claim that $j_*(\mathcal{L})$ does the job. Indeed, since f is faithfully flat, it's enough to show that $f^*(j_*(\mathcal{L}))$ is a line bundle on X. Since f is flat, by the trivial case of the projection formula, we have: $f^*(j_*(\mathcal{L})) \simeq \tilde{j}_*(f^*(\mathcal{L}))$, where $\tilde{j} : \overset{\circ}{X} \hookrightarrow X$, where $\overset{\circ}{X} := f^{-1}(\overset{\circ}{Y})$. However, by Proposition 1.4.1, $\mathrm{codim}(X - \overset{\circ}{X}) = \mathrm{codim}(Y - \overset{\circ}{Y})$, so we know that $f^*(\mathcal{L})$ admits an extension to a line bundle $\mathcal{L}'$ on X. Since X is normal, we also know that this extension must coincide with $\tilde{j}_*(f^*(\mathcal{L}))$. □

2. Smoothness over a field

2.1. Before we proceed, let's give a refresher on the behavior of dimension for schemes of finite type over a field.

Here are two important facts:

Proposition 2.1.1. *Let X be a scheme of finite type over a field k. Then:*

(i) If X irreducible, then $\dim(X)_{x_1} = \dim(X)_{x_2}$ for any two closed points $x_1, x_2 \in X$.

(ii) If X is integral, then $\dim(X)$ equals $\mathrm{tr.deg.}(K(X)/k)$, where $K(X)$ is the field of fractions of X.

2.2. Let first k be an algebraically closed field, and X a scheme of finite type over k.

Definition 2.2.1. *We say that X is smooth of dimension n if it is regular as a scheme and has dimension n.*

Lemma 2.2.2. *Assume that every irreducible component of X is of dimension $\geq n$. Then the following conditions are equivalent:*

(i) *X is smooth of dimension n.*

(ii) *$\Omega^1_{X/k}$ is a locally free sheaf of* $\dim n$.

The proof follows the fact that for any closed point $x \in X$, the natural map

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \to (\Omega^1_{X/k})_x$$

is an isomorphism. □

2.3. A digression on the latter map:

Exercise 2.3.1. *Let A be a local ring over a field k, such that k_A, the residue field of A is a finite extension of k.*

(i) *Construct the natural map $\mathfrak{m}/\mathfrak{m}^2 \to (\Omega^1_{A/k}) \underset{A}{\otimes} k_A$.*

(ii) *Give an example of how this map fails to be an isomorphism even when A is regular.*

(iii) *Prove (by induction) that if k_A/k is perfect, then for any n the projection $A/\mathfrak{m}^n \to k_A$ admits a unique section (i.e., this makes $A/\mathfrak{m}^n$ into a k_A-algebra).*

(iv) *Deduce that if k_A/k is perfect, then the map in (i) is an isomorphism.*

2.4. Let k be an arbitrary field. Let X be a scheme of finite type over X.

Definition 2.4.1. *We say that X is smooth of dimension n over k if $X \underset{k}{\times} \overline{k}$ is smooth of dimension n over $\overline{k}$.*

From Lemma 2.2.2 we obtain:

Corollary 2.4.2. *A scheme X is smooth of dimension n over k if and only if each of its irreducible components has dimension $\geq n$, and $\Omega_{X/k}$ is locally free of rank n.*

As we saw in class, an imperfect field extension k'/k provides an example of a scheme over k, which is regular, but not smooth.

Exercise 2.4.3. *Let X be a scheme over k.*

(i) *Show that any scheme smooth over a field is regular.*

(ii) *Let k'/k be a separable field extension. Show that if X is regular, then so is $X' := X \underset{k}{\times} k'$.*

(iii) *Show that over a perfect field, a scheme is smooth if and only if it's regular.*

2.5. Let's recall the following assertion about regular local rings:

Proposition 2.5.1. *Let $A' \to A$ be a surjection of local rings with A' regular of dimension m. Then the following conditions are equivalent:*

(a) *A is regular of dimension n.*

(b) *The ideal $\ker(A' \to A)$ can be generated by $m-n$ elements $f_1, ..., f_{m-n}$, whose images in $\mathfrak{m}'/\mathfrak{m}'^2$ are linearly independent.*

Proof. Exercise. □

We shall now discuss an analog of this assertion when "regular" is replaced by "smooth".

Theorem 2.5.2. *Let X be a scheme of finite type over k. Then the following conditions are equivalent:*

(a) *X is smooth over k.*

(b) *For any closed embedding $X \hookrightarrow X'$ with with a sheaf of ideals $\mathfrak{I}$ and X' smooth, the sequence*

$$\mathfrak{I}/\mathfrak{I}^2 \to \Omega_{X'/k}|_X \to \Omega_{X/k} \to 0$$

is a short exact sequence of vector bundles.

Proof. The impication (b) $\Rightarrow$ (a) follows from Lemma 2.4.2. For the implication in the other direction, we can base change to $\overline{k}$. In the latter case we deduce it from Proposition 2.5.1 using the following lemma:

Lemma 2.5.3. *Let $\alpha : \mathcal{F} \to \mathcal{E}$ be a map of coherent sheaves on a locally Noetherian scheme X, where $\mathcal{E}$ is locally free. Assume that for every $x \in X$, the map $\mathcal{F}_x \to \mathcal{E}_x$ is injective. Then: (i) the map α is injective, (ii) $\mathcal{F}$ is locally free, (iii)* $\operatorname{coker}(\alpha)$ *is locally free. Moreover, the above condition is sufficient to check for closed points only.*

□

Note the difference in the conditions of Proposition 2.5.1 and Theorem 2.5.2: for a closed point $x \in X$, the former requires that we can generate

$$\mathcal{I}_x := \ker(\mathcal{O}_{X',x} \to \mathcal{O}_{X,x})$$

by elements $f_1, ..., f_{n-m}$ such that their images $\overline{f}_1, ..., \overline{f}_{m-n}$ in $\mathfrak{m}'/\mathfrak{m}'^2$ are linearly independent. The latter requires that their further images $df_i(x) \in (\Omega_{X'/x})_x$ be linearly independent.

2.6. Generic smoothness.

Definition 2.6.1. *We say that X is generically smooth of dimension n over k if X contains a dense Zariski open $\overset{\circ}{X}$, which is smooth of dimension n over k.*

Lemma 2.6.2. *Let X be an irreducible scheme of finite type over k.*

(1) *The generic rank of $\Omega_{X/k}$ is $\geq \dim(X)$.*

(2) *The equality in* (1) *holds if and only if X is generically smooth.*

The proof follows from Lemma 2.2.2 by base change. □

Definition 2.6.3. *A finitely generated field extension K/k is said to be separably generated if it can be written in the form $K \subset K_0 = k(x_1, ..., x_n)$, where K/K_0 is a finite separable extension.*

Lemma 2.6.4. *Let K/k be a finitely generated field extension.*

(1) *We have* $\dim_K(\Omega_{K/k}) \geq \operatorname{tr.deg.}(K/k)$.

(2) *If K/k is separably generated, then the inequality in* (1) *is an equality.*

NB: We'll soon see that the assertion of (2) is in fact "if and only if".

Proof. To prove point (1) choose an integral scheme of finite type over X with field of fractions K. Then the assertion follows from Lemma 2.6.2

To prove point (2), consider the exact sequence:

$$K \underset{K_0}{\otimes} \Omega_{K_0/k} \to \Omega_{K/k} \to \Omega_K/K_0,$$

and the assertion follows.

□

Lemma 2.6.5. *If k is perfect, then any finitely generated field extension is separably generated.*

For the proof, see references in Hartshorne, Theorem 4.8A.

Corollary 2.6.6. *Any integral scheme over a perfect field is generically smooth.*

3. Smooth morphisms between schemes

3.1. Let $f : X \to S$ be a morphism of schemes. Untill the end of this write-up, we'll assume that the base S is locally Noetherian.

Assume that f is of finite type (smoothness is only defined for morphisms of finite type).

Definition 3.1.1. *We say that f is smooth of relative dimension n if the following conditions hold:*

(i) *X is flat over S.*

(ii) *For every point $s \in S$, the base change $X \underset{S}{\times} k_S$ is a smooth scheme of dimension n over the residue field k_s.*

Theorem 3.1.2. *Let $f : X \to S$ be a morphism of finite type. The following conditions are equivalent:*

(a) *Condition* (ii) *in the definition of smoothness holds, and $\Omega_{X/S}$ is locally free of rank n.*

(b) *f is smooth of* $\mathrm{rel.\,dim.} = n$.

(c) *X is flat and locally on X, we can find a closed embedding over $X \hookrightarrow X' := \mathbb{A}^n_S$ (compatible with the projection to S), so that if we denote by $\mathcal{I}_{X,X'}$ the corresponding sheaf of ideals on X', the sequence*

$$\mathcal{I}_{X,X'}/\mathcal{I}^2_{X,X'} \to \Omega_{X'/S}|_X \to \Omega_{X/S} \to 0$$

is a short exact sequence of vector bundles.

Proof. The fact that (a) implies (b) follows from Lemma 2.2.2. The implications (c) $\Rightarrow$ (a) is tautological. So, it's enough to show that (b) implies (c). We'll use the following generalization of Lemma 2.5.3:

Lemma 3.1.3. *Let $\alpha : \mathcal{F} \to \mathcal{E}$ be a map of coherent sheaves on a locally Noetherian scheme X, where $\mathcal{E}$ is locally free. Assume that for every $x \in X$, there exists a closed subscheme $Z_x \subset X$ containing x, such that the resulting map $\alpha|_{Z_x} : \mathcal{F}|_{Z_x} \to \mathcal{F}|_{Z_x}$ is injective and the quotient is locally free on Z_x. Then: (i) the map α is injective, (ii) $\mathcal{F}$ is locally free, (iii)* $\mathrm{coker}(\alpha)$ *is locally free. In fact, it is sufficient to check this condition for closed points only.*

We apply this lemma to the map $\mathcal{I}_{X,X'}/\mathcal{I}^2_{X,X'} \to \Omega_{X'/S}|_X$. For every $x \in X$ we take $Z_x := X_{f(x)}$. We need to verify that the conditions of the lemma hold. First, since X is flat over Y, for every $y \in Y$, the natural map $\mathcal{I}|_{X'_y} \to \mathcal{I}_{X'_y,X_y}$ is an isomorphism, and hence

$$(\mathcal{I}_{X,X'}/\mathcal{I}^2_{X,X'})_{X_y} \simeq \mathcal{I}_{X_y,X'_y}/\mathcal{I}^2_{X_y,X'_y}.$$

Now, the assertion follows from Theorem 2.5.2.

□

NB: Note that from what we proved it follows that Theorem 3.1.2 can be rephrased as follows: $X \to S$ is smooth if and only it is flat, and for any closed embedding $X \hookrightarrow X'$ with X' smooth over S, the sequence in point (3) of the theorem is short exact. In one of the problems for this week you'll show that the condition that $X \to S$ is in fact automatic.

3.2. We'll now discuss a differential criterion for smoothness of a map:

Let X, Y be two schemes, smooth over a base S, and $f : X \to Y$ a map between them.

Theorem 3.2.1. *The following conditions are equivalent:*
(a) f *is smooth of relative dimension* $\dim.\mathrm{rel}(X, S) - \dim.\mathrm{rel}(Y, S)$.
(b) $\Omega_{X/Y}$ *is locally free of rank* $\dim.\mathrm{rel}(X, S) - \dim.\mathrm{rel}(Y, S)$.
(c) *The map* $f^*(\Omega_{Y/S}) \to \Omega_{X/S}$ *is injective, and the quotient is locally free.*