SHEAVES OF CATEGORIES AND THE NOTION OF 1-AFFINENESS

DENNIS GAITSGORY

Abstract. We define the notion of 1-affineness for a prestack, and prove an array of results that establish 1-affineness of certain types of prestacks.

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Introduction

0.1. **The notion of sheaf of categories.**

0.1.1. Before we define what we mean by a *sheaf of categories*, let us specify what these are sheaves on: we will consider sheaves of categories over arbitrary *prestacks*.

In this paper we work in the framework of derived algebraic geometry, as developed by J. Lurie. For a brief summary of our conventions, the reader is referred to the paper [GL:Stacks]. Throughout this paper we will be working over a fixed ground field k of characteristic 0.

Let $\mathrm{DGSch}^{\mathrm{aff}}$ be the category of affine DG schemes. By definition, a *prestack* is an arbitrary functor [1] of ∞-categories

$$\mathcal{Y} : (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Grpd}\,.$$

I.e., a prestack is given by its functor of points on affine DG schemes (with the only condition being of set-theoretic nature, referred to in the footnote).

0.1.2. Informally, a sheaf of categories $\mathcal{C}$ over a prestack $\mathcal{Y}$ is a functorial assignment for every affine DG scheme S, mapping to $\mathcal{Y}$, of a DG category $\mathbf{\Gamma}(S, \mathcal{C})$, which is acted on by the monoidal DG category $\mathrm{QCoh}(S)$ of quasi-coherent sheaves on S. I.e.,

$$(S \to \mathcal{Y}) \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}} \rightsquigarrow \mathbf{\Gamma}(S, \mathcal{C}) \in \mathrm{QCoh}(S)\text{-}\mathbf{mod}, \tag{0.1}$$

where we denote by $\mathrm{QCoh}(S)\text{-}\mathbf{mod}$ the ∞-category of $\mathrm{QCoh}(S)$-module categories.

The assignment (0.1) must be functorial in S in the sense that for a map $f : S_1 \to S_2$ in $\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}$ we must be given an isomorphism in $\mathrm{QCoh}(S_1)\text{-}\mathbf{mod}$:

$$\mathrm{QCoh}(S_1) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathbf{\Gamma}(S_2, \mathcal{C}) \to \mathbf{\Gamma}(S_1, \mathcal{C}),$$

together with a homotopy-coherent system of compatibilities for compositions of morphisms.

A precise definition of the ∞-category $\mathrm{ShvCat}(\mathcal{Y})$ is given in Sect. 1.1.

[1] Technically, we require our prestacks to be accessible as functors.

0.1.3. As is often the case with subjects such as the present one, a natural question to ask is why we should care about the notion of sheaf of categories, and especially in such generailty.

The author was led to the study of this notion by the (still highly conjectural) local geometric Langlands program. Namely, the object of study of this program is the notion of category, equipped with an action of the loop group $G(\!(t)\!)$.

Now, the ∞-category of categories acted on by $G(\!(t)\!)$ is (more or less by definition) the same as $\mathrm{ShvCat}(\mathcal{Y})$, for

$$\mathcal{Y} := B\left(G(\!(t)\!)_{\mathrm{dR}}\right),$$

where $B(\mathcal{G})$ denotes the classifying prestack of a given group-prestack $\mathcal{G}$, and $(-)_{\mathrm{dR}}$ is the de Rham prestack of a given stack.

It turns out that the case of $B\left(G(\!(t)\!)_{\mathrm{dR}}\right)$ contains the complexity of all the examples considered in this paper combined (algebraic stacks, indschemes, classifying stacks of formal groups, de Rham prestacks).

0.2. **Quasi-coherent sheaves on a prestack.** We shall now take a slightly different approach to what a sheaf of categories over a prestack might mean.

0.2.1. For every prestack $\mathcal{Y}$ we have the DG category $\mathrm{QCoh}(\mathcal{Y})$. By definition, its objects are assigments

$$(S \to \mathcal{Y}) \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}} \rightsquigarrow \mathcal{F}_S \in \mathrm{QCoh}(S),$$

endowed with the data of

$$f^*(\mathcal{F}_{S_2}) \simeq \mathcal{F}_{S_1}, \quad (f : S_1 \to S_2) \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}},$$

together with a homotopy-coherent system of compatibilities for compositions of morphisms. I.e., informally, a quasi-coherent sheaf on $\mathcal{Y}$ is a compatible family of quasi-coherent sheaves on affine DG schemes mapping to $\mathcal{Y}$.

The DG category $\mathrm{QCoh}(\mathcal{Y})$ has a natural (symmetric) monoidal structure given by term-wise tensor product:

$$(\mathcal{F}^1 \otimes \mathcal{F}^2)_S := \mathcal{F}^1_S \underset{\mathcal{O}_S}{\otimes} \mathcal{F}^2_S.$$

0.2.2. Can consider the ∞-category $\mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$ of module categories over $\mathrm{QCoh}(\mathcal{Y})$. The goal of this paper is to study the connection between the ∞-categories

$$\mathrm{ShvCat}(\mathcal{Y}) \text{ and } \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}.$$

The above two ∞-categories are tautologically equivalent if $\mathcal{Y}$ is an affine DG scheme.

0.2.3. The first observation is that the categories $\mathrm{ShvCat}(\mathcal{Y})$ and $\mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$ are related by a pair of adjoint functors:

Given $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y})$, we can take the DG category $\mathbf{\Gamma}(\mathcal{Y}, \mathcal{C})$ of its global sections over $\mathcal{Y}$. It will be naturally acted on by $\mathrm{QCoh}(\mathcal{Y})$. Thus, we obtain an object

$$\mathbf{\Gamma}^{\mathrm{enh}}(\mathcal{Y}, \mathcal{C}) \in \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod},$$

and we obtain a functor

$$\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}} := \mathbf{\Gamma}^{\mathrm{enh}}(\mathcal{Y}, -) : \mathrm{ShvCat}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}.$$

The functor $\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}}$ admits a left adjoint, denoted $\mathbf{Loc}_{\mathcal{Y}}$, given by tensoring up. Namely, for $\mathbf{C} \in \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$ we let $\mathbf{Loc}_{\mathcal{Y}}(\mathbf{C})$ be the sheaf of categories, whose value on $S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}$ is

$$\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C}.$$

0.2.4. We can now give the definition central for this paper: we shall say that a prestack $\mathcal{Y}$ is 1-affine if the functors $\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}}$ and $\mathbf{Loc}_{\mathcal{Y}}$ are (mutually inverse) equivalences of ∞-categories.

Thus, $\mathcal{Y}$ is 1-affine, if and only if the category $\mathrm{ShvCat}(\mathcal{Y})$ can be completely recovered from the monoidal DG category $\mathrm{QCoh}(\mathcal{Y})$.

As was mentioned above, an affine DG scheme is tautologically 1-affine.

0.2.5. The origin of the name is the following: let us say that a prestack $\mathcal{Y}$ is *weakly 0-affine* if the functor

$$\Gamma(\mathcal{Y}, -) : \mathrm{QCoh}(\mathcal{Y}) \to \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})\text{-mod}$$

is an equivalence of categories. Here $\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is the (DG) algebra of global sections of $\mathcal{O}_{\mathcal{Y}}$, and $\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$-mod is DG category of its modules.

Tautologically, an affine DG scheme is weakly 0-affine. However, the class of weakly 0-affine prestacks is much larger than just affine DG schemes. For example, any quasi-affine DG scheme is weakly 0-affine. In addition, the algebraic stack $\mathrm{pt}/\mathbb{G}_a$ is also weakly 0-affine.

The notion of 1-affineness is a higher-categorical analog, where instead of modules over DG algebras, we consider module categories over monoidal DG categories.

0.3. **Main results.** This paper aims to determine which prestacks are 1-affine.

Remark 0.3.1. Let us say right away that it is "much easier" for a prestack to be 1-affine than weakly 0-affine. We shall see multiple manifestations of this phenomenon below (however, it is *not* true that every weakly 0-affine prestack is 1-affine).

0.3.2. First, one shows that any (quasi-compact, quasi-separated) DG scheme is 1-affine (Theorem 2.1.1).

Furthermore, we show that algebraic stacks (under some not too restrictive technical conditions) are also 1-affine (Theorem 2.2.6). [2]

One of the technical conditions for 1-affineness of algebraic stacks is that the inertia group of points be of finite type. This condition turns out to be necessary. Namely, we show (Theorem 2.2.3) that the classifying stack of a group-scheme of infinite type is typically *not* 1-affine.

0.3.3. One can wonder whether it is reasonable to expect higher Artin stacks to be 1-affine. Unfortunately, we did not find a principle that governs the answer:

Consider the iterated classifying spaces $B\mathbb{G}_a$, $B^2\mathbb{G}_a$, $B^3\mathbb{G}_a$. We prove (Theorem 2.5.7) that they are all 1-affine. However, we also prove that $B^4\mathbb{G}_a$ is *not* 1-affine.

0.3.4. Another class of prestacks of interest for us is (DG) indschemes. These turn out *not* to be 1-affine, even in the nicest cases, such as $\mathbb{A}^\infty$.

[2]Unfortunately, the only proof of this result that we could come up with for arbitrary algebraic stacks is rather complicated. On the other hand, a much simpler proof can be given for algebraic stacks that are global quotients (Theorem 2.2.4). The core idea of both proofs is due to J. Lurie.

0.3.5. A third class of primary interest is prestacks of the form Z_{dR}, where Z is a scheme of finite type. We can think of $\mathrm{ShvCat}(Z_{\mathrm{dR}})$ as the category of *crystals of categories* over Z.

We prove (Theorem 2.6.3) that Z_{dR} is 1-affine.

Note, however, that if Z is not a scheme but an algebraic stack, then Z_{dR} is no longer 1-affine.

0.3.6. *Methods.* Let us say a few words about what goes into proving that a given class of prestacks is or is not 1-affine. Invariably, this question reduces to that of whether a certain functor between two very concrete DG categories is *monadic* (see Sect. 0.5.1 for what this means).

Usually, the monadicity of a functor is established using the Barr-Beck-Lurie theorem ([Lu2, Theorem 6.2.2.5]). In order to apply this theorem, one needs to check two conditions. One is that the functor in question is conservative (usually, this is fairly easy). The second condition is that the functor commutes with certain geometric realizations. This condition is much harder to check in practice, unless our functor happens to commute with *all* colimits (i.e., is continuous), while the latter is not always the case.

Verifying this second condition constitutes the bulk of the technical work in this paper. Let us emphasize again that, although our main assertions are initially about *continuous* functors between DG categories (i.e., functors that commute with all colimits), the core of the proofs involves non-continuous functors.

So, one can say that at the end of the day, the proofs consist of showing that certain colimits commute with certain limits, i.e., we deal with convergence problems. In this sense, what we do in this paper can be called "functional analysis within homological algebra."

0.4. **Organization of the paper.** The paper can be loosely divided into three parts.

0.4.1. In Part I we give the definitions and discuss some general constructions.

In Sect. 1 we define sheaves of categories, the property of 1-affineness, and discuss some basic results.

In Sect. 2 we state the main results of this paper pertaining to 1-affineness and non 1-affineness of certain classes of prestacks.

In Sect. 3 we discuss the functors of direct and inverse image of sheaves of categories, and study how these functors interact with the functors $\mathbf{\Gamma}^{\mathrm{enh}}$ and **Loc** mentioned earlier.

In Sect. 4 we show that the property of 1-affineness survives the operation of taking the formal completion of a prestack along a closed subset.

0.4.2. In Part II we consider the question of 1-affineness of algebraic spaces and algebraic stacks.

In Sect. 5 we prove that (quasi-compact, quasi-separated) algebraic spaces are 1-affine. In addition, we single out a class of prestacks (we call them *passable*; this class includes algebraic stacks satisfying certain technical hypotheses) for which the functor $\mathbf{\Gamma}^{\mathrm{enh}}$ is fully faithful.

In Sect. 6 we give several equivalent conditions for a (passable) algebraic stack to be 1-affine. Essentially, these conditions reduce the verification of 1-affineness of a given algebraic stack to the question of monadicity of a certain very concrete functor.

In Sect. 7 we show that the classifying stack of a (classical) algebraic group of finite type is 1-affine. The proof is based on the criterion of 1-affineness developed in Sect. 6. The idea of the proof belongs to J. Lurie. From the 1-affineness of the classifying stack we (easily) deduce the 1-affineness of algebraic stacks that are global quotients.

Finallly, in Sect. 8 we prove that algebraic stacks (under certain technical hypotheses) are 1-affine. As was mentioned above, the proof is rather long. It consists of checking the monadacity of a functor when the conditions of the Barr-Beck-Lurie theorem could not be checked directly (or, rather, the author did not find a way to do so).

0.4.3. In Part III we treat the question of 1-affineness of a host of cases: (DG) indschemes, classifying prestacks of general group-prestacks, classifying prestacks of formal groups, de Rham prestacks, and other related types of prestacks.

In Sect. 9 we specify a class of (DG) indschemes, for which the functor **Loc** is fully faithful. This class includes formally smooth indcshemes locally almost of finite type. We also show that (DG) indschemes are typically not 1-affine.

In Sect. 10 we study sheaves of categories over prestacks of the form $B\mathcal{G}$, where $\mathcal{G}$ is a group-object in the category of prestacks. We explain how the theory of sheaves of categories over such prestacks can be viewed as "higher representation theory," i.e., as the theory of categories acted on by $\mathcal{G}$.

In Sect. 11 we show that prestacks of the form $B\mathcal{G}$, where $\mathcal{G}$ is a formal group, which as a formal scheme is isomorphic to $\mathrm{Spf}(k[t_1, ..., t_n])$, is 1-affine.

In Sect. 12 we study the question of 1-affineness of prestacks of the form Z_{dR}, where Z is a scheme or algebraic stack. The proof of 1-affineness in the case of schemes relies on 1-affineness of formal classifying spaces, developed in the previous section.

In Sect. 13 we study the following problem: we start with a DG scheme Z with a point $z \in Z(k)$, and consider the (derived) inertia group $\Omega(Z, z)$ (a.k.a. the infinitesimal loop group of Z at z). We study the connection between sheaves of categories on the prestack $B(\Omega(Z, z))$ and sheaves of categories on the formal completion of Z at z.

In Sect. 14 we study the question of 1-affineness of iterated classifying prestacks of the form $B^k(\mathbb{G}_a)$, and of classifying prestacks of iterated loop groups $\Omega^k(Z, z)$.

0.4.4. This paper contains several appendices, included for the reader's convenience in order to make the exposition more self-contained.

In Sect. A we reproduce the proof of the result of J. Lurie that the assigment

$$\mathcal{Y} \rightsquigarrow \mathrm{ShvCat}(\mathcal{Y})$$

is itself a sheaf in the fppf topology.

In Sect. B we reproduce proofs of several statements from [GL:QCoh] pertaining to the behavior of quasi-affine morphisms from the point of view of tensor products of categories.

In Sect. C we review the (monadic and co-monadic) Beck-Chevalley conditions for co-simplicial categories. These conditions make the totalization of the given co-simplicial category calculable: namely the forgetful functor of evaluation on 0-simplices turns out to be monadic (resp., co-monadic), and the corresponding monad (resp., co-monad) can be described explicitly.

In Sect. D we review the notion of rigidity for a monoidal DG category. [3] This notion turns out to be very convenient, as it allows for explicit control of the operation of tensor product of module categories over our monoidal DG category.

[3]For compactly generated monoidal DG categories, the condition of rigidity is equivalent to requiring that every compact object admit a left and right monoidal duals.

In Sect. E we prove a certain basic result about commutative Hopf algebras in symmetric monoidal ∞-categories (its version in ordinary categories is easy to prove by hand, and so is often passed by, without being stated explicitly).

0.5. **Conventions.**

0.5.1. This paper relies on the theory of ∞-categories as developed by J. Lurie in [Lu1] and [Lu2]. By a slight abuse of terminology we shall sometimes say "category", when we actually mean ∞-category.

0.5.2. The following terminology is used throughout the paper. If M is a monad acting on an ∞-category $\mathbf{C}$, we let $\mathsf{M}\text{-mod}(\mathbf{C})$ denote the category of M-modules (sometimes also called M-algebras) in $\mathbf{C}$. We let

$$\mathrm{ind}_{\mathsf{M}} : \mathbf{C} \rightleftarrows \mathsf{M}\text{-mod}(\mathbf{C}) : \mathrm{oblv}_{\mathsf{M}}$$

the resulting adjoint pair of functors ("oblv" stands for the forgetful functor, and "ind" for the induction functor).

Let

$$\mathbf{C} \leftarrow \mathbf{D} : \mathsf{G}$$

be a functor between ∞-categories. We shall say that G is *monadic* if it admits a left adjoint, denoted F, and when we view the composition $\mathsf{G} \circ \mathsf{F}$ as a monad acting on $\mathbf{C}$, the resulting functor

$$(\mathsf{G} \circ \mathsf{F})\text{-mod}(\mathbf{C}) \leftarrow \mathbf{D} : \mathsf{G}^{\mathrm{enh}}$$

is an equivalence.

Replacing "left" by "right", we obtain the notion of *co-monadic* functor.

0.5.3. Our conventions regarding DG categories follow those adopted in [GL:DG].

We let Vect denote the DG category of chain complexes of k-vector spaces. [4] In this paper all DG categories will be assumed presentable (in particular, cocomplete, i.e., closed under arbitrary direct sums).

We let DGCat denote the ∞-category of DG categories and accessible exact functors. We let $\mathrm{DGCat}_{\mathrm{cont}}$ denote the category with the same objects, but where we restrict 1-morphisms to be continuous (i.e., commuting with all direct sums, equivalently, with all colimits).

In multiple places of the paper we will use the result of [Lu1, Corollary 5.5.3.3] that says the colimit of a diagram in $\mathrm{DGCat}_{\mathrm{cont}}$, can be computed as a DG category, as the limit in DGCat of the diagram obtained by passage to right adjoint functors. For a sktech of the proof of this result the reader is referred to [GL:DG, Lemma 1.3.3].

[4] The reader can substitute the notion of DG category by a better documented notion of presentable stable ∞-category, tensored over Vect.

0.5.4. The ∞-category $\mathrm{DGCat}_{\mathrm{cont}}$ carries a natural symmetric monoidal structure given by tensor product of DG categories. (We emphasize that we live in the world of cocomplete DG categories and continuous functors.)

If $\mathbf{O}$ is an algebra object in $\mathrm{DGCat}_{\mathrm{cont}}$, i.e., a monoidal DG category, we let $\mathbf{O}\text{-}\mathbf{mod}$ denote the category of $\mathbf{O}$-modules in $\mathrm{DGCat}_{\mathrm{cont}}$, i.e., the ∞-category of $\mathbf{O}$-module categories.

In general, throughout the paper, we use boldface symbols for "higher" objects and functors. E.g., if $\mathcal{G}$ is an affine DG group-scheme, we use $\mathrm{inv}^{\mathcal{G}}$ to denote the functor

$$\mathrm{Rep}(\mathcal{G}) \to \mathrm{Vect},$$

of invariants on the category of $\mathcal{G}$-representations, and we use $\mathbf{inv}^{\mathcal{G}}$ to denote the functor

$$\mathcal{G}\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}$$

that sends a DG category acted on by $\mathcal{G}$ to the category of $\mathcal{G}$-equivariant objects.

For a pair of DG categories $\mathbf{D}_1, \mathbf{D}_2$, we let $\underline{\mathrm{Hom}}(\mathbf{D}_1, \mathbf{D}_2)$ denote their "internal Hom", i.e., the DG category of continuous functors $\mathbf{D}_1 \to \mathbf{D}_2$. Similarly, for $\mathbf{D}_1, \mathbf{D}_2 \in \mathbf{O}\text{-}\mathbf{mod}$ we will use the notation $\underline{\mathrm{Hom}}_{\mathbf{O}}(\mathbf{D}_1, \mathbf{D}_2)$ for the DG category of functors compatible with the $\mathbf{O}$-module structure.

0.5.5. Our conventions regarding derived algebraic geometry follow those of [GL:Stacks]. We let $\mathrm{DGSch}^{\mathrm{aff}}$ denote the ∞-category of affine DG schemes, which is by definition the opposite category to that of connective k-algebras.

We let PreStk denote the ∞-category of all prestacks, i.e., the category of accessible functors

$$(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-}\mathrm{Grpd},$$

where ∞-Grpd is the ∞-ategory of ∞-groupoids (a.k.a., spaces).

In the main body of the paper, we will need the notion of fppf morphism between DG schemes, for which the reader is referred to [GL:Stacks, Sect. 2.1][5].

We will need the notion of what it means for a DG scheme or Artin stack to be classical (resp., eventually coconnective). By definition, an affine DG scheme is classical (resp., eventually coconnective) if its DG ring of functions has no (resp., finitely many) non-zero cohomology groups. An Artin stack is classical (resp., eventually coconnective) if it admits an fppf cover by an affine DG scheme which is classical (resp., eventually coconnective). For further details see [GL:Stacks, Sects. 1.1, 2.4 and 4.6].

We let $\mathrm{PreStk}_{\mathrm{laft}}$ denote the full subcategory of PreStk formed by prestacks that are *locally almost of finite type*, see [GL:Stacks, Sect. 1.3.9].

0.5.6. In some proofs, we will need to use the category of *ind-coherent sheaves*, developed in [Ga]. This category is defined on prestacks that belong to $\mathrm{PreStk}_{\mathrm{laft}}$.

[5]In this paper we use the more common "fppf" rather than "fpppf."

0.6. **Acknowledgements.** The problems such as those addressed in this paper were brought to the author's awareness by J. Lurie, so this paper can be regarded as a research project carried out under his guidance, and the influence of his ideas is evident everywhere in the text.

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Part I: Generalities

1. Quasi-coherent sheaves of categories

1.1. Definition of a quasi-coherent sheaf of categories.

1.1.1. Consider the functor

$$\mathrm{ShvCat}_{\mathrm{DGSch}^{\mathrm{aff}}} : (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Cat}, \quad S \mapsto \mathrm{QCoh}(S)\text{-}\mathbf{mod},$$

that assigns to an affine DG scheme the ∞-category of module categories over the monoidal DG category $\mathrm{QCoh}(S)$.

Let

$$\mathrm{ShvCat}_{\mathrm{PreStk}} : (\mathrm{PreStk})^{\mathrm{op}} \to \infty\text{-Cat}$$

be the right Kan extension of $\mathrm{ShvCat}_{\mathrm{DGSch}^{\mathrm{aff}}}$ along the Yoneda embedding

$$(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow \mathrm{PreStk}^{\mathrm{op}}.$$

For a prestack $\mathcal{Y}$, we let

$$\mathrm{ShvCat}(\mathcal{Y}) \in \infty\text{-Cat}$$

denote the value of $\mathrm{ShvCat}_{\mathrm{PreStk}}$ on $\mathcal{Y} \in \mathrm{PreStk}$.

We shall refer to objects of $\mathrm{ShvCat}(\mathcal{Y})$ as a "quasi-coherent sheaves of DG categories on $\mathcal{Y}$."

1.1.2. In other words, for $\mathcal{Y} \in \mathrm{PreStk}$, an object $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y})$ is an assignment

$$S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}} \rightsquigarrow \mathbf{\Gamma}(S, \mathcal{C}) \in \mathrm{QCoh}(S)\text{-}\mathbf{mod},$$

and for an arrow $g : S_1 \to S_2$ in $\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}$ of an equivalence

$$\mathrm{QCoh}(S_1) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathbf{\Gamma}(S_2, \mathcal{C}) \simeq \mathbf{\Gamma}(S_1, \mathcal{C}),$$

along with a homotopy-coherent system of compatibilities.

Morphisms between sheaves of categories are defined naturally.

From the definition of $\mathrm{ShvCat}(-)$ as the right Kan extension we obtain:

Lemma 1.1.3. *The functor* $\mathrm{ShvCat}(-)$ *takes colimits in* PreStk *to limits in* ∞-Cat.

1.1.4. The basic example of an object of $\mathrm{ShvCat}(\mathcal{Y})$ is $\mathrm{QCoh}_{/\mathcal{Y}}$, whose value on $S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}$ is $\mathrm{QCoh}(S)$.

The category $\mathrm{ShvCat}(\mathcal{Y})$ carries a symmetric monoidal structure given by component-wise tensor product, and $\mathrm{QCoh}_{/\mathcal{Y}}$ is its unit object.

1.1.5. The category $\mathrm{ShvCat}(\mathcal{Y})$ contains colimits that are computed value-wise.

The category $\mathrm{ShvCat}(\mathcal{Y})$ contains limits, which are computed by

$$\mathbf{\Gamma}\left(S, \underset{\leftarrow i}{lim}\,(\mathcal{C}_i)\right) \simeq \underset{\leftarrow i}{lim}\,\mathbf{\Gamma}(S, \mathcal{C}_i).$$

Indeed, this follows from the fact that for a morphism $f : S_1 \to S_2$ in $\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}$, the functor

$$\mathrm{QCoh}(S_1) \underset{\mathrm{QCoh}(S_2)}{\otimes} - : \mathrm{QCoh}(S_2)\text{-}\mathbf{mod} \to \mathrm{QCoh}(S_1)\text{-}\mathbf{mod}$$

commutes with *limits*, which in turn follows from the fact that the category $\mathrm{QCoh}(S_1)$ is dualizable as an object of $\mathrm{QCoh}(S_2)\text{-}\mathbf{mod}$, see Lemma 1.4.7.

1.2. **Global sections.**

1.2.1. For a given $\mathcal{Y}$ and $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y})$, we can right-Kan-extend the functor

$$\mathbf{\Gamma}(-, \mathcal{C}) : (\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

to a functor

$$(\mathrm{PreStk}_{/\mathcal{Y}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}; \quad \mathcal{Z} \mapsto \mathbf{\Gamma}(\mathcal{Z}, \mathcal{C}).$$

I.e.,

$$\mathbf{\Gamma}(\mathcal{Z}, \mathcal{C}) := \underset{\leftarrow S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Z}}}{lim} \mathbf{\Gamma}(S, \mathcal{C}).$$

For example, it is clear that

$$\mathbf{\Gamma}(\mathcal{Z}, \mathrm{QCoh}_{/\mathcal{Y}}) \simeq \mathrm{QCoh}(\mathcal{Z}).$$

In particular, we obtain a DG category $\mathbf{\Gamma}(\mathcal{Y}, \mathcal{C})$.

1.2.2. It is clear that the functor

$$\mathcal{Z} \mapsto \mathbf{\Gamma}(\mathcal{Z}, \mathcal{C})$$

takes colimits in $\mathrm{PreStk}_{/\mathcal{Y}}$ to limits in $\mathrm{DGCat}_{\mathrm{cont}}$.

1.2.3. The functor

$$\mathbf{\Gamma}(\mathcal{Z}, -) : \mathrm{ShvCat}(\mathcal{Y}) \to \mathrm{DGCat}_{\mathrm{cont}}$$

is lax symmetric monoidal.

In particular, we obtain that it natually upgrades to a functor

$$\mathrm{ShvCat}(\mathcal{Y}) \to \mathbf{\Gamma}(\mathcal{Z}, \mathrm{QCoh}_{/\mathcal{Y}})\text{-}\mathbf{mod} \simeq \mathrm{QCoh}(\mathcal{Z})\text{-}\mathbf{mod}.$$

We shall denote the resulting functor

$$\mathrm{ShvCat}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Z})\text{-}\mathbf{mod}$$

by $\mathbf{\Gamma}^{\mathrm{enh}}(\mathcal{Z}, -)$.

When $\mathcal{Z} = \mathcal{Y}$, we shall sometimes write

$$\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}} : \mathrm{ShvCat}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$$

instead of $\mathbf{\Gamma}^{\mathrm{enh}}(\mathcal{Y}, -)$.

1.3. **Posing the problem.**

1.3.1. We note that the functor

$$\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}} : \mathrm{ShvCat}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$$

admits a left adjoint; we denote it by $\mathbf{Loc}_{\mathcal{Y}}$.

Namely, for $\mathbf{C} \in \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$ we have

$$\mathbf{\Gamma}(S, \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C})) = \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C}, \quad S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}.$$

It clear from the construction that the functor

$$\mathbf{Loc}_{\mathcal{Y}} : \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod} \to \mathrm{ShvCat}(\mathcal{Y})$$

is symmetric monoidal.

1.3.2. The questions that we want to address in this paper are the following:

Question 1.3.3.

(1) *Under what conditions, for* $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y})$ *is the co-unit map*

$$\mathbf{Loc}_{\mathcal{Y}}\left(\mathbf{\Gamma}^{\mathrm{enh}}(\mathcal{Y}, \mathcal{C})\right) \to \mathcal{C} \tag{1.1}$$

an equivalence?

(2) *Under what conditions on* $\mathcal{Y}$ *is* (1.1) *an equivalence for* all $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y})$*? I.e., when is* $\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}}$ *fully faithful?*

(3) *Under what conditions, for* $\mathbf{C} \in \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$ *is the unit map*

$$\mathbf{C} \to \mathbf{\Gamma}^{\mathrm{enh}}(\mathcal{Y}, \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C})) \tag{1.2}$$

an equivalence?

(4) *Under what conditions on* $\mathcal{Y}$ *is* (1.2) *an equivalence for* all $\mathbf{C} \in \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$*? I.e., when is* $\mathbf{Loc}_{\mathcal{Y}}$ *fully faithful?*

1.3.4. In some cases, the answer is very easy:

Lemma 1.3.5. *Suppose that* $\mathbf{C}$ *is dualizable as an object of* $\mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$*. Then the adjunction map*

$$\mathbf{C} \to \mathbf{\Gamma}^{\mathrm{enh}}(\mathcal{Y}, \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C}))$$

is an equivalence.

Proof. This follows from the fact that for $\mathbf{C} \in \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$ dualizable, the functor

$$- \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} : \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}$$

commutes with limits. Indeed,

$$\mathbf{\Gamma}(\mathcal{Y}, \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C})) \simeq \underset{S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}}{\underset{\longleftarrow}{lim}} (\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C}) \simeq$$

$$\left(\underset{S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}}{\underset{\longleftarrow}{lim}} \mathrm{QCoh}(S)\right) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \simeq \mathrm{QCoh}(\mathcal{Y}) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \simeq \mathbf{C}.$$

□

1.3.6. We give the following definition:

Definition 1.3.7. *We shall say that $\mathcal{Y}$ is 1-affine if the functors $\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}}$ and $\mathbf{Loc}_{\mathcal{Y}}$ are mutually inverse equivalences.*

The main results of this paper will amount to saying that certain classes of prestacks are (or are not) 1-affine.

1.4. **Dualizability and compact generation.**

1.4.1. Recall that in any symmetric monoidal ∞-category we can talk about the property of an object to be dualizable.

Since the functor $\mathbf{Loc}_{\mathcal{Y}}$ is symmetric monoidal, it automatically sends dualizable objects in $\mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$ to dualizable objects in $\mathrm{ShvCat}(\mathcal{Y})$.

The following is also tautological:

Lemma 1.4.2. *If $\mathcal{Y}$ is 1-affine, then the lax symmetric monoidal structure on $\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}}$ is strict (i.e., non-lax).*

From here we obtain:

Corollary 1.4.3. *If $\mathcal{Y}$ is 1-affine, and $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y})$ is dualiable, then $\mathbf{\Gamma}(\mathcal{Y}, \mathcal{C})$ is dualizable as an object of $\mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$.*

1.4.4. Let us make the notion of being dualizable as an object of $\mathrm{ShvCat}(\mathcal{Y})$ more explicit:

Proposition 1.4.5. *An object $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y})$ is dualizable if and only if for every $S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}$, the category $\mathbf{\Gamma}(S, \mathcal{C})$ is dualizable as a plain DG category.*

Proof. The proof follows from the combination of the following two lemmas:

Lemma 1.4.6 (Lurie)**.** *Let a symmetric monoidal category $\mathbf{O}$ be equal to the limit*

$$\underset{\alpha}{\overleftarrow{lim}}\, \mathbf{O}_\alpha$$

of a diagram $\alpha \mapsto \mathbf{O}_\alpha$ of symmetric monoidal categories. Then an object $\mathbf{o} \in \mathbf{O}$ is dualizable if and only if its projection $\mathbf{o}_\alpha \in \mathbf{O}_\alpha$ is dualizable for every index α.

For the next lemma recall the notion of *rigid* monoidal DG category, see Sect. D.1. For example, for an affine DG scheme S, the monoidal DG category $\mathrm{QCoh}(S)$ is rigid (this is the trivial case of Lemma B.2.3).

We have (see Sect. D.5.3):

Lemma 1.4.7. *Let $\mathbf{A}$ is a symmetric monoidal DG category, which is rigid as a monoidal DG category. Then $\mathbf{C} \in \mathbf{A}\text{-}\mathbf{mod}$ is dualizable as an object of the symmetric monoidal category $\mathbf{A}\text{-}\mathbf{mod}$ if and only if $\mathbf{C}$ is dualizable as a plain DG category (i.e., as an object of $\mathrm{DGCat}_{\mathrm{cont}}$).*

□

We also notice the following corollary of Lemma 1.4.7:

Corollary 1.4.8. *Let $\mathcal{Y}$ be such that $\mathrm{QCoh}(\mathcal{Y})$ is rigid. Then:*

(a) *An object $\mathbf{C} \in \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$ is dualizable if and only if it is dualizable as a plain category.*

(b) *Then the functor $\mathbf{Loc}_{\mathcal{Y}}$ commutes with limits.*

1.4.9. One can also ask the following questions:

Question 1.4.10.

(1) *Suppose that* $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y})$ *is such that for all* $S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}$*, the category* $\mathbf{\Gamma}(S, \mathcal{C})$ *is compactly generated. When can we guarantee that* $\mathbf{\Gamma}(\mathcal{Y}, \mathcal{C})$ *is compactly generated as a plain category?*

(2) *Let* $\mathbf{C} \in \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$ *be compactly generated as a plain DG category. When can we guarantee that* $\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C}$ *is compactly generated for any* $S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}$.

These questions appear to be more subtle. For example, to the best of our knowledge, it is not known whether the category $\mathrm{QCoh}(\mathcal{Y})$ is compactly generated when $\mathcal{Y}$ is an algebraic stack (when $\mathcal{Y}$ is neither smooth nor a global quotient).

1.5. **Descent.** Before we proceed to the discussion of main results of this paper, let us remark that the questions such as those in Question 1.3.3 are insensitive to fppf sheafification:

1.5.1. First, we recall that following result, which is essentially established in [Lu3, Theorem 5.4]:

Theorem 1.5.2. *Let* Y *be an affine DG scheme, and let* $\mathbf{C}$ *be an object* $\mathrm{QCoh}(Y)\text{-}\mathbf{mod}$*. Then the functor*

$$(\mathrm{DGSch}^{\mathrm{aff}}_{/Y})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}, \quad S \rightsquigarrow \mathbf{\Gamma}(S, \mathbf{Loc}_Y(\mathbf{C})) = \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}$$

satisfies fppf descent.

This theorem is proved in [Lu3] when instead of the fppf topology one considers the étale topology. The remanining step is easy (and well-known), since

"fppf descent"="Nisnevich descent"+"finite flat descent."

For the sake of completeness, we will prove Theorem 1.5.2 in Appendix A.

1.5.3. As a formal consequence, we obtain:

Corollary 1.5.4. *For a prestack* $\mathcal{Y}$ *and* $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y})$*, the functor*

$$(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}, \quad S \rightsquigarrow \mathbf{\Gamma}(S, \mathcal{C})$$

satisfies fppf descent.

From here:

Corollary 1.5.5. *Let* $\mathcal{C}$ *be an object of* $\mathrm{ShvCat}(\mathcal{Y})$.

(a) *If* $\mathcal{Z} \to \mathcal{W}$ *is an fppf surjection in* $\mathrm{PreStk}_{/\mathcal{Y}}$*, then the pullback functor*

$$\mathbf{\Gamma}(\mathcal{W}, \mathcal{C}) \to \mathrm{Tot}(\mathbf{\Gamma}(\mathcal{Z}^\bullet/\mathcal{W}, \mathcal{C}))$$

is an equivalence, where $\mathcal{Z}^\bullet/\mathcal{W}$ *is the Čech nerve of the map* $\mathcal{Z} \to \mathcal{W}$.

(b) *For* $\mathcal{Z} \in \mathrm{PreStk}_{/\mathcal{Y}}$*, the pullback functor*

$$\mathbf{\Gamma}(L_{/\mathcal{Y}}(\mathcal{Z}), \mathcal{C}) \to \mathbf{\Gamma}(\mathcal{Z}, \mathcal{C})$$

is an equivalence, where $L_{/\mathcal{Y}}(-)$ *is fppf sheafification in the category* $\mathrm{PreStk}_{/\mathcal{Y}}$.

(b') *If* $\mathcal{Y}$ *is an fppf stack, then for* $\mathcal{Z} \in \mathrm{PreStk}_{/\mathcal{Y}}$*, the pullback functor*

$$\mathbf{\Gamma}(L(\mathcal{Z}), \mathcal{C}) \to \mathbf{\Gamma}(\mathcal{Z}, \mathcal{C})$$

is an equivalence, where $L(-)$ *is fppf sheafification in the category* PreStk.

1.5.6. Next, from Theorem 1.5.2, one formally deduces the next result (this is [Lu3, Theorem 5.13]):

Theorem 1.5.7. *The functor* $\mathrm{ShvCat}_{\mathrm{PreStk}} : (\mathrm{PreStk})^{\mathrm{op}} \to \infty\text{-Cat}$ *satisfies fppf descent.*

For the reader's convenience, we will supply the derivation Theorem 1.5.2 $\Rightarrow$ Theorem 1.5.7 in Appendix A.

As a formal consequence of Theorem 1.5.7, we obtain:

Corollary 1.5.8.
(a) *If* $\mathcal{Z} \to \mathcal{Y}$ *is an fppf surjection in* PreStk*, the pullback functor*

$$\mathrm{ShvCat}(\mathcal{Y}) \to \mathrm{Tot}(\mathrm{ShvCat}(\mathcal{Z}^\bullet/\mathcal{Y}))$$

is an equivalence, where $\mathcal{Z}^\bullet/\mathcal{Y}$ *is the Čech nerve of the map* $\mathcal{Z} \to \mathcal{Y}$.
(b) *For* $\mathcal{Y} \in \mathrm{PreStk}$*, the pullback functor*

$$\mathrm{ShvCat}(L(\mathcal{Y})) \to \mathrm{ShvCat}(\mathcal{Y})$$

is an equivalence, where $L(-)$ *is fppf sheafification.*

2. Statements of the results

2.1. **Algebraic spaces and schemes.** The following will not be difficult (see Sect. 5.3):

Theorem 2.1.1. *Let* $\mathcal{Y}$ *be a quasi-compact quasi-separated algebraic space. Then* $\mathcal{Y}$ *is 1-affine.*

2.2. **Algebraic stacks.** Our conventions regarding algebraic stacks follow those of [DrGa, Sect. 1.3.3]. In particular, we assume that the diagonal morphism is representable, quasi-compact and quasi-separated.

2.2.1. In Sect. 7 we will prove:

Theorem 2.2.2. *The stack* pt/G*, where* G *is a classical affine algebraic group of finite type, is 1-affine.*

The assumption that the group G be of finite type is important. Namely, in Sect. 7.3 we will prove:

Theorem 2.2.3. *The stack* pt/G *for* $G = \underset{\longleftarrow n}{lim}\, (\mathbb{G}_a)^{\times n}$ *is* not *1-affine.*

In Sect. 7.1.1, from Theorem 2.2.2 we will deduce:

Theorem 2.2.4. *An algebraic stack that can be realized as* Z/G*, where* Z *is a quasi-compact quasi-separated algebraic space and* G *is a classical affine algebraic group of finite type, is* 1-*affine.*

2.2.5. Finally, in Sect. 8 we will prove:

Theorem 2.2.6. *An eventually coconnective quasi-compact algebraic stack locally almost of finite type with an affine diagonal is 1-affine.*

We conjecture that in Theorem 2.2.6, the assumption that $\mathcal{Y}$ be eventually coconnective is superfluous.

2.3. **Formal completions.** In Sect. 4 we will prove:

Theorem 2.3.1. *Suppose that $\mathcal{Y}$ is obtained as a formal completion* [6] *of a* 1*-affine prestack along a closed subfunctor such that the embedding of its complement is quasi-compact. Then $\mathcal{Y}$ is* 1*-affine.*

In particular, combining with Theorem 2.1.1, we obtain:

Corollary 2.3.2. *Let $\mathcal{Y}$ be the formal completion of a quasi-compact quasi-separated algebraic space, along a closed subset whose complement is quasi-compact. Then $\mathcal{Y}$ is* 1*-affine.*

2.4. **(DG) ind-schemes.** We refer the reader to [GR1] for our conventions regarding DG indschemes. In particular, we will need the notions of a *weakly* $\aleph_0$ DG indscheme (see [GR1, 1.4.11]) and of *formally smooth* DG indscheme (see [GR1, 8.1.3]).

2.4.1. In Sect. 9.2 we will prove:

Theorem 2.4.2. *Let $\mathcal{Y}$ be a weakly $\aleph_0$ formally smooth DG indscheme localy almost of finite type. Then the functor $\mathbf{Loc}_{\mathcal{Y}}$ is fully faithful.*

From here in Sect. 9.2.2 we will deduce:

Theorem 2.4.3. *Let G be a classical affine algebraic group of finite type. Then for the DG indscheme $G(\!(t)\!)$, the functor $\mathbf{Loc}_{G(\!(t)\!)}$ is fully faithful.*

2.4.4. We should note that even the "nicest" DG indschemes are typically *not* 1-affine. As a manifestation of this, in Sect. 9.3 we will prove:

Theorem 2.4.5. *Let $\mathcal{Y} = \mathbb{A}^\infty := \underset{\longrightarrow n}{colim}\, \mathbb{A}^n$. Then $\mathcal{Y}$ is* not *1-affine.*

2.4.6. We do not know whether the functor $\mathbf{Loc}_{\mathcal{Y}}$ is fully faithful for more general DG indschemes. For example, we do not know this in the example of

$$\mathcal{Y} := \mathrm{pt} \underset{\mathbb{A}^\infty}{\times} \mathrm{pt}\,.$$

2.5. **Classifying prestacks.**

2.5.1. Let $\mathcal{G}$ be a group-object in PreStk. We let $B^\bullet\mathcal{G}$ denote the standard simplicial object of PreStk associated with $\mathcal{G}$, i.e., the usual simplicial model for the classifying space.

We let $B\mathcal{G}$ denote the geometric realization,

$$B\mathcal{G} := |B^\bullet\mathcal{G}| \in \mathrm{PreStk}\,.$$

We remark that if G is an algebraic group, then the algebraic stack pt/G, mentioned earlier, is by definition the fppf sheafification of BG.

If $\mathcal{G}$ is such that $\mathbf{Loc}_{\mathcal{G}}$ is fully faithful, the category $\mathrm{ShvCat}(B\mathcal{G})$ can be described more explicitly, see Sect. 10.1.

Remark 2.5.2. Note that Theorem 2.2.2 (combined with Corollary 1.5.8(b)) implies that if G is a classical an affine algebraic group of finite type, then BG is 1-affine. Note also that according to Theorem 2.2.3, if G is an affine group-scheme of infinite type, then BG is typically not 1-affine. Below we will discuss several more cases when $B\mathcal{G}$ is (or is not) 1-affine.

[6] The definition of formal completion will be recalled in Sect. 4.1.2.

2.5.3. In Sect. 11.3 we will prove:

Theorem 2.5.4. *Let $\mathcal{G}$ be a group-object in* PreStk*, which as a prestack is a weakly $\aleph_0$ formally smooth DG indscheme locally almost of finite type with $({}^{cl}\mathcal{G})_{red} = \mathrm{pt}$. In this case:*

(a) *The functor $\mathbf{Loc}_{B\mathcal{G}}$ is fully faithful.*

(b) *The prestack $B\mathcal{G}$ is 1-affine if and only if the tangent space of $\mathcal{G}$ at the origin is finite-dimensional.*

In addition, in Sect. 12.3, we will prove:

Theorem 2.5.5. *Let G be a classical affine algebraic group of finite type, and $H \subset G$ a subgroup. Let $\mathcal{G}$ be the formal completion of G along H. Then $B\mathcal{G}$ is 1-affine.*

2.5.6. Let $0 \neq V \in \mathrm{Vect}^{\heartsuit}$ be finite-dimensional, regarded as a commutative (i.e., E_∞) group-object of PreStk.

Recall that by Remark 2.5.2, the prestack BV is 1-affine. As another series of examples of group-objects $\mathcal{G}$ for which $B\mathcal{G}$ is (or is not) 1-affine, in Sect. 14 we will prove:

Theorem 2.5.7.

(a) *The prestack $B^2(V)$ is 1-affine.*

(b) *The prestack $B^3(V)$ is 1-affine.*

(c) *The prestack $B^4(V)$ is* not *1-affine.*

(d) *The prestack $B^2(V_0^\wedge)$ is* not *1-affine, where $V_0^\wedge$ is the completion of V at the origin, regarded as a commutative group-object of* PreStk.

2.5.8. Let now $\mathcal{G}$ be a group-object of $\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$. (Note that the fppf sheafification of $B\mathcal{G}$ is not an algebraic stack, so the discussion in Sect. 2.2 is not applicable to it.)

We propose the following conjecture:

Conjecture 2.5.9. *The classifying prestack $B\mathcal{G}$ is 1-affine.*

As a piece of evidence toward this conjecture, in Sect. 14.2 we will prove:

Theorem 2.5.10. *Let $V \in \mathrm{Vect}^{\heartsuit}$ be finite-dimensional, and for $n \in \mathbb{Z}^+$ let us view $\mathcal{G} = \mathrm{Spec}(\mathrm{Sym}(V[n]))$ as a group-object of $\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$. Then $B\mathcal{G}$ is 1-affine.*

2.5.11. Finally, let us take

$$\mathcal{G} = \underset{\overrightarrow{n}}{colim}\, (\mathbb{G}_a)^{\times n}.$$

In Sect. 12.1.5 we will show that $B\mathcal{G}$ is not 1-affine. In this example, the functor $\mathbf{\Gamma}^{\mathrm{enh}}_{B\mathcal{G}}$ fails to be fully faithful. We do not know whether $\mathbf{Loc}_{B\mathcal{G}}$ is fully faithful.

2.6. **De Rham prestacks.**

2.6.1. Let $\mathcal{Z}$ be a prestack. Recall that the prestack $\mathcal{Z}_{\mathrm{dR}}$ is defined by

$$\mathrm{Maps}(S, \mathcal{Z}_{\mathrm{dR}}) := \mathrm{Maps}(({}^{cl}S)_{red}, \mathcal{Z}).$$

2.6.2. In Sect. 12.1 we will prove:

Theorem 2.6.3. *Let $\mathcal{Y}$ be of the form $\mathcal{Z}_{\mathrm{dR}}$, where $\mathcal{Z}$ is an indscheme* [7] *locally of finite type. Then $\mathcal{Y}$ is 1-affine.*

[7] We say "indscheme" instead of "DG indscheme", because for a prestack $\mathcal{Z}$, the prestack $\mathcal{Z}_{\mathrm{dR}}$ only depends on the underlying classical prestack.

2.6.4. However, in Sect. 12.2 we will show:

Proposition 2.6.5. *Let $\mathcal{Y} = \mathcal{Z}_{\mathrm{dR}}$, where $\mathcal{Z} = \mathrm{pt}/\mathbb{G}_a$. Then $\mathcal{Y}$ is* not *1-affine.*

Hence, if $\mathcal{Y} = \mathcal{Z}_{\mathrm{dR}}$, where $\mathcal{Z}$ is quasi-compact algebraic stack locally of finite type with an affine diagonal, it is not in general true that $\mathcal{Y}$ is 1-affine.

However, we propose:

Conjecture 2.6.6. *Let $\mathcal{Z}$ is quasi-compact algebraic stack locally of finite type with an affine diagonal. Then the functor $\mathbf{Loc}_{\mathcal{Z}_{\mathrm{dR}}}$ is fully faithful.*

In fact, we conjecture that in the situation of Conjecture 2.6.6 it should be possible to describe the essential image of the functor $\mathbf{Loc}_{\mathcal{Z}_{\mathrm{dR}}}$ in terms of the groups of automorphisms of geometric points of $\mathcal{Z}$.

3. Direct and inverse images for sheaves of categories

3.1. **Definition of functors.** Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism between prestacks.

3.1.1. The monoidal functor

$$f^* : \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1)$$

defines a forgetful functor

$$\mathbf{res}_f : \mathrm{QCoh}(\mathcal{Y}_1)\text{-}\mathbf{mod} \to \mathrm{QCoh}(\mathcal{Y}_2)\text{-}\mathbf{mod}.$$

It has a left adjoint, denoted $\mathbf{ind}_f$, given by

$$\mathbf{C}_2 \mapsto \mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathbf{C}_2.$$

3.1.2. We also have a tautological functor

$$\mathbf{cores}_f : \mathrm{ShvCat}(\mathcal{Y}_2) \to \mathrm{ShvCat}(\mathcal{Y}_1).$$

Namely, for $\mathcal{C}_2 \in \mathrm{ShvCat}(\mathcal{Y}_2)$, we restrict the assignment $S \mapsto \mathbf{\Gamma}(S, \mathcal{C}_2)$ from $\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2}$ to $\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_1}$.

We shall sometimes denote this functor by

$$\mathcal{C}_2 \in \mathrm{ShvCat}(\mathcal{Y}_2) \mapsto \mathcal{C}_2|_{\mathcal{Y}_1} \in \mathrm{ShvCat}(\mathcal{Y}_1).$$

3.1.3. We claim that the functor $\mathbf{cores}_f$ admits a right adjoint (to be denoted $\mathbf{coind}_f$). Namely, for $\mathcal{C}_1 \in \mathrm{ShvCat}(\mathcal{Y}_2)$ and $S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2}$ we set

$$\mathbf{\Gamma}(S, \mathbf{coind}_f(\mathcal{C}_1)) := \mathbf{\Gamma}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \mathcal{C}_1).$$

The fact that this is indeed a quasi-coherent sheaf of categories follows from the next lemma:

Lemma 3.1.4. *For a map $S' \to S$ in $\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2}$, the functor*

$$\mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{\Gamma}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \mathcal{C}_1) \to \mathbf{\Gamma}(S' \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \mathcal{C}_1)$$

is an equivalence.

Proof. By definition, we calculate $\mathbf{\Gamma}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \mathcal{C}_1)$ as

$$\underset{T \in \mathrm{DGSch}^{\mathrm{aff}}_{/S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1}}{\underset{\longleftarrow}{lim}} \mathbf{\Gamma}(T, \mathcal{C}_1).$$

Since $\mathrm{QCoh}(S')$ is dualizable as a $\mathrm{QCoh}(S)$-module (by Lemma 1.4.7), tensoring with it commutes with limits in the category $\mathrm{QCoh}(S)$ - **mod**. Hence, we obtain:

$$\mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{\Gamma}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \mathcal{C}_1) \simeq \underset{T \in \mathrm{DGSch}^{\mathrm{aff}}_{/S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1}}{\underset{\longleftarrow}{lim}} \left(\mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{\Gamma}(T, \mathcal{C}_1) \right).$$

Note that the functor

$$\mathrm{DGSch}^{\mathrm{aff}}_{/S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1} \to \mathrm{DGSch}^{\mathrm{aff}}_{/S' \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1}, \quad T \mapsto T \underset{S}{\times} S'$$

is cofinal. Hence, we can calculate $\mathbf{\Gamma}(S' \underset{\mathcal{Y}_1}{\times} \mathcal{Y}_1, \mathcal{C}_1)$ as

$$\underset{T \in \mathrm{DGSch}^{\mathrm{aff}}_{/S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1}}{\underset{\longleftarrow}{lim}} \left(\mathbf{\Gamma}(S' \underset{S}{\times} T, \mathcal{C}_1) \right),$$

and the two expressions are manifestly isomorphic.

□

3.1.5. Let

$$\mathcal{Y}_1 \xrightarrow{f_{1,2}} \mathcal{Y}_2 \xrightarrow{f_{2,3}} \mathcal{Y}_3$$

be a pair of morphisms. We have an obvious isomorphism

$$\mathbf{cores}_{f_{1,2}} \circ \mathbf{cores}_{f_{2,3}} \simeq \mathbf{cores}_{f_{1,3}}.$$

By passing to right adjoints we obtain a canonical isomorphism

$$\mathbf{coind}_{f_{2,3}} \circ \mathbf{coind}_{f_{1,2}} \simeq \mathbf{coind}_{f_{1,3}}.$$

3.1.6. We note that the functor

$$\mathbf{\Gamma}(\mathcal{Y}, -) : \mathrm{ShvCat}(\mathcal{Y}) \to \mathrm{DGCat}_{\mathrm{cont}}$$

is a particular case of **coind**, namely, for for the morphism $p_{\mathcal{Y}} : \mathcal{Y} \to \mathrm{pt}$.

In particular, we obtain:

Lemma 3.1.7. *For a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ and $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y}_1)$ there is a canonical isomorphism*

$$\mathbf{\Gamma}(\mathcal{Y}_2, \mathbf{coind}_f(\mathcal{C})) \simeq \mathbf{\Gamma}(\mathcal{Y}_1, \mathcal{C}).$$

3.1.8. Let us note the following property of prestacks for which $\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}}$ is fully faithful:

Proposition 3.1.9. *Suppose $\mathcal{Y}$ is such that $\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}}$ is fully faithful. Then for $S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}$ and $f : \mathcal{Y}' \to \mathcal{Y}$, the map*

$$\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathrm{QCoh}(\mathcal{Y}') \to \mathrm{QCoh}(S \underset{\mathcal{Y}}{\times} \mathcal{Y}')$$

is an equivalence.

Proof. Consider the object

$$\mathcal{C} := \mathbf{coind}_f(\mathrm{QCoh}_{/\mathcal{Y}'}) \in \mathrm{ShvCat}(\mathcal{Y}).$$

Now, the two sides in the proposition are obtained by evaluation the two sides in

$$\mathbf{Loc}_{\mathcal{Y}}(\mathbf{\Gamma}^{\mathrm{enh}}(\mathcal{Y}, \mathcal{C})) \to \mathcal{C}$$

on $S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}$. □

3.2. **Commutation of diagrams.** Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism of presracks.

3.2.1. We note that the following diagram is commutative by construction

$$\begin{array}{ccc} \mathrm{QCoh}(\mathcal{Y}_1)\text{-}\mathbf{mod} & \xrightarrow{\mathbf{Loc}_{\mathcal{Y}_1}} & \mathrm{ShvCat}(\mathcal{Y}_1) \\ \mathbf{ind}_f \Big\uparrow & & \Big\uparrow \mathbf{cores}_f \\ \mathrm{QCoh}(\mathcal{Y}_2)\text{-}\mathbf{mod} & \xrightarrow{\mathbf{Loc}_{\mathcal{Y}_2}} & \mathrm{ShvCat}(\mathcal{Y}_2). \end{array} \tag{3.1}$$

By adjunction, the following diagram is commutative as well:

$$\begin{array}{ccc} \mathrm{QCoh}(\mathcal{Y}_1)\text{-}\mathbf{mod} & \xleftarrow{\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}_1}} & \mathrm{ShvCat}(\mathcal{Y}_1) \\ \mathbf{res}_f \Big\downarrow & & \Big\downarrow \mathbf{coind}_f \\ \mathrm{QCoh}(\mathcal{Y}_2)\text{-}\mathbf{mod} & \xleftarrow{\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}_2}} & \mathrm{ShvCat}(\mathcal{Y}_2). \end{array} \tag{3.2}$$

3.2.2. Hence, each of the following two diagrams

$$\begin{array}{ccc} \mathrm{QCoh}(\mathcal{Y}_1)\text{-}\mathbf{mod} & \xrightarrow{\mathbf{Loc}_{\mathcal{Y}_1}} & \mathrm{ShvCat}(\mathcal{Y}_1) \\ \mathbf{res}_f \Big\downarrow & & \Big\downarrow \mathbf{coind}_f \\ \mathrm{QCoh}(\mathcal{Y}_2)\text{-}\mathbf{mod} & \xrightarrow{\mathbf{Loc}_{\mathcal{Y}_2}} & \mathrm{ShvCat}(\mathcal{Y}_2) \end{array} \tag{3.3}$$

and

$$\begin{array}{ccc} \mathrm{QCoh}(\mathcal{Y}_1)\text{-}\mathbf{mod} & \xleftarrow{\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}_1}} & \mathrm{ShvCat}(\mathcal{Y}_1) \\ \mathbf{ind}_f \Big\uparrow & & \Big\uparrow \mathbf{cores}_f \\ \mathrm{QCoh}(\mathcal{Y}_2)\text{-}\mathbf{mod} & \xleftarrow{\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}_2}} & \mathrm{ShvCat}(\mathcal{Y}_2) \end{array} \tag{3.4}$$

commutes *up to a natural transformation.*

3.2.3. For future use, let us record the following:

Lemma 3.2.4. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism between 1-affine prestacks. Then for $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y}_2)$, the functor*

$$\mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathbf{\Gamma}^{\mathrm{enh}}(\mathcal{Y}_2, \mathcal{C}) \to \mathbf{\Gamma}^{\mathrm{enh}}(\mathcal{Y}_1, \mathbf{cores}_f(\mathcal{C}))$$

is an equivalence, i.e., the natural transformation in the diagram (3.4) *is an isomorphism.*

Proof. Follows from the commutation of (3.1), as the horizontal arrows are equivalences. □

3.2.5. The following assertion will be a key tool for many proofs:

Proposition 3.2.6. *Assume that for any $S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2}$ the map*

$$\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathrm{QCoh}(\mathcal{Y}_1) \to \mathrm{QCoh}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1)$$

is an equivalence.

(a) *Suppose that f is such that its base change by every affine DG scheme yields a prestack for which* **Loc** *is fully faithful. Then:*

(i) *The diagram* (3.3) *commutes, i.e., the natural transformation is an isomorphism.*

(ii) *If* $\mathrm{QCoh}(\mathcal{Y}_1)$ *is dualizable as an object of* $\mathrm{QCoh}(\mathcal{Y}_2)$ - **mod**, *then the diagram* (3.4) *commutes, i.e., the natural transformation is an isomorphism.*

(iii) *If* $\mathbf{Loc}_{\mathcal{Y}_2}$ *is fully faithful, then so is* $\mathbf{Loc}_{\mathcal{Y}_1}$.

(b) *Suppose that f is such that its base change by every affine DG scheme yields a 1-affine prestack. Then if* $\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}_2}$ *is fully faithful, then so is* $\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}_1}$.

Corollary 3.2.7. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map, where $\mathcal{Y}_2$ is 1-affine, and the base change of f by an affine DG scheme yields a 1-affine prestack. Then $\mathcal{Y}_1$ is 1-affine.*

Proof. Follows from Proposition 3.2.6, points (a,iii) and (b). The condition of the proposition holds because of Proposition 3.1.9. □

Corollary 3.2.8. *The product of two 1-affine prestacks is 1-affine.*

3.3. Proof of Proposition 3.2.6.

3.3.1. *Proof of point* (a,i).

Fix $\mathbf{C}_1 \in \mathrm{QCoh}(\mathcal{Y}_1)$ - **mod** and $S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2}$. By definition:

$$\mathbf{\Gamma}(S, \mathbf{Loc}_{\mathcal{Y}_2} \circ \mathbf{res}_f(\mathbf{C}_1)) = \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathbf{C}_1 \simeq$$
$$\simeq (\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathrm{QCoh}(\mathcal{Y}_1)) \underset{\mathrm{QCoh}(\mathcal{Y}_1)}{\otimes} \mathbf{C}_1,$$

while the latter maps isomorphically to

$$\mathrm{QCoh}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y}_1)}{\otimes} \mathbf{C}_1,$$

by the assumption of the proposition.

Also,

$$\mathbf{\Gamma}(S, \mathbf{coind}_f \circ \mathbf{Loc}_{\mathcal{Y}_1}(\mathbf{C}_1)) \simeq \mathbf{\Gamma}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \mathbf{Loc}_{\mathcal{Y}_1}(\mathbf{C}_1)).$$

Set

$$\mathbf{C} := \mathrm{QCoh}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y}_1)}{\otimes} \mathbf{C}_1 \in \mathrm{QCoh}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1) \text{ - } \mathbf{mod}.$$

We have

$$\mathbf{Loc}_{\mathcal{Y}_1}(\mathbf{C}_1)|_{S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1} \simeq \mathbf{Loc}_{S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1}(\mathbf{C}),$$

and hence

$$\mathbf{\Gamma}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \mathbf{Loc}_{\mathcal{Y}_1}(\mathbf{C}_1)) \simeq \mathbf{\Gamma}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \mathbf{Loc}_{S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1}(\mathbf{C})).$$

Now, the assumtion in (a) implies that the latter is isomorphic to $\mathbf{C}$ itself, as desired.

□

3.3.2. For the proof of point (a,ii) we will need the following assertion:

Lemma 3.3.3. *Under the assumption of (a) for $S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2}$ we have:*

(1) *For $\mathbf{C} \in \mathrm{QCoh}(S)\text{-}\mathbf{mod}$, the natural map*

$$\mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathbf{C} \to \mathbf{\Gamma}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \mathbf{Loc}_S(\mathbf{C})) \tag{3.5}$$

is an isomorphism.

(2) *For $\mathcal{C}_2 \in \mathrm{ShvCat}(\mathcal{Y}_2)$, the natural map*

$$\mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathbf{\Gamma}(S, \mathcal{C}_2) \to \mathbf{\Gamma}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \mathcal{C}_2) \tag{3.6}$$

is an isomorphism.

Proof. We rewrite the left-hand side in (3.5) as

$$(\mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathrm{QCoh}(S)) \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C},$$

which by the assumption of the proposition maps isomorphically to

$$\mathrm{QCoh}(S \underset{\mathcal{Y}_1}{\times} \mathcal{Y}_2) \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C}.$$

So, in order to prove that (3.5) is an isomorphism, we have to show that the natural map

$$\mathrm{QCoh}(S \underset{\mathcal{Y}_1}{\times} \mathcal{Y}_2) \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C} \to \mathbf{\Gamma}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \mathbf{Loc}_S(\mathbf{C})) \tag{3.7}$$

is an isomorphism.

Set

$$\mathbf{C}' := \mathrm{QCoh}(S \underset{\mathcal{Y}_1}{\times} \mathcal{Y}_2) \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C} \in \mathrm{QCoh}(S \underset{\mathcal{Y}_1}{\times} \mathcal{Y}_2)\text{-}\mathbf{mod}.$$

We have:

$$\mathbf{Loc}_S(\mathbf{C})|_{S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1} \simeq \mathbf{Loc}_{S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1}(\mathbf{C}'),$$

and the map in (3.7) identifies with

$$\mathbf{C}' \to \mathbf{\Gamma}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \mathbf{Loc}_{S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1}(\mathbf{C}')),$$

which is an isomorphism by the assumption in (a). This shows that (3.5) is an isomorphism.

To prove that (3.6) is an isomorphism, we note that the two sides identify with the corresponding sides in (3.5) for

$$\mathbf{C} := \mathbf{\Gamma}(S, \mathcal{C}_2),$$

using the fact that

$$\mathcal{C}_2|_S \simeq \mathbf{Loc}_S(\mathbf{\Gamma}(S, \mathcal{C}_2)).$$

□

3.3.4. *Proof of point* (a,ii). For $\mathcal{C}_2 \in \mathrm{ShvCat}(\mathcal{Y}_2)$, we have

$$\mathbf{\Gamma}(\mathcal{Y}_1, \mathbf{cores}_f(\mathcal{C}_2)) \simeq \mathbf{\Gamma}(\mathcal{Y}_2, \mathbf{coind}_f \circ \mathbf{cores}_f(\mathcal{C}_2)) \simeq \underset{S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2}}{\underset{\longleftarrow}{lim}} \mathbf{\Gamma}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \mathcal{C}_2). \tag{3.8}$$

By Lemma 3.3.3(2), we have

$$\mathbf{\Gamma}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \mathcal{C}_2) \simeq \mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathbf{\Gamma}(S, \mathcal{C}_2).$$

Hence, the expression in (3.8) identifies with

$$\underset{S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2}}{\underset{\longleftarrow}{lim}} \mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathbf{\Gamma}(S, \mathcal{C}_2). \tag{3.9}$$

Now, the assumption that $\mathrm{QCoh}(\mathcal{Y}_1)$ is dualizable as an object of $\mathrm{QCoh}(\mathcal{Y}_2)\text{-}\mathbf{mod}$ implies that

$$\mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} - : \mathrm{QCoh}(\mathcal{Y}_2)\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}$$

commutes with limits, so we can rewrite the expression in (3.9) as

$$\mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \left(\underset{S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2}}{\underset{\longleftarrow}{lim}} \mathbf{\Gamma}(S, \mathcal{C}_2) \right) \simeq \mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathbf{\Gamma}(\mathcal{Y}_2, \mathcal{C}_2),$$

as desired.

3.3.5. *Proof of point* (a,iii). We need to show that the unit map

$$\mathbf{C}_1 \to \mathbf{\Gamma}^{\mathrm{enh}}(\mathcal{Y}_1, \mathbf{Loc}_{\mathcal{Y}_1}(\mathbf{C}_1))$$

is an isomorphism. Note that the functor $\mathbf{res}_f$ is conservative. Hence, it suffices to show that

$$\mathbf{res}_f(\mathbf{C}_1) \to \mathbf{res}_f\left(\mathbf{\Gamma}^{\mathrm{enh}}(\mathcal{Y}_1, \mathbf{Loc}_{\mathcal{Y}_1}(\mathbf{C}_1))\right)$$

is an isomorphism. However, we have a commutative diagram

$$\begin{array}{ccc} \mathbf{res}_f\left(\mathbf{\Gamma}^{\mathrm{enh}}(\mathcal{Y}_1, \mathbf{Loc}_{\mathcal{Y}_1}(\mathbf{C}_1))\right) & \xrightarrow{\sim} & \mathbf{\Gamma}^{\mathrm{enh}}\left(\mathcal{Y}_2, \mathbf{coind}_f(\mathbf{Loc}_{\mathcal{Y}_1}(\mathbf{C}_1))\right) \\ \uparrow & & \uparrow \\ \mathbf{res}_f(\mathbf{C}_1) & \longrightarrow & \mathbf{\Gamma}^{\mathrm{enh}}\left(\mathcal{Y}_2, \mathbf{Loc}_{\mathcal{Y}_2}(\mathbf{res}_f(\mathbf{C}_1))\right), \end{array}$$

where the right vertical arrow is an isomorphism by point (a,i). Hence, if $\mathbf{Loc}_{\mathcal{Y}_2}$ is fully faithful, the bottom horizontal arrow is an isomorphism, and hence so is the left vertical arrow.

3.3.6. *Proof of point* (b). By an argument similar that in point (a,iii), it suffices to show that under the assumption of point (b), the functor $\mathbf{coind}_f$ is conservative.

Let $\phi : \mathcal{C}'_1 \to \mathcal{C}''_1$ be a morphism in $\mathrm{ShvCat}(\mathcal{Y}_1)$, such that $\mathbf{coind}_f(\phi)$ is an isomorphism. We need to show that for every $T \in \mathrm{DGSch}^{\mathrm{aff}}_{\mathcal{Y}_1}$, the resulting map

$$\mathbf{\Gamma}(T, \mathcal{C}'_1) \to \mathbf{\Gamma}(T, \mathcal{C}''_1) \tag{3.10}$$

is an isomorphism (under the assumption of the proposition).

Set $\mathcal{Z} := T \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$, considered as a prestack over $\mathcal{Y}_1$. Consider the corresponding map

$$\mathcal{C}'_1|_{\mathcal{Z}} \to \mathcal{C}''_1|_{\mathcal{Z}}. \tag{3.11}$$

The assumption that $\mathbf{coind}_f(\phi)$ is an isomorphism implies that the induced map

$$\mathbf{\Gamma}(\mathcal{Z}, \mathcal{C}_1'|_{\mathcal{Z}}) \to \mathbf{\Gamma}(\mathcal{Z}, \mathcal{C}_1''|_{\mathcal{Z}})$$

is an isomorphism. Now, the fact that $\mathcal{Z}$ is 1-affine implies that (3.11) is an isomorphism as well.

Evaluating (3.11) on $T \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Z}}$, we obtain that (3.10) is an isomorphism, as required. $\square$

4. The case of formal completions

Let $\mathcal{Y}$ be a 1-affine prestack, $\mathcal{Y}' \overset{\iota}{\to} \mathcal{Y}$ a closed embedding, and let $\mathcal{Y}_0 \overset{\jmath}{\hookrightarrow} \mathcal{Y}$ be the complementary open. Throughout this section, we will be assuming that $\jmath$ is quasi-compact.

4.1. QCoh-modules on a formal completion.

4.1.1. We have an adjoint pair of functors

$$\jmath^* : \mathrm{QCoh}(\mathcal{Y}) \rightleftarrows \mathrm{QCoh}(\mathcal{Y}_0) : \jmath_*.$$

The assumption that $\jmath$ is quasi-compact implies that $\jmath_*$ is continuous (see, e.g., [GL:QCoh, Proposition 2.1.1]).

Let $\mathrm{QCoh}(\mathcal{Y})_{\mathcal{Y}'}$ be the full subcategory of $\mathrm{QCoh}(\mathcal{Y})$ consisting of objects set-theoretically supported on $\mathcal{Y}'$, i.e., $\mathrm{QCoh}(\mathcal{Y})_{\mathcal{Y}'} = \ker(\jmath^*)$.

Let

$$\widehat{\imath}_!^{\mathrm{QCoh}} : \mathrm{QCoh}(\mathcal{Y})_{\mathcal{Y}'} \hookrightarrow \mathrm{QCoh}(\mathcal{Y})$$

denote the tautological embedding.

The functor $\widehat{\imath}_!^{\mathrm{QCoh}}$ admits a continuous right adjoint, denoted by $\widehat{\imath}^{\mathrm{QCoh},!}$, and given by

$$\mathcal{F} \mapsto \mathrm{Cone}(\mathcal{F} \to \jmath_* \circ \jmath^*(\mathcal{F}))[-1].$$

We obtain a localization sequence:

$$\mathrm{QCoh}(\mathcal{Y})_{\mathcal{Y}'} \underset{\widehat{\imath}^{\mathrm{QCoh},!}}{\overset{\widehat{\imath}_!^{\mathrm{QCoh}}}{\rightleftarrows}} \mathrm{QCoh}(\mathcal{Y}) \underset{\jmath_*}{\overset{\jmath^*}{\rightleftarrows}} \mathrm{QCoh}(\mathcal{Y}_0). \tag{4.1}$$

4.1.2. Let $\mathcal{Y}^\wedge_{\mathcal{Y}'}$ the formal completion of $\mathcal{Y}$ along $\mathcal{Y}'$, see [GR1, Defn. 6.1.2]. I.e., $\mathcal{Y}^\wedge_{\mathcal{Y}'}$ is the prestack defined by

$$\mathrm{Maps}(S, \mathcal{Y}^\wedge_{\mathcal{Y}'}) := \mathrm{Maps}(S, \mathcal{Y}) \underset{\mathrm{Maps}((^{cl}S)_{red}, \mathcal{Y})}{\times} \mathrm{Maps}((^{cl}S)_{red}, \mathcal{Y}').$$

Let $\widehat{\imath} : \mathcal{Y}^\wedge_{\mathcal{Y}'} \to \mathcal{Y}$ denote the tautological map.

The following results from [Ga, Proposition 7.1.3] by base change:

Proposition 4.1.3. *The functor*

$$\widehat{\imath}^* : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}^\wedge_{\mathcal{Y}'})$$

factors as

$$\mathrm{QCoh}(\mathcal{Y}) \overset{\widehat{\imath}^{\mathrm{QCoh},!}}{\longrightarrow} \mathrm{QCoh}(\mathcal{Y})_{\mathcal{Y}'} \to \mathrm{QCoh}(\mathcal{Y}^\wedge_{\mathcal{Y}'}),$$

where the second arrow is an equivalence.

4.1.4. Consider now the adjoint functors

$$\mathbf{ind}_{\widehat{i}} : \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod} \rightleftarrows \mathrm{QCoh}(\mathcal{Y}^{\wedge}_{\mathcal{Y}'})\text{-}\mathbf{mod} : \mathbf{res}_{\widehat{i}}.$$

Proposition 4.1.5. *The functor* $\mathbf{res}_{\widehat{i}}$ *is fully faithful; its essential image consists of those* $\mathbf{C}_{\mathcal{Y}} \in \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$*, on which* $\ker(\widehat{i}^*) = \mathrm{Im}(\jmath_*) \subset \mathrm{QCoh}(\mathcal{Y})$ *acts trivially (i.e., by zero).*

The assertion of the proposition follows from Proposition 4.1.3 and the next general assertion:

Let $\mathbf{O}$ be a monoidal DG category, and let $\mathsf{F} : \mathbf{O} \to \mathbf{O}'$ be a monoidal functor. Let $\mathbf{C}$ be an $\mathbf{O}$-module category, on which $\ker(\mathsf{F})$ acts by zero.

Lemma 4.1.6. *Assume that* F *admits a fully faithful continuous right or left adjoint, which is a map of right* $\mathbf{O}$*-module categories. Then the canonical map*

$$\mathbf{C} \simeq \mathbf{O} \underset{\mathbf{O}}{\otimes} \mathbf{C} \to \mathbf{O}' \underset{\mathbf{O}}{\otimes} \mathbf{C}$$

is an equivalence.

Proof. Let F have a fully continuous right adjoint. The assumption on F implies that in the localization sequence

$$\mathbf{O}' \overset{\mathsf{F}}{\rightleftarrows} \mathbf{O} \rightleftarrows \ker(\mathsf{F}),$$

all functors are maps of right $\mathbf{O}$-module categories. Hence, tensoring up by $\mathbf{C}$ over $\mathbf{O}$ on the right, we obtain a localization sequence of DG categories:

$$\mathbf{O}' \underset{\mathbf{O}}{\otimes} \mathbf{C} \overset{\mathsf{F}\otimes \mathrm{Id}_{\mathbf{C}}}{\rightleftarrows} \mathbf{C} \rightleftarrows \ker(\mathsf{F}) \underset{\mathbf{O}}{\otimes} \mathbf{C}.$$

However, the assumption on $\mathbf{C}$ says that the functor $\mathbf{C} \leftarrow \ker(\mathsf{F}) \underset{\mathbf{O}}{\otimes} \mathbf{C}$ is zero, and since this functor is fully faithful, we obtain that $\ker(\mathsf{F}) \underset{\mathbf{O}}{\otimes} \mathbf{C} = 0$. Hence, the functor

$$\mathbf{O}' \underset{\mathbf{O}}{\otimes} \mathbf{C} \leftarrow \mathbf{C},$$

is an equivalence, as desired.

The proof when F admits a fully faithful left adjoint is similar.

□

4.2. **Sheaves of categories on a formal completion.**

4.2.1. Consider now the pair of adjoint functors

$$\mathbf{cores}_{\widehat{i}} : \mathrm{ShvCat}(\mathcal{Y}) \rightleftarrows \mathrm{ShvCat}(\mathcal{Y}^{\wedge}_{\mathcal{Y}'}) : \mathbf{coind}_{\widehat{i}},$$

see Sect. 3.1.3.

Since $\mathcal{Y}^{\wedge}_{\mathcal{Y}'} \underset{\mathcal{Y}}{\times} \mathcal{Y}^{\wedge}_{\mathcal{Y}'} \simeq \mathcal{Y}^{\wedge}_{\mathcal{Y}'}$, the adjunction map

$$\mathbf{cores}_{\widehat{i}} \circ \mathbf{coind}_{\widehat{i}} \to \mathrm{Id}$$

is an isomorphism. Hence, the functor $\mathbf{coind}_{\widehat{i}}$ is fully faithful. We now claim:

Proposition 4.2.2. *The essential image of* $\mathbf{coind}_{\widehat{i}}$ *consists of those* $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y})$*, for which* $\mathcal{C}|_{\mathcal{Y}_0} = 0$.

Proof. The assertion readily reduces to the case when $\mathcal{Y} = S \in \mathrm{DGSch}^{\mathrm{aff}}$. Let $S' \subset S$ be a closed DG subscheme whose complement $S_0 \overset{\jmath}{\hookrightarrow} S$ is quasi-compact. We need to show that for

$$\mathbf{C} \in \mathrm{QCoh}(S)\text{-}\mathbf{mod},$$

on which the action of $\mathrm{QCoh}(S)$ factors through the restriction functor $\mathrm{QCoh}(S) \to \mathrm{QCoh}(S^\wedge_{S'})$, the map

$$\mathbf{C} = \mathbf{\Gamma}(S, \mathbf{Loc}_S(\mathbf{C})) \to \mathbf{\Gamma}(S^\wedge_{S'}, \mathbf{Loc}_S(\mathbf{C}))$$

is an equivalence.

By [GR1, Proposition 6.7.4], we can exhibit $S^\wedge_{S'}$ as

$$\underset{\longrightarrow}{lim}_n S_n,$$

where S_n are closed subschemes of S, and the transition maps $\iota_{n_1,n_2} : S_{n_1} \to S_{n_2}$ are such that the functors $\iota^*_{n_1,n_2}$ admit *left* adjoints.

In this case, by [GL:DG, Lemma 1.3.3], we calculate

$$\mathrm{QCoh}(S^\wedge_{S'}) := \underset{\longleftarrow}{lim}_n \mathrm{QCoh}(S_n) \simeq \underset{\longrightarrow}{colim}_n \mathrm{QCoh}(S_n),$$

where the *limit* is taken with respect to the transition functors $\iota^*_{n_1,n_2}$, and the *colimit* is taken with respect to the transition functors $(\iota^*_{n_1,n_2})^L$.

Similarly,

$$\mathbf{\Gamma}(S^\wedge_{S'}, \mathbf{Loc}_S(\mathbf{C})) := \underset{\longleftarrow}{lim}_n \left(\mathbf{C} \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{QCoh}(S_n)\right) \simeq \underset{\longrightarrow}{colim}_n \left(\mathbf{C} \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{QCoh}(S_n)\right) \simeq$$

$$\simeq \mathbf{C} \underset{\mathrm{QCoh}(S)}{\otimes} \left(\underset{\longrightarrow}{colim}_n \mathrm{QCoh}(S_n)\right) \simeq \mathbf{C} \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{QCoh}(S^\wedge_{S'}) \simeq \mathbf{C},$$

where the latter is isomorphism holds by Proposition 4.1.3 and Lemma 4.1.6.

□

4.3. **Proof of Theorem 2.3.1.**

4.3.1. Recall the commutative diagram:

$$\begin{array}{ccc} \mathrm{ShvCat}(\mathcal{Y}^\wedge_{\mathcal{Y}'}) & \xrightarrow{\mathbf{coind}_{\hat{\imath}}} & \mathrm{ShvCat}(\mathcal{Y}) \\ \mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}^\wedge_{\mathcal{Y}'}} \downarrow & & \downarrow \mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}} \\ \mathrm{QCoh}(\mathcal{Y}^\wedge_{\mathcal{Y}'})\text{-}\mathbf{mod} & \xrightarrow{\mathbf{res}_{\hat{\imath}}} & \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}. \end{array}$$

By Propositions 4.1.5 and 4.2.2, the horizontal arrows in this diagram are fully faithful. Hence, we obtain that if $\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}}$ is fully faithful, then so is $\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}^\wedge_{\mathcal{Y}'}}$.

4.3.2. We now claim that the map

$$\widehat{i} : \mathcal{Y}^{\wedge}_{\mathcal{Y}'} \to \mathcal{Y}$$

satisfies the conditions of Proposition 3.2.6(a).

Indeed, the assumption of Proposition 3.2.6 is satisfied by the localization sequence (4.1) and [GL:QCoh, Proposition 3.2.1], applied to $\jmath$. The assumption of Proposition 3.2.6(a) is satisfied by Sect. 4.3.1 above, applied to an affine DG scheme and its formal completion.

Hence, by Proposition 3.2.6(a,i), we obtain that the diagram

$$\begin{array}{ccc} \mathrm{ShvCat}(\mathcal{Y}^{\wedge}_{\mathcal{Y}'}) & \xrightarrow{\mathbf{coind}_{\widehat{i}}} & \mathrm{ShvCat}(\mathcal{Y}) \\ {\scriptstyle \mathbf{Loc}_{\mathcal{Y}^{\wedge}_{\mathcal{Y}'}}} \uparrow & & \uparrow {\scriptstyle \mathbf{Loc}_{\mathcal{Y}}} \\ \mathrm{QCoh}(\mathcal{Y}^{\wedge}_{\mathcal{Y}'})\text{-}\mathbf{mod} & \xrightarrow{\mathbf{res}_{\widehat{i}}} & \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}. \end{array}$$

commutes as well.

4.3.3. From the above diagram, we conlcude that if the functor $\mathbf{Loc}_{\mathcal{Y}}$ is fully faithful, then so is $\mathbf{Loc}_{\mathcal{Y}^{\wedge}_{\mathcal{Y}'}}$.

□

Part II: 1-Affinness of Algebraic Stacks

5. Algebraic stacks: preparations

5.1. Passable prestacks.

5.1.1. We give the following definition, taken from [GL:QCoh, Sect. 3.3]:

Definition 5.1.2. *A prestack $\mathcal{Y}$ is called* passable *if it satisfies:*

- *The diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is schematic, quasi-affine and quasi-compact;*
- $\mathcal{O}_{\mathcal{Y}} \in \mathrm{QCoh}(\mathcal{Y})$ *is compact;*
- $\mathrm{QCoh}(\mathcal{Y})$ *is dualizable as a plain DG category.*

As our initial observation, we will prove:

Proposition 5.1.3. *Let $\mathcal{Y}$ be a passable prestack. Then the functor*

$$\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}} : \mathrm{ShvCat}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$$

is fully faithful.

5.1.4. Many algebraic stacks are passable as prestacks. In particular, any algebraic stack, which in the terminology of [BFN] is *perfect*, is passable.

The following assertion is proved in [DrGa, Theorems 4.3.1 and 1.4.2]:

Theorem 5.1.5. *An eventually coconnective QCA* [8] *algebraic stack locally almost of finite type is passable.*

We conjecture that in Theorem 5.1.5 the hypothesis that $\mathcal{Y}$ should be eventually coconnective is unnecessary.

[8] Under the "locally almost of finite type" assumption, QCA means "quasi-compact and the automorphism group of every field-valued point is affine."

5.1.6. The proof of Proposition 5.1.3 will use the following ingredient (see [GL:QCoh, Proposition 2.3.2]; the proof is reproduced in Sect. B.2.4 for the reader's convenience):

Proposition 5.1.7. *If $\mathcal{Y}$ is passable, the category* $\mathrm{QCoh}(\mathcal{Y})$ *is rigid as a monoidal DG category.*

Proof of Proposition 5.1.3. We need to show that for $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y})$ and $T \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}$, the canonical map

$$\mathrm{QCoh}(T) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{\Gamma}(\mathcal{Y}, \mathcal{C}) \to \mathbf{\Gamma}(T, \mathcal{C})$$

is an isomorphism. We will prove this by applying Proposition 3.2.6(a,ii) to the morphism $T \to \mathcal{Y}$.

The condition of Proposition 3.2.6 holds by [GL:QCoh, Proposition 3.3.3] (for the reader's convenience we will reproduce the proof in Proposition B.2.2).

The condition of Proposition 3.2.6(a) holds by Theorem 2.1.1 in the case of the quasi-affine schemes (which will be proved independently).

The condition of Proposition 3.2.6(a,ii) holds by Lemma 1.4.7, using Proposition 5.1.7.

□

5.2. **A corollary of fully faithfulness of Loc.**

5.2.1. Let $\mathcal{Y}_1, \mathcal{Y}_2$ be two objects of PreStk, and recall that we have a canonically defined (symmetric monoidal) functor

$$\mathrm{QCoh}(\mathcal{Y}_1) \otimes \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1 \times \mathcal{Y}_2). \tag{5.1}$$

Recall also that the map (5.1) is an equivalence if for one of the prestacks, the category $\mathrm{QCoh}(\mathcal{Y}_i)$ is dualizable (for the proof see, e.g., [GL:QCoh, Proposition 1.4.4]).

5.2.2. We shall now prove:

Proposition 5.2.3. *Suppose that $\mathcal{Y}_1$ is such that $\mathbf{Loc}_{\mathcal{Y}_1}$ is fully faithful. Then* (5.1) *is an equivalence.*

Proof. Consider

$$\mathbf{C} := \mathrm{QCoh}(\mathcal{Y}_1) \otimes \mathrm{QCoh}(\mathcal{Y}_2) \in \mathrm{QCoh}(\mathcal{Y}_1)\text{-}\mathbf{mod}.$$

The value of $\mathbf{Loc}_{\mathcal{Y}_1}(\mathbf{C})$ on $S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_1}$ is

$$\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathcal{Y}_1)}{\otimes} (\mathrm{QCoh}(\mathcal{Y}_1) \otimes \mathrm{QCoh}(\mathcal{Y}_2)) \simeq \mathrm{QCoh}(S) \otimes \mathrm{QCoh}(\mathcal{Y}_2),$$

and the latter is isomorphic to $\mathrm{QCoh}(S \times \mathcal{Y}_2)$, since $\mathrm{QCoh}(S)$ is dualiazable.

Hence, the category $\mathbf{\Gamma}(\mathcal{Y}_1, \mathbf{Loc}_{\mathcal{Y}_1}(\mathbf{C}))$ is

$$\underset{S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_1}}{\underset{\longleftarrow}{lim}} \mathrm{QCoh}(S \times \mathcal{Y}_2),$$

and the latter is isomorphic to $\mathrm{QCoh}(\mathcal{Y}_1 \times \mathcal{Y}_2)$.

Hence, if $\mathbf{C} \to \mathbf{\Gamma}(\mathcal{Y}_1, \mathbf{Loc}_{\mathcal{Y}_1}(\mathbf{C}))$ is an equivalence, then so is

$$\mathrm{QCoh}(\mathcal{Y}_1) \otimes \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1 \times \mathcal{Y}_2).$$

□

5.2.4. As a corollary of Proposition 5.2.3, we obtain:

Corollary 5.2.5. *Let* $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathrm{PreStk}$ *such that the functors* $\mathbf{Loc}_{\mathcal{Y}_i}$ $i = 1, 2$ *are fully faithful. Then* $\mathbf{Loc}_{\mathcal{Y}_1 \times \mathcal{Y}_2}$ *is also fully faithful.*

Proof. Follows from Proposition 3.2.6(a,iii), applied to the map $\mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathcal{Y}_1$. □

5.2.6. As another corollary of Proposition 5.2.3, we obtain:

Corollary 5.2.7. *Let* $\mathcal{Y}$ *be prestack that satisfies:*

- *The diagonal morphism* $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ *is representable, quasi-compact and quasi-separated;*
- $\mathcal{O}_{\mathcal{Y}} \in \mathrm{QCoh}(\mathcal{Y})$ *is compact;*
- $\mathbf{Loc}_{\mathcal{Y}}$ *is fully faithful.*

Then $\mathrm{QCoh}(\mathcal{Y})$ *is rigid as a monoidal DG category, and in particular, dualizable as a plain DG category.*

Proof. Follows from [GL:QCoh, Proposition 2.3.2]. □

5.2.8. Hence, we we obtain:

Corollary 5.2.9. *Let* $\mathcal{Y} \in \mathrm{PreStk}$ *be such that:*

- *The diagonal morphism* $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ *is schematic, quasi-affine and quasi-compact;*
- $\mathcal{O}_{\mathcal{Y}} \in \mathrm{QCoh}(\mathcal{Y})$ *is compact.*

Then we have the following implications:

$$\mathbf{Loc}_{\mathcal{Y}} \text{ is fully faithful} \Rightarrow \mathcal{Y} \text{ is passable} \Rightarrow \mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}} \text{ is fully faithful.}$$

5.3. **Algebraic spaces: proof of Theorem 2.1.1.** We shall first prove Theorem 2.1.1, assuming its validity in the case of quasi-compact quasi-affine schemes. In particular, we assume the validity of Proposition 5.1.3.

5.3.1. First, we know that any quasi-compact quasi-separated algebraic space $\mathcal{Y}$ is passable as a prestack: this is given by [GL:QCoh, Propositions 2.2.2 and 2.3.6]. Hence, the functor $\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}}$ is fully faithful by Proposition 5.1.3. It remains to show that $\mathbf{Loc}_{\mathcal{Y}}$ is fully faithful.

By [GL:QCoh, Lemma 2.2.4], we can exhibit $\mathcal{Y}$ is a finite union

$$\emptyset = \mathcal{Y}_0 \subset \mathcal{Y}_1 \subset \ldots \subset \mathcal{Y}_{k-1} \subset \mathcal{Y}_k = \mathcal{Y}$$

of open subsets, such that for each $1 \leq i \leq k$ there exists a quasi-affine quasi-compact scheme U_k equipped with étale map $f_k : U_k \to \mathcal{Y}_k$, which is one-to-one over $\mathcal{Y}_k - \mathcal{Y}_{k-1}$.

By induction, we can assume that $\mathcal{Y}_{k-1}$ is 1-affine. Thus, we can assume having a Cartesian diagram of algebraic spaces

$$\begin{array}{ccc} U' & \xrightarrow{j_U} & U \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{j_{\mathcal{Y}}} & \mathcal{Y}, \end{array}$$

where the horizontal maps are open embeddings, the vertical maps are étale, f is one-to-one over $\mathcal{Y} - \mathcal{Y}'$, and $U, U', \mathcal{Y}'$ are 1-affine. We wish to deduce that the functor $\mathbf{Loc}_{\mathcal{Y}}$ is fully faithful.

5.3.2. In Sect. A.1.2 it is shown that for $\mathbf{C} \in \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$ the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(U') \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} & \xleftarrow{(j_U)^* \otimes \mathrm{Id}_{\mathbf{C}}} & \mathrm{QCoh}(U) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \\ {\scriptstyle (f')^* \otimes \mathrm{Id}_{\mathbf{C}}} \uparrow & & \uparrow {\scriptstyle f^* \otimes \mathrm{Id}_{\mathbf{C}}} \\ \mathrm{QCoh}(\mathcal{Y}') \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} & \xleftarrow{(j_{\mathcal{Y}})^* \otimes \mathrm{Id}_{\mathbf{C}}} & \mathbf{C} \end{array} \tag{5.2}$$

is a pull-back diagram of DG categories.

For $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y})$ consider the diagram

$$\begin{array}{ccc} \mathbf{\Gamma}(U', \mathcal{C}) & \xleftarrow{(j_U)^*_{\mathcal{C}}} & \mathbf{\Gamma}(U, \mathcal{C}) \\ {\scriptstyle (f')^*_{\mathcal{C}}} \uparrow & & \uparrow {\scriptstyle f^*_{\mathcal{C}}} \\ \mathbf{\Gamma}(\mathcal{Y}', \mathcal{C}) & \xleftarrow{(j_{\mathcal{Y}})^*_{\mathcal{C}}} & \mathbf{\Gamma}(\mathcal{Y}, \mathcal{C}), \end{array} \tag{5.3}$$

which is a pull-back diagram by Corollary 1.5.5(a) (applied to Nisnevich covers).

Taking $\mathcal{C} := \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C})$, and using the fact that $\mathcal{Y}'$ is 1-affine, we obtain that the map

$$\mathrm{QCoh}(\mathcal{Y}') \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \to \mathbf{\Gamma}\left(\mathcal{Y}', \mathbf{Loc}_{\mathcal{Y}'}(\mathrm{QCoh}(\mathcal{Y}') \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C})\right) \simeq \mathbf{\Gamma}(\mathcal{Y}', \mathcal{C})$$

is an equivalence, and the same holds for $\mathcal{Y}'$ replaced by U' and U.

Hence, we obtain a map from the diagram (5.2) to (5.3), which induces equivalences

$$\mathrm{QCoh}(U) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \to \mathbf{\Gamma}(U, \mathcal{C}), \quad \mathrm{QCoh}(\mathcal{Y}') \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \to \mathbf{\Gamma}(\mathcal{Y}', \mathcal{C})$$

and

$$\mathrm{QCoh}(U') \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \to \mathbf{\Gamma}(U', \mathcal{C}).$$

Hence, the map

$$\mathbf{C} \to \mathbf{\Gamma}(\mathcal{Y}, \mathcal{C})$$

is also an equivalence, as required.

5.3.3. To finish the proof of Theorem 2.1.1, it remains to give an *a priori* proof in the case when $\mathcal{Y}$ is a quasi-compact quasi-affine scheme. More generally, let us assume that $\mathcal{Y}$ is a quasi-compact separated scheme.

Note that Proposition 5.1.3 is valid for $\mathcal{Y}$, because the diagonal of $\mathcal{Y}$ is a closed embedding, and hence is affine (in the proof of Proposition 5.1.3 we used the fact that the base change of the diagonal morphism of $\mathcal{Y}$ by an affine scheme yields a prestack which is 1-affine, which is tautological if the diagonal morphism is affine).

The proof proceeds by induction on the number of affine opens by which we can cover $\mathcal{Y}$. Namely, we repeat the proof of Theorem 2.1.1, given above, replacing the word "étale morphism" by "open embedding", and where the schemes U_k are affine. In the induction step, $\mathcal{Y}'$ and U' are both 1-affine by the induction hypothesis.

□

5.4. **Direct image for representable morphisms.**

5.4.1. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a representable, quasi-compact and quasi-separated morphism between prestacks. For $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y}_2)$, we have a canonically defined functor

$$f^*_{\mathcal{C}} : \mathbf{\Gamma}(\mathcal{Y}_2, \mathcal{C}) \to \mathbf{\Gamma}(\mathcal{Y}_1, \mathcal{C}) = \mathbf{\Gamma}(\mathcal{Y}_1, \mathbf{cores}_f(\mathcal{C})).$$

We claim:

Proposition 5.4.2. *The above functor $f^*_{\mathcal{C}} : \mathbf{\Gamma}(\mathcal{Y}_2, \mathcal{C}) \to \mathbf{\Gamma}(\mathcal{Y}_1, \mathbf{cores}_f(\mathcal{C}))$ admits a continuous right adjoint (to be denoted $f_{\mathcal{C},*}$). For a Cartesian diagram of prestacks*

$$\begin{array}{ccc} \mathcal{Y}'_1 & \xrightarrow{g_1} & \mathcal{Y}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}'_2 & \xrightarrow{g_2} & \mathcal{Y}_2, \end{array}$$

the diagram

$$\begin{array}{ccc} \mathbf{\Gamma}(\mathcal{Y}'_1, \mathcal{C}) & \xleftarrow{(g_1)^*_{\mathcal{C}}} & \mathbf{\Gamma}(\mathcal{Y}_1, \mathcal{C}) \\ {\scriptstyle f'_{\mathcal{C},*}}\downarrow & & \downarrow{\scriptstyle f_{\mathcal{C},*}} \\ \mathbf{\Gamma}(\mathcal{Y}'_2, \mathcal{C}) & \xleftarrow{(g_2)^*_{\mathcal{C}}} & \mathbf{\Gamma}(\mathcal{Y}_2, \mathcal{C}), \end{array}$$

which a priori *commutes up to a natural transformation, actually commutes.*

5.4.3. *Proof of Proposition 5.4.2, Step 1.* Let us first consider the case when $\mathcal{Y}_2$ is a quasi-compact and quasi-separated algebrac space. In this case, $\mathcal{Y}_1$ has the same properties.

By Lemma 3.2.4, we have

$$\mathbf{\Gamma}(\mathcal{Y}_1, \mathbf{cores}_f(\mathcal{C})) \simeq \mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathbf{\Gamma}(\mathcal{Y}_2, \mathcal{C}).$$

In this case, the sought-for functor $f_{\mathcal{C},*}$ identifies with

$$\mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathbf{\Gamma}(\mathcal{Y}_2, \mathcal{C}) \xrightarrow{f_* \otimes \mathrm{Id}_{\mathbf{\Gamma}(\mathcal{Y}_2, \mathcal{C})}} \mathrm{QCoh}(\mathcal{Y}_2) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathbf{\Gamma}(\mathcal{Y}_2, \mathcal{C}) = \mathbf{\Gamma}(\mathcal{Y}_2, \mathcal{C}).$$

5.4.4. *Proof of Proposition 5.4.2, Step 2.* It suffices to show that for any $S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2}$, the functor

$$\mathbf{\Gamma}(S, \mathcal{C}) \to \mathbf{\Gamma}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \mathcal{C})$$

admits a right adjoint, and that for a map $g : S' \to S$, the corresponding diagram

$$\begin{array}{ccc} \mathbf{\Gamma}(S' \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \mathcal{C}) & \longleftarrow & \mathbf{\Gamma}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathbf{\Gamma}(S', \mathcal{C}) & \longleftarrow & \mathbf{\Gamma}(S, \mathcal{C}), \end{array}$$

commutes.

However, this follows from Step 1 using base change for QCoh for maps between quasi-compact and quasi-separated algebraic spaces (see, e.g., [DrGa, Corollary 1.4.5] for the latter assertion).

□

5.5. **Global sections via a Čech cover.** Let $\mathcal{Y}$ be a quasi-compact algebraic stack. In this subsection we will describe a more economical way to compute the functor $\mathbf{\Gamma}(\mathcal{Y}, -)$.

5.5.1. Let $U \to \mathcal{Y}$ be an fppf cover, where U is a quasi-compact and quasi-separated algebraic space. Note that all the terms of the Čech nerve $U^\bullet/\mathcal{Y}$ are also quasi-compact and quasi-separated algebraic spaces.

Consider the co-simplicial category $\mathbf{\Gamma}(U^\bullet/\mathcal{Y}, \mathcal{C})^*$: for $[i] \in \mathbf{\Delta}$ the category of i-th simplices is $\mathbf{\Gamma}(U^i/\mathcal{Y}, \mathcal{C})$, and for a map $\alpha : [j] \to [i]$ in $\mathbf{\Delta}$, and the corresponding map $f^\alpha : U^i/\mathcal{Y} \to U^j/\mathcal{Y}$, the functor

$$\mathbf{\Gamma}(U^j/\mathcal{Y}, \mathcal{C}) \to \mathbf{\Gamma}(U^i/\mathcal{Y}, \mathcal{C})$$

is $(f^\alpha)^*_{\mathcal{C}}$.

The following results from Corollary 1.5.5(a):

Lemma 5.5.2. *For $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y})$, the restriction map*

$$\Gamma(\mathcal{Y}, \mathcal{C}) \to \mathrm{Tot}\,(\mathbf{\Gamma}(U^\bullet/\mathcal{Y}, \mathcal{C})^*)$$

is an equivalence.

We shall now describe the category $\mathrm{Tot}\,(\mathbf{\Gamma}(U^\bullet/\mathcal{Y}, \mathcal{C})^*)$ as co-modules over a co-monad acting on $\mathbf{\Gamma}(U, \mathcal{C})$.

5.5.3. We claim that the co-simplicial category $\mathbf{\Gamma}(U^\bullet/\mathcal{Y}, \mathcal{C})$ satisfies the *co-monadic* Beck-Chevalley condition (see Sect. C.1 for what this means). Indeed, this follows from Proposition 5.4.2.

Hence, from Lemma C.1.9, we obtain:

Lemma 5.5.4.

(a) *The functor of evaluation on* 0*-simplices*

$$\mathrm{ev}^0 : \mathrm{Tot}\,(\mathbf{\Gamma}(U^\bullet/\mathcal{Y}, \mathcal{C})^*) \to \mathbf{\Gamma}(U, \mathcal{C})$$

admits a (continuous) right adjoint; to be denoted $(\mathrm{ev}^0)^R$.

(b) *The co-monad* $\mathrm{Av}_*^{U/\mathcal{Y}} := \mathrm{ev}^0 \circ (\mathrm{ev}^0)^R$, *acting on* $\mathbf{\Gamma}(U, \mathcal{C})$, *is isomorphic, as a plain endofunctor, to* $(\mathrm{pr}_2)_{\mathcal{C},*} \circ (\mathrm{pr}_1)^*_{\mathcal{C}}$, *where* $\mathrm{pr}_1, \mathrm{pr}_2$ *denote the two projections*

$$U \underset{\mathcal{Y}}{\times} U \rightrightarrows U.$$

(c) *The functor*

$$\mathrm{ev}^0 : \mathrm{Tot}\,(\mathbf{\Gamma}(U^\bullet/\mathcal{Y}, \mathcal{C})^*) \to \mathbf{\Gamma}(U, \mathcal{C})$$

is co-monadic, i.e., the natural functor

$$\mathrm{Tot}\,(\mathbf{\Gamma}(U^\bullet/\mathcal{Y}, \mathcal{C})) \to \mathrm{Av}_*^{U/\mathcal{Y}}\text{-comod}(\mathbf{\Gamma}(U, \mathcal{C}))$$

is an equivalence.

Combining with Lemma 5.5.2, we obtain:

Corollary 5.5.5. *The functor*

$$f^*_{\mathcal{C}} : \mathbf{\Gamma}(\mathcal{Y}, \mathcal{C}) \to \mathbf{\Gamma}(U, \mathcal{C})$$

is co-monadic.

5.6. **The Čech picture for sheaves of categories.** We retain the notations of Sect. 5.5.

5.6.1. Consider the following co-simplicial object of ∞-Cat, denoted $\mathrm{ShvCat}(U^\bullet/\mathcal{Y})$. For $[i] \in \mathbf{\Delta}$, its category of i-simplices is $\mathrm{ShvCat}(U^i/\mathcal{Y})$. For a map $\alpha : [j] \to [i]$ in $\mathbf{\Delta}$, the functor

$$\mathrm{ShvCat}(U^i/\mathcal{Y}) \to \mathrm{ShvCat}(U^i/\mathcal{Y})$$

is $\mathbf{cores}_{f^\alpha}$.

The following results from Corollary 1.5.8(a):

Lemma 5.6.2. *The pullback functor*

$$\mathrm{ShvCat}(\mathcal{Y}) \to \mathrm{Tot}\left(\mathrm{ShvCat}(U^\bullet/\mathcal{Y})\right)$$

is an equivalence.

5.6.3. We now claim that $\mathrm{ShvCat}(U^\bullet/\mathcal{Y}) \in \infty\text{-Cat}^{\mathbf{\Delta}}$ satsfies the co-monadic Beck-Chevalley condition. This amounts to the fact that the diagram

$$\begin{array}{ccc} \mathrm{ShvCat}(U^i/\mathcal{Y}) & \xleftarrow{\mathbf{coind}_{f^{\partial_i}}} & \mathrm{ShvCat}(U^{i+1}/\mathcal{Y}) \\ {\scriptstyle \mathbf{cores}_{f^\alpha}}\uparrow & & {\scriptstyle \mathbf{cores}_{f^{\alpha+1}}}\uparrow \\ \mathrm{ShvCat}(U^j/\mathcal{Y}) & \xleftarrow{\mathbf{coind}_{f^{\partial_i}}} & \mathrm{ShvCat}(U^{j+1}/\mathcal{Y}) \end{array}$$

being commutative, which follows from the definitions.

Hence, from Lemma C.1.9, we obtain:

Lemma 5.6.4.
(a) *The functor of evaluation on* 0*-simplices*

$$\mathrm{ev}^0 : \mathrm{Tot}\left(\mathrm{ShvCat}(U^\bullet/\mathcal{Y})\right) \to \mathrm{ShvCat}(U)$$

admits a right adjoint; to be denoted $(\mathrm{ev}^0)^R$.
(b) *The co-monad* $\mathrm{ev}^0 \circ (\mathrm{ev}^0)^R$*, acting on* $\mathrm{ShvCat}(U)$*, is isomorphic, as a plain endo-functor, to* $\mathbf{coind}_{\mathrm{pr}_2} \circ \mathbf{cores}_{\mathrm{pr}_1}$.
(c) *The functor*

$$\mathrm{ev}^0 : \mathrm{Tot}\left(\mathrm{ShvCat}(U^\bullet/\mathcal{Y})\right) \to \mathrm{ShvCat}(U)$$

is co-monadic, i.e., the natural functor

$$\mathrm{Tot}\left(\mathrm{ShvCat}(U^\bullet/\mathcal{Y})\right) \to \left(\mathrm{ev}^0 \circ (\mathrm{ev}^0)^R\right)\text{-comod}(\mathrm{ShvCat}(U))$$

is an equivalence.

Combining with Lemma 5.6.2, we obtain:

Corollary 5.6.5. *The functor*

$$\mathbf{cores}_f : \mathrm{ShvCat}(\mathcal{Y}) \to \mathrm{ShvCat}(U)$$

is co-monadic.

6. Algebraic stacks: criteria for 1-affineness

In this section we let $\mathcal{Y}$ be a quasi-compact algebraic stack, which is *passable* as a prestack (see Sect. 5.1 for what this means). Note that in view of Corollary 5.2.9, the assumption that $\mathcal{Y}$ be passable is not very restrictive.

We will give a series of equivalent conditions for $\mathcal{Y}$ to be 1-affine.

6.1. **The 1st criterion for 1-affineness.**

6.1.1. Note that under our assumptions on $\mathcal{Y}$, the functor $\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}}$ is fully faithful, by Proposition 5.1.3. Hence, the question of 1-affineness for $\mathcal{Y}$ is equivalent to that of fully faithfulness of the functor $\mathbf{Loc}_{\mathcal{Y}}$.

We are going to prove:

Proposition 6.1.2. *The following conditions are equivalent:*

(i) $\mathcal{Y}$ *is* 1-*affine.*

(ii) *The functor* $\mathbf{Loc}_{\mathcal{Y}}$ *is fully faithful;*

(iii) *The functor* $\mathbf{Loc}_{\mathcal{Y}}$ *is conservative;*

(iv) *The functor* $\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}}$ *commutes with tensor products by objects of* $\mathrm{DGCat}_{\mathrm{cont}}$.

(v) *The unit morphism* $\mathrm{Id} \to \mathbf{\Gamma}^{\mathrm{enh}}(\mathcal{Y}, \mathbf{Loc}_{\mathcal{Y}}(-))$ *is an equivalence on objects of the form*

$$\mathrm{QCoh}(\mathcal{Y}) \otimes \mathbf{D} \in \mathrm{QCoh}(\mathcal{Y}) \text{-}\,\mathbf{mod}, \quad \mathbf{D} \in \mathrm{DGCat}_{\mathrm{cont}}\,.$$

Proof. Since $\mathbf{\Gamma}_{\mathcal{Y}}$ is fully faithful, the equivalence of (i), (ii) and (iii) is evident. It is also clear that (i) implies (iv) and that (iv) implies (v).

Let assume (v) and deduce (ii). We need to show that for $\mathbf{C} \in \mathrm{QCoh}(\mathcal{Y}) \text{-}\,\mathbf{mod}$, the unit of the adjunction

$$\mathbf{C} \to \mathbf{\Gamma}^{\mathrm{enh}}(\mathcal{Y}, \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C}))$$

is an equivalence. Since $\mathrm{QCoh}(\mathcal{Y})$ is rigid, an arbitrary object of $\mathrm{QCoh}(\mathcal{Y}) \text{-}\,\mathbf{mod}$ can be written as a *limit* of objects of the form $\mathrm{QCoh}(\mathcal{Y}) \otimes \mathbf{D}$ for $\mathbf{D} \in \mathrm{DGCat}_{\mathrm{cont}}$, see Corollary D.4.7.

Now, the assertion follows from the fact that both $\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}}$ and $\mathbf{Loc}_{\mathcal{Y}}$ commute with limits: the former because $\mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Y}}$ admits a left adjoint, and the latter by Corollary 1.4.8(b).

□

6.2. **The 2nd criterion for 1-affineness.** We shall now give another set of criteria for $\mathcal{Y}$ to be 1-affine. The idea of this approach goes back to Jacob Lurie.

We shall consider another functor

$$\text{co-}\,\mathbf{\Gamma}^{\mathrm{enh}}(\mathcal{Y}, -) : \mathrm{ShvCat}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}) \text{-}\,\mathbf{mod}.$$

6.2.1. Choose a fppf cover $U \to \mathcal{Y}$, where U is a quasi-compact and quasi-separated algebraic space.

For $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y})$ we consider a *simplicial* object

$$\mathbf{\Gamma}(U^\bullet/\mathcal{Y}, \mathcal{C})_* \in \mathrm{QCoh}(\mathcal{Y}) \text{-}\,\mathbf{mod}.$$

Namely, for $[i] \in \mathbf{\Delta}$, the category of i-simplices in $\mathbf{\Gamma}(U^i/\mathcal{Y}, \mathcal{C})$. For a map $[j] \to [i]$ in $\mathbf{\Delta}$, the corresponding functor

$$\mathbf{\Gamma}(U^i/\mathcal{Y}, \mathcal{C}) \to \mathbf{\Gamma}(U^j/\mathcal{Y}, \mathcal{C})$$

is $(f^\alpha)_{\mathcal{C},*}$, see Sect. 5.4.

6.2.2. We set

$$\text{co-}\mathbf{\Gamma}_U(\mathcal{Y},\mathcal{C}) := |\mathbf{\Gamma}(U^\bullet/\mathcal{Y},\mathcal{C})_*|.$$

Note that the simplicial object $\mathbf{\Gamma}(U^\bullet/\mathcal{Y},\mathcal{C})_*$ of $\mathrm{DGCat}_{\mathrm{cont}}$ naturally upgrades to one in $\mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$; we will denote it by $\mathbf{\Gamma}^{\mathrm{enh}}(U^\bullet/\mathcal{Y},\mathcal{C})_*$

We denote the resulting functor $\mathrm{ShvCat}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$ by $\text{co-}\mathbf{\Gamma}^{\mathrm{enh}}_U(\mathcal{Y},\mathcal{C})$.

Note that the functors $\text{co-}\mathbf{\Gamma}_U(\mathcal{Y},\mathcal{C})$ and $\text{co-}\mathbf{\Gamma}^{\mathrm{enh}}_U(\mathcal{Y},\mathcal{C})$ commute with colimits and tensor products by objects of $\mathrm{DGCat}_{\mathrm{cont}}$, by construction.

Remark 6.2.3. The functor

$$\mathcal{C} \mapsto \text{co-}\mathbf{\Gamma}^{\mathrm{enh}}_U(\mathcal{Y},\mathcal{C})$$

a priori depends on the choice of the cover. However, it follows from Proposition 6.2.7 below (specifically, from the fact that Proposition 6.2.7(b) holds for quasi-compact quasi-separated algebraic spaces) that this definition can be rewritten invariantly as

$$\underset{X\in(\mathrm{AlgSpc}_{\mathrm{qc,qs}})_{/\mathcal{Y}}}{\underset{\longrightarrow}{colim}} \mathbf{\Gamma}(X,\mathcal{C}) \simeq \underset{S\in\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}}{\underset{\longrightarrow}{colim}} \mathbf{\Gamma}(S,\mathcal{C}).$$

So, in fact, we have a well-defined functor

$$\text{co-}\mathbf{\Gamma}^{\mathrm{enh}}_U(\mathcal{Y},-) : \mathrm{ShvCat}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}.$$

6.2.4. We claim now that there exists a canonically defined natural transformation

$$\text{co-}\mathbf{\Gamma}_U(\mathcal{Y},-) \to \mathbf{\Gamma}(\mathcal{Y},-). \tag{6.1}$$

Indeed, if f^i denotes the morphism $U^i \to \mathcal{Y}$, the corresponding compatible family of functors

$$\mathbf{\Gamma}(U^i,\mathcal{C}) \to \mathbf{\Gamma}(\mathcal{Y},\mathcal{C})$$

is given by $(f^i)_{\mathcal{C},*}$.

The natural transformation (6.1) can be interpreted in the framework of the following general paradigm (specifically, it is a particular case of the map (6.2) below):

Let $\mathbf{C}^\bullet$ be a co-simplicial category, in which all functors admit right adjoints. Let $\mathbf{C}^{\bullet,R}$ be the simplicial category obtained by passing to the right adjoint functors. Then each of the evaluation functors

$$\mathrm{ev}^i : \mathrm{Tot}(\mathbf{C}^\bullet) \to \mathbf{C}^i$$

admits a right adjoint, and these right adjoints together define a functor

$$|\mathbf{C}^{\bullet,R}| \to \mathrm{Tot}(\mathbf{C}^\bullet). \tag{6.2}$$

Note also that by replacing the word "right" by "left" we obtain a functor

$$|\mathbf{C}^{\bullet,L}| \to \mathrm{Tot}(\mathbf{C}^\bullet),$$

which is an equivalence by [GL:DG, Lemma 1.3.3].

In the above constructions, the category of indices $\mathbf{\Delta}$ can be replaced by any other index category.

6.2.5. The natural transformation (6.1) upgrades to

$$\text{co-}\mathbf{\Gamma}^{\mathrm{enh}}_U(\mathcal{Y},-) \to \mathbf{\Gamma}^{\mathrm{enh}}(\mathcal{Y},-). \tag{6.3}$$

In particular, by evaluating on $\mathcal{C} := \mathrm{QCoh}_{/\mathcal{Y}}$, we obtain a functor

$$|\,\mathrm{QCoh}(U^\bullet/\mathcal{Y})_*| \to \mathrm{QCoh}(\mathcal{Y}). \tag{6.4}$$

6.2.6. We claim:

Proposition 6.2.7. *The following conditions are equivalent:*

(a) $\mathcal{Y}$ *is* 1-*affine.*

(b) *The natural transformation in* (6.1) *is an isomorphism.*

(c) *The functor* $\text{co-}\mathbf{\Gamma}_U^{\text{enh}}(\mathcal{Y},-)$ *is* a *left inverse of* $\mathbf{Loc}_{\mathcal{Y}}$.

(d) *The functor in* (6.4) *is an equivalence.*

(e) *There exists* an *isomorphism* $|\operatorname{QCoh}(U^\bullet/\mathcal{Y})_*| \simeq \operatorname{QCoh}(\mathcal{Y})$ *as* $\operatorname{QCoh}(\mathcal{Y})$-*module categories.*

(f) *The category* $|\operatorname{QCoh}(U^\bullet/\mathcal{Y})_*|$ *is dualizable.*

Proof. First note that following tautological implications: (b) $\Rightarrow$ (d), (d) $\Rightarrow$ (e), and (e) $\Rightarrow$ (f).

The implication (b) $\Rightarrow$ (a) follows from the implication (iv) $\Rightarrow$ (i) in Proposition 6.1.2. The implication (c) $\Rightarrow$ (a) follows from the implication (iii) $\Rightarrow$ (i) in Proposition 6.1.2. Clearly, (a) and (b) together imply (c); hence (b) implies (c).

Let us show that (e) implies (c). Given $\mathbf{C} \in \operatorname{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$, we have:

$$\text{co-}\mathbf{\Gamma}_U^{\text{enh}}(\mathcal{Y}, \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C})) \simeq \text{co-}\mathbf{\Gamma}_U^{\text{enh}}(\mathcal{Y}, \operatorname{QCoh}_{/\mathcal{Y}}) \underset{\operatorname{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C},$$

which identifies with $\mathbf{C}$ by assumption.

Let us now show that (a) implies (d). The assumption in (a) implies that $\mathbf{Loc}_{\mathcal{Y}}$ is conservative, i.e., it suffices to show that the map

$$\mathbf{Loc}_{\mathcal{Y}}\left(\text{co-}\mathbf{\Gamma}_U^{\text{enh}}(\mathcal{Y}, \operatorname{QCoh}_{/\mathcal{Y}})\right) \to \operatorname{QCoh}_{/\mathcal{Y}}$$

is an isomorphism. By Lemma 5.6.2, it suffices to show that the map

$$\mathbf{Loc}_{\mathcal{Y}}\left(\text{co-}\mathbf{\Gamma}_U^{\text{enh}}(\mathcal{Y}, \operatorname{QCoh}_{/\mathcal{Y}})\right)|_U \to \operatorname{QCoh}_{/U}$$

is an isomorphism. Since U is 1-affine, it suffices to show that

$$\mathbf{\Gamma}\left(U, \mathbf{Loc}_{\mathcal{Y}}\left(\text{co-}\mathbf{\Gamma}_U^{\text{enh}}(\mathcal{Y}, \operatorname{QCoh}_{/\mathcal{Y}})\right)|_U\right) \to \operatorname{QCoh}(U)$$

is an isomorphism. Using Proposition 3.2.6(a,ii), we have:

$$\mathbf{\Gamma}\left(U, \mathbf{Loc}_{\mathcal{Y}}\left(\text{co-}\mathbf{\Gamma}_U^{\text{enh}}(\mathcal{Y}, \operatorname{QCoh}_{/\mathcal{Y}})\right)|_U\right) \simeq \operatorname{QCoh}(U) \underset{\operatorname{QCoh}(\mathcal{Y})}{\otimes} \text{co-}\mathbf{\Gamma}_U^{\text{enh}}(\mathcal{Y}, \operatorname{QCoh}_{/\mathcal{Y}}),$$

and the latter identifies with

$$|\operatorname{QCoh}(U) \underset{\operatorname{QCoh}(\mathcal{Y})}{\otimes} \operatorname{QCoh}(U^\bullet/\mathcal{Y})_*|.$$

However, the simplicial category $\operatorname{QCoh}(U) \underset{\operatorname{QCoh}(\mathcal{Y})}{\otimes} \operatorname{QCoh}(U^\bullet/\mathcal{Y})_*$ identifies with

$$\operatorname{QCoh}(U^{\bullet+1}/\mathcal{Y})_*,$$

(e.g., because $\mathcal{Y}$ is passable), and hence is split by $\operatorname{QCoh}(U)$. This implies the required assertion.

Thus, we obtain that (a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e). Let us show that these conditions imply (b). It is sufficient to evaluate both sides of (6.1) on objects of the form $\mathbf{Loc}_{\mathcal{Y}}(\mathbf{C})$ for $\mathbf{C} \in \operatorname{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$. By (a), both sides in (6.1) commute with colimits and tensor products by objects of $\operatorname{DGCat}_{\text{cont}}$; hence, we can take $\mathbf{C} = \operatorname{QCoh}(\mathcal{Y})$. In this case, the required isomorphism is supplied by (d).

Finally, let us show that (f) implies (e). This follows from the fact that

$$\operatorname{Funct}_{\text{cont}}(|\operatorname{QCoh}(U^\bullet/\mathcal{Y})_*|, \operatorname{Vect}) \simeq \operatorname{Tot}(\operatorname{QCoh}(U^\bullet/\mathcal{Y})^*) \simeq \operatorname{QCoh}(\mathcal{Y}),$$

while $\mathrm{QCoh}(\mathcal{Y})$ is its own dual.

□

6.3. **The 3rd criterion for 1-affineness.** We will now give an explicit criterion for condition (d) of Proposition 6.2.7 to hold. This will provide a 3rd criterion for 1-affiness of an algebraic stack.

6.3.1. We define the *co-simplicial* object of DGCat, denoted $\mathrm{QCoh}(U^\bullet/\mathcal{Y})^?$ as follows. For $[i] \in \mathbf{\Delta}$, the category of i-simplices is $\mathrm{QCoh}(U^i/\mathcal{Y})$. For a map $[j] \to [i]$, the corresponding functor

$$\mathrm{QCoh}(U^j/\mathcal{Y}) \to \mathrm{QCoh}(U^i/\mathcal{Y})$$

is $(f^\alpha)^?$, *right* adjoint to $f^\alpha_* : \mathrm{QCoh}(U^i/\mathcal{Y}) \to \mathrm{QCoh}(U^j/\mathcal{Y})$.

We note that the functors $(f^\alpha)^?$ are typically *non-continuous*, so $\mathrm{QCoh}(U^\bullet/\mathcal{Y})^?$ is a co-simplicial object of DGCat, but not in $\mathrm{DGCat}_{\mathrm{cont}}$.

Note that by [GL:DG, Lemma 1.3.3], we have a canonical equivalence

$$|\,\mathrm{QCoh}(U^\bullet/\mathcal{Y})_*| \to \mathrm{Tot}\left(\mathrm{QCoh}(U^\bullet/\mathcal{Y})^?\right). \tag{6.5}$$

6.3.2. We now claim that the co-simplicial category $\mathrm{QCoh}(U^\bullet/\mathcal{Y})^?$ satsifies the *monadic* Beck-Chevalley condition (see Sect. C.1 for what this means). Indeed, this follows by applying Lemma C.1.6 to the simplicial category $\mathrm{QCoh}(U^\bullet/\mathcal{Y})_*$.

Hence, from from Lemma C.1.8, we obtain:

Lemma 6.3.3.
(a) *The functor of evaluation on* 0*-simplices*

$$\mathrm{ev}^0 : \mathrm{Tot}\left(\mathrm{QCoh}(U^\bullet/\mathcal{Y})^?\right) \to \mathrm{QCoh}(U)$$

admits a left adjoint; to be denoted $(\mathrm{ev}^0)^L$.
(b) *The monad* $\mathrm{Av}_?^{U/\mathcal{Y}} := \mathrm{ev}^0 \circ (\mathrm{ev}^0)^L$, *acting on* $\mathrm{QCoh}(U)$, *is isomorphic, as a plain endo-functor, to* $(\mathrm{pr}_2)_* \circ (\mathrm{pr}_1)^?$.
(c) *The functor*

$$\mathrm{ev}^0 : \mathrm{Tot}\left(\mathrm{QCoh}(U^\bullet/\mathcal{Y})^?\right) \to \mathrm{QCoh}(U)$$

is monadic, i.e., the natural functor

$$\mathrm{Tot}\left(\mathrm{QCoh}(U^\bullet/\mathcal{Y})^?\right) \to \mathrm{Av}_?^{U/\mathcal{Y}}\text{-mod}(\mathrm{QCoh}(U))$$

is an equivalence.

6.3.4. Consider now the functor $f_* : \mathrm{QCoh}(U) \to \mathrm{QCoh}(\mathcal{Y})$. Let $f^?$ denote its *right adjoint*. Consider the monad $f^? \circ f_*$ acting on $\mathrm{QCoh}(U)$, and the corresponding functor

$$(f^?)^{\mathrm{enh}} : \mathrm{QCoh}(\mathcal{Y}) \to (f^? \circ f_*)\text{-mod}(\mathrm{QCoh}(U)). \tag{6.6}$$

Note now that ?-pullback defines a functor

$$\mathrm{QCoh}(\mathcal{Y}) \to \mathrm{Tot}\left(\mathrm{QCoh}(U^\bullet/\mathcal{Y})^?\right).$$

which under the equivalence of (6.5), identifies with the right adjoint to the functor

$$|\,\mathrm{QCoh}(U^\bullet/\mathcal{Y})_*| \to \mathrm{QCoh}(U)$$

of (6.4).

Note also that the following diagram tautologically commutes:

$$\begin{tikzcd} \mathrm{QCoh}(\mathcal{Y}) \arrow[rr] \arrow[dr, "f^?"'] & & \mathrm{Tot}(\mathrm{QCoh}(U^\bullet/\mathcal{Y})^?) \arrow[dl, "\mathrm{ev}^0"] \\ & \mathrm{QCoh}(U). & \end{tikzcd} \tag{6.7}$$

Hence, we obtain a homomorphism of monads acting on $\mathrm{QCoh}(U)$

$$\mathrm{Av}_?^{U/\mathcal{Y}} := \left((\mathrm{ev}^0)^L \circ \mathrm{ev}^0\right) \to (f^? \circ f_*). \tag{6.8}$$

Lemma 6.3.5. *The map* (6.8) *is an isomorphism.*

Proof. It is enough to show that the map in question is an isomorphism as plain endo-functors. Using Lemma 6.3.3(b), the map (6.8) identifies with

$$(\mathrm{pr}_2)_* \circ (\mathrm{pr}_1)^? \to f^? \circ f_*,$$

which is obtained by passing to right adjoints from the base change map

$$f^* \circ f_* \to (\mathrm{pr}_1)_* \circ (\mathrm{pr}_2)^*,$$

while the latter is an isomorphism.

□

6.3.6. Hence, in view of Lemma 6.3.3(c), we can identify the diagram (6.7) with

$$\begin{tikzcd} \mathrm{QCoh}(\mathcal{Y}) \arrow[rr, "(f^?)^{\mathrm{enh}}"] \arrow[dr, "f^?"'] & & (f^? \circ f_*)\text{-mod}(\mathrm{QCoh}(U)) \arrow[dl] \\ & \mathrm{QCoh}(U). & \end{tikzcd} \tag{6.9}$$

From here we obtain:

Proposition 6.3.7. *The following conditions are equivalent:*

(1) $\mathcal{Y}$ *is* 1-*affine.*

(2) *The functor* $|\mathrm{QCoh}(U^\bullet/\mathcal{Y})_*| \to \mathrm{QCoh}(\mathcal{Y})$ *is an equivalence.*

(3) *The functor* $\mathrm{QCoh}(\mathcal{Y}) \to \mathrm{Tot}\left(\mathrm{QCoh}(U^\bullet/\mathcal{Y})^?\right)$ *is an equivalence.*

(4) *The functor* $f^? : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(U)$ *is monadic.*

7. Classifying stacks of algebraic groups

The goal of this section is to prove Theorems 2.2.2, 2.2.4 and 2.2.3.

7.1. **Reduction steps.** Let G be a classical affine algebraic group of finite type.

7.1.1. First, let us note that Theorem 2.2.2 implies Theorem 2.2.4. Indeed, apply Corollary 3.2.7 to the morphism

$$Z/G \to \mathrm{pt}\,/G.$$

7.1.2. Let $G_1 \hookrightarrow G_2$ be a closed embedding. Note that the corresponding map

$$\mathrm{pt}\,/G_1 \to \mathrm{pt}\,/G_2$$

is schematic, quasi-compact and quasi-separated.

Hence, by Corollary 3.2.7, if G_2 is such that $\mathrm{pt}\,/G_2$ is 1-affine, the same will be true for $\mathrm{pt}\,/G_1$.

7.1.3. Choose a closed embedding $G \hookrightarrow GL_n$. We obtain that in order to prove Theorem 2.2.2, it is enough to consider the case of $G = GL_n$. I.e., we can assume that G is reductive.

7.2. **Proof of Theorem 2.2.2 in the reductive case.** The idea of the proof belongs to Jacob Lurie.

7.2.1. Note that for $\mathcal{Y} = \mathrm{pt}\,/G$, the category $\mathrm{QCoh}(\mathcal{Y})$ identifies with $\mathrm{Rep}(G)$, i.e., the category of G-representations. Under this identification, the functor

$$f^* : \mathrm{QCoh}(\mathrm{pt}\,/G) \to \mathrm{QCoh}(\mathrm{pt}) \simeq \mathrm{Vect},$$

corresponding to $f : \mathrm{pt} \to \mathrm{pt}\,/G$, is the forgetful functor $\mathrm{oblv}_G : \mathrm{Rep}(G) \to \mathrm{Vect}$.

The right adjoint f_* of f^* is the (usual) functor of co-induction

$$\mathrm{coind}_G : \mathrm{Vect} \to \mathrm{Rep}(G),$$

right adjoint to the forgetful functor oblv_G.

7.2.2. By Proposition 6.3.7, it is enough to show that the functor

$$(\mathrm{coind}_G)^R : \mathrm{Rep}(G) \to \mathrm{Vect}$$

is monadic, where $(\mathrm{coind}_G)^R$ is the (discontinuous) *right* adjoint of coind_G.

Remark 7.2.3. Note that, according to Proposition 6.3.7, the assertion of Theorem 2.2.2 is equivalent to the fact that the functor $(\mathrm{coind}_G)^R$ is monadic for any classical affine algebraic group of finite type (i.e., not necessarily reductive).

7.2.4. Let A be the set of irreducible representations of G. Since G is reductive (and we are over a field of characteristic 0), choosing representatives, we obtain an equivalence

$$\mathrm{Rep}(G) \simeq \mathrm{Vect}^A,$$

where Vect^A is the product of copies of Vect, indexed by the set A.

The functor

$$\mathrm{coind}_G : \mathrm{Vect} \to \mathrm{Rep}(G)$$

is the functor

$$\mathsf{S} : \mathrm{Vect} \to \mathrm{Vect}^A, \quad V \rightsquigarrow V^A \in \mathrm{Vect}^A,$$

(i.e., V in each component).

The right adjoint functor $(\mathrm{coind}_G)^R$ is

$$\mathsf{T} : \mathrm{Vect}^A \to \mathrm{Vect}, \quad \{W_a, a \in A\} \in \mathrm{Vect}^A \rightsquigarrow \underset{a \in A}{\Pi} W_a \in \mathrm{Vect}\,.$$

7.2.5. We recall that a DG category $\mathbf{C}$ equipped with a t-structure is said to be *right-complete* with respect to this t-structure if the functor

$$\mathbf{C} \to \underset{n \in \mathbb{Z}^+}{\underset{\longleftarrow}{lim}} \mathbf{C}^{\leq n}, \quad \mathbf{c} \mapsto \{\tau^{\leq n}(\mathbf{c})\}$$

is an equivalence.

If this happens, the inverse equivalence is given by

$$\{\mathbf{c}_n \in \mathbf{C}^{\leq n}\} \mapsto \underset{n \in \mathbb{Z}^+}{\underset{\longrightarrow}{lim}} \mathbf{c}_n.$$

Recall also $\mathbf{C}$ is said to be *left-complete* with respect to its t-structure if the functor

$$\mathbf{C} \to \underset{n \in \mathbb{Z}^+}{\underset{\longleftarrow}{lim}} \mathbf{C}^{\geq -n}, \quad \mathbf{c} \mapsto \{\tau^{\geq -n}(\mathbf{c})\}$$

is an equivalence.

If this happens, the inverse equivalence is given by

$$\{\mathbf{c}_n \in \mathbf{C}^{\geq -n}\} \mapsto \underset{n \in \mathbb{Z}^+}{\underset{\longleftarrow}{lim}} \mathbf{c}_n.$$

7.2.6. Note that both categories Vect and Vect^A carry t-structures, in which they are both right-complete and left-complete.

The functors S and T are t-exact. In particular, they define a pair of adjoint functors

$$\mathsf{S}^{\leq n} : \mathrm{Vect}^{\leq n} \rightleftarrows (\mathrm{Vect}^A)^{\leq n} : \mathsf{T}^{\leq n}$$

for every n.

7.2.7. Consider the following general paradigm. Let I be an index category, and let

$$i \rightsquigarrow \mathbf{C}_i \text{ and } i \rightsquigarrow \mathbf{D}_i$$

be two family of categories. Let

$$\mathbf{C} := \underset{i \in I}{lim}\, \mathbf{C}_i \text{ and } \mathbf{D} := \underset{i \in I}{lim}\, \mathbf{D}_i$$

be the limits.

Let us be given a compatible system of adjoint functors

$$\mathsf{S}_i : \mathbf{C}_i \rightleftarrows \mathbf{D}_i : \mathsf{T}_i.$$

Denote by

$$\mathsf{S} : \mathbf{C} \rightleftarrows \mathbf{D} : \mathsf{T}$$

the resulting adjoint pair.

We have the following general lemma:

Lemma 7.2.8. *Suppose that for every i, the pair $\mathsf{S}_i : \mathbf{C}_i \rightleftarrows \mathbf{D}_i : \mathsf{T}_i$ is monadic. Then the pair $\mathsf{S} : \mathbf{C} \rightleftarrows \mathbf{D} : \mathsf{T}$ is also monadic.*

7.2.9. Applying Lemma 7.2.8, we obtain that it suffices to show that the pair of adjoint functors

$$\mathsf{S}^{\leq n} : \mathrm{Vect}^{\leq n} \rightleftarrows (\mathrm{Vect}^A)^{\leq n} : \mathsf{T}^{\leq n}$$

is monadic. With no restriction of generality, we can assume that $n = 0$.

We will prove that the pair

$$\mathsf{S}^{\leq 0} : \mathrm{Vect}^{\leq 0} \rightleftarrows (\mathrm{Vect}^A)^{\leq 0} : \mathsf{T}^{\leq 0}$$

is monadic by verifying that the functor $\mathsf{T}^{\leq 0}$ satsifies the conditions of the Barr-Beck-Lurie theorem ([Lu2, Theorem 6.2.2.5]).

The fact that T is conservative is manifest from the explicit description of the functor in question.

The fact that the functor T, restricted to $(\mathrm{Vect}^A)^{\leq 0}$, commutes with G-split geometric realizations follows from the next general assertion:

Lemma 7.2.10. *Let $\mathsf{T} : \mathbf{C}_1 \to \mathbf{C}_2$ be a functor between DG categories. Assume that both categories are equipped with a t-structure and that T sends $\mathbf{C}_1^{\leq 0}$ to $\mathbf{C}_2^{\leq k}$ for some k. Assume that $\mathbf{C}_2$ is* left*-complete in its t-structure. Then T commutes with all geometric realizations of objects in $\mathbf{C}_1^{\leq 0}$.*

7.3. **Classifying stacks of group-schemes of infinite type.** In this subsection we will prove Theorem 2.2.3.

7.3.1. Let G be the affine group-scheme $\underset{\leftarrow n}{lim}\, (\mathbb{G}_a)^{\times n}$. Note that the map

$$BG \to \mathrm{pt}\,/G \tag{7.1}$$

is an isomorphism (indeed, on an affine DG scheme S there are no non-trivial G-torsors).

7.3.2. Note that since G is of infinite type, the map $\mathrm{pt} \to \mathrm{pt}\,/G$ is not an fppf cover (it is an fpqc cover). However, since (7.1) is an isomorphism, the map

$$\mathrm{ShvCat}(\mathrm{pt}\,/G) \to \mathrm{ShvCat}(BG) \simeq \mathrm{Tot}(\mathrm{ShvCat}(B^\bullet G) = \mathrm{Tot}\,(\mathrm{QCoh}(B^\bullet G)\text{-}\mathbf{mod})$$

is an equivalence.

Denote

$$\mathrm{Rep}(G) := \mathrm{QCoh}(\mathrm{pt}\,/G).$$

Let f denote the tautological morphism $\mathrm{pt} \to \mathrm{pt}\,/G$. Set

$$\mathrm{oblv}_G := f^* : \mathrm{Rep}(G) \to \mathrm{Vect}, \quad \mathrm{coind}_G := f_* : \mathrm{Rep}(G) \to \mathrm{Vect}\,.$$

Suppose, for the sake of contradiction that $\mathrm{pt}\,/G$ was 1-affine. Then by the same logic as in Sect. 7.2.2, we would obtain that the functor $(\mathrm{coind}_G)^R$, right adjoint to coind_G would be monadic.

However, we claim that $(\mathrm{coind}_G)^R$ fails to be conservative:

7.3.3. Note that the functor coind_G sends $k \in \mathrm{Vect}$ to the regular representation $\mathcal{O}_G \in \mathrm{Rep}(G)$. We claim that

$$\mathcal{M}aps_{\mathrm{Rep}(G)}(\mathcal{O}_G, k) = 0,$$

where $k \in \mathrm{Rep}(G)$ is the trivial representation.

Indeed, if $G = \mathrm{Spec}(\mathrm{Sym}(W))$, where W is a countable-dimensional vector space, then the object $k \in \mathrm{Rep}(G)$ admits a resolution whose n-term is $\mathrm{coind}_G(\Lambda^n(W))$.

Hence, $\mathcal{M}aps_{\mathrm{Rep}(G)}(\mathcal{O}_G, k)$ is computed by the complex whose n-th term is

$$\mathrm{Hom}_{\mathrm{Vect}^\heartsuit}(\mathrm{Sym}(W), \Lambda^n(W)),$$

which is easily seen to be acyclic.

□

8. Algebraic stacks: proof of Theorem 2.2.6

Let $\mathcal{Y}$ be as in Theorem 2.2.6. I.e., $\mathcal{Y}$ is a quasi-compact algebraic stack, locally almost of finite type, which is eventually coconnective and has an affine diagonal.

We know that $\mathcal{Y}$ is passable by Theorem 5.1.5. We will prove that $\mathcal{Y}$ is 1-affine by verifying condition (4) of Proposition 6.3.7.

8.1. **Strategy.** Let $f : U \to \mathcal{Y}$ be an smooth cover, where U is an affine DG scheme. We consider the functor

$$f^? : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(U),$$

and the resulting monad $f^? \circ f_*$ acting on $\mathrm{QCoh}(U)$. We denote the resulting pair of adjoint functors by

$$(f_*)^{\mathrm{enh}} : (f^? \circ f_*)\text{-mod}(\mathrm{QCoh}(U)) \rightleftarrows \mathrm{QCoh}(\mathcal{Y}) : (f^?)^{\mathrm{enh}}.$$

We will deduce Theorem 2.2.6 from the combination of the following two statements:

Proposition 8.1.1.

(a) *The functor $f^?$ is conservative.*

(b) *The functor $(f_*)^{\mathrm{enh}}$ is conservative.*

Proposition 8.1.2. *There exists a constant n that depends only on $\mathcal{Y}$, such that for any flat map $f : U \to \mathcal{Y}$ with $U \in \mathrm{DGSch}^{\mathrm{aff}}$, the functor*

$$f^? : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(U),$$

right adjoint to f_*, has a cohomological amplitude bounded on the right by n.

8.2. **Proof of Theorem 2.2.6.** Let us assume both Propositions 8.1.1 and 8.1.2 and deduce Theorem 2.2.6.

8.2.1. Let $f : U \to \mathcal{Y}$ be an fppf cover, where $U \in \mathrm{DGSch}^{\mathrm{aff}}$.

By Proposition 8.1.1(b), we only have to show that co-unit map

$$(f_*)^{\mathrm{enh}} \circ (f^?)^{\mathrm{enh}} \to \mathrm{Id}_{\mathrm{QCoh}(\mathcal{Y})}$$

is an isomorphism.

For an object $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$, the object $(f_*)^{\mathrm{enh}} \circ (f^?)^{\mathrm{enh}}(\mathcal{F})$ is the gemetric realization of the simplicial object given by

$$[i] \mapsto (f^i)_* \circ (f^i)^?(\mathcal{F}),$$

where $f^i : U^i \to \mathcal{Y}$, and where U^i is the i-th term of the Čech nerve of $f : U \to \mathcal{Y}$.

The map

$$|(f^\bullet)_* \circ (f^\bullet)^?(\mathcal{F})| \to \mathcal{F}$$

is the natural augmentation map.

8.2.2. *Step 1.* We first consider the case when $\mathcal{F}$ is bounded above with respect to the standard t-structure on $\mathrm{QCoh}(\mathcal{Y})$. With no restriction of generality, let us assume that $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^{\leq 0}$.

Since the functor $f^?$ is conservative (by Proposition 8.1.1(a)), it suffices to show that the map

$$f^? \left(|(f^\bullet)_* \circ (f^\bullet)^?(\mathcal{F})| \right) \to f^?(\mathcal{F})$$

is an isomorphism.

Consider the composition

$$|f^?((f^\bullet)_* \circ (f^\bullet)^?(\mathcal{F}))| \to f^? \left(|(f^\bullet)_* \circ (f^\bullet)^?(\mathcal{F})| \right) \to f^?(\mathcal{F}).$$

The composed map is an isomorphism, since the simplicial object $f^?((f^\bullet)_* \circ (f^\bullet)^?(\mathcal{F}))$ of $\mathrm{QCoh}(U)$ is split by $f^?(\mathcal{F})$. Hence, it suffices to show that the map

$$|f^?((f^\bullet)_* \circ (f^\bullet)^?(\mathcal{F}))| \to f^? \left(|(f^\bullet)_* \circ (f^\bullet)^?(\mathcal{F})| \right)$$

is an isomorphism.

Since the t-structure on $\mathrm{QCoh}(U)$ is left-complete, it suffices to show that the map

$$\tau^{\geq -k} \left(|f^?((f^\bullet)_* \circ (f^\bullet)^?(\mathcal{F}))| \right) \to \tau^{\geq -k} \left(f^? \left(|(f^\bullet)_* \circ (f^\bullet)^?(\mathcal{F})| \right) \right)$$

is an isomorphism for every $k \in \mathbb{Z}^{\geq 0}$.

Let n be the integer from Proposition 8.1.2. Consider the commutative diagram

$$\begin{array}{ccc} \tau^{\geq -k} \left(|f^?((f^\bullet)_* \circ (f^\bullet)^?(\mathcal{F}))| \right) & \longrightarrow & \tau^{\geq -k} \left(f^? \left(|(f^\bullet)_* \circ (f^\bullet)^?(\mathcal{F})| \right) \right) \\ \uparrow & & \uparrow \\ \tau^{\geq -k} \left(|f^?((f^\bullet)_* \circ (f^\bullet)^?(\mathcal{F}))|_{k+2n} \right) & \longrightarrow & \tau^{\geq -k} \left(f^? \left(|(f^\bullet)_* \circ (f^\bullet)^?(\mathcal{F})|_{k+2n} \right) \right), \end{array} \tag{8.1}$$

where for $m \in \mathbb{Z}^{\geq m}$, we denote by $|-|_{\leq m}$ the gemetric realization of the m-skeleton (i.e., the colimit over the subcategory of $\mathbf{\Delta}^{\mathrm{op}}$ corresponding to $[i]$ with $i \leq m$).

We claim that both vertical arrows and the bottom horizontal arrow in the diagram (8.1) are isomorphisms.

The assertion regarding the bottom horizontal arrow follows from the fact that $|-|_{\leq k+2n}$ is a *finite* colimit, and hence commutes with $f^?$.

Note that since the diagonal of $\mathcal{Y}$ is affine, all of the maps f^i are affine. In particular, each of the functors $(f^i)_*$ is t-exact. Furthermore, by the assumption on n, each of the functors $(f^i)^?$ has a cohomological amplitude bounded on the right by n.

In particular, each of the terms $(f^i)_* \circ (f^i)^?(\mathcal{F})$ lies in $\mathrm{QCoh}(\mathcal{Y})^{\leq n}$. Hence, we obtain that for any k', the map

$$\tau^{\geq -k'}\left(|(f^\bullet)_* \circ (f^\bullet)^?(\mathcal{F})|\right) \to \tau^{\geq -k'}\left(|(f^\bullet)_* \circ (f^\bullet)^?(\mathcal{F})|_{\leq k'+n}\right)$$

is an isomorphism.

Taking $k' = k+n$, and using the fact that the cohomological amplitude of $f^?$ is bounded on the right by n, we obtain that the right vertical arrow in (8.1) is an isomorphism.

Similarly, the terms of the simplicial object $f^?((f^\bullet)_* \circ (f^\bullet)^?(\mathcal{F}))$ lie in $\mathrm{QCoh}(\mathcal{Y})^{\leq 2n}$. Hence, the left vertical arrow in (8.1) is an isomorphism, as required.

8.2.3. *Step 2.* Let now $\mathcal{F}$ be arbitrary. Consider the commutative diagram

$$\begin{array}{ccc} |(f^\bullet)_* \circ (f^\bullet)^?(\mathcal{F})| & \longrightarrow & \mathcal{F} \\ \uparrow & & \uparrow \\ \underset{\longrightarrow n}{colim}\, |(f^\bullet)_* \circ (f^\bullet)^?(\tau^{\leq n}(\mathcal{F}))| & \longrightarrow & \underset{\longrightarrow n}{colim}\, \tau^{\leq n}(\mathcal{F}). \end{array} \tag{8.2}$$

We need to show that the top horizontal arrow in (8.2) is an isomorphism. We will do so by showing that the bottom horizontal arrow, as well as the vertical arrows, are isomorphisms.

The assertion regarding the bottom horizontal arrow follows from Step 1. The assertion regarding the right vertical arrow expresses the fact that the t-structure on $\mathrm{QCoh}(\mathcal{Y})$ is right-complete.

To show that the left vertical arrow in (8.2) is an isomorphism, it suffices to show that for every i, the map

$$\underset{\longrightarrow n}{colim}\, (f^i)_* \circ (f^i)^?(\tau^{\leq n}(\mathcal{F})) \to (f^i)_* \circ (f^i)^?(\mathcal{F})$$

is an isomorphism.

Since the functor $(f^i)_*$ commutes with colimits, it suffices to show that the map

$$\underset{\longrightarrow n}{colim}\, (f^i)^?(\tau^{\leq n}(\mathcal{F})) \to (f^i)^?(\mathcal{F})$$

is an isomorphism in $\mathrm{QCoh}(U^i)$. Since the t-structure on $\mathrm{QCoh}(U^i)$ is right-complete and compatible with filtered colimits, it suffices to show that for every $k \in \mathbb{Z}^{\geq 0}$, the map

$$\underset{\longrightarrow n}{colim}\, \tau^{\leq k}\left((f^i)^?(\tau^{\leq n}(\mathcal{F}))\right) \to \tau^{\leq k}\left((f^i)^?(\mathcal{F})\right)$$

is an isomorphism. However, the map

$$\tau^{\leq k}\left((f^i)^?(\tau^{\leq n}(\mathcal{F}))\right) \to \tau^{\leq k}\left((f^i)^?(\mathcal{F})\right)$$

is already an isomorphism for any $n \geq k$, since the functor $(f^i)^?$ is left t-exact (the latter is because its left adjoint, namely $(f^i)_*$, is t-exact, and in particular, right t-exact.)

□

8.3. **Proof of Proposition 8.1.2.**

8.3.1. Let A be a connective k-algebra. Let us recall that an object $M \in A\text{-mod}$ is said to be *flat* if

- $M \in A\text{-mod}^{\leq 0}$;
- $H^0(M)$ is flat as an $H^0(A)$-module;
- the natural maps $H^{-i}(A) \underset{H^0(A)}{\otimes} H^0(M) \to H^{-i}(M)$ are isomorphisms.

For a prestack $\mathcal{Y}$ and $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$, we shall say that $\mathcal{F}$ is flat if its pullback to any affine scheme is flat.

Note that the assumption of the proposition implies that the object $f_*(\mathcal{O}_U) \in \mathrm{QCoh}(\mathcal{Y})$ is flat. Hence, the assertion of Proposition 8.1.2 follows from the next general result:

Proposition 8.3.2. *Let $\mathcal{Y}$ be a QCA*[9] *stack, locally almost of finite type. There exists an integer n, such that for any flat $\mathcal{E} \in \mathrm{QCoh}(\mathcal{Y})$, the functor*

$$\mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{E}, -) : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{Vect}$$

has a cohomological amplitude bounded on the right by n.

The proof of this proposition will occupy the rest of this subsection.

8.3.3. *Step 1.* We claim that it is sufficient to show that there exists an integer n such that

$$\mathrm{Hom}_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{E}, \mathcal{F}[i]) = 0 \text{ for all } i > n \text{ and } \mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^{\heartsuit}.$$

The proof is the same as that of [DrGa, Lemma 2.1.3].

As in [DrGa, Sect. 2.2.1], this allows to assume that $\mathcal{Y}$ is classical: if n is the integer that works for ${}^{cl}\mathcal{Y}$, then it also works for $\mathcal{Y}$.

Since $\mathcal{Y}$ is Noetherian, we can further replace ${}^{cl}\mathcal{Y}$ by $({}^{cl}\mathcal{Y})_{red}$: if an integer n works for $({}^{cl}\mathcal{Y})_{red}$, then it works also for ${}^{cl}\mathcal{Y}$.

8.3.4. *Step 2.* Recall the setting and notations of Sect. 4.1.1.

We consider the (discontinuous) functor

$$\widehat{\imath}_?^{\mathrm{QCoh}} : \mathrm{QCoh}(\mathcal{Y})_{\mathcal{Y}'} \to \mathrm{QCoh}(\mathcal{Y}),$$

right adjoint to $\widehat{\imath}_!^{\mathrm{QCoh}}$.

Let

$$\jmath^? : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}_0)$$

denote the (discontinuous) right adjoint of $\jmath_*$. For $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ we have a distinguished triangle

$$\jmath_* \circ \jmath^?(\mathcal{F}) \to \mathcal{F} \to \widehat{\imath}_?^{\mathrm{QCoh}} \circ \widehat{\imath}^{\mathrm{QCoh},!}(\mathcal{F}). \tag{8.3}$$

We will prove:

Lemma 8.3.5. *Let $\mathcal{Y}$ satisfy the assumption of Proposition 8.3.2, and assume that $\mathcal{Y}$ is classical.*

(a) *The functor $\widehat{\imath}_?^{\mathrm{QCoh}} \circ \widehat{\imath}^{\mathrm{QCoh},!}$ is right t-exact.*

(b) *Assume that $\mathcal{Y}'$ satisfies the conclusion of Proposition 8.3.2 with an integer n'. Then for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^{\leq 0}$, we have*

$$\mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y})}\left(\mathcal{E}, \widehat{\imath}_?^{\mathrm{QCoh}} \circ \widehat{\imath}^{\mathrm{QCoh},!}(\mathcal{F})\right) \in \mathrm{Vect}^{\leq n'+1}.$$

[9] See [DrGa, Definition 1.1.8] for what this means.

8.3.6. *Step 3.* Let us assume Lemma 8.3.5 and finish the proof of Proposition 8.3.2.

By Step 1, we can assume that $\mathcal{Y}$ is classical and reduced. We can choose a closed substack $\mathcal{Y}' \subset \mathcal{Y}$ such that the complementary open $\mathcal{Y}_0$ is smooth. In Step 4 we will show that a smooth QCA stack satisfies the conclusion of Proposition 8.3.2. Let n_0 denote the corresponding integer.

We claim that $n := \max(n', n_0) + 1$ will work for $\mathcal{Y}$. Indeed, let $\mathcal{F}$ be an object $\mathrm{QCoh}(\mathcal{Y})^{\leq 0}$, and let us show that

$$\mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{E}, \mathcal{F}) \in \mathrm{Vect}^{\leq n+1}.$$

By (8.3) and Lemma 8.3.5(b), it suffices to show that

$$\mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y})}\left(\mathcal{E}, \jmath_* \circ \jmath^?(\mathcal{F})\right) \simeq \mathcal{M}aps_{\mathrm{QCoh}(\mathcal{U})}\left(\jmath^*(\mathcal{E}), \jmath^?(\mathcal{F})\right) \in \mathrm{Vect}^{\leq n_0+1}.$$

By the construction of n_0, it suffices to show that $\jmath^?(\mathcal{F}) \in \mathrm{QCoh}(\mathcal{U})^{\leq 1}$. Since $\jmath^* \circ \jmath_* \simeq \mathrm{Id}_{\mathrm{QCoh}(\mathcal{U})}$ and $\jmath^*$ is t-exact, it suffices to show that

$$\jmath_* \circ \jmath^?(\mathcal{F}) \in \mathrm{QCoh}(\mathcal{Y})^{\leq 1}.$$

However, the latter follows from Lemma 8.3.5(a).

8.3.7. *Step 4.* Assume that $\mathcal{Y}$ is a smooth QCA stack. Let us prove Proposition 8.3.2 directly in this case. We claim that in this case the category $\mathrm{QCoh}(\mathcal{Y})$ is of bounded cohomological dimension.

Indeed, as in Step 1, it is sufficient to show that there exists an integer n such that

$$\mathrm{Hom}_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F}', \mathcal{F}[i]) = 0 \text{ for all } i > n \text{ and } \mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^{\heartsuit}, \mathcal{F}' \in \mathrm{Coh}(\mathcal{Y})^{\heartsuit}.$$

Since $\mathcal{Y}$ is smooth and quasi-compact, there exists an integer n', such that any $\mathcal{F}' \in \mathrm{QCoh}(\mathcal{Y})^{\heartsuit}$ admits a resolution of length n' consisting of locally free $\mathcal{O}_{\mathcal{Y}}$-modules of finite rank. Hence, it suffices to show that there exists an integer n'' such that

$$\mathrm{Hom}_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{E}', \mathcal{F}[i]) = 0 \text{ for all } i > n'' \text{ and } \mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^{\heartsuit}$$

for $\mathcal{E}'$ locally free of finite rank (we will then set $n = n' + n''$).

However,

$$\mathrm{Hom}_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{E}', \mathcal{F}[i]) \simeq \Gamma(\mathcal{Y}, \mathcal{F}' \otimes (\mathcal{E}')^{\vee}),$$

and the existence of n'' follows from [DrGa, Theorem 1.4.2(ii)].

□

8.4. **Proof of Lemma 8.3.5.**

8.4.1. Note that from Proposition 4.1.3 we obtain:

Corollary 8.4.2. *The endo-functor* $\widehat{\imath}_?^{\mathrm{QCoh}} \circ \widehat{\imath}^{\mathrm{QCoh},!}$ *is canonically isomorphic to*

$$\widehat{\imath}_* \circ \widehat{\imath}^*,$$

where

$$\widehat{\imath}_* : \mathrm{QCoh}(\mathcal{Y}^{\wedge}_{\mathcal{Y}'}) \to \mathrm{QCoh}(\mathcal{Y})$$

is the right adjoint to $\widehat{\imath}^*$.

8.4.3. Let $\mathcal{Y}'_k \overset{\imath_k}{\hookrightarrow} \mathcal{Y}$ denote the k-th *classical* infinitesimal neighborhood of $\mathcal{Y}'$ inside $\mathcal{Y}$. The following results from [GR1, Proposition 6.8.2]:

Lemma 8.4.4. *The map* $\underset{k}{colim}\, \mathcal{Y}'_k \to \mathcal{Y}^{\wedge}_{\mathcal{Y}'}$ *becomes an isomorphism after fppf sheafification.*

Corollary 8.4.5.

(a) *We have a canonical equivalence*

$$\mathrm{QCoh}(\mathcal{Y}^{\wedge}_{\mathcal{Y}'}) \simeq \underset{k\in\mathbb{Z}^+}{\underset{\longleftarrow}{lim}}\ \mathrm{QCoh}(\mathcal{Y}'_k).$$

(b) *The endo-functor* $\widehat{\imath}_* \circ \widehat{\imath}^*$ *of* $\mathrm{QCoh}(\mathcal{Y})$ *is canonically isomorphic to*

$$\mathcal{F} \mapsto \underset{k\in\mathbb{Z}^+}{\underset{\longleftarrow}{lim}}\ (\imath_k)_* \circ (\imath_k)^*(\mathcal{F}).$$

8.4.6. We are now ready to prove Lemma 8.3.5. Point (a) of Lemma 8.3.5. follows immediately from Corollaries 8.4.2 and 8.4.5(b).

To prove point (b) of Lemma 8.3.5, we note by Proposition 4.1.3 and Corollary 8.4.5(a), that for $\mathcal{E}, \mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$, we have:

$$\mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y})}\left(\mathcal{E}, \widehat{\imath}_?^{\mathrm{QCoh}} \circ \widehat{\imath}^{\mathrm{QCoh},!}(\mathcal{F})\right) \simeq \underset{k\in\mathbb{Z}^+}{\underset{\longleftarrow}{lim}}\ \mathcal{M}aps_{\mathcal{Y}'_k}(\imath_k^*(\mathcal{E}), \imath_k^*(\mathcal{F})).$$

Since the functor of projective limit over $\mathbb{Z}^+$ in Vect has cohomological amplitude 1, it is sufficient to show that for $\mathcal{E}$ flat and $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^{\leq 0}$, each term

$$\mathcal{M}aps_{\mathcal{Y}'_k}(\imath_k^*(\mathcal{E}), \imath_k^*(\mathcal{F}))$$

belongs to $\mathrm{Vect}^{\leq n'}$.

However, $\imath_k^*(\mathcal{E})$ is flat on $\mathcal{Y}'_k$, and $\imath_k^*(\mathcal{F}) \in \mathrm{QCoh}(\mathcal{Y}'_k)^{\leq 0}$, and the assertion follows from the definition of n', combined with Step 1 of the proof of Proposition 8.3.2.

□

8.5. **Proof of Proposition 8.1.1(a).**

8.5.1. *Step 1.* Let $\mathcal{Y}' \overset{\imath}{\hookrightarrow} \mathcal{Y}$ be a closed substack, and let $\mathcal{Y}_0 \overset{\jmath}{\hookrightarrow} \mathcal{Y}$ be a complementary open. Denote $U' := U \underset{\mathcal{Y}}{\times} \mathcal{Y}'$ and $U_0 := U \underset{\mathcal{Y}}{\times} \mathcal{Y}_0$, and let $\imath_U$ and $\jmath_U$ denote the corresponding maps, and

$$f' : U' \to \mathcal{Y}' \text{ and } f_0 : U_0 \to \mathcal{Y}_0.$$

Consider the commutative diagrams

$$\begin{array}{ccc} \mathrm{QCoh}(U) & \xrightarrow{(\jmath_U)^*} & \mathrm{QCoh}(U_0) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle (f_0)_*} \\ \mathrm{QCoh}(\mathcal{Y}) & \xrightarrow{\jmath^*} & \mathrm{QCoh}(\mathcal{Y}_0) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{QCoh}(U) & \xleftarrow{(\jmath_U)_*} & \mathrm{QCoh}(U_0) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle (f_0)_*} \\ \mathrm{QCoh}(\mathcal{Y}) & \xleftarrow{\jmath_*} & \mathrm{QCoh}(\mathcal{Y}_0). \end{array}$$

By passing to right adjoints we obtain the commutative diagrams

$$\begin{array}{ccc} \mathrm{QCoh}(U) & \xleftarrow{(\jmath_U)_*} & \mathrm{QCoh}(U_0) \\ {\scriptstyle f^?}\uparrow & & \uparrow{\scriptstyle f_0^?} \\ \mathrm{QCoh}(\mathcal{Y}) & \xleftarrow{\jmath_*} & \mathrm{QCoh}(\mathcal{Y}_0) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{QCoh}(U) & \xrightarrow{(\jmath_U)^?} & \mathrm{QCoh}(U_0) \\ {\scriptstyle f^?}\uparrow & & \uparrow{\scriptstyle f_0^?} \\ \mathrm{QCoh}(\mathcal{Y}) & \xrightarrow{\jmath^?} & \mathrm{QCoh}(\mathcal{Y}_0). \end{array}$$

Hence, from the exact triangle (8.3), we obtain that $f^?$ defines a functor

$$(\widehat{f'})^? : \mathrm{QCoh}(\mathcal{Y})_{\mathcal{Y}'} \to \mathrm{QCoh}(U)_{U'}$$

that makes the diagrams

$$\begin{array}{ccc} \mathrm{QCoh}(U)_{U'} & \xleftarrow{(\widehat{\imath}_U)^{\mathrm{QCoh},!}} & \mathrm{QCoh}(U) \\ {\scriptstyle (\widehat{f'})^?}\uparrow & & \uparrow{\scriptstyle f^?} \\ \mathrm{QCoh}(\mathcal{Y})_{\mathcal{Y}'} & \xleftarrow{\widehat{\imath}^{\mathrm{QCoh},!}} & \mathrm{QCoh}(\mathcal{Y}) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{QCoh}(U)_{U'} & \xrightarrow{(\widehat{\imath}_U)_?^{\mathrm{QCoh}}} & \mathrm{QCoh}(U) \\ {\scriptstyle (\widehat{f'})^?}\uparrow & & \uparrow{\scriptstyle f^?} \\ \mathrm{QCoh}(\mathcal{Y})_{\mathcal{Y}'} & \xrightarrow{\widehat{\imath}_?^{\mathrm{QCoh}}} & \mathrm{QCoh}(\mathcal{Y}) \end{array}$$

commute.

From the exact triangle (8.3), we obtain that if the functors $f_0^?$ and $(\widehat{f'})^?$ are both conservative, then so is $f^?$.

8.5.2. *Step 2.* We will now show that if the functor $(f')^? : \mathrm{QCoh}(\mathcal{Y}') \to \mathrm{QCoh}(U')$ is conservative, then so is

$$(\widehat{f'})^? : \mathrm{QCoh}(\mathcal{Y})_{\mathcal{Y}'} \to \mathrm{QCoh}(U)_{U'}.$$

We note that the functor $\imath_* : \mathrm{QCoh}(\mathcal{Y}') \to \mathrm{QCoh}(\mathcal{Y})$ factors canonically as

$$\mathrm{QCoh}(\mathcal{Y}') \to \mathrm{QCoh}(\mathcal{Y})_{\mathcal{Y}'} \overset{\widehat{\imath}_!^{\mathrm{QCoh}}}{\hookrightarrow} \mathrm{QCoh}(\mathcal{Y}),$$

and the diagram

$$\begin{array}{ccccc} \mathrm{QCoh}(U') & \longrightarrow & \mathrm{QCoh}(U)_{U'} & \xrightarrow{(\widehat{\imath}_U)_!^{\mathrm{QCoh}}} & \mathrm{QCoh}(U) \\ {\scriptstyle f'_*}\downarrow & & {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{QCoh}(\mathcal{Y}') & \longrightarrow & \mathrm{QCoh}(\mathcal{Y})_{\mathcal{Y}'} & \xrightarrow{\widehat{\imath}_!^{\mathrm{QCoh}}} & \mathrm{QCoh}(\mathcal{Y}) \end{array}$$

commutes. Hence, the diagram

$$\begin{array}{ccccc} \mathrm{QCoh}(U') & \longleftarrow & \mathrm{QCoh}(U)_{U'} & \xleftarrow{(\widehat{\imath}_U)^{\mathrm{QCoh},!}} & \mathrm{QCoh}(U) \\ {\scriptstyle (f')^?}\uparrow & & {\scriptstyle (\widehat{f}')^?}\uparrow & & \downarrow{\scriptstyle f^?} \\ \mathrm{QCoh}(\mathcal{Y}') & \longleftarrow & \mathrm{QCoh}(\mathcal{Y})_{\mathcal{Y}'} & \xleftarrow{\widehat{\imath}^{\mathrm{QCoh},!}} & \mathrm{QCoh}(\mathcal{Y}) \end{array}$$

commutes, as well, where the left horizontal arrows are obtained by restricting the (discontinuous) functor $\imath^{\mathrm{QCoh},!}$ (resp., $(\imath_U)^{\mathrm{QCoh},!}$), right adjoint to $\imath_*$ (resp., $(\imath_U)_*$), to $\mathrm{QCoh}(\mathcal{Y})_{\mathcal{Y}'} \subset \mathrm{QCoh}(\mathcal{Y})$ (resp., $\mathrm{QCoh}(U)_{U'} \subset \mathrm{QCoh}(U)$).

Hence, in order to show that $(\widehat{f}')^?$ is conservative, it is sufficient to show that the restriction of the functor

$$\imath^{\mathrm{QCoh},!} : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}')$$

to $\mathrm{QCoh}(\mathcal{Y})_{\mathcal{Y}'} \subset \mathrm{QCoh}(\mathcal{Y})$ is conservative.

I.e., we have to show that the essential image of the functor

$$\imath_* : \mathrm{QCoh}(\mathcal{Y}') \to \mathrm{QCoh}(\mathcal{Y})$$

generates $\mathrm{QCoh}(\mathcal{Y})_{\mathcal{Y}'}$.

However, that latter is established in [DrGa, Sect. 2.6.8]. [10]

8.5.3. *Step 3.* By Step 2, we can assume that $\mathcal{Y}$ is classical and reduced (take $\mathcal{Y}' := ({}^{cl}\mathcal{Y})_{red}$). By Step 1 and Noetherian induction, it suffices to show that $\mathcal{Y}$ contains a non-empty open substack $\mathcal{Y}_0$, for which Proposition 8.1.1(a) holds.

Hence, by [DrGa, Proposition 2.3.4], we can assume that $\mathcal{Y}$ admits a finite étale cover $\pi : \widetilde{\mathcal{Y}} \to \mathcal{Y}$, such that $\widetilde{\mathcal{Y}}$ is isomorphic to the quotient of a quasi-compact quasi-separated classical reduced scheme by an action of a classical affine algebraic group of finite type.

For $\mathcal{Y}$ of this form we will establish the assertion of Proposition 8.1.1(a) directly. First, since the right adjoint of π_* is isomorphic to π^*, we can replace $\mathcal{Y}$ by $\widetilde{\mathcal{Y}}$. So, we can assume that $\mathcal{Y}$ itself is of the form Y/G, where Y is a quasi-compact quasi-separated classical scheme and G is an algebraic group of finite type.

In this case, the assertion of Proposition 8.1.1(a) follows from Theorem 2.2.4 and Proposition 6.3.7. Here is an argument independent of Theorem 2.2.4:

We need to show that the essential image of the functor $f_* : \mathrm{QCoh}(U) \to \mathrm{QCoh}(\mathcal{Y})$ generates $\mathrm{QCoh}(\mathcal{Y})$, where $\mathcal{Y} = Y/G$.

With no restriction of generality, we can assume that G is reductive. Let p denote the projection $Y \to Y/G$. Since G is reductive, its regular representation contains the trivial representation as a direct summand. Hence, any $\mathcal{F} \in \mathrm{QCoh}(Y/G)$ is a direct summand of $p_* \circ p^*(\mathcal{F})$. This reduces the assertion to the case when Y/G is replaced by Y, i.e., we can assume that $\mathcal{Y}$ is a quasi-compact quasi-separated classical reduced scheme Y. One further easily reduces to the case when Y is affine.

Now, for a map $f : U \to Y$, where Y is an affine classical reduced scheme, the assertion of Proposition 8.1.1(a) is easy: the full subcategory generated by the essential image of f_* is a tensor ideal, and since f is faithfully flat, it contains the structure sheaf of the generic point of any irreducible subscheme of Y, and thus equals all of $\mathrm{QCoh}(Y)$.

[10] Here the assumption that $\mathcal{Y}$ be eventually coconnective is crucial.

Alternatively, the assertion of Proposition 8.1.1(a) for any affine DG scheme follows from Proposition 6.3.7.

8.6. **Passing from** QCoh **to** IndCoh**.** In order to prove Proposition 8.1.1(b), we will need to replace the categories $\mathrm{QCoh}(\mathcal{Y})$ and $\mathrm{QCoh}(U)$ by $\mathrm{IndCoh}(\mathcal{Y})$ and $\mathrm{IndCoh}(U)$, respectively. The reason for this will be explained in Step 2 of the proof of Proposition 8.6.5, which is an IndCoh version of Proposition 8.1.1(b).

8.6.1. Let us recall that for an algebraic stack $\mathcal{Y}$ locally almost of finite type, there exists a canonically defined natural transformation

$$\Psi_{\mathcal{Y}} : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}),$$

see [Ga, Sect. 11.7].

Moreover, when $\mathcal{Y}$ is eventually coconnective, the functor $\Psi_{\mathcal{Y}}$ admits a left adjoint $\Xi_{\mathcal{Y}}$, see [Ga, Sect. 11.7.3]. The interactions of the pair $(\Xi_{\mathcal{Y}}, \Psi_{\mathcal{Y}})$ with the functors arising from schematic maps between stacks are the same as those for maps between DG schemes, see [Ga, Sect. 3].

8.6.2. Consider the commutative diagram:

$$\begin{array}{ccc} \mathrm{QCoh}(U) & \xleftarrow{\Psi_U} & \mathrm{IndCoh}(U) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*^{\mathrm{IndCoh}}} \\ \mathrm{QCoh}(\mathcal{Y}) & \xleftarrow{\Psi_{\mathcal{Y}}} & \mathrm{IndCoh}(\mathcal{Y}). \end{array} \tag{8.4}$$

According to [Ga, Proposition 3.6.7], since f is fppf, the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(U) & \xrightarrow{\Xi_U} & \mathrm{IndCoh}(U) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*^{\mathrm{IndCoh}}} \\ \mathrm{QCoh}(\mathcal{Y}) & \xrightarrow{\Xi_{\mathcal{Y}}} & \mathrm{IndCoh}(\mathcal{Y}), \end{array} \tag{8.5}$$

obtained from (8.4) by passing to left adjoints along the horizontal arrows is also commutative.

8.6.3. Let $f^{\mathrm{IndCoh},?}$ denote the (discontinuous) right adjoint to f_*^{IndCoh}. By passing to right adjoints along all arrows in (8.5), we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(U) & \xleftarrow{\Psi_U} & \mathrm{IndCoh}(U) \\ {\scriptstyle f^?}\uparrow & & \uparrow{\scriptstyle f^{\mathrm{IndCoh},?}} \\ \mathrm{QCoh}(\mathcal{Y}) & \xleftarrow{\Psi_{\mathcal{Y}}} & \mathrm{IndCoh}(\mathcal{Y}). \end{array} \tag{8.6}$$

Consider the monad $\mathrm{Av}_?^{\mathrm{IndCoh},U/\mathcal{Y}} := f^{\mathrm{IndCoh},?} \circ f_*^{\mathrm{IndCoh}}$ acting on $\mathrm{IndCoh}(U)$. We obtain that the functor Ψ_U intertwines the actions of the monads $\mathrm{Av}_?^{\mathrm{IndCoh}}$ on $\mathrm{IndCoh}(U)$ and $\mathrm{Av}_?^{U/\mathcal{Y}} := f^? \circ f_*$ on $\mathrm{IndCoh}(\mathcal{Y})$, respectively.

In particular, we obtain commutative diagrams

$$\begin{array}{ccc} \mathrm{Av}_?^{U/\mathcal{Y}}\text{-mod}(\mathrm{QCoh}(U)) & \xleftarrow{\Psi_U^{\mathrm{Av}}} & \mathrm{Av}_?^{\mathrm{IndCoh},U/\mathcal{Y}}\text{-mod}(\mathrm{IndCoh}(U)) \\ {\scriptstyle \mathrm{ind}}\Big\uparrow\Big\downarrow{\scriptstyle \mathrm{oblv}} & & {\scriptstyle \mathrm{ind}}\Big\uparrow\Big\downarrow{\scriptstyle \mathrm{oblv}} \\ \mathrm{QCoh}(U) & \xleftarrow{\Psi_U} & \mathrm{IndCoh}(U) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Av}_?^{U/\mathcal{Y}}\text{-mod}(\mathrm{QCoh}(U)) & \xleftarrow{\Psi_U^{\mathrm{Av}}} & \mathrm{Av}_?^{\mathrm{IndCoh},U/\mathcal{Y}}\text{-mod}(\mathrm{IndCoh}(U)) \\ {\scriptstyle (f^?)^{\mathrm{enh}}}\uparrow & & \uparrow{\scriptstyle (f^{\mathrm{IndCoh},?})^{\mathrm{enh}}} \\ \mathrm{QCoh}(\mathcal{Y}) & \xleftarrow{\Psi_{\mathcal{Y}}} & \mathrm{IndCoh}(\mathcal{Y}). \end{array}$$

8.6.4. In Proposition 8.1.1(b) we need to show that the left adjoint $(f_*)^{\mathrm{enh}}$ of $(f^?)^{\mathrm{enh}}$ is conservative.

In the remainder of this subsection, we will deduce this assertion from the next one:

Proposition 8.6.5. *The functor*

$$(f_*^{\mathrm{IndCoh}})^{\mathrm{enh}} : \mathrm{Av}_?^{\mathrm{IndCoh},U/\mathcal{Y}}\text{-mod}(\mathrm{IndCoh}(U)) \to \mathrm{IndCoh}(\mathcal{Y}),$$

left adjoint to $(f^{\mathrm{IndCoh},?})^{\mathrm{enh}}$*, is conservative.*

8.6.6. *Proof of Proposition 8.1.1(b).* We need to show that the essential image of the functor $(f^?)^{\mathrm{enh}}$ *co-generates* $\mathrm{Av}_?^{U/\mathcal{Y}}$-mod$(\mathrm{QCoh}(U))$, i.e., generates under the operation of taking *limits*.

The assertion of Proposition 8.6.5 is equivalent to the fact that the essential image of the functor $(f^{\mathrm{IndCoh},?})^{\mathrm{enh}}$ co-generates $\mathrm{Av}_?^{\mathrm{IndCoh},U/\mathcal{Y}}$-mod$(\mathrm{IndCoh}(U))$.

Hence, it is sufficient to show that the functor Ψ_U^{Av} is essentially surjective. We will show that the functor Ψ_U^{Av} admits a fully faithful *right* adjoint.

First, since the functor Ψ_U is a co-localization (see [Ga, Proposition 1.5.3]) and is continuous, it admits a right adjoint, denoted Φ_U, which is also fully faithful. Hence, it suffices to show that the functor Φ_U intertwines the actions of the monads $\mathrm{Av}_?^{U/\mathcal{Y}}$ on $\mathrm{QCoh}(U)$ and $\mathrm{Av}_?^{\mathrm{IndCoh},U/\mathcal{Y}}$ on $\mathrm{IndCoh}(U)$, respectively. For that, it is sufficient to show that the diagrams

$$\begin{array}{ccc} \mathrm{QCoh}(U) & \xrightarrow{\Phi_U} & \mathrm{IndCoh}(U) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*^{\mathrm{IndCoh}}} \\ \mathrm{QCoh}(\mathcal{Y}) & \xrightarrow{\Phi_{\mathcal{Y}}} & \mathrm{IndCoh}(\mathcal{Y}) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{QCoh}(U) & \xrightarrow{\Phi_U} & \mathrm{IndCoh}(U) \\ {\scriptstyle f^?}\uparrow & & \uparrow{\scriptstyle f^{\mathrm{IndCoh},?}} \\ \mathrm{QCoh}(\mathcal{Y}) & \xrightarrow{\Phi_{\mathcal{Y}}} & \mathrm{IndCoh}(\mathcal{Y}) \end{array}$$

commute.

The commutation of these diagrams is obtained by passing to right adjoints in the diagrams

$$\begin{array}{ccc} \mathrm{QCoh}(U) & \xleftarrow{\Psi_U} & \mathrm{IndCoh}(U) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^{\mathrm{IndCoh},*}} \\ \mathrm{QCoh}(\mathcal{Y}) & \xleftarrow{\Psi_{\mathcal{Y}}} & \mathrm{IndCoh}(\mathcal{Y}) \end{array}$$

(see [Ga, Proposition 3.5.4]) and (8.4), respectively.

□

8.7. **Proof of the** IndCoh**-version of Proposition 8.1.1(b).** In this subsection we will prove Proposition 8.6.5. We need to show that the essential image of the functor $(f^{\mathrm{IndCoh},?})^{\mathrm{enh}}$ co-generates $\mathrm{Av}_?^{\mathrm{IndCoh},U/\mathcal{Y}}$-mod$(\mathrm{IndCoh}(U))$.

8.7.1. *Step 1.* Let $\mathcal{Y}' \overset{\imath}{\hookrightarrow} \mathcal{Y}$ be a closed substack, and let $\mathcal{Y}_0 \overset{\jmath}{\hookrightarrow} \mathcal{Y}$ be the complementary open. Consider the corresponding adjoint pair of functors

$$\jmath^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(\mathcal{Y}) \rightleftarrows \mathrm{IndCoh}(\mathcal{Y}_0) : \jmath_*^{\mathrm{IndCoh}}.$$

Recall the notation

$$\mathrm{IndCoh}(\mathcal{Y})_{\mathcal{Y}'} := \ker(\jmath^{\mathrm{IndCoh},*}),$$

see [Ga, Sect. 4.1.2]. Let

$$\widehat{\imath}_! : \mathrm{IndCoh}(\mathcal{Y})_{\mathcal{Y}'} \rightleftarrows \mathrm{IndCoh}(\mathcal{Y}) : \widehat{\imath}^!$$

be the resulting adjoint pair of functors. Let $\widehat{\imath}_?$ denote the (discontinuous) right adjoint of $\widehat{\imath}^!$. We will use similar notations for the corresponding objects on U.

We have a commutative diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(U_0) \arrow[r, shift left, "(\jmath_U)^{\mathrm{IndCoh},*}"] \arrow[d, "(f_0)_*^{\mathrm{IndCoh}}"'] & \mathrm{IndCoh}(U) \arrow[l, shift left, "(\jmath_U)_*^{\mathrm{IndCoh}}"] \arrow[d, "f_*^{\mathrm{IndCoh}}"] \\
\mathrm{IndCoh}(\mathcal{Y}_0) \arrow[r, shift left, "\jmath^{\mathrm{IndCoh},*}"] & \mathrm{IndCoh}(\mathcal{Y}) \arrow[l, shift left, "\jmath_*^{\mathrm{IndCoh}}"]
\end{tikzcd}$$

and the diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(U_0) \arrow[r, "(\jmath_U)_*"] & \mathrm{IndCoh}(U) \\
\mathrm{IndCoh}(\mathcal{Y}_0) \arrow[r, "\jmath_*"] \arrow[u, "f_0^{\mathrm{IndCoh},*}"] & \mathrm{IndCoh}(\mathcal{Y}) \arrow[u, "f^{\mathrm{IndCoh},*}"']
\end{tikzcd}$$

is also commutative, see [Ga, Lemma 3.6.9].

Hence, by passing to the right adjoint functors, we obtain that both functors $(\jmath_U)_*^{\mathrm{IndCoh}}$ and $(\jmath_U)^{\mathrm{IndCoh},?}$ intertwine the monads $\mathrm{Av}_?^{\mathrm{IndCoh},U/\mathcal{Y}}$ and $\mathrm{Av}_?^{\mathrm{IndCoh},U_0/\mathcal{Y}_0}$ acting on $\mathrm{IndCoh}(U)$ and $\mathrm{IndCoh}(U_0)$, respectively.

In addition, we obtain a monad $\widehat{\mathrm{Av}}_?^{U'/\mathcal{Y}'}$, acting on $\mathrm{IndCoh}(U)_{U'}$, such that both functors $(\widehat{\imath}_U)^!$ and $(\widehat{\imath}_U)_?$ intertwine the monads $\mathrm{Av}_?^{\mathrm{IndCoh},U/\mathcal{Y}}$ and $\widehat{\mathrm{Av}}_?^{\mathrm{IndCoh},U'/\mathcal{Y}'}$ acting on $\mathrm{IndCoh}(U)$ and $\mathrm{IndCoh}(U)_{U'}$, respectively.

Thus, we obtain a localization sequence of categories

$$\mathrm{Av}_?^{\mathrm{IndCoh},U_0/\mathcal{Y}_0}\text{-mod} \rightleftarrows \mathrm{Av}_?^{\mathrm{IndCoh},U/\mathcal{Y}}\text{-mod} \rightleftarrows \widehat{\mathrm{Av}}_?^{\mathrm{IndCoh},U'/\mathcal{Y}'}\text{-mod},$$

which makes the diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathcal{Y}_0) \arrow[r, shift left, "\jmath_*^{\mathrm{IndCoh}}"] \arrow[d, "(f_0^{\mathrm{IndCoh},?})^{\mathrm{enh}}"'] & \mathrm{IndCoh}(\mathcal{Y}) \arrow[l, shift left, "\jmath^{\mathrm{IndCoh},?}"] \arrow[r, shift left, "\widehat{\imath}^!"] \arrow[d, "(f^{\mathrm{IndCoh},?})^{\mathrm{enh}}"'] & \mathrm{IndCoh}(\mathcal{Y})_{\mathcal{Y}'} \arrow[l, shift left, "\widehat{\imath}_?"] \arrow[d, "((\widehat{f}')^{\mathrm{IndCoh},?})^{\mathrm{enh}}"] \\
\mathrm{Av}_?^{\mathrm{IndCoh},U_0/\mathcal{Y}_0}\text{-mod} \arrow[r, shift left] \arrow[d, "\mathrm{oblv}"'] & \mathrm{Av}_?^{\mathrm{IndCoh},U/\mathcal{Y}}\text{-mod} \arrow[l, shift left] \arrow[r, shift left] \arrow[d, "\mathrm{oblv}"'] & \widehat{\mathrm{Av}}_?^{\mathrm{IndCoh},U'/\mathcal{Y}'}\text{-mod} \arrow[l, shift left] \arrow[d, "\mathrm{oblv}"] \\
\mathrm{IndCoh}(U_0) \arrow[r, shift left, "(\jmath_U)_*^{\mathrm{IndCoh}}"] & \mathrm{IndCoh}(U) \arrow[l, shift left, "(\jmath_U)^{\mathrm{IndCoh},?}"] \arrow[r, shift left, "(\widehat{\imath}_U)^!"] & \mathrm{IndCoh}(U)_{U'} \arrow[l, shift left, "(\widehat{\imath}_U)_?"]
\end{tikzcd}$$

commute.

Therefore we obtain that if the essential image of $(f_0^{\mathrm{IndCoh},?})^{\mathrm{enh}}$ co-generates the category $\mathrm{Av}_?^{\mathrm{IndCoh},U_0/\mathcal{Y}_0}\text{-mod}(\mathrm{IndCoh}(U_0))$, and the essential image of $((\widehat{f}')^{\mathrm{IndCoh},?})^{\mathrm{enh}}$ co-generates the category $\widehat{\mathrm{Av}}_?^{\mathrm{IndCoh},U'/\mathcal{Y}'}\text{-mod}(\mathrm{IndCoh}(U)_{U'})$, then the essential image of $(f^{\mathrm{IndCoh},?})^{\mathrm{enh}}$ co-generates $\mathrm{Av}_?^{\mathrm{IndCoh},U/\mathcal{Y}}\text{-mod}(\mathrm{IndCoh}(U))$.

8.7.2. *Step 2.* We will now show that if the essential image of

$$(f')^{\mathrm{IndCoh},?})^{\mathrm{enh}} : \mathrm{IndCoh}(\mathcal{Y}') \to \mathrm{Av}_?^{\mathrm{IndCoh},U'/\mathcal{Y}'}\text{-mod}(\mathrm{IndCoh}(U'))$$

co-generates $\mathrm{Av}_?^{\mathrm{IndCoh},U'/\mathcal{Y}'}\text{-mod}(\mathrm{IndCoh}(U'))$, then the essential image of

$$((\widehat{f}')^{\mathrm{IndCoh},?})^{\mathrm{enh}} : \mathrm{IndCoh}(\mathcal{Y})_{\mathcal{Y}'} \to \widehat{\mathrm{Av}}_?^{\mathrm{IndCoh},U'/\mathcal{Y}'}\text{-mod}(\mathrm{IndCoh}(U)_{U'})$$

co-generates $\widehat{\mathrm{Av}}_?^{\mathrm{IndCoh},U'/\mathcal{Y}'}\text{-mod}(\mathrm{IndCoh}(U)_{U'})$.

We have a commutative diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathcal{Y})_{\mathcal{Y}'} \arrow[r, "\imath^!"] & \mathrm{IndCoh}(\mathcal{Y}') \\
\mathrm{IndCoh}(U)_{U'} \arrow[u, "(\widehat{f}')_*^{\mathrm{IndCoh}}"] \arrow[r, "(\imath_U)^!"] & \mathrm{IndCoh}(U') \arrow[u, "(f')_*^{\mathrm{IndCoh}}"']
\end{tikzcd} \tag{8.7}$$

(which expresses the base change isomorphism) and the commutative diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathcal{Y})_{\mathcal{Y}'} \arrow[r, "\imath^!"] \arrow[d, "(\widehat{f}')^{\mathrm{IndCoh},*}"'] & \mathrm{IndCoh}(\mathcal{Y}') \arrow[d, "(f')^{\mathrm{IndCoh},*}"] \\
\mathrm{IndCoh}(U)_{U'} \arrow[r, "(\imath_U)^!"] & \mathrm{IndCoh}(U'),
\end{tikzcd} \tag{8.8}$$

see [Ga, Proposition 7.1.6].

Note that since the functor $\imath^!$ is continuous, it admits a (discontinuous) right adjoint [11], denoted $\imath_? : \mathrm{IndCoh}(\mathcal{Y})_{\mathcal{Y}'} \to \mathrm{IndCoh}(\mathcal{Y}')$. By passing to right adjoints in the diagrams (8.7) and

[11] This manipulation is the reason for replacing QCoh by IndCoh in the proof of Proposition 8.1.1(b).

(8.8), we obtain commutative diagrams

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathcal{Y})_{\mathcal{Y}'} & \xleftarrow{\imath_?} & \mathrm{IndCoh}(\mathcal{Y}') \\ {\scriptstyle (\widehat{f}')^{\mathrm{IndCoh},?}}\downarrow & & \downarrow{\scriptstyle (f')^{\mathrm{IndCoh},?}} \\ \mathrm{IndCoh}(U)_{U'} & \xleftarrow{(\imath_U)_?} & \mathrm{IndCoh}(U') \end{array}$$

and

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathcal{Y})_{\mathcal{Y}'} & \xleftarrow{\imath_?} & \mathrm{IndCoh}(\mathcal{Y}') \\ {\scriptstyle (\widehat{f}')_*^{\mathrm{IndCoh}}}\uparrow & & \uparrow{\scriptstyle (f')_*^{\mathrm{IndCoh}}} \\ \mathrm{IndCoh}(U)_{U'} & \xleftarrow{(\imath_U)_?} & \mathrm{IndCoh}(U'). \end{array}$$

In particular, we obtain that the functor $(\imath_U)_?$ intertwines monads $\mathrm{Av}_?^{\mathrm{IndCoh},U'/\mathcal{Y}'}$ acting on $\mathrm{IndCoh}(U')$ and $\widehat{\mathrm{Av}}_?^{\mathrm{IndCoh},U'/\mathcal{Y}'}$ acting on $\mathrm{IndCoh}(U)_{U'}$, and we have a commutative diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathcal{Y})_{\mathcal{Y}'} & \xleftarrow{\imath_?} & \mathrm{IndCoh}(\mathcal{Y}') \\ {\scriptstyle ((\widehat{f}')^{\mathrm{IndCoh},?})^{\mathrm{enh}}}\downarrow & & \downarrow{\scriptstyle ((f')^{\mathrm{IndCoh},?})^{\mathrm{enh}}} \\ \widehat{\mathrm{Av}}_?^{\mathrm{IndCoh},U'/\mathcal{Y}'}\text{-mod}(\mathrm{IndCoh}(U)_{U'}) & \xleftarrow{(\imath_U^{\mathrm{Av}})_?} & \mathrm{Av}_?^{\mathrm{IndCoh},U'/\mathcal{Y}'}\text{-mod}(\mathrm{IndCoh}(U')). \end{array}$$

Hence, to carry out Step 2, it remains to show that the essential image of $(\imath_U^{\mathrm{Av}})_?$ co-generates $\widehat{\mathrm{Av}}_?^{\mathrm{IndCoh},U'/\mathcal{Y}'}$-mod$(\mathrm{IndCoh}(U)_{U'})$. For this, it suffices to show that $(\imath_U^{\mathrm{Av}})_?$ admits a left adjoint, to be denoted $(\imath_U^{\mathrm{Av}})^!$, which is conservative.

We claim that $(\imath_U^{\mathrm{Av}})^!$ exists and makes the diagram

$$\begin{array}{ccc} \widehat{\mathrm{Av}}_?^{\mathrm{IndCoh},U'/\mathcal{Y}'}\text{-mod}(\mathrm{IndCoh}(U)_{U'}) & \xrightarrow{(\imath_U^{\mathrm{Av}})^!} & \mathrm{Av}_?^{\mathrm{IndCoh},U'/\mathcal{Y}'}\text{-mod}(\mathrm{IndCoh}(U')) \\ {\scriptstyle \mathrm{oblv}}\downarrow & & \downarrow{\scriptstyle \mathrm{oblv}} \\ \mathrm{IndCoh}(U)_{U'} & \xrightarrow{(\imath_U)^!} & \mathrm{IndCoh}(U') \end{array}$$

commutative. This would also imply that $(\imath_U^{\mathrm{Av}})^!$ is conservative, since $(\imath_U)^!$ is conservative, by [Ga, Proposition 4.1.7(a)].

To prove the existence of $(\imath_U^{\mathrm{Av}})^!$ with the required property, it suffices to show that the functor $(\imath_U)^!$ intertwines the monads $\widehat{\mathrm{Av}}_?^{\mathrm{IndCoh},U'/\mathcal{Y}'}$ acting on $\mathrm{IndCoh}(U)_{U'}$ and $\mathrm{Av}_?^{\mathrm{IndCoh},U'/\mathcal{Y}'}$ acting on $\mathrm{IndCoh}(U')$.

The latter follows from the commutativity of the diagram (8.8) and the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathcal{Y})_{\mathcal{Y}'} & \xrightarrow{\imath^!} & \mathrm{IndCoh}(\mathcal{Y}') \\ {\scriptstyle (\widehat{f}')^{\mathrm{IndCoh},?}}\downarrow & & \downarrow{\scriptstyle (f')^{\mathrm{IndCoh},?}} \\ \mathrm{IndCoh}(U)_{U'} & \xrightarrow{(\imath_U)^!} & \mathrm{IndCoh}(U'), \end{array}$$

which is obtained by passing to right adjoints in the commutative diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathcal{Y})_{\mathcal{Y}'} & \xleftarrow{\imath_*} & \mathrm{IndCoh}(\mathcal{Y}') \\ {\scriptstyle (\widehat{f}')^{\mathrm{IndCoh}}_*}\Big\uparrow & & \Big\uparrow{\scriptstyle (f')^{\mathrm{IndCoh}}_*} \\ \mathrm{IndCoh}(U)_{U'} & \xleftarrow{(\imath_U)_*} & \mathrm{IndCoh}(U'). \end{array}$$

8.7.3. *Step 3.* By Step 2, we can assume that $\mathcal{Y}$ is classical and reduced (take $\mathcal{Y}' := ({}^{cl}\mathcal{Y})_{red}$). By Step 1 and Noetherian induction, it suffices to show that $\mathcal{Y}$ contains a non-empty open substack $\mathcal{Y}_0$, for which Proposition 8.6.5 holds.

In particular, we can replace $\mathcal{Y}$ by its open substack, which is smooth. By assumption, the morphism $f : U \to \mathcal{Y}$ is smooth, so U is smooth as well. Now, for smooth schemes, there is no difference between IndCoh and QCoh, and the conclusion of Proposition 8.6.5 is identical to that of Proposition 8.1.1(b).

Thus, it suffices to show that any classical reduced $\mathcal{Y}$ contains a non-empty open that satisfies the conclusion of Proposition 8.1.1(b).

The rest of the proof is the same as that of Step 3 in the proof of Proposition 8.1.1(a):

We reduce the assertion to the case when $\mathcal{Y}$ is of the form Y/G, where Y is a quasi-compact quasi-separated scheme, and G is a classical affine algebraic group of finite type. However, $\mathcal{Y}$ if this form satisfies the conclusion of Proposition 8.1.1(b) because of Theorem 2.2.4 and Proposition 6.3.7.

Part III: (DG) Indschemes, Classifying Prestacks and De Rham Prestacks

9. DG indschemes

9.1. **A key proposition.**

9.1.1. Recall that $\mathrm{PreStk}_{\mathrm{laft}}$ denotes the full subcategory of PreStk formed by prestacks *locally almost of finite type*, and that on this category we have a well-defined functor

$$\mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{laft}}} : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

see [Ga, Sect. 10.1].

In addition, we recall that the functor $\mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{laft}}}$, comes equipped with a natural transformation, denoted

$$\Upsilon_{\mathrm{PreStk}_{\mathrm{laft}}} : \mathrm{QCoh}_{\mathrm{PreStk}_{\mathrm{laft}}} \to \mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{laft}}},$$

see [Ga, Sect. 10.3].

For an individual object $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$, the corresponding functor

$$\Upsilon_{\mathcal{Y}} : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y})$$

is given by

$$\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y}) \mapsto \mathcal{F} \otimes \omega_{\mathcal{Y}} \in \mathrm{IndCoh}(\mathcal{Y}),$$

where $\omega_{\mathcal{Y}} \in \mathrm{IndCoh}(\mathcal{Y})$ is the dualizing object, and where $\otimes$ is the canonical action on $\mathrm{QCoh}(\mathcal{Y})$ on $\mathrm{IndCoh}(\mathcal{Y})$.

9.1.2. A key technical tool that will allow us to establish the results pertaining to 1-affineness of DG indschemes (as well as formal classifying spaces and de Rham prestacks) is the following assetion:

Proposition 9.1.3. *Let $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$ be such that the unctor $\Upsilon_{\mathcal{Y}} : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y})$ is an equivalence. Suppose also that $\mathcal{Y}$ can be written as a colimit of $\underset{i \in I}{\underset{\longrightarrow}{colim}}\, \mathcal{Z}_i$ with $\mathcal{Z}_i \in \mathrm{PreStk}_{\mathrm{laft}}$, where I is some index category, such that:*

(1) *For every i, the functor $\mathbf{Loc}_{\mathcal{Z}_i}$ is fully faithful;*
(2) *For every i, the functor $\Upsilon_{\mathcal{Z}_i} : \mathrm{QCoh}(\mathcal{Z}_i) \to \mathrm{IndCoh}(\mathcal{Z}_i)$ is fully faithful and admits a continuous right adjoint, compatible with the action of $\mathrm{QCoh}(\mathcal{Z}_i)$;*
(3) *For every arrow $(i \to j) \in I$ and the corresponding map $f_{i,j} : \mathcal{Z}_i \to \mathcal{Z}_j$, the functor $f^!_{i,j} : \mathrm{IndCoh}(\mathcal{Z}_j) \to \mathrm{IndCoh}(\mathcal{Z}_i)$ admits a left adjoint, compatible with the action of $\mathrm{QCoh}(\mathcal{Z}_j)$;*

Then the functor $\mathbf{Loc}_{\mathcal{Y}}$ is fully faithful.

The rest of this subsection is devoted to the proof of Proposition 9.1.3.

9.1.4. *Step 1.* Let $\mathcal{Y} \simeq \underset{i}{\underset{\longrightarrow}{colim}}\, \mathcal{Z}_i$ be a presentation of $\mathcal{Y}$ as in Proposition 9.1.3.

For $\mathbf{C} \in \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$, we have:

$$\mathbf{\Gamma}(\mathcal{Y}, \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C})) \simeq \underset{i}{\underset{\longleftarrow}{lim}}\, \mathbf{\Gamma}(\mathcal{Z}_i, \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C})).$$

Note, however, that

$$\mathbf{Loc}_{\mathcal{Y}}(\mathbf{C})|_{\mathcal{Z}_i} \simeq \mathbf{Loc}_{\mathcal{Z}_i}(\mathrm{QCoh}(\mathcal{Z}_i) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C}).$$

Hence, the assumption that the functor $\mathbf{Loc}_{\mathcal{Z}_i}$ is fully faithful implies that

$$\mathbf{\Gamma}(\mathcal{Z}_i, \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C})) \simeq \mathrm{QCoh}(\mathcal{Z}_i) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C}.$$

Hence, we conclude that

$$\mathbf{\Gamma}(\mathcal{Y}, \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C})) \simeq \underset{i}{\underset{\longleftarrow}{lim}} \left(\mathrm{QCoh}(\mathcal{Z}_i) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \right).$$

9.1.5. *Step 2.* Consider also the category

$$\mathbf{\Gamma}^!(\mathcal{Y}, \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C})) := \underset{i}{\underset{\longleftarrow}{lim}} \left(\mathrm{IndCoh}(\mathcal{Z}_i) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \right), \tag{9.1}$$

where the functors

$$\mathrm{IndCoh}(\mathcal{Z}_j) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \to \mathrm{IndCoh}(\mathcal{Z}_i) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C}$$

are $f^!_{i,j} \otimes \mathrm{id}_{\mathbf{C}}$, where for an arrow $(i \to j) \in I$, we denote by $f_{i,j}$ the corresponding map $\mathcal{Z}_i \to \mathcal{Z}_j$.

Since in the formation of (9.1), the transition functors admit left adjoints, by [GL:DG, Lemma 1.3.3], we can rewrite $\mathbf{\Gamma}^!(\mathcal{Y}, \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C}))$ also as

$$\underset{i}{\underset{\longrightarrow}{colim}} \left(\mathrm{IndCoh}(\mathcal{Z}_i) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \right),$$

where the transition functors

$$\mathrm{IndCoh}(\mathcal{Z}_i) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \to \mathrm{IndCoh}(\mathcal{Z}_j) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C}$$

are now $(f^!_{i,j})^L \otimes \mathrm{Id}_{\mathbf{C}}$.

Commuting the colimit with the tensor product we obtain:

$$\mathbf{\Gamma}^!(\mathcal{Y}, \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C})) \simeq \underset{\underset{i}{\longrightarrow}}{colim} \left(\mathrm{IndCoh}(\mathcal{Z}_i) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \right) \simeq$$

$$\simeq \left(\underset{\underset{i}{\longrightarrow}}{colim}\, \mathrm{IndCoh}(\mathcal{Z}_i) \right) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \simeq \left(\underset{\underset{i}{\longleftarrow}}{lim}\, \mathrm{IndCoh}(\mathcal{Z}_i) \right) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \simeq \mathrm{IndCoh}(\mathcal{Y}) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C}.$$

9.1.6. *Step 3.* Recall now the natural transformation $\Upsilon_{\mathrm{PreStk}_{\mathrm{laft}}}$.

For any $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$, this gives rise to a functor, denoted $\Upsilon_{\mathcal{Y},\mathbf{C}}$,

$$\mathbf{\Gamma}(\mathcal{Y}, \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C})) \simeq \underset{\underset{i}{\longleftarrow}}{lim} \left(\mathrm{QCoh}(\mathcal{Z}_i) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \right) \to$$

$$\to \underset{\underset{i}{\longleftarrow}}{lim} \left(\mathrm{IndCoh}(\mathcal{Z}_i) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \right) \simeq \mathbf{\Gamma}^!(\mathcal{Y}, \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C})),$$

and we have a commutative diagram

$$\begin{array}{ccccc} \mathbf{C} & \xrightarrow{\sim} & \mathrm{QCoh}(\mathcal{Y}) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} & \longrightarrow & \mathbf{\Gamma}(\mathcal{Y}, \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C})) \\ & & \Upsilon_{\mathcal{Y}} \otimes \mathrm{Id}_{\mathbf{C}} \downarrow & & \downarrow \Upsilon_{\mathcal{Y},\mathbf{C}} \\ & & \mathrm{IndCoh}(\mathcal{Y}) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} & \xrightarrow{\sim} & \mathbf{\Gamma}^!(\mathcal{Y}, \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C})). \end{array} \tag{9.2}$$

By assumption, $\Upsilon_{\mathcal{Y}}$ is an equivalence, and hence so is the left vertical arrow in diagram (9.2).

Thus, we obtain that $\mathbf{C}$ is a retract of $\mathbf{\Gamma}(\mathcal{Y}, \mathbf{Loc}_{\mathcal{Y}}(\mathbf{C}))$. To prove the proposition, it remains to show that the functor $\Upsilon_{\mathcal{Y},\mathbf{C}}$ is fully faithful.

9.1.7. *Step 4.* To prove that $\Upsilon_{\mathcal{Y},\mathbf{C}}$ is fully faithful, it is sufficient to show that each of the functors

$$\Upsilon_{\mathcal{Z}_i} \otimes \mathrm{Id}_{\mathbf{C}} : \mathrm{QCoh}(\mathcal{Z}_i) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \to \mathrm{IndCoh}(\mathcal{Z}_i) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C}$$

is fully faithful.

However, this follows from the fact that $\Upsilon_{\mathcal{Z}_i}$ is fully faithful and admits a continuous right adjoint, compatible with the action of $\mathrm{QCoh}(\mathcal{Z}_i)$. Indeed, in this case, the functor $\Upsilon_{\mathcal{Z}_i} \otimes \mathrm{Id}_{\mathbf{C}}$ admits a right adjoint, given by $(\Upsilon_{\mathcal{Z}_i})^R \otimes \mathrm{Id}_{\mathbf{C}}$, and the unit of the adjunction

$$\mathrm{Id}_{\mathrm{QCoh}(\mathcal{Z}_i)} \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \to \left((\Upsilon_{\mathcal{Z}_i})^R \otimes \mathrm{Id}_{\mathbf{C}} \right) \circ (\Upsilon_{\mathcal{Z}_i} \otimes \mathrm{Id}_{\mathbf{C}}) \simeq \left((\Upsilon_{\mathcal{Z}_i})^R \circ \Upsilon_{\mathcal{Z}_i} \right) \otimes \mathrm{Id}_{\mathbf{C}}$$

is an isomorphism.

□

9.2. **Fully faithfulness of Loc.** In this subsection we will prove Theorems 2.4.2 and 2.4.3.

9.2.1. *Proof of Theorem 2.4.2.* Let $\mathcal{Y}$ be a DG indscheme, which is weakly $\aleph_0$, locally almost of finite type and formally smooth. First, by [GR1, Theorem 10.1.1], the functor $\Psi_{\mathcal{Y}}$ is an equivalence.

Next, by [GR1, Theorem 9.1.6], we obtain that $\mathcal{Y}$ is *classical*, i.e., it can be written (up to fppf sheafification) as

$$\underset{\alpha}{\underset{\longrightarrow}{lim}} Z_\alpha,$$

where Z_α are classical schemes of finite type, and the maps $Z_{\alpha_1} \to Z_{\alpha_2}$ are closed embeddings.

Let us show that this presentation satisfies that conditions of Proposition 9.1.3. Indeed, Condition (1) is satisfied by Theorem 2.1.1. Condition (2) is satisfied by [Ga, Corollary 9.6.3]. Condition (3) is satisfied because the left adjoints are given by $(f_{i,j})_*^{\mathrm{IndCoh}}$.

9.2.2. *Proof of Theorem 2.4.3.* Consider the affine Grassmannian Gr_G and the canonical projection $\pi : G(\!(t)\!) \to \mathrm{Gr}_G$.

By Theorem 2.4.2, the functor $\mathbf{Loc}_{\mathrm{Gr}_G}$ is fully faithful. Hence, in order to show that $\mathbf{Loc}_{G(\!(t)\!)}$ is also fully faithful, it suffices to show that π satisfies the assumptions of Proposition 3.2.6(a).

However, since π is affine this follows from [GL:QCoh, Proposition 3.2.1] (reproduced for the reader's convenience in the Appendix as Proposition B.1.3).

9.3. **Non 1-affineness of $\mathbb{A}^\infty$.** In this subsection we will prove Theorem 2.4.5. I.e., we will show that the functor $\mathbf{\Gamma}_{\mathbb{A}^\infty}$ fails to be fully faithful.

9.3.1. Let ι denote the map $\mathrm{pt} \to \mathbb{A}^\infty$ corresponding to $0 \in \mathbb{A}^\infty$. Consider

$$\mathcal{C} := \mathbf{coind}_\iota(\mathrm{Vect}).$$

We will show that the counit map

$$\mathbf{\Gamma}\left(\mathrm{pt}, \mathbf{Loc}_{\mathbb{A}^\infty} \circ \mathbf{\Gamma}^{\mathrm{enh}}_{\mathbb{A}^\infty}(\mathcal{C})\right) \to \mathbf{\Gamma}(\mathrm{pt}, \mathcal{C})$$

is not an equivalence.

By definition

$$\mathbf{\Gamma}(\mathrm{pt}, \mathcal{C}) \simeq \mathrm{QCoh}(\mathrm{pt} \underset{\mathbb{A}^\infty}{\times} \mathrm{pt}),$$

and

$$\mathbf{\Gamma}\left(\mathrm{pt}, \mathbf{Loc}_{\mathbb{A}^\infty} \circ \mathbf{\Gamma}^{\mathrm{enh}}_{\mathbb{A}^\infty}(\mathcal{C})\right) \simeq \mathrm{Vect} \underset{\mathrm{QCoh}(\mathbb{A}^\infty)}{\otimes} \mathrm{Vect}\,.$$

Thus, we want to show that the naturally defined functor

$$\mathrm{Vect} \underset{\mathrm{QCoh}(\mathbb{A}^\infty)}{\otimes} \mathrm{Vect} \to \mathrm{QCoh}(\mathrm{pt} \underset{\mathbb{A}^\infty}{\times} \mathrm{pt}) \tag{9.3}$$

is not an equivalence.

9.3.2. Let S denote the tautological functor

$$\mathrm{Vect} \to \mathrm{Vect} \underset{\mathrm{QCoh}(\mathbb{A}^\infty)}{\otimes} \mathrm{Vect},$$

and let T be its (discontinuous) right adjoint.

The composed functor

$$\mathrm{Vect} \xrightarrow{\mathsf{S}} \mathrm{Vect} \underset{\mathrm{QCoh}(\mathbb{A}^\infty)}{\otimes} \mathrm{Vect} \to \mathrm{QCoh}(\mathrm{pt} \underset{\mathbb{A}^\infty}{\times} \mathrm{pt})$$

is the pull-back functor p^*, where p denotes the map $\mathrm{pt} \underset{\mathbb{A}^\infty}{\times} \mathrm{pt} \to \mathrm{pt}$. The right adjoint of p^* is the usual direct image functor p_*.

We will show:

Lemma 9.3.3. *The functor* T *is monadic.*

Lemma 9.3.4. *The functor* p_* *is* not *monadic.*

Together, these two lemmas imply that (9.3) is not an equivalence.

9.3.5. *Proof of Lemma 9.3.3.* To analyze (S, T) we identify $\mathrm{QCoh}(\mathbb{A}^\infty)$ with $\mathrm{IndCoh}(\mathbb{A}^\infty)$ via the functor $\Psi_{\mathbb{A}^\infty}$. The usual monoidal structure on $\mathrm{QCoh}(\mathbb{A}^\infty)$ goes over to the $\overset{!}{\otimes}$ monoidal structure on $\mathrm{IndCoh}(\mathbb{A}^\infty)$. Similarly, $\iota^* : \mathrm{QCoh}(\mathbb{A}^\infty) \to \mathrm{Vect}$ goes over to the functor $\iota^!$.

Note in this case the tensor product functor

$$\mathrm{IndCoh}(\mathbb{A}^\infty) \otimes \mathrm{IndCoh}(\mathbb{A}^\infty) \to \mathrm{IndCoh}(\mathbb{A}^\infty)$$

admits a *left* adjoint, given by $(\Delta_{\mathbb{A}^\infty})_*^{\mathrm{IndCoh}}$. Similarly, the functor $\iota^!$ admits a *left* adjoint given by $\iota_*^{\mathrm{IndCoh}}$, which is a map of $\mathrm{IndCoh}(\mathbb{A}^\infty)$-module categories.

This implies that T is monadic by Corollary C.2.3.

□

9.4. **Proof of Lemma 9.3.4.**

9.4.1. By definition:

$$\mathrm{QCoh}(\mathrm{pt} \underset{\mathbb{A}^\infty}{\times} \mathrm{pt}) \simeq \underset{n}{\overleftarrow{lim}}\, \mathrm{QCoh}(\mathrm{pt} \underset{V_n}{\times} \mathrm{pt}),$$

where $V_n = \mathbb{A}^n$. Set $V := \underset{n}{\overrightarrow{lim}}\, V_n$.

For each n, we have:

$$\mathrm{QCoh}(\mathrm{pt} \underset{V_n}{\times} \mathrm{pt}) \simeq \mathrm{Sym}(V_n^*[1])\text{-mod}.$$

Hence, the monad $\mathsf{H} := p_* \circ p^*$ is given by

$$M \mapsto \underset{n}{\overleftarrow{lim}}\, (\mathrm{Sym}(V_n^*[1]) \otimes M)\,.$$

Consider the corresponding functor

$$(p_*)^{\mathrm{enh}} : \mathrm{QCoh}(\mathrm{pt} \underset{\mathbb{A}^\infty}{\times} \mathrm{pt}) \to \mathsf{H}\text{-mod}(\mathrm{Vect}).$$

We need to show that $(p_*)^{\mathrm{enh}}$ is *not* an equivalence. We will do so by showing that it does not send a certain direct sum to the direct sum.

9.4.2. Consider the object N of

$$\mathrm{QCoh}(\mathrm{pt} \underset{\mathbb{A}^\infty}{\times} \mathrm{pt}) \simeq \underset{n}{\overleftarrow{lim}}\, \mathrm{Sym}(V_n^*[1])\text{-mod},$$

whose n-th term is

$$\underset{m \geq n}{\overleftarrow{lim}}\, \mathrm{Sym}\left(\ker(V_m^* \to V_n^*)[2]\right),$$

viewed as an object of $\mathrm{Sym}(V_n^*[1])$-mod via the trivial action.

We have $p_*(N) = k \in \mathrm{Vect}$. It is also easy to see that $(p_*)^{\mathrm{enh}} = k \in \mathsf{H}\text{-mod}$, where the action of H on k is trivial.

Consider now the object $\underset{i}{\oplus} N[-2i] \in \mathrm{QCoh}(\mathrm{pt} \underset{\mathbb{A}^\infty}{\times} \mathrm{pt})$. We will show that the map

$$\bigoplus_i (p_*)^{\mathrm{enh}}(N)[-2i] \to (p_*)^{\mathrm{enh}}\left(\underset{i}{\oplus} N[-2i]\right)$$

is *not* an isomorphism.

9.4.3. On the one hand, it is easy to see that the object of Vect, underlying the object

$$\underset{i}{\oplus} k[-2i] \in \mathsf{H}\text{-mod}$$

is isomorphic to $\underset{i}{\oplus} k[-2i]$ (although the forgetful functor $\mathsf{H}\text{-mod} \to \mathrm{Vect}$ does not commute with arbitrarty direct sums).

9.4.4. On the other hand, we will show that the object of Vect underlying

$$(p_*)^{\mathrm{enh}}\left(\underset{i}{\oplus} N[-2i]\right)$$

has a non-trivial cohomology in degree 1.

Namely, the 1st cohomology in question is equal to $R^1(\mathit{limproj})$ of the following inverse family of vector spaces

$$n \mapsto \bigoplus_i \left(\underset{m \geq n}{\overleftarrow{lim}}\, \left(\mathrm{Sym}^i(\ker(V_m^* \to V_n^*))\right)\right).$$

We compute $R(\mathit{limproj})$ of the above family by embedding it into the constant family with value

$$\bigoplus_i \left(\underset{m}{\overleftarrow{lim}}\, \mathrm{Sym}^i(V_m^*)\right).$$

To see that $R^1(\mathit{limproj}) \neq 0$, we need to show that the map

$$\bigoplus_i \underset{m}{\overleftarrow{lim}}\, \mathrm{Sym}^i(V_m^*) \to \tag{9.4}$$

$$\to \underset{n}{\mathit{limproj}}\left(\bigoplus_i \mathrm{coker}\left(\underset{m \geq n}{\overleftarrow{lim}}\, \mathrm{Sym}^i(\ker(V_m^* \to V_n^*)) \to \underset{m}{\overleftarrow{lim}}\, \mathrm{Sym}^i(V_m^*)\right)\right)$$

is *not* surjective.

9.4.5. Choose a basis $v_1, v_2, \ldots$ of V so that $\{v_1, ..., v_n\}$ is a basis of V_n. For every m, let $\{v^*_{m,1}, ..., v^*_{m,m}\}$ be the corresponding dual basis of V^*_m.

The following element in the right-hand side of (9.4) does not lie in the image of the left-hand side:

Its n-th component, i.e., the corresponding element of

$$\bigoplus_i \operatorname{coker}\left(\underset{\leftarrow}{lim}_{m\geq n} \operatorname{Sym}^i(\ker(V^*_m \to V^*_n)) \to \underset{\leftarrow}{lim}_m \operatorname{Sym}^i(V^*_m)\right) \tag{9.5}$$

equals the sum over $i = 1, 2, \ldots$ of the elements $w_{n,i}$, where each $w_{n,i}$ is the image under

$$\underset{\leftarrow}{lim}_m \operatorname{Sym}^i(V^*_m) \to \operatorname{coker}\left(\underset{\leftarrow}{lim}_{m\geq n} \operatorname{Sym}^i(\ker(V^*_m \to V^*_n)) \to \underset{\leftarrow}{lim}_m \operatorname{Sym}^i(V^*_m)\right)$$

of the family of elements

$$m \mapsto (v^*_{m,i})^{\otimes i} \in \operatorname{Sym}^i(V^*_m), \quad m \geq i.$$

Note that $w_{n,i} = 0$ if $i \geq n$, because in this case

$$(v^*_{m,i})^{\otimes i} \in \operatorname{Sym}^i(\ker(V^*_m \to V^*_n)).$$

Hence, the sum $\sum_i w_{n,i}$ is finite, i.e., gives rise to a well-defined element in (9.5).

□

10. Classifying prestacks

10.1. **Sheaves of categories over classifying prestacks.**

10.1.1. In this section we let $\mathcal{G}$ be a group-object of PreStk such that the functor $\mathbf{Loc}_{\mathcal{G}}$ is fully faithful.

Note that by Corollary 5.2.5, the functor $\mathbf{Loc}_{\mathcal{G}^n}$ is fully faithful for any n.

We will give a more explicit description of the category $\operatorname{ShvCat}(B\mathcal{G})$ as well as the functors $\mathbf{\Gamma}_{B\mathcal{G}}$ and $\mathbf{Loc}_{B\mathcal{G}}$.

10.1.2. First, by definition, we have:

$$\operatorname{ShvCat}(B\mathcal{G}) \simeq \operatorname{Tot}(\operatorname{ShvCat}(B^\bullet\mathcal{G})).$$

We now claim:

Proposition 10.1.3. *The term-wise* **Loc** *functor*

$$\operatorname{Tot}(\operatorname{QCoh}(B^\bullet\mathcal{G})\text{-}\mathbf{mod}) \to \operatorname{Tot}(\operatorname{ShvCat}(B^\bullet\mathcal{G}))$$

is an equivalence.

Proof. The functor in question is fully faithful since each $\mathbf{Loc}_{\mathcal{G}^{\times n}}$ is fully faithful. To prove that it is essentially surjective, we need to show that for $\mathbf{C}^\bullet \in \operatorname{Tot}(\operatorname{ShvCat}(B^\bullet\mathcal{G}))$, each term $\mathbf{C}^n$ lies in the essential image of the functor $\mathbf{Loc}_{\mathcal{G}^{\times n}}$.

Choosing any map $[0] \to [n]$ in $\mathbf{\Delta}$, from the commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathcal{G}^{\times n})\text{-}\mathbf{mod} & \xrightarrow{\mathbf{Loc}_{\mathcal{G}^{\times n}}} & \mathrm{ShvCat}(\mathcal{G}^{\times n}) \\ \uparrow & & \uparrow \\ \mathrm{QCoh}(\mathcal{G}^{\times 0})\text{-}\mathbf{mod} & \xrightarrow{\mathbf{Loc}_{\mathcal{G}^{\times 0}}} & \mathrm{ShvCat}(\mathcal{G}^{\times 0}), \end{array}$$

we obtain that it is sufficient to consider the case of $n = 0$. However, since $\mathcal{G}^{\times 0} = \mathrm{pt}$, the latter case is evident. □

10.1.4. Note that, by Proposition 5.2.3, the assumption that $\mathbf{Loc}_{\mathcal{G}}$ be fully faithful implies that for any n, the functor

$$\mathrm{QCoh}(\mathcal{G})^{\otimes n} \to \mathrm{QCoh}(\mathcal{G}^{\times n})$$

is an equivalence.

In particular, the structure on $\mathcal{G}$ is group-object of PreStk defines on $\mathrm{QCoh}(\mathcal{G})$ a structure of augmented co-monoidal DG category, such that the corresponding co-simplicial category $\text{co-Bar}^\bullet(\mathrm{QCoh}(\mathcal{G}))$ identifies with $\mathrm{QCoh}(B^\bullet\mathcal{G})$.

We note, however, that the above co-monoidal structure on $\mathrm{QCoh}(\mathcal{G})$ naturally extends to a commutative Hopf algebra structure (see Sect. E for what this means) as an object of $\mathrm{DGCat}_{\mathrm{cont}}$, via *pointwise* (symmetric) monoidal structure. In particular, $\text{co-Bar}^\bullet(\mathrm{QCoh}(\mathcal{G}))$ is naturally a co-simplicial (symmetric) monoidal DG category.

From Proposition E.1.4 we obtain:

Corollary 10.1.5. *The category* $\mathrm{Tot}(\mathrm{QCoh}(B^\bullet\mathcal{G})\text{-}\mathbf{mod})$ *is canonically equivalent to the category* $\mathrm{QCoh}(\mathcal{G})\text{-}\mathbf{comod}$. *Under this identification, for* $\mathbf{D} \in \mathrm{QCoh}(\mathcal{G})\text{-}\mathbf{comod}$, *the corresponding object of* $\mathrm{Tot}(\mathrm{QCoh}(B^\bullet\mathcal{G})\text{-}\mathbf{mod})$ *identifies with*

$$\text{co-Bar}^\bullet(\mathrm{QCoh}(\mathcal{G}), \mathbf{D}) \in \text{co-Bar}^\bullet(\mathrm{QCoh}(\mathcal{G}))\text{-}\mathbf{mod} \simeq \mathrm{QCoh}(B^\bullet\mathcal{G})\text{-}\mathbf{mod}.$$

10.2. **Categories acted on by $\mathcal{G}$.**

10.2.1. Let $\mathcal{G}\text{-}\mathbf{mod}$ denote the category $\mathrm{QCoh}(\mathcal{G})\text{-}\mathbf{comod}$.

Combining Corollary 10.1.5 and Proposition 10.1.3, we obtain an equivalence

$$\mathrm{ShvCat}(B\mathcal{G}) \simeq \mathcal{G}\text{-}\mathbf{mod}. \tag{10.1}$$

In what follows we shall refer to objects of $\mathcal{G}\text{-}\mathbf{mod}$ as *categories endowed with an action of the group-prestack* $\mathcal{G}$.

10.2.2. Under the equivalence (10.1) the category $\mathrm{QCoh}(B\mathcal{G})$ identifies with

$$\mathrm{Rep}(\mathcal{G}) := \underline{\mathrm{Hom}}_{\mathcal{G}}(\mathrm{Vect}, \mathrm{Vect}).$$

The functor

$$\mathbf{\Gamma}_{B\mathcal{G}} : \mathrm{ShvCat}(B\mathcal{G}) \to \mathrm{DGCat}_{\mathrm{cont}}$$

identifies with the functor

$$\mathbf{inv}_{\mathcal{G}} : \mathcal{G}\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}, \quad \mathbf{D} \mapsto \underline{\mathrm{Hom}}_{\mathcal{G}}(\mathrm{Vect}, \mathbf{D}) \simeq \mathrm{Tot}\left(\text{co-Bar}^\bullet(\mathrm{QCoh}(\mathcal{G}), \mathbf{D})\right).$$

This functor naturally upgrades to the functor

$$\mathbf{inv}_{\mathcal{G}}^{\mathrm{enh}} : \mathcal{G}\text{-}\mathbf{mod} \to \mathrm{Rep}(\mathcal{G})\text{-}\mathbf{mod},$$

and the latter identifies with $\mathbf{\Gamma}_{B\mathcal{G}}^{\mathrm{enh}}$.

Remark 10.2.3. We regard $\mathrm{Rep}(\mathcal{G})$ as being equipped with a monoidal structure resulting from its definition as $\underline{\mathrm{Hom}}_{\mathcal{G}}(\mathrm{Vect},\mathrm{Vect})$. However, this structure naturally extends to a symmetric (i.e., E_∞) monoidal structure:

The right-lax symmetric monoidal structure on the functor $\mathbf{inv}_{\mathcal{G}}$ defines on

$$\mathrm{Rep}(\mathcal{G}) \simeq \mathbf{inv}_{\mathcal{G}}(\mathrm{Vect})$$

a structure of unital symmetric monoidal DG category, which is compatible (=distributive) with the monoidal structure given by composition. Hence, by Eckmann-Hilton, the latter structure is induced by the former.

10.2.4. The functor

$$\mathbf{Loc}_{B\mathcal{G}} : \mathrm{Rep}(\mathcal{G})\text{-}\mathbf{mod} \to \mathrm{ShvCat}(B\mathcal{G})$$

identifies with the functor

$$\mathbf{rec}^{\mathrm{enh}}_{\mathcal{G}} : \mathrm{Rep}(\mathcal{G})\text{-}\mathbf{mod} \to \mathcal{G}\text{-}\mathbf{mod}, \quad \mathbf{C} \mapsto \mathrm{Vect} \underset{\mathrm{Rep}(\mathcal{G})}{\otimes} \mathbf{C},$$

where [12] the right hand side naturally acquires a structure of an object of $\mathcal{G}\text{-}\mathbf{mod}$ via the commuting $\mathcal{G}$- and $\mathrm{Rep}(\mathcal{G})$-actions on Vect.

Here the action of $\mathrm{Rep}(\mathcal{G})$ on Vect is given by the augmentation (forgetful) functor

$$\mathrm{oblv}_{\mathcal{G}} : \mathrm{Rep}(\mathcal{G}) \to \mathrm{Vect}, \quad \underline{\mathrm{Hom}}_{\mathcal{G}}(\mathrm{Vect},\mathrm{Vect}) \to \underline{\mathrm{Hom}}(\mathrm{Vect},\mathrm{Vect}).$$

In what follows we shall denote by $\mathbf{rec}_{\mathcal{G}}$ the composition of $\mathbf{rec}^{\mathrm{enh}}_{\mathcal{G}}$ and the forgetful functor

$$\mathbf{oblv}_{\mathcal{G}} : \mathcal{G}\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

I.e.,

$$\mathbf{rec}_{\mathcal{G}}(\mathbf{C}) = \mathrm{Vect} \underset{\mathrm{Rep}(\mathcal{G})}{\otimes} \mathbf{C} \in \mathrm{DGCat}_{\mathrm{cont}}\,.$$

10.2.5. It is easy to see that $\mathbf{inv}_{\mathcal{G}}(\mathrm{QCoh}(\mathcal{G})) \simeq \mathrm{Vect}$, and moreover, this equivalence extends to an isomorphism

$$\mathbf{inv}^{\mathrm{enh}}_{\mathcal{G}}(\mathrm{QCoh}(\mathcal{G})) \simeq \mathrm{Vect}$$

in $\mathrm{Rep}(\mathcal{G})\text{-}\mathbf{mod}$, where $\mathrm{Rep}(\mathcal{G})$ acts on Vect "trivially", i.e., via the functor $\mathrm{oblv}_{\mathcal{G}}$.

In particlar, by adjunction, we obtain a map in $\mathcal{G}\text{-}\mathbf{mod}$

$$\mathbf{rec}^{\mathrm{enh}}_{\mathcal{G}}(\mathrm{Vect}) \to \mathrm{QCoh}(\mathcal{G}). \tag{10.2}$$

At the level of plain DG categories, the map (10.2) identifies with the functor

$$p^* : \mathrm{Vect} \underset{\mathrm{Rep}(\mathcal{G})}{\otimes} \mathrm{Vect} \to \mathrm{QCoh}(\mathcal{G}). \tag{10.3}$$

[12] The symbol $\mathbf{rec}^{\mathrm{enh}}$ is for "reconstruction."

10.2.6. Assume for a moment that $\mathrm{QCoh}(\mathcal{G})$ is dualizable as a plain DG category. Consider its dual, $\mathrm{QCoh}(\mathcal{G})^\vee$, as a monoidal DG category (the monoidal structure on $\mathrm{QCoh}(\mathcal{G})^\vee$ is the dual of the co-monoidal structure on $\mathrm{QCoh}(\mathcal{G})$).

In this case, by taking the dual of the co-action of $\mathrm{QCoh}(\mathcal{G})^\vee$, we can identify

$$\mathrm{QCoh}(\mathcal{G})\text{-}\mathbf{comod} \simeq \mathrm{QCoh}(\mathcal{G})^\vee\text{-}\mathbf{mod},$$

and hence

$$\mathcal{G}\text{-}\mathbf{mod} \simeq \mathrm{QCoh}(\mathcal{G})^\vee\text{-}\mathbf{mod}. \tag{10.4}$$

Let us note that in this case, in addition to the functors $\mathbf{inv}_{\mathcal{G}}$ and $\mathbf{inv}_{\mathcal{G}}^{\mathrm{enh}}$ we also have the functor

$$\mathbf{coinv}_{\mathcal{G}} : \mathcal{G}\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}} \text{ and } \mathbf{coinv}_{\mathcal{G}}^{\mathrm{enh}} : \mathcal{G}\text{-}\mathbf{mod} \to \mathrm{Rep}(\mathcal{G})\text{-}\mathbf{mod},$$

defined by

$$\mathbf{D} \mapsto \mathrm{Vect} \underset{\mathrm{QCoh}(\mathcal{G})^\vee}{\otimes} \mathbf{D} \simeq |\operatorname{Bar}^\bullet(\mathrm{QCoh}(\mathcal{G})^\vee, \mathbf{D})|.$$

10.3. **Affine group DG schemes.** In this subsection we let $\mathcal{G}$ be a group-object of $\mathrm{DGSch}^{\mathrm{aff}}$.

10.3.1. For $\mathbf{D} \in \mathcal{G}\text{-}\mathbf{mod}$ we consider the co-simplicial category $\text{co-Bar}^\bullet(\mathrm{QCoh}(\mathcal{G}), \mathbf{D})$. As in Lemma 5.5.4, we note that $\text{co-Bar}^\bullet(\mathrm{QCoh}(\mathcal{G}), \mathbf{D})$ satisfies the co-monadic Beck-Chevalley condition.

In particular, the forgetful functor

$$\mathrm{oblv}_{\mathcal{G}} : \mathbf{inv}_{\mathcal{G}}(\mathbf{D}) \to \mathbf{D}$$

admits a right adjoint, denoted $\mathrm{coind}_{\mathcal{G}}$, and the co-monad

$$\mathrm{oblv}_{\mathcal{G}} \circ \mathrm{coind}_{\mathcal{G}}$$

identifies, when viewed as a plain endo-functor of $\mathbf{D}$, with the composition

$$\mathbf{D} \overset{\text{co-action}}{\longrightarrow} \mathbf{D} \otimes \mathrm{QCoh}(\mathcal{G}) \overset{\mathrm{Id}_{\mathbf{D}} \otimes p_*}{\to} \mathbf{D},$$

where $p : \mathcal{G} \to \mathrm{pt}$.

10.3.2. Note that in the affine case we have a canonical identification

$$\mathrm{QCoh}(\mathcal{G})^\vee \simeq \mathrm{QCoh}(\mathcal{G}). \tag{10.5}$$

Moreover, we note that the *monoidal* structure on $\mathrm{QCoh}(\mathcal{G})$, induced by (10.5) and the *co-monoidal* structure on $\mathrm{QCoh}(\mathcal{G})$, is canonically equivalent to that given by the structure on $\mathcal{G}$ of an algebra-object in $\mathrm{DGSch}^{\mathrm{aff}}$ (with respect to the monoidal structure on $\mathrm{DGSch}^{\mathrm{aff}}$ given by the Cartesian product) and the monoidal functor

$$\mathrm{QCoh}_* : \mathrm{DGSch}^{\mathrm{aff}} \to \mathrm{DGCat}_{\mathrm{cont}}, \quad S \rightsquigarrow \mathrm{QCoh}(S), \quad (f : S_1 \to S_2) \rightsquigarrow f_*.$$

We shall denote the resulting monoidal DG category by $\mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}}$.

When we consider $\mathrm{QCoh}(\mathcal{G})$ with the (symmetric) monoidal structure given by the pointwise tensor product, we shall denote it by $\mathrm{QCoh}(\mathcal{G})_{\mathrm{ptw}}$.

10.4. **A criterion for 1-affineness in the affine case case.** In this subsection we continue to assume that $\mathcal{G}$ is an affine group DG scheme.

10.4.1. Consider the simplicial category $\mathrm{Bar}^\bullet(\mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}})$. It is obtained by applying the functor QCoh_* to the simplicial object $B^\bullet\mathcal{G} \in \mathrm{DGSch}^{\mathrm{aff}}$. In particular, we have a canonical identification

$$|\,\mathrm{QCoh}(B^\bullet\mathcal{G})_*| \simeq \mathrm{Vect} \underset{\mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}}}{\otimes} \mathrm{Vect}\,.$$

As in Lemma 6.3.3, from Lemma C.1.6 and [GL:DG, Lemma 1.3.3], we obtain:

Lemma 10.4.2. *The right adjoint of the tautological functor*

$$\mathrm{Vect} \to \mathrm{Vect} \underset{\mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}}}{\otimes} \mathrm{Vect}$$

is monadic, and the corresponding monad on Vect*, viewed as a plain endo-functor, identifies canonically with* $p_* \circ (p_*)^R$*, where*

$$p : \mathcal{G} \to \mathrm{pt}\,.$$

10.4.3. Finally, we note that as in Sect. 6.2.4, from the natural transformation (6.2) we obtain that there exists a canonically defined functor

$$\mathrm{coind}_{\mathcal{G}} : \mathrm{Vect} \underset{\mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}}}{\otimes} \mathrm{Vect} \to \mathrm{Rep}(\mathcal{G}), \tag{10.6}$$

which lifts to a map

$$\mathbf{coinv}^{\mathrm{enh}}_{\mathcal{G}}(\mathrm{Vect}) \to \mathbf{inv}^{\mathrm{enh}}_{\mathcal{G}}(\mathrm{Vect}) = \mathrm{Rep}(\mathcal{G})$$

in $\mathrm{Rep}(\mathcal{G})\text{-}\mathbf{mod}$.

We claim:

Proposition 10.4.4. *The following conditions are equivalent:*

(a) $B\mathcal{G}$ *is 1-affine;*

(b) *The following two conditions hold:*

  (1) *The functor* (10.6) *is an equivalence;*
  (2) *The functor* (10.3) *is an equivalence.*

(b') *The following two conditions hold:*

  (1) *There exists* some *isomorphism of objects of* $\mathrm{Rep}(\mathcal{G})\text{-}\mathbf{mod}$

$$\mathbf{coinv}^{\mathrm{enh}}_{\mathcal{G}}(\mathrm{Vect}) \simeq \mathrm{Rep}(\mathcal{G}).$$

  (2) *There exists* some *isomorphism of objects of objects of* $\mathcal{G}\text{-}\mathbf{mod}$

$$\mathbf{rec}^{\mathrm{enh}}_{\mathcal{G}}(\mathrm{Vect}) \simeq \mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}}.$$

Proof. Let assume (a) and deduce (b). The fact that the map (10.3) is an equivalence holds is the expression of the fact that the functor $\mathbf{\Gamma}^{\mathrm{enh}}_{B\mathcal{G}}$ is fully faithful.

Let us view Vect as object of $\mathcal{G}\text{-}\mathbf{mod}$, endowed with a commuting action of $\mathrm{Rep}(\mathcal{G})$. Tensoring up the map (10.6) on the right by Vect over $\mathrm{Rep}(\mathcal{G})$, we obtain a map

$$\begin{aligned}\mathrm{Vect} \underset{\mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}}}{\otimes} \left(\mathrm{Vect} \underset{\mathrm{Rep}(\mathcal{G})}{\otimes} \mathrm{Vect}\right) \simeq \\ \simeq \left(\mathrm{Vect} \underset{\mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}}}{\otimes} \mathrm{Vect}\right) \underset{\mathrm{Rep}(\mathcal{G})}{\otimes} \mathrm{Vect} \to \mathrm{Rep}(\mathcal{G}) \underset{\mathrm{Rep}(\mathcal{G})}{\otimes} \mathrm{Vect},\end{aligned} \tag{10.7}$$

so that the diagram

$$\begin{array}{ccc} \mathrm{Vect} \underset{\mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}}}{\otimes} \left(\mathrm{Vect} \underset{\mathrm{Rep}(\mathcal{G})}{\otimes} \mathrm{Vect}\right) & \longrightarrow & \mathrm{Rep}(\mathcal{G}) \underset{\mathrm{Rep}(\mathcal{G})}{\otimes} \mathrm{Vect} \\ \downarrow{\scriptstyle \mathrm{Id}_{\mathrm{Vect}} \otimes p^*} & & \downarrow{\scriptstyle \sim} \\ \mathrm{Vect} \underset{\mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}}}{\otimes} \mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}} & \xrightarrow{\sim} & \mathrm{Vect} \end{array}$$

commutes. Since the left vertical arrow is an isomorphism (by the above), we obtain that so is the top horizontal map.

Now, since $\mathbf{rec}_{\mathcal{G}}$ is conservative, the fact that (10.7) is an equivalence, implies that so is the map (10.6).

The fact that (b) implies (b') is tautological.

Let us show that (b') implies (a). However, this is obvious: (b') implies that the functors $\mathbf{coinv}_{\mathcal{G}}^{\mathrm{enh}}$ and $\mathbf{rec}_{\mathcal{G}}^{\mathrm{enh}}$ are mutually inverse on the nose.

□

11. Groups with a rigid convolution category

11.1. **The rigidity condition.** We return to the conext of Sect. 10.2.6. In this subsection we will assume that $\mathrm{QCoh}(\mathcal{G})$ is dualizable as a plain category, and, moreover, that the *monoidal* category $\mathrm{QCoh}(\mathcal{G})^\vee$ is rigid (see Sect. D.1 for what this means).

11.1.1. The self-duality of $\mathrm{QCoh}(\mathcal{G})^\vee$ induced by its rigid monoidal structure (see Sect. D.1.2) defines, in particular, an identification

$$\mathrm{QCoh}(\mathcal{G})^\vee \simeq \mathrm{QCoh}(\mathcal{G}),$$

as plain categories.

Thus, we can again think of $\mathrm{QCoh}(\mathcal{G})$ is a monoidal DG category; when considered as such, it will be denoted by $\mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}^L}$. This monoidal structure should not be confused with the pointwise (symmetric) monoidal structure; the latter is denoted by $\mathrm{QCoh}(\mathcal{G})_{\mathrm{ptw}}$.

11.1.2. By Sect. D.3, we obtain that the monoidal structure on $\mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}^L}$ is obtained from the co-monoidal structure on $\mathrm{QCoh}(\mathcal{G})$ by passage to the *left* adjoint functors.

By construction, the unit in $\mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}^L}$ is given by the functor

$$(e^*)^L : \mathrm{Vect} \to \mathrm{QCoh}(\mathcal{G}),$$

which is both the *left* adjoint and the dual of $e^* : \mathrm{QCoh}(\mathcal{G}) \to \mathrm{Vect}$, where $e : \mathrm{pt} \to \mathcal{G}$ is the unit point.

By Proposition D.3.6, the augmentation on $\mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}^L}$ is given by the functor

$$(p^*)^L : \mathrm{QCoh}(\mathcal{G}) \to \mathrm{Vect},$$

which is both the *left* adjoint and the dual of $p^* : \mathrm{Vect} \to \mathrm{QCoh}(\mathcal{G})$.

11.1.3. Let now $\mathbf{D}$ be an object of $\mathcal{G}\text{-}\mathbf{mod}$, thought of as an object of $\mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}^L}\text{-}\mathbf{mod}$. From Sect. D.3 we obtain:

Lemma 11.1.4.

(a) *The* right adjoint *of the action data*

$$\mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}^L} \otimes \mathbf{D} \to \mathbf{D}$$

identifies with the data of co-action

$$\mathbf{D} \to \mathrm{QCoh}(\mathcal{G}) \otimes \mathbf{D}.$$

(b) *The simplicial category* $\mathrm{Bar}^\bullet(\mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}^L}, \mathbf{D})$ *is obtained from* $\text{co-Bar}^\bullet(\mathrm{QCoh}(\mathcal{G}), \mathbf{D})$ *by passage to the left adjoint functors.*

From Corollary D.4.9, we obtain:

Corollary 11.1.5.

(a) *The co-simplicial category* $\text{co-Bar}^\bullet(\mathrm{QCoh}(\mathcal{G}), \mathbf{D})$ *satisfies the monadic Beck-Chevalley condition.*

(b) *The forgetful functor*

$$\mathrm{oblv}_{\mathcal{G}} : \mathbf{inv}_{\mathcal{G}}(\mathbf{D}) \to \mathbf{D}$$

admits a left adjoint (denoted $\mathrm{ind}_{\mathcal{G}}$*) and is monadic. The monad*

$$\mathrm{oblv}_{\mathcal{G}} \circ \mathrm{ind}_{\mathcal{G}},$$

viewed as a plain endo-functor of $\mathbf{D}$*, identifies with the composition*

$$\mathbf{D} \overset{\text{co-action}}{\longrightarrow} \mathrm{QCoh}(\mathcal{G}) \otimes \mathbf{D} \overset{(p^*)^L \otimes \mathrm{Id}_{\mathbf{D}}}{\longrightarrow} \mathbf{D},$$

where $(p^*)^L$ *is the left adjoint of the functor* $p^* : \mathrm{Vect} \to \mathrm{QCoh}(\mathcal{G})$.

11.1.6. Combining Lemma 11.1.4(b) with [GL:DG, Lemma 1.3.3], we obtain:

Corollary 11.1.7. *There exists a canonical isomorphism of functors*

$$\mathbf{coinv}_{\mathcal{G}} \simeq \mathbf{inv}_{\mathcal{G}} : \mathcal{G}\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

It follows from the construction, the isomorphism of Corollary 11.1.7 lifts to an isomorphism

$$\mathbf{coinv}^{\mathrm{enh}}_{\mathcal{G}} \simeq \mathbf{inv}^{\mathrm{enh}}_{\mathcal{G}} \tag{11.1}$$

as functors $\mathcal{G}\text{-}\mathbf{mod} \to \mathrm{Rep}(\mathcal{G})\text{-}\mathbf{mod}$.

11.2. **A criterion for 1-affineness in the rigid case.** The goal of this subsection is to prove the following assertion:

Proposition 11.2.1. *Let* $\mathcal{G}$ *be as in Sect. 11.1. Then the following conditions are equivalent:*

(a) $B\mathcal{G}$ *is 1-affine;*

(b) *The functor* (10.3) *is an equivalence.*

(b') *There exists* an *equivalence in* $\mathcal{G}\text{-}\mathbf{mod}$*:*

$$\mathbf{rec}^{\mathrm{enh}}_{\mathcal{G}}(\mathrm{Vect}) \simeq \mathrm{QCoh}(\mathcal{G}).$$

(c) *The functor* $p_* : \mathrm{QCoh}(\mathcal{G}) \to \mathrm{Vect}$*, right adjoint to* p^**, is monadic.*

11.2.2. *Step 1.* Let us assume (a). Then (b) expresses the fact that the functor $\mathbf{\Gamma}^{\mathrm{enh}}_{B\mathcal{G}}$ is fully faithful.

The implication (b) $\Rightarrow$ (b') is tautological. The implication (b') $\Rightarrow$ (a) is easy: we obtain that the functors $\mathbf{coinv}^{\mathrm{enh}}_{\mathcal{G}}$ and $\mathbf{rec}^{\mathrm{enh}}_{\mathcal{G}}$ are mutually inverse on the nose.

11.2.3. *Step 2.* It remains to establish the equivalence of (b) and (c). We claim that the functor

$$\mathrm{Vect} \to \mathrm{Vect} \underset{\mathrm{Rep}(\mathcal{G})}{\otimes} \mathrm{Vect}$$

is monadic, and the corresponding monad on Vect maps isomorphically, as a plain endo-functor, to $p_* \circ p^*$.

We will prove this by applying Corollary C.2.3. In Step 3 we will show that the monoidal operation

$$\mathrm{Rep}(\mathcal{G}) \otimes \mathrm{Rep}(\mathcal{G}) \to \mathrm{Rep}(\mathcal{G})$$

admits a left adjoint. Assuming this, the required assertion follows from Corollary C.2.3, combined with Lemma C.2.5:

Indeed, it remains to show that the map

$$\mathrm{oblv}_{\mathcal{G}} \circ (\mathrm{oblv}_{\mathcal{G}})^R \to p_* \circ p^*$$

is an isomorphism, which follows from the fact that the corresponding map of left adjoints

$$(p^*)^L \circ p^* \to \mathrm{oblv}_{\mathcal{G}} \circ \mathrm{ind}_{\mathcal{G}}$$

is an isomorphism, by Corollary 11.1.5(b).

11.2.4. *Step 3.* Using Corollary 11.1.7, we interpret $\mathrm{Rep}(\mathcal{G})$ as

$$\mathrm{Vect} \underset{\mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}^L}}{\otimes} \mathrm{Vect}\,.$$

Hence, we can identify

$$\mathrm{Rep}(\mathcal{G}) \otimes \mathrm{Rep}(\mathcal{G}) \simeq \mathrm{Rep}(\mathcal{G} \times \mathcal{G}),$$

so that the monoidal operation on $\mathrm{Rep}(\mathcal{G})$ identifies with restriction under the diagonal map.

We now claim that if $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ is any homomorphism between group-objects of PreStk, satisfying the assumption of Sect. 11.1, then the restriction functor $\mathrm{Rep}(\mathcal{G}_2) \to \mathrm{Rep}(\mathcal{G}_1)$ admits a left adjoint.

Indeed, interpreting $\mathrm{Rep}(\mathcal{G}_i)$ as $\mathrm{Vect} \underset{\mathrm{QCoh}(\mathcal{G}_i)_{\mathrm{conv}^L}}{\otimes} \mathrm{Vect}$, the left adjoint in question is given by the homomorphism

$$\mathrm{QCoh}(\mathcal{G}_1)_{\mathrm{conv}^L} \to \mathrm{QCoh}(\mathcal{G}_2)_{\mathrm{conv}^L},$$

which is the left adjoint (and simultaneously dual, see Proposition D.3.6) of the restriction map $\phi^* : \mathrm{QCoh}(\mathcal{G}_2) \to \mathrm{QCoh}(\mathcal{G}_1)$ of the corresponding co-monoidal categories.

□

11.3. **Classifying prestacks of formal groups.** In this subsection we will prove Theorem 2.5.4. Let $\mathcal{G}$ be a weakly $\aleph_0$ formally smooth formal group locally almost of finite type.

11.3.1. *Proof of point (a).* We will deduce the required assertion by applying Proposition 9.1.3.

By definition,

$$B\mathcal{G} := |B^\bullet \mathcal{G}|,$$

and we claim that this presentation satisfies the conditions of Proposition 9.1.3. Note that each term of $B^\bullet \mathcal{G}$ is of the form $\mathcal{G}^{\times n}$.

Condition (1) is satisfied by Theorem 2.4.2.

Condition (2) is satisfied by [GR1, Theorem 10.1.1]: indeed, each of the functors

$$\Upsilon_{\mathcal{G}^{\times n}} : \mathrm{QCoh}(\mathcal{G}^{\times n}) \to \mathrm{IndCoh}(\mathcal{G}^{\times n})$$

is an equivalence. This also shows that

$$\mathrm{QCoh}(B\mathcal{G}) \simeq \mathrm{Tot}(\mathrm{QCoh}(B^\bullet \mathcal{G})) \to \mathrm{Tot}(\mathrm{IndCoh}(B^\bullet \mathcal{G})) \simeq \mathrm{IndCoh}(B\mathcal{G})$$

is an equivalence.

Finally, condition (3) is satisfied because all the maps in $B^\bullet \mathcal{G}$ are *ind-proper* (see [GR1, Sect. 2.7.4 and Corollary 2.8.3]).

□[Theorem 2.5.4(a)]

11.3.2. *Proof of point (b).* In Sect. 11.3.3, we will show that $\mathcal{G}$ satisfies the assumption of Sect. 11.1.

Assuming this, in order to prove point (b) of the theorem, by Proposition 11.2.1, it remains to show that the functor p_*, right adjoint to p^* is monadic if and only if the the tangent space of $\mathcal{G}$ at the origin is finite-dimensional. We will do this in Sect. 11.4.

11.3.3. Using [GR1, Theorem 10.1.1], we identify $\mathrm{QCoh}(\mathcal{G}) \simeq \mathrm{IndCoh}(\mathcal{G})$ via the functor $\Upsilon_{\mathcal{G}}$ as co-monoidal categories, where the co-monoidal structure on $\mathrm{IndCoh}(\mathcal{G})$ is induced by the structure on $\mathcal{G}$ of group-object in $\mathrm{PreStk}_{\mathrm{laft}}$ via the operation of !-pullback.

Recall now the self-duality

$$\mathrm{IndCoh}(\mathcal{G}) \simeq \mathrm{IndCoh}(\mathcal{G})^\vee$$

(see [GR1, Corollary 2.6.2]).

The above co-monoidal structure on $\mathrm{IndCoh}(\mathcal{G})$ defines via duality a monoidal structure on $\mathrm{IndCoh}(\mathcal{G})$; we shall denote the resulting monoidal DG category by $\mathrm{IndCoh}(\mathcal{G})_{\mathrm{conv}}$. By construction, the monoidal operation on $\mathrm{IndCoh}(\mathcal{G})_{\mathrm{conv}}$ is given by the operation of $(\mathrm{IndCoh}, *)$-direct image, i.e., it is obtained by applying the (symmetric) monoidal functor

$$\mathrm{IndCoh}_{\mathrm{DGindSch}_{\mathrm{laft}}} : \mathrm{DGindSch}_{\mathrm{laft}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

of [GR1, Sect. 2.7] to the algebra object $\mathcal{G} \in \mathrm{DGindSch}_{\mathrm{laft}}$.

We claim:

Lemma 11.3.4. *The monoidal DG category* $\mathrm{IndCoh}(\mathcal{G})_{\mathrm{conv}}$ *is rigid.*

Proof. Follows from the fact that $\mathcal{G}$ is ind-proper and the base change isomorphism of [GR1, Proposition 2.9.2]. □

This implies that $\mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}^L}$ is rigid as a monoidal DG category, as

$$\mathrm{IndCoh}(\mathcal{G})_{\mathrm{conv}} \simeq \mathrm{QCoh}(\mathcal{G})_{\mathrm{conv}^L},$$

as monoidal DG categories, by construction.

11.4. **Computation of the monad.** Let $\mathcal{G}$ be as in Theorem 2.5.4.

11.4.1. We identity $\mathrm{QCoh}(\mathcal{G}) \simeq \mathrm{IndCoh}(\mathcal{G})$ by means of the functor $\Upsilon_{\mathcal{G}}$, so that the functor p^* corresponds to

$$p^! : \mathrm{Vect} \to \mathrm{IndCoh}(\mathcal{G}),$$

and p_* corresponds to the right adjoint of $p^!$, denoted $p_?$.

We will show that the functor $p_? : \mathrm{IndCoh}(\mathcal{G}) \to \mathrm{Vect}$ is monadic if and only if the tangent space of $\mathcal{G}$ at the origin is finite-dimensional.

11.4.2. Note that this assertion does not involve the group structure on $\mathcal{G}$. By [BD, Propositions 7.12.22 and 7.12.23], the assumption on $\mathcal{G}$ implies that, as a DG indscheme, it can be described as follows:

- If the tangent space of $\mathcal{G}$ at the origin is finite-dimensional, then $\mathcal{G}$ is isomorphic to the completion of a the vector group V at the origin, where $V \in \mathrm{Vect}^{\heartsuit}$ is finite-dimensional.
- If the tangent space of $\mathcal{G}$ at the origin is infinite-dimensional, then $\mathcal{G}$ is isomorphic to

$$\underset{n}{\underset{\longrightarrow}{colim}}\, (\mathbb{A}^n)^{\wedge}_0.$$

11.4.3. Let $\mathcal{G} = V_0^{\wedge}$. In this case the category $\mathrm{IndCoh}(\mathcal{G})$ identifies with $\mathrm{Rep}(V^*)$, where V^* is the dual vector space.

Under this identification $p^!$ identifies with the functor coind_{V^*}, the *right* adjoint to the forgetful functor $\mathrm{oblv}_G : \mathrm{Rep}(V^*) \to \mathrm{Vect}$. The functor $p_?$ is the (discontinuous) *right* adjoint $(\mathrm{coind}_{V^*})^R$ of coind_{V^*}.

Hence, the functor $p_?$ is monadic by Remark Sect. 7.2.3.

Remark 11.4.4. Let us note that for $\mathcal{G} = V_0^{\wedge}$, the assertion of Theorem 2.5.4(b) is equivalent to that of Theorem 2.2.2 for the group V^*. Indeed, in this case

$$\mathrm{IndCoh}(\mathcal{G}) \simeq \mathrm{Rep}(V^*),$$

as monoidal categories, so that $\mathcal{G}\text{-}\mathbf{mod} \simeq \mathrm{Rep}(V^*)\text{-}\mathbf{mod}$ and the functor $\mathbf{inv}^{\mathrm{enh}}_{\mathcal{G}} \simeq \mathbf{coinv}^{\mathrm{enh}}_{\mathcal{G}}$ identifies with $\mathbf{rec}^{\mathrm{enh}}_{V^*}$. In addition,

$$\mathrm{Rep}(\mathcal{G}) \simeq \mathrm{QCoh}(V^*)_{\mathrm{conv}},$$

also as monoidal categories, so

$$\mathrm{Rep}(\mathcal{G})\text{-}\mathbf{mod} \simeq V^*\text{-}\mathbf{mod},$$

and the functor $\mathbf{rec}^{\mathrm{enh}}_{\mathcal{G}}$ idenitifies with $\mathbf{coinv}^{\mathrm{enh}}_{V^*}$.

11.4.5. Let

$$\mathcal{G} := \underset{n}{\underset{\longrightarrow}{colim}}\, (\mathbb{A}^n)^{\wedge}_0.$$

We claim that in this case the functor $p_?$ fails to be conservative. Indeed, let ι_n denote the embedding

$$(\mathbb{A}^n)^{\wedge}_0 \hookrightarrow \mathcal{G}.$$

We claim that the functor $p_?$ annihilates $(\iota_0)^{\mathrm{IndCoh}}_*(k)$. To prove this we have to show that

$$\mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{G})}(\omega_{\mathcal{G}}, (\iota_0)^{\mathrm{IndCoh}}_*(k)) = 0.$$

We note that

$$\omega_{\mathcal{G}} \simeq \underset{n}{\underset{\longrightarrow}{colim}}\, (\iota_n)^{\mathrm{IndCoh}}_*(\omega_{(\mathbb{A}^n)^{\wedge}_0}),$$

and so

$$\mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{G})}(\omega_{\mathcal{G}}, (\iota_0)_*^{\mathrm{IndCoh}}(k)) \simeq \underset{\overleftarrow{n}}{lim}\, \mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{G})}((\iota_n)_*^{\mathrm{IndCoh}}(\omega_{(\mathbb{A}^n)_0^\wedge}), (\iota_0)_*^{\mathrm{IndCoh}}(k)).$$

Now, for every n,

$$(\iota_n)_*^{\mathrm{IndCoh}}(\omega_{(\mathbb{A}^n)_0^\wedge}) \in \mathrm{IndCoh}(\mathbb{A}^\infty)^{\leq -n},$$

so

$$\mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{G})}((\iota_n)_*^{\mathrm{IndCoh}}(\omega_{(\mathbb{A}^n)_0^\wedge}), (\iota_0)_*^{\mathrm{IndCoh}}(k)) \in \mathrm{Vect}^{\geq n},$$

and hence the above limit vanishes.

12. De Rham prestacks

12.1. **De Rham prestacks of indschemes.**

The goal of this subsection is to prove Theorem 2.6.3. Recall that we fix a DG indscheme $\mathcal{Z}$ locally almost of finite type, and we want to show that the prestack $\mathcal{Z}_{\mathrm{dR}}$ is 1-affine.

12.1.1. *Step 1.* We will first prove that the prestack Z_{dR} is 1-affine, where Z is an affine scheme of finite type. We can embed Z into $\mathbb{A}^n$. Since Z_{dR} identifies with its formal completion inside $(\mathbb{A}^n)_{\mathrm{dR}}$, by Theorem 2.3.1, it is enough to consider the case of $Z = \mathbb{A}^n$.

Let $\mathcal{G}$ be the formal completion of $\mathbb{A}^n$ at the origin, considered as a formal group. Note that the prestack quotient of $\mathbb{A}^n$ by $\mathcal{G}$ identifies with $(\mathbb{A}^n)_{\mathrm{dR}}$. Hence, we have a canonical map

$$(\mathbb{A}^n)_{\mathrm{dR}} \to B\mathcal{G},$$

and for any $S \in (\mathrm{DGSch}^{\mathrm{aff}})_{B\mathcal{G}}$, the fiber product

$$S \underset{B\mathcal{G}}{\times} (\mathbb{A}^n)_{\mathrm{dR}}$$

identifies with $S \times \mathbb{A}^n$.

Applying Theorem 2.5.4(b) and Corollary 3.2.7, we deduce that $(\mathbb{A}^n)_{\mathrm{dR}}$ is 1-affine.

12.1.2. *Step 2.* We now claim that for an arbitrary scheme of finite type Z, the prestack Z_{dR} is 1-affine. Indeed, the reduction to the affine case is routine and is left to the reader.

12.1.3. *Step 3.* Let $\mathcal{Z}$ be an indscheme written as

$$\underset{\overrightarrow{i \in I}}{colim}\, Z_i,$$

where Z_i are schemes of finite type, and the maps $f_{i,j} : Z_i \to Z_j$ are closed embeddings.

The fact that the functor $\mathbf{Loc}_{\mathcal{Z}_{\mathrm{dR}}}$ is fully faithful follows from Proposition 9.1.3: indeed, the functor

$$\Psi_{\mathcal{Z}_{\mathrm{dR}}} : \mathrm{QCoh}(\mathcal{Z}_{\mathrm{dR}}) \to \mathrm{IndCoh}(\mathcal{Z}_{\mathrm{dR}})$$

is an equivalence for *any* $\mathcal{Z} \in \mathrm{PreStk}_{\mathrm{laft}}$, see [GR2, Proposition 2.4.4].

12.1.4. *Step 4.* It remains to show that for $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Z}_{\mathrm{dR}})$, the co-unit of the adjunction

$$\mathbf{Loc}_{\mathcal{Z}_{\mathrm{dR}}} \circ \mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Z}_{\mathrm{dR}}}(\mathcal{C}) \to \mathcal{C}$$

is an equivalence.

Since the theorem has been established for schemes, it is sufficient to show that for every index $i_0 \in I$, the functor

$$\mathrm{QCoh}((Z_{i_0})_{\mathrm{dR}}) \underset{\mathrm{QCoh}(\mathcal{Z}_{\mathrm{dR}})}{\otimes} \mathbf{\Gamma}_{\mathcal{Z}_{\mathrm{dR}}}(\mathcal{C}) \simeq \mathbf{\Gamma}\left((Z_{i_0})_{\mathrm{dR}}, \mathbf{Loc}_{\mathcal{Z}_{\mathrm{dR}}} \circ \mathbf{\Gamma}^{\mathrm{enh}}_{\mathcal{Z}_{\mathrm{dR}}}(\mathcal{C})\right) \to \mathbf{\Gamma}((Z_{i_0})_{\mathrm{dR}}, \mathcal{C}) \tag{12.1}$$

is an equivalence.

As in the proof of Proposition 9.1.3, we can express $\mathbf{\Gamma}_{\mathcal{Z}_{\mathrm{dR}}}(\mathcal{C})$ as

$$\underset{i \in I}{\underset{\longrightarrow}{colim}}\, \mathbf{\Gamma}((Z_i)_{\mathrm{dR}}, \mathcal{C}).$$

Since I is filtered, the map $I_{i_0/} \to I$ is cofinal. Hence,

$$\mathbf{\Gamma}_{\mathcal{Z}_{\mathrm{dR}}}(\mathcal{C}) \simeq \underset{i \in I_{i_0/}}{\underset{\longrightarrow}{colim}}\, \mathbf{\Gamma}((Z_i)_{\mathrm{dR}}, \mathcal{C}).$$

Therefore, the left-hand side in (12.1) identifies with

$$\underset{i \in I_{i_0/}}{\underset{\longrightarrow}{colim}} \left(\mathrm{QCoh}((Z_{i_0})_{\mathrm{dR}}) \underset{\mathrm{QCoh}(\mathcal{Z}_{\mathrm{dR}})}{\otimes} \mathrm{QCoh}((Z_i)_{\mathrm{dR}})\right) \underset{\mathrm{QCoh}((Z_i)_{\mathrm{dR}})}{\otimes} \mathbf{\Gamma}((Z_i)_{\mathrm{dR}}, \mathcal{C}). \tag{12.2}$$

Note, however, that for every $i \in I_{i_0/}$, the map

$$\mathrm{QCoh}((Z_{i_0})_{\mathrm{dR}}) \underset{\mathrm{QCoh}(\mathcal{Z}_{\mathrm{dR}})}{\otimes} \mathrm{QCoh}((Z_i)_{\mathrm{dR}}) \to \mathrm{QCoh}((Z_{i_0})_{\mathrm{dR}})$$

is an equivalence. Indeed, this follows by Lemma 4.1.6 from the fact that the restriction functor

$$\mathrm{QCoh}((Z_i)_{\mathrm{dR}}) \to \mathrm{QCoh}((Z_{i_0})_{\mathrm{dR}})$$

admits a left adjoint that commutes with the $\mathrm{QCoh}((Z_i)_{\mathrm{dR}})$-action.

Furthermore, the fact that Theorem 2.6.3 holds for schemes implies that

$$\mathrm{QCoh}((Z_{i_0})_{\mathrm{dR}}) \underset{\mathrm{QCoh}((Z_i)_{\mathrm{dR}})}{\otimes} \mathbf{\Gamma}((Z_i)_{\mathrm{dR}}, \mathcal{C}) \to \mathbf{\Gamma}((Z_{i_0})_{\mathrm{dR}}, \mathcal{C})$$

is an equivalence.

Hence, the expression in (12.2) identifies with

$$\underset{i \in I_{i_0/}}{\underset{\longrightarrow}{colim}}\, \mathbf{\Gamma}((Z_{i_0})_{\mathrm{dR}}, \mathcal{C}).$$

However, since the category of indices is contractible, the resulting colimit is isomorphic to $\mathbf{\Gamma}((Z_{i_0})_{\mathrm{dR}}, \mathcal{C})$, as required.

□

12.1.5. To conclude this subsection, consider the group prestack

$$\mathcal{G} = \underset{\longrightarrow n}{colim}\, (\mathbb{G}_a)^{\times n},$$

see Sect. 2.5.11. Let us show that $B\mathcal{G}$ is not 1-affine.

Let $\mathcal{G}'$ be the formal completion of $\mathcal{G}$ at the origin. I.e.,

$$\mathcal{G}' := \underset{\longrightarrow n}{colim}\, ((\mathbb{G}_a)^{\times n})^{\wedge}_0.$$

Consider the natural map

$$B\mathcal{G}' \to B\mathcal{G}.$$

Note that its base change by any $S \in \mathrm{DGSch}^{\mathrm{aff}}_{/B\mathcal{G}}$ yields the prestack

$$S \times (\mathcal{G})_{\mathrm{dR}},$$

which is 1-affine by Corollary 3.2.8 and Theorem 2.6.3.

Assume for the sake of contradiction that $\mathbf{\Gamma}^{\mathrm{enh}}_{B\mathcal{G}}$ was fully faithful. Then by Proposition 3.1.9 and Proposition 3.2.6(b), we would obtain that $\mathbf{\Gamma}^{\mathrm{enh}}_{B\mathcal{G}'}$ is also fully faithful. However, the latter is false by Theorem 2.5.4(b).

12.2. **De Rham prestacks of classifying stacks.** In this subsection we let G be a classical affine algebraic group of finite type.

12.2.1. Note that we tautologically have:

$$B(G_{\mathrm{dR}}) \simeq (BG)_{\mathrm{dR}}.$$

Next, we note that since the canonical map

$$BG \to \mathrm{pt}\,/G$$

becomes an isomorphism after the étale sheafification, the same is true for the map

$$(BG)_{\mathrm{dR}} \to (\mathrm{pt}\,/G)_{\mathrm{dR}}.$$

Hence,

$$\mathrm{ShvCat}((\mathrm{pt}\,/G)_{\mathrm{dR}}) \simeq \mathrm{ShvCat}((BG)_{\mathrm{dR}}) \simeq \mathrm{ShvCat}(B(G_{\mathrm{dR}})),$$

and by Theorem 2.6.3 and Sect. 10.1, we have

$$\mathrm{ShvCat}(B(G_{\mathrm{dR}})) \simeq G_{\mathrm{dR}}\text{-}\mathbf{mod}.$$

12.2.2. Let us now prove Proposition 2.6.5. We will show that the functor $\mathbf{\Gamma}_{B(G_{\mathrm{dR}})}$ fails to be conservative for $G = \mathbb{G}_a$.

Consider the following two objects $\mathbf{D}_1, \mathbf{D}_2 \in G_{\mathrm{dR}}\text{-}\mathbf{mod}$. Namely, we take $\mathbf{D}_1 = \mathrm{Vect}$, with the trivial action, and $\mathbf{D}_2 := \mathrm{QCoh}(G_{\mathrm{dR}})$. There is a canonical map $\mathbf{D}_2 \to \mathbf{D}_1$, which is *not* an equivalence. However, we claim that it becomes an equivalence after applying the functor $\mathbf{\Gamma}_{B(G_{\mathrm{dR}})}$.

Indeed, it is easy to see that $\mathbf{inv}_{G_{\mathrm{dR}}}(\mathrm{QCoh}(G_{\mathrm{dR}})) \simeq \mathrm{Vect}$. Note that

$$\mathbf{inv}_{G_{\mathrm{dR}}}(\mathrm{Vect}) \simeq \mathrm{QCoh}(B(G_{\mathrm{dR}})).$$

Thus, it remains to show that the natural functor

$$\mathrm{Vect} \to \mathrm{QCoh}(B(G_{\mathrm{dR}}))$$

is an equivalence for $G = \mathbb{G}_a$.

12.2.3. We calculate $\mathrm{QCoh}(B(G_{\mathrm{dR}}))$ as

$$\mathrm{Tot}(\mathrm{QCoh}((G^\bullet)_{\mathrm{dR}})).$$

Note, however, that for $G = \mathbb{G}_a$, for any n, the pullback functor

$$\mathrm{Vect} \simeq \mathrm{QCoh}(\mathrm{pt}) \to \mathrm{QCoh}((G^n)_{\mathrm{dR}})$$

is fully faithful. Since it is an equivalence on 0-simplices, we obtain that

$$\mathrm{Tot}(\mathrm{Vect}^\bullet) \to \mathrm{Tot}(\mathrm{QCoh}((G^\bullet)_{\mathrm{dR}}))$$

is an equivalence, where $\mathrm{Vect}^\bullet$ is the constant co-simplicial category with value Vect.

Since the category $\mathbf{\Delta}$ os contractible, we obtain that

$$\mathrm{Vect} \to \mathrm{Tot}(\mathrm{Vect}^\bullet)$$

is also an equivalence, implying the desired assertion.

12.3. **Classifying prestack of a formal completion: Proof of Theorem 2.5.5.**

12.3.1. Let G be a classical affine algebraic group of finite type, and let $H \subset G$ be a closed subgroup. Let $\mathcal{G}$ be denotes the formal completion of H in G.

We need to show that the prestack $B\mathcal{G}$ is 1-affine.

12.3.2. Consider the tautological homomorphism $\mathcal{G} \to G$, and the resulting map

$$B\mathcal{G} \to BG.$$

Since BG is 1-affine (by Theorem 2.2.2 and Corollary 1.5.8(b)), by Corollary 3.2.7, in order to show that $B\mathcal{G}$ is 1-affine, it suffices to show that for $S \in (\mathrm{DGSch}^{\mathrm{aff}})_{/BG}$, the prestack

$$S \underset{BG}{\times} B\mathcal{G}$$

is 1-affine.

We note that any map $S \to BG$ factors as $S \to \mathrm{pt} \to BG$, so

$$S \underset{BG}{\times} B\mathcal{G} \simeq S \times (\mathrm{pt} \underset{BG}{\times} B\mathcal{G}).$$

By Corollary 3.2.8, we obtain that it suffices to show that the prestack $\mathrm{pt} \underset{BG}{\times} B\mathcal{G}$ is 1-affine.

12.3.3. Note now that we have a canonical map

$$\mathrm{pt} \underset{BG}{\times} B\mathcal{G} \to (G/H)_{\mathrm{dR}},$$

which becomes an isomorphism after étale sheafification.

This implies that $\mathrm{pt} \underset{BG}{\times} B\mathcal{G}$ is 1-affine by Theorem 2.6.3 and Corollary 1.5.8(b).

13. Infinitesimal loop spaces

13.1. **The setting.**

13.1.1. Consider the following situation. Let Z be an affine DG scheme locally almost of finite type, and $\iota : \mathrm{pt} \to Z$ a point with image z.

Consider the adjoint pairs of functors:

$$\iota^* : \mathrm{QCoh}(Z)_{\{z\}} \rightleftarrows \mathrm{Vect} : \iota_*$$

and

$$\iota_* : \mathrm{Vect} \rightleftarrows \mathrm{QCoh}(Z)_{\{z\}} : \iota^{\mathrm{QCoh},!}.$$

Conjecture 13.1.2. *Assume that Z is eventually coconnective. Then the functor $\iota^{\mathrm{QCoh},!}$ is monadic.*

In Sect. 13.3 we will prove:

Proposition 13.1.3.

(1) *Conjecture 13.1.2 holds if Z is smooth.*

(2) *Conjecture 13.1.2 holds if Z is of the form* $\mathrm{pt} \underset{\mathbb{A}^n}{\times} \mathrm{pt}$.

Remark 13.1.4. One can show that Proposition 13.1.3 implies that Conjecture 13.1.2 holds for any Z, which is *quasi-smooth*.

13.2. **Consequences of Conjecture 13.1.2.** In this subsection we will assume that Conjecture 13.1.2 holds for a given (Z, z), and deduce some consequences.

13.2.1. Consider the group-object of $\mathrm{DGSch}^{\mathrm{aff}}$

$$\Omega(Z, z) := \mathrm{pt} \underset{Z}{\times} \mathrm{pt},$$

i.e., the (derived!) inertia group of Z at z.

We will prove:

Theorem 13.2.2. *Assume that (Z, z) satisfies Conjecture 13.1.2. Then the prestack $B(\Omega(Z, z))$ is 1-affine and we have a canonical equivalence of symmetric monodical categories*

$$\mathrm{Rep}(\Omega(Z, z)) \simeq \mathrm{QCoh}(Z)_{\{z\}}.$$

The rest of this subsection is devoted to the proof of this theorem.

13.2.3. Note that the functor $\iota_* : \mathrm{Vect} \to \mathrm{QCoh}(Z)_{\{z\}}$ naturally upgrades to a functor:

$$\mathrm{Vect} \underset{\mathrm{QCoh}(\Omega(Z,z))_{\mathrm{conv}}}{\otimes} \mathrm{Vect} \to \mathrm{QCoh}(X)_{\{z\}}. \tag{13.1}$$

Moreover, the functor (13.1) upgrades to a map in $\mathrm{QCoh}(Z)$ - **mod**, where $\mathrm{QCoh}(Z)$ acts on Vect via ι^*.

We claim (assuming that the pair (Z, z) satisfies Conjecture 13.1.2):

Proposition 13.2.4. *The functor* (13.1) *is an equivalence.*

Proof. By Lemma 10.4.2, the right adjoint of the functor

$$\mathrm{Vect} \to \mathrm{Vect} \underset{\mathrm{QCoh}(\Omega(Z,z))_{\mathrm{conv}}}{\otimes} \mathrm{Vect}$$

is monadic. Hence, to prove the assertion of the proposition, it remains to show that the functor (13.1) induces an isomorphism of the resulting monads on Vect, regarded as plain endo-functors.

However, unwinding the definitions, we obtain that the resulting map of endo-functors is

$$p_* \circ (p_*)^R \simeq (p_* \circ p^*)^R \simeq (\iota^* \circ \iota_*)^R \simeq (\iota_*)^R \circ \iota_*,$$

where the isomorphism $p_* \circ p^* \simeq \iota^* \circ \iota_*$ comes from base change along the Cartesian diagram

$$\begin{array}{ccc} \mathrm{pt} \underset{Z}{\times} \mathrm{pt} & \xrightarrow{p} & \mathrm{pt} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \iota} \\ \mathrm{pt} & \xrightarrow{\iota} & Z. \end{array}$$

□

13.2.5. Consider now the category

$$\mathrm{ShvCat}(Z^\wedge_{\{z\}}),$$

which according to Theorem 2.3.1, identifies with

$$\mathrm{QCoh}(Z^\wedge_{\{z\}})\text{-}\mathbf{mod} \simeq \mathrm{QCoh}(Z)_{\{z\}}\text{-}\mathbf{mod}.$$

Consider the functor

$$\mathrm{QCoh}(Z)_{\{z\}}\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}, \quad \mathbf{C} \mapsto \mathrm{Vect} \underset{\mathrm{QCoh}(Z)_{\{z\}}}{\otimes} \mathbf{C} \simeq \mathrm{Vect} \underset{\mathrm{QCoh}(Z)}{\otimes} \mathbf{C} \tag{13.2}$$

(the last equivalence is due to Lemma 4.1.6).

Note that since $\mathrm{QCoh}(Z)$ is rigid, by Corollary D.4.5, we can rewrite the functor (13.2) also as

$$\underline{\mathrm{Hom}}_{\mathrm{QCoh}(Z)}(\mathrm{Vect}, \mathbf{C}) \simeq \underline{\mathrm{Hom}}_{\mathrm{QCoh}(Z)_{\{z\}}}(\mathrm{Vect}, \mathbf{C}).$$

We note that the functor (13.2) naturally upgrades to a functor

$$\mathrm{QCoh}(Z^\wedge_{\{z\}})\text{-}\mathbf{mod} \to \Omega(Z,z)\text{-}\mathbf{mod} \tag{13.3}$$

by regarding Vect as equipped with the trivial action of $\Omega(Z,z)$ that commutes with one of $\mathrm{QCoh}(Z)$.

13.2.6. We claim (assuming that the pair (Z,z) satisfies Conjecture 13.1.2):

Proposition 13.2.7. *The functor* (13.3) *is an equivalence.*

Proof. We construct a functor

$$\Omega(Z,z)\text{-}\mathbf{mod} \to \mathrm{QCoh}(Z)\text{-}\mathbf{mod}$$

by

$$\mathbf{D} \mapsto \mathrm{Vect} \underset{\mathrm{QCoh}(\Omega(Z,z))_{\mathrm{conv}}}{\otimes} \mathbf{D}; \tag{13.4}$$

It is easy to see that the essential image of (13.4) lies in the full subcategory

$$\mathrm{QCoh}(Z^\wedge_{\{z\}})\text{-}\mathbf{mod} \subset \mathrm{QCoh}(Z)\text{-}\mathbf{mod}.$$

We claim that the functors (13.2) and (13.4) are mutually inverse. Indeed, this follows from the (tautological) equivalence

$$\mathrm{Vect} \underset{\mathrm{QCoh}(Z)_{\{z\}}}{\otimes} \mathrm{Vect} \simeq \mathrm{Vect} \underset{\mathrm{QCoh}(Z)}{\otimes} \mathrm{Vect} \simeq \mathrm{QCoh}(\Omega(Z,z)) \tag{13.5}$$

combined with that of (13.1).

□

13.2.8. Note that the functor ι^* naturally upgrades to a functor

$$\mathrm{QCoh}(X)_{\{z\}} \to \underline{\mathrm{Hom}}_{\mathrm{QCoh}(\Omega(Z,z))_{\mathrm{conv}}}(\mathrm{Vect}, \mathrm{Vect}). \tag{13.6}$$

We claim (still assuming that the pair (Z, z) satisfies Conjecture 13.1.2):

Proposition 13.2.9. *The functor* (13.6) *is an equivalence.*

Proof. Follows from Proposition 13.2.7. □

Corollary 13.2.10.

(a) *There exists a canonical equivalence*

$$\mathrm{Rep}(\Omega(Z, z)) \simeq \mathrm{QCoh}(Z)_{\{z\}}.$$

(b) *The map* (10.6) *is an equivalence for* $\Omega(Z, z)$.

Proof. Point (a) is a reformulation of Proposition 13.2.9. Point (b) follows by combining point (a) with Proposition 13.2.4. □

Finally, we obtain (always assuming that the pair (Z, z) satisfies Conjecture 13.1.2):

Corollary 13.2.11. *The prestack* $B(\Omega(Z, z))$ *is 1-affine.*

Proof. Follows using Proposition 10.4.4(b') from Corollary 13.2.10(b) and the equivalence (13.5). □

Note that Corollaries 13.2.11 and 13.2.10(a) together amount to the statement of Theorem 13.2.2.

13.3. **Towards Conjecture 13.1.2.**

13.3.1. Consider the pair of adjoint functors

$$\iota_{\mathrm{IndCoh},*} : \mathrm{Vect} \rightleftarrows \mathrm{IndCoh}(Z)_{\{z\}} : \iota^!.$$

We claim that the functor $\iota^!$ is monadic. Indeed, it is conservative by [Ga, Proposition 4.1.7(a)], and is continuous.

Note that this implies the statement of Proposition 13.1.3(1), as in the smooth case there is no difference between IndCoh and QCoh.

13.3.2. Let us show that the functor $\iota^{\mathrm{QCoh},!}$ is conservative for Z eventually coconnective. Recall the functor

$$\Phi_Z : \mathrm{QCoh}(Z) \to \mathrm{IndCoh}(Z),$$

right adjoint to the functor $\Psi_Z : \mathrm{IndCoh}(Z) \to \mathrm{QCoh}(Z)$.

Note that

$$\iota_* \simeq \Psi_Z \circ \iota_{\mathrm{IndCoh},*},$$

and hence

$$\iota^{\mathrm{QCoh},!} \simeq \iota^! \circ \Phi_Z.$$

We have just seen that the functor $\iota^!$ is conservative. Hence, it is enough to show that Z eventually coconnective, the functor Φ_Z is conservative.

We claim that Φ_Z is in fact fully faithful. Indeed, this follows from the fact that Ψ_Z admits a fully faithful *left* adjoint (see [Ga, Proposition 1.5.3]).

Hence, we obtain that the monadicity of $\iota^{\mathrm{QCoh},!}$ is equivalent to it satisfying the second condition in the Barr-Beck-Lurie theorem.

13.3.3. Let us show that $\iota^{\mathrm{QCoh},!}$ fails to be monadic for the non-eventually coconnective DG scheme $Z = \mathrm{pt} \underset{\mathrm{pt} \times \mathbb{A}^1}{\times} \mathrm{pt}$. In fact, we claim that in this case, it fails to be conservative.

Indeed, $Z = \mathrm{Spec}(k[\xi])$, where $\deg(\xi) = -2$. The functor $\iota^{\mathrm{QCoh},!}$ annihilates the module $k[\xi, \xi^{-1}]$.

13.4. **Shift of grading and proof of Proposition 13.1.3(2).**

13.4.1. *Shift of grading.* In order to prove Proposition 13.1.3(2), we will use the "shift of grading" trick (see, e.g., [AG, Sect. A.2]).

Consider the symmetric monoidal DG category $\mathrm{Rep}(\mathbb{G}_m)$, i.e., the category chain complexes of $\mathbb{Z}$-graded vector spaces. It carries a canonical (symmetric monoidal self-equivalence), denoted

$$M \mapsto M^{\mathrm{shift}}.$$

Namely, the n-graded piece of the m-th cohomology of M^{shift} equals by definition the n-th graded piece of the $(m+2n)$-th cohomology of M.

If A is an algebra object of $\mathrm{Rep}(\mathbb{G}_m)$, we obtain an equivalence

$$A\text{-mod}(\mathrm{Rep}(\mathbb{G}_m)) \simeq A^{\mathrm{shift}}\text{-mod}(\mathrm{Rep}(\mathbb{G}_m)). \tag{13.7}$$

Note, however, that the equivalence (13.7) *does not* commute with the forgetful funcors

$$A\text{-mod}(\mathrm{Rep}(\mathbb{G}_m)) \to A\text{-mod}(\mathrm{Vect}) \text{ and } A^{\mathrm{shift}}\text{-mod}(\mathrm{Rep}(\mathbb{G}_m)) \to A^{\mathrm{shift}}\text{-mod}(\mathrm{Vect}).$$

13.4.2. Let $\mathbf{O}$ be an algebra object in the symmetric monoidal category $\mathrm{ShvCat}(B\mathbb{G}_m) \simeq \mathbb{G}_m\text{-}\mathbf{mod}$, and let $\mathbf{C}_1$ and $\mathbf{C}_2$ be right and left $\mathbf{O}$-modules, respectively. Let

$$\mathbf{C}_1 \underset{\mathbf{O}}{\otimes} \mathbf{C}_2 \to \mathbf{C}$$

be a map in $\mathbb{G}_m\text{-}\mathbf{mod}$.

The following results from Theorem 2.2.2 applied to $G = \mathbb{G}_m$:

Lemma 13.4.3. *The functor $\mathbf{C}_1 \underset{\mathbf{O}}{\otimes} \mathbf{C}_2 \to \mathbf{C}$ is an equivalence as plain DG categories if and only if the functor*

$$\mathbf{inv}_{\mathbb{G}_m}(\mathbf{C}_1) \underset{\mathbf{inv}_{\mathbb{G}_m}(\mathbf{O})}{\otimes} \mathbf{inv}_{\mathbb{G}_m}(\mathbf{C}_2) \to \mathbf{inv}_{\mathbb{G}_m}(\mathbf{C})$$

is an equivalence of plain DG categories.

13.4.4. *Proof of Proposition 13.1.3(2).* Let V be a finite-dimensional vector space so that $\mathbb{A}^n = \mathrm{Spec}(\mathrm{Sym}(V))$.

By Lemma 10.4.2, it is enough to show that the functor

$$\mathrm{Vect} \underset{\mathrm{Sym}(V[2])\text{-mod}}{\otimes} \mathrm{Vect} \to \mathrm{Sym}(V[1])\text{-mod} \tag{13.8}$$

is an equivalence.

We will deduce this from the fact that the functor

$$\mathrm{Vect} \underset{\mathrm{Sym}(V)\text{-mod}}{\otimes} \mathrm{Vect} \to \mathrm{Sym}(V[-1])\text{-mod} \tag{13.9}$$

is an equivalence; the latter is due to Theorem 2.2.2 and Proposition 10.4.4, once we identify

$$\mathrm{QCoh}(V^*)_{\mathrm{conv}} \simeq \mathrm{Sym}(V)\text{-mod} \text{ and } \mathrm{Rep}(V^*) \simeq \mathrm{Sym}(V[-1])\text{-mod}.$$

Indeed, by Lemma 13.4.3, to show that (13.8) is an equivalence, it is enough to show that

$$\mathrm{Rep}(\mathbb{G}_m) \underset{\mathrm{Sym}(V[2])\text{-mod}(\mathrm{Rep}(\mathbb{G}_m))}{\otimes} \mathrm{Rep}(\mathbb{G}_m) \to \mathrm{Sym}(V[1])\text{-mod}(\mathrm{Rep}(\mathbb{G}_m))$$

is an equivalence.

By (13.7), the latter is equivalent to

$$\mathrm{Rep}(\mathbb{G}_m) \underset{\mathrm{Sym}(V)\text{-mod}(\mathrm{Rep}(\mathbb{G}_m))}{\otimes} \mathrm{Rep}(\mathbb{G}_m) \to \mathrm{Sym}(V[-1])\text{-mod}(\mathrm{Rep}(\mathbb{G}_m))$$

being an equivalence, which, again by Lemma 13.4.3, follows from the fact that (13.9) is an equivalence.

14. Classifying prestacks of (co)-affine group-prestacks

14.1. **The iterated classifying prestack.** In this subsection we will prove Theorem 2.5.7.

14.1.1. *Proof of Theorem 2.5.7(a).* Let $V \in \mathrm{Vect}^{\heartsuit}$ be finite-dimensional. Set $\mathcal{G} := BV$; we already know that $\mathcal{G}$ is 1-affine, so the category $\mathrm{ShvCat}(B\mathcal{G})$ can be described as in Sect. 10.1.

We note that there is a canonical equivalence

$$\mathrm{QCoh}(\mathcal{G}) \simeq \mathrm{QCoh}(V^*)_{\{0\}}, \tag{14.1}$$

under which the co-monoidal structure on $\mathrm{QCoh}(\mathcal{G})$ (induced by the group structure on $\mathcal{G}$) is obtained via the duality

$$\left(\mathrm{QCoh}(V^*)_{\{0\}}\right)^{\vee} \simeq \mathrm{QCoh}(V^*)_{\{0\}}$$

from the pointwise monoidal structure on $\mathrm{QCoh}(V^*)_{\{0\}}$.

Thus, the category $\mathcal{G}\text{-}\mathbf{mod}$ can be identified with

$$\mathrm{QCoh}(V^*)_{\{0\}}\text{-}\mathbf{mod},$$

with the functor $\mathbf{inv}_{\mathcal{G}}$ being

$$\mathbf{D} \mapsto \underline{\mathrm{Hom}}_{\mathrm{QCoh}(V^*)_{\{0\}}}(\mathrm{Vect}, \mathbf{D}),$$

where $\mathrm{QCoh}(V^*)_{\{0\}}$ acts on Vect via the functor ι^*, where $\iota : \mathrm{pt} \to V^*$ corresponds to $0 \in V^*$.

Now, the assertion of Theorem 2.5.7(a) follows from Proposition 13.2.7 for $(Z, z) = (V^*, 0)$.

14.1.2. *Proof of Theorem 2.5.7(b).* Let $\mathcal{G}_1 = B^2(V)$. According to Theorem 2.5.7(a), proved above, $\mathcal{G}_1$ is 1-affine. Hence, the category $\mathrm{ShvCat}(B\mathcal{G}_1)$ can be described as in Sect. 10.1.

Moreover, we have identified $\mathrm{QCoh}(\mathcal{G}_1)$ as a plain category with $\mathrm{QCoh}(Z_1)$, where

$$Z_1 := \mathrm{pt} \underset{V^*}{\times} \mathrm{pt}\,.$$

Under this identification, the co-monoidal structure on $\mathrm{QCoh}(\mathcal{G}_1)$ (induced by the group structure on $\mathcal{G}_1$) is obtained via the duality

$$\mathrm{QCoh}(Z_1) \simeq \mathrm{QCoh}(Z_1)^{\vee},$$

from the pointwise monoidal structure on $\mathrm{QCoh}(Z_1)$. Since $\mathrm{QCoh}(Z_1)_{\mathrm{ptw}}$ is rigid, the group $\mathcal{G}_1$ satisfies the assumption of Sect. 11.1.

Thus, the category $\mathcal{G}_1\text{-}\mathbf{mod}$ can be identified with

$$\mathrm{QCoh}(Z_1)\text{-}\mathbf{mod},$$

with the functor $\mathbf{inv}_{\mathcal{G}_1}$ being

$$\mathbf{D} \mapsto \underline{\mathrm{Hom}}_{\mathrm{QCoh}(Z_1)}(\mathrm{Vect}, \mathbf{D}),$$

where $\mathrm{QCoh}(Z_1)$ acts on Vect via the functor ι_1^*, where $\iota_1 : \mathrm{pt} \to Z_1$ is the unique k-point z_1 of Z_1.

The category $\mathrm{Rep}(\mathcal{G}_1)$ therefore identifies with $\mathrm{QCoh}(\Omega(Z_1, z_1))_{\mathrm{conv}}$. We will show that $B\mathcal{G}_1$ is 1-affine by applying Proposition 11.2.1(b'). Indeed, the condition that

$$\mathrm{Vect} \underset{\mathrm{Rep}(\mathcal{G}_1)}{\otimes} \mathrm{Vect} \simeq \mathrm{QCoh}(\mathcal{G}_1)$$

translates as

$$\mathrm{Vect} \underset{\mathrm{QCoh}(\Omega(Z_1,z_1))_{\mathrm{conv}}}{\otimes} \mathrm{Vect} \simeq \mathrm{QCoh}(Z_1),$$

and follows from Proposition 13.2.4 for $(Z, z) = (Z_1, z_1)$.

14.1.3. *Proof of Theorem 2.5.7(c).* Denote $\mathcal{G}_2 := B^3(V)$. According to Theorem 2.5.7(a), proved above, $\mathcal{G}_2$ is 1-affine. Hence, the category $\mathrm{ShvCat}(B\mathcal{G}_2)$ can be described as in Sect. 10.1.

Moreover, we have identified $\mathrm{QCoh}(\mathcal{G}_2)$ as a plain DG category with $\mathrm{QCoh}(Z_2)$, where

$$Z_2 := \mathrm{pt} \underset{Z_1}{\times} \mathrm{pt}, \text{ where } Z_1 := \mathrm{pt} \underset{V^*}{\times} \mathrm{pt}\,.$$

Under this identification, the co-monoidal structure on $\mathrm{QCoh}(\mathcal{G}_2)$ (induced by the group structure on $\mathcal{G}_2$) is obtained via the duality

$$\mathrm{QCoh}(Z_2) \simeq \mathrm{QCoh}(Z_2)^\vee,$$

from the pointwise monoidal structure on $\mathrm{QCoh}(Z_2)$. Since $\mathrm{QCoh}(Z_2)_{\mathrm{ptw}}$ is rigid, the group $\mathcal{G}_2$ satisfies the assumption of Sect. 11.1.

Thus, the category $\mathcal{G}_2\text{-}\mathbf{mod}$ can be identified with

$$\mathrm{QCoh}(Z_2)\text{-}\mathbf{mod},$$

with the functor $\mathbf{inv}_{\mathcal{G}_2}$ being

$$\mathbf{D} \mapsto \underline{\mathrm{Hom}}_{\mathrm{QCoh}(Z_2)}(\mathrm{Vect}, \mathbf{D}),$$

where $\mathrm{QCoh}(Z_2)$ acts on Vect via the functor ι_2^*, where $\iota_2 : \mathrm{pt} \to Z_2$ is the unique k-point z_2 of Z_2.

The category $\mathrm{Rep}(\mathcal{G}_2)$ therefore identifies with $\mathrm{QCoh}(\Omega(Z_2, z_2))_{\mathrm{conv}}$. We will show that $B\mathcal{G}_2$ is *not* 1-affine by applying Proposition 11.2.1(c).

The functor $p^* : \mathrm{Vect} \to \mathrm{QCoh}(\mathcal{G}_2)$ is the right adjoint of the functor $\mathrm{QCoh}(\mathcal{G}_2)_{\mathrm{conv}^L} \to \mathrm{Vect}$ that defines the trivial action of $\mathcal{G}_2$ on Vect. Thus, under the identification

$$\mathrm{QCoh}(\mathcal{G}_2)_{\mathrm{conv}^L} \simeq \mathrm{QCoh}(Z_2)_{\mathrm{ptw}},$$

the functor p^* corresponds to $(\iota_2)_*$. Hence the functor $(p_*)^R$ translates as $\iota_2^{\mathrm{QCoh},!}$. However, we claim that $\iota_2^{\mathrm{QCoh},!}$ is *not* monadic. In fact, we claim that $\iota_2^{\mathrm{QCoh},!}$ is *not* conservative. Indeed, this follows from Sect. 13.3.3.

14.1.4. *Proof of Theorem 2.5.7(d).* Set $\mathcal{G}_{\mathrm{inf}} := B(V^{\wedge}_{\{0\}})$. We know that $\mathcal{G}_{\mathrm{inf}}$ is 1-affine by Theorem 2.5.4.

We note that $\mathcal{G}_{\mathrm{inf}}$ falls into the paradigm of Sect. 11.1, where

$$\mathrm{QCoh}(\mathcal{G}_{\mathrm{inf}})_{\mathrm{conv}^L} \simeq \mathrm{QCoh}(V^*)_{\mathrm{ptw}}.$$

Under this identification, the trivial action of $\mathrm{QCoh}(\mathcal{G}_{\mathrm{inf}})_{\mathrm{conv}^L}$ on Vect corresponds to

$$\iota^* : \mathrm{QCoh}(V^*) \to \mathrm{Vect},$$

where $\iota : \mathrm{pt} \to V^*$ corresponds to $0 \in V^*$.

We will show that $B\mathcal{G}_{\mathrm{inf}}$ is *not* 1-affine by applying Proposition 11.2.1(c). We note that the functor

$$p^* : \mathrm{Vect} \to \mathrm{QCoh}(\mathcal{G}_{\mathrm{inf}})$$

is the *right* adjoint to one corresponding to the augmentation $\mathrm{QCoh}(\mathcal{G}_{\mathrm{inf}}) \to \mathrm{Vect}$. I.e., p^* indentifies with the functor

$$\iota_* : \mathrm{Vect} \to \mathrm{QCoh}(V^*).$$

Its right adjoint is, therefore, the functor

$$\iota^{\mathrm{QCoh},!} : \mathrm{QCoh}(V^*) \to \mathrm{Vect}\,.$$

However, it is clear that $\iota^{\mathrm{QCoh},!}$ is not conservative: it annihilates any object that comes as direct image from $V^* - \{0\}$.

14.2. **Group DG schemes.** In this subsection we will prove Theorem 2.5.10.

14.2.1. Denote $\mathcal{G}_n = \mathrm{Spec}(\mathrm{Sym}(V[n]))$. The case $n = 0$ follows from Theorem 2.2.2 for $G = V^*$. For $n > 0$, we will consider the cases of n even and odd separately.

14.2.2. Let first $n = 2$. In this case, the assertion follows from Corollary 13.2.11 applied to $Z := \mathrm{Spec}(\mathrm{Sym}(V[1]))$.

Let now n be an arbitrary even integer. The required assertion follows Proposition 10.4.4(b) and the case of $n = 2$, using the "shift of grading" trick, as in Sect. 13.4.4.

14.2.3. Let us now take $n = 1$. In this case the assertion follows from Corollary 13.2.11 applied to $Z := \mathrm{Spec}(\mathrm{Sym}(V))$.

Let now n be an arbitrary odd integer. The required assertion follows Proposition 10.4.4(b) and the case of $n = 1$, using the "shift of grading" trick, as in Sect. 13.4.4.

Appendix A. Descent theorems

A.1. **Descent for module categories.** In this subsection we will prove Theorem 1.5.2. Let Y be an affine DG scheme.

A.1.1. *Step 1.* Let

$$\mathcal{P} : (\mathrm{DGSch}^{\mathrm{aff}}_{/Y})^{\mathrm{op}} \to \mathcal{T}$$

be a functor, where $\mathcal{T}$ is some ∞-category. It is well-known that if $\mathcal{P}$ satisfies descent with respect to finite flat maps and Nisnevich covers, then $\mathcal{P}$ satisfies fppf descent. This follows from the fact that, Nisnevich-locally, any fppf map admits a section after a finite flat base change.

We let $\mathcal{T} = \mathrm{DGCat}_{\mathrm{cont}}$ and $\mathcal{P}$ be the functor

$$S \in \mathrm{DGSch}^{\mathrm{aff}}_{/Y} \rightsquigarrow \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}.$$

In Step 2 we will show that $\mathcal{T}$ satisfies Nisnevich descent, and in Step 3 we will show that $\mathcal{T}$ satisfies finite flat descent. This will prove Theorem 1.5.2.

A.1.2. *Step 2.* For Nisnevich descent, it is enough to consider the case of *basic Nisnevich covers.* I.e., let $\overset{\circ}{S} \overset{\jmath}{\hookrightarrow} S$ be an open embedding, and $\pi : S_1 \to S$ an étale map, such that π is one-to-one over $S - \overset{\circ}{S}$. Set $\overset{\circ}{S}_1 := \overset{\circ}{S} \underset{S}{\times} S_1$. Let $\overset{\circ}{\pi} : \overset{\circ}{S}_1 \to \overset{\circ}{S}$ and $\jmath_1 : \overset{\circ}{S}_1 \to S_1$ denote the corresponding morphisms.

We need to show that

$$\begin{array}{ccc} \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C} & \xrightarrow{\jmath^* \otimes \mathrm{Id}_{\mathbf{C}}} & \mathrm{QCoh}(\overset{\circ}{S}) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C} \\ \pi^* \otimes \mathrm{Id}_{\mathbf{C}} \Big\downarrow & & \Big\downarrow \overset{\circ}{\pi}{}^* \otimes \mathrm{Id}_{\mathbf{C}} \\ \mathrm{QCoh}(S_1) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C} & \xrightarrow{\jmath_1^* \otimes \mathrm{Id}_{\mathbf{C}}} & \mathrm{QCoh}(\overset{\circ}{S}_1) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C} \end{array}$$

is a pull-back square.

By definition, the pull-back

$$\left(\mathrm{QCoh}(S_1) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}\right) \underset{\mathrm{QCoh}(\overset{\circ}{S}_1) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}}{\times} \left(\mathrm{QCoh}(\overset{\circ}{S}) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}\right)$$

is the category of quintuples

$$\{\mathbf{c}_1 \in \mathrm{QCoh}(S_1) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}, \quad \overline{\mathbf{c}}_1 \in \mathrm{QCoh}(\overset{\circ}{S}_1) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}, \quad \overline{\mathbf{c}} \in \mathrm{QCoh}(\overset{\circ}{S}) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}, \tag{A.1}$$
$$\alpha : (\jmath_1^* \otimes \mathrm{Id}_{\mathbf{C}})(\mathbf{c}_1) \simeq \overline{\mathbf{c}}_1, \qquad \beta : (\overset{\circ}{\pi}{}^* \otimes \mathrm{Id}_{\mathbf{C}})(\overline{\mathbf{c}}) \simeq \overline{\mathbf{c}}_1.\}$$

We define the right adjoint to the natural functor

$$\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C} \to \mathrm{QCoh}(S_1) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C} \underset{\mathrm{QCoh}(\overset{\circ}{S}_1) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}}{\times} \mathrm{QCoh}(\overset{\circ}{S}) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}$$

by sending a quintuple as in (A.1) to

$$\mathrm{Cone}\left((\pi_* \otimes \mathrm{Id}_{\mathbf{C}})(\mathbf{c}_1) \oplus (\jmath_* \otimes \mathrm{Id}_{\mathbf{C}})(\overline{\mathbf{c}}) \to ((\jmath \circ \overset{\circ}{\pi})_* \otimes \mathrm{Id}_{\mathbf{C}})(\overline{\mathbf{c}}_1)\right)[-1].$$

Thus, we obtain that the above right adjoint is obtained from the right adjoint for $\mathbf{C} := \mathrm{QCoh}(Y)$ by tensoring by $- \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}$. Therefore, since the unit and the co-unit of the adjunction are isomorphisms in the former case (by Nisnevich descent for QCoh), they are isomorphisms for any $\mathbf{C}$.

Remark A.1.3. It is very easy to prove directly that for $\mathrm{QCoh}(Y)$, the unit and co-unit map are isomorphisms, see [GL:QCoh, Lemma 2.2.6].

A.1.4. *Step 3.* Let now $\pi : T \to S$ be a finite faithfully flat map, and let $T^\bullet/S$ be its Čech nerve. We need to show that the functor

$$\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C} \to \mathrm{Tot}\left(\mathrm{QCoh}(T^\bullet/S)^* \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}\right)$$

is an equivalence.

We note that for every map $\alpha : [j] \to [i]$ and the corresponding map $f^\alpha : T^i/S \to T^j/S$, the functor $(f^\alpha)^*$ admits a *left* adjoint, denoted $f^\alpha_\sharp$. Namely, for any finite flat map f, the functor $f_\sharp$ is the ind-extension of the functor on $\mathrm{QCoh}(-)^{\mathrm{perf}}$, defined as

$$\mathcal{E} \mapsto (f_*(\mathcal{E}^\vee))^\vee.$$

In particular, we obtain that the functors $(f^\alpha)^* \otimes \mathrm{Id}_{\mathbf{C}}$ all admit left adjoints, given by $f^\alpha_\sharp \otimes \mathrm{Id}_{\mathbf{C}}$. Hence, by [GL:DG, Lemma 1.3.3], we have

$$\mathrm{Tot}\left(\mathrm{QCoh}(T^\bullet/S)^* \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}\right) \simeq |\,\mathrm{QCoh}(T^\bullet/S)_\sharp \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}|.$$

From the commutative diagram

$$\begin{array}{ccc}
\mathrm{Tot}\,(\mathrm{QCoh}(T^\bullet/S)^*) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C} & \longrightarrow & |\,\mathrm{QCoh}(T^\bullet/S)_\sharp| \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C} \\
\downarrow & & \downarrow \sim \\
\mathrm{Tot}\left(\mathrm{QCoh}(T^\bullet/S)^* \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}\right) & \longrightarrow & |\,\mathrm{QCoh}(T^\bullet/S)_\sharp \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}|
\end{array}$$

we obtain that the functor

$$\mathrm{Tot}\,(\mathrm{QCoh}(T^\bullet/S)^*) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C} \to \mathrm{Tot}\left(\mathrm{QCoh}(T^\bullet/S)^* \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}\right)$$

is an equivalence.

Now, the required assertion follows from the commutative diagram

$$\begin{array}{ccc}
\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C} & \longrightarrow & \mathrm{Tot}\,(\mathrm{QCoh}(T^\bullet/S)^*) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C} \\
{\scriptstyle \mathrm{Id}}\downarrow & & \downarrow \sim \\
\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C} & \longrightarrow & \mathrm{Tot}\left(\mathrm{QCoh}(T^\bullet/S)^* \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}\right)
\end{array}$$

and the fact that

$$\mathrm{QCoh}(S) \to \mathrm{Tot}\,(\mathrm{QCoh}(T^\bullet/S)^*)$$

is an equivalence, by the fppf descent for QCoh.

Remark A.1.5. One can show directly that $\mathrm{QCoh}(S) \to \mathrm{Tot}\,(\mathrm{QCoh}(T^\bullet/S)^*)$ is an equivalence (so that together with Remark A.1.3 one obtains an alternative proof of fppf descent for QCoh, without apealing to the t-structures).

Proof of finite flat descent for QCoh.
By Lemma C.1.8, the functor of evaluation on 0-simplices

$$\mathrm{Tot}\,(\mathrm{QCoh}(T^\bullet/S)^*) \to \mathrm{QCoh}(T)$$

is monadic, and the corresponding monad on $\mathrm{QCoh}(T)$, viewed as a plain endo-functor, is canonically isomorphic to $(\mathrm{pr}_2)_\sharp \circ \mathrm{pr}_1^*$, where $\mathrm{pr}_1, \mathrm{pr}_2 : T \underset{S}{\times} T \rightrightarrows T$ are the two projections. Note also that we have a canonical isomorphism of endo-functors:

$$(\mathrm{pr}_2)_\sharp \circ \mathrm{pr}_1^* \simeq ((\mathrm{pr}_1)_* \circ \mathrm{pr}_2^*)^L \simeq (\pi^* \circ \pi_*)^L \simeq \pi^* \circ \pi_\sharp.$$

Hence, it remains to show that the functor $\pi^* : \mathrm{QCoh}(S) \to \mathrm{QCoh}(T)$ is monadic (we have just seen that the corresponding monad $\pi^* \circ \pi_\sharp$ maps isomorphically to one defining $\mathrm{Tot}\,(\mathrm{QCoh}(T^\bullet/S)^*)$).

Since π is faithfully flat, it is easy to see that π^* is conservative. Now, since π^* is continuous, it commutes with all geometric realizations. Hence, the monadicity of π^* follows from the Barr-Beck-Lurie theorem.

□

A.2. **Descent for sheaves of categories.** In this subsection we will prove Theorem 1.5.7.

A.2.1. *Step 0.* Let $\pi : T \to S$ be an fppf cover, and let $T^\bullet/S$ be its Čech nerve. We need to show that the functor

$$\mathrm{ShvCat}(S) \to \mathrm{Tot}(\mathrm{ShvCat}(T^\bullet/S)) \tag{A.2}$$

is an equivalence.

Since all the DG schemes involved are affine, we have

$$\mathrm{ShvCat}(S) = \mathrm{QCoh}(S)\text{-}\mathbf{mod} \text{ and } \mathrm{ShvCat}(T^\bullet/S) = \mathrm{QCoh}(T^\bullet/S)\text{-}\mathbf{mod}.$$

The right adjoint to the functor (A.2) sends

$$\mathbf{C}^\bullet \in \mathrm{Tot}(\mathrm{QCoh}\,(T^\bullet/S)\text{-}\mathbf{mod}) \rightsquigarrow \mathrm{Tot}(\mathbf{C}^\bullet),$$

where the totalization is taken in $\mathrm{QCoh}(S)\text{-}\mathbf{mod}$.

We need to show that the functor (A.2) and its right adjoint are mutually inverse. We will check that the unit and the co-unit of the adjunction are isomorphisms.

A.2.2. *Step 1.* The fact that the unit of the adjunction is an isomorphism follows immediately from Theorem 1.5.2.

A.2.3. *Step 2.* To say that the co-unit of the adjunction is an isomorphism is equivalent to saying that for $\mathbf{C}^\bullet \in \mathrm{Tot}\,(\mathrm{QCoh}(T^\bullet/S)\text{-}\mathbf{mod})$ the functor

$$\mathrm{QCoh}(T) \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{Tot}(\mathbf{C}^\bullet) \to \mathbf{C}^0$$

is an equivalence.

By Lemma 1.4.7, $\mathrm{QCoh}(T)$ is dualiazable as an object of $\mathrm{QCoh}(S)\text{-}\mathbf{mod}$. Hence, the functor

$$\mathrm{QCoh}(T) \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{Tot}(\mathbf{C}^\bullet) \to \mathrm{Tot}(\mathrm{QCoh}(T) \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C}^\bullet)$$

is an equivalence. Hence, it remains to show that the functor

$$\mathrm{Tot}(\mathrm{QCoh}(T) \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C}^\bullet) \to \mathbf{C}^0 \tag{A.3}$$

is an equivalence.

Now, we note that the co-simplicial category $\mathrm{QCoh}(T) \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C}^\bullet$ identifies with $\mathbf{C}^{\bullet+1}$, which is split, and its map to $\mathbf{C}^0$ is the augmentation. Hence, (A.3) is an equivalence, as desired.

Appendix B. Quasi-affine morphisms

B.1. **Fiber products of prestacks vs. tensor products of categories.**

B.1.1. Let

$$\begin{array}{ccc} \mathcal{Y}_1' & \xrightarrow{g_1} & \mathcal{Y}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}_2' & \xrightarrow{g_2} & \mathcal{Y}_2 \end{array}$$

be a Cartesian diagram in PreStk. It gives rise to a (symmetric monoidal) functor

$$\mathrm{QCoh}(\mathcal{Y}_2') \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathrm{QCoh}(\mathcal{Y}_1) \to \mathrm{QCoh}(\mathcal{Y}_1'). \tag{B.1}$$

In this section we will discuss two instances in which the functor (B.1) is an equivalence.

B.1.2. We will prove:

Proposition B.1.3. *Assume that f (and hence f') is quasi-affine and quasi-compact. Then* (B.1) *is an equivalence.*

The proof will use the following lemma, proved in Sect. B.1.7:

Lemma B.1.4. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a quasi-affine quasi-compact map, and consider $f_*(\mathcal{O}_{\mathcal{Y}_1})$ as an assciative algebra in* $\mathrm{QCoh}(\mathcal{Y}_2)$*. Then the functor*

$$\mathrm{QCoh}(\mathcal{Y}_1) \to f_*(\mathcal{O}_{\mathcal{Y}_1})\text{-mod}(\mathrm{QCoh}(\mathcal{Y}_2))$$

is an equivalence.

Proof of Proposition B.1.3. By Lemma B.1.4, we have:

$$\mathrm{QCoh}(\mathcal{Y}_1) \simeq f_*(\mathcal{O}_{\mathcal{Y}_1})\text{-mod}(\mathrm{QCoh}(\mathcal{Y}_2)) \text{ and } \mathrm{QCoh}(\mathcal{Y}_1') \simeq f'_*(\mathcal{O}_{\mathcal{Y}_1'})\text{-mod}(\mathrm{QCoh}(\mathcal{Y}_2')).$$

Hence, it is sufficient to show that the functor g_2^* induces an equivalence

$$\mathrm{QCoh}(\mathcal{Y}_2') \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} (f_*(\mathcal{O}_{\mathcal{Y}_1})\text{-mod}(\mathrm{QCoh}(\mathcal{Y}_2))) \to f'_*(\mathcal{O}_{\mathcal{Y}_1'})\text{-mod}(\mathrm{QCoh}(\mathcal{Y}_2')).$$

By base change, $f'_*(\mathcal{O}_{\mathcal{Y}_1'}) \simeq g^*(f_*(\mathcal{O}_{\mathcal{Y}_1}))$. Now, the required assertion follows from the following general lemma:

Lemma B.1.5. *Let $\mathbf{O}$ be a monoidal DG category, $\mathbf{C}$ a left $\mathbf{O}$-module category, and $A \in \mathbf{O}$ an algebra. Consider A-mod$(\mathbf{O})$ as a right $\mathbf{O}$-module category. Then the natural functor*

$$A\text{-mod}(\mathbf{O}) \underset{\mathbf{O}}{\otimes} \mathbf{C} \to A\text{-mod}(\mathbf{C})$$

is an equivalence.

□

B.1.6. *Proof of Lemma B.1.5.* We have an adjoint pair

$$\mathrm{ind}_{A,\mathbf{O}} : \mathbf{O} \rightleftarrows A\text{-mod}(\mathbf{O}) : \mathrm{oblv}_{A,\mathbf{O}}$$

as right $\mathbf{O}$-module categories. Tensoring up on the right with $\mathbf{C}$ over $\mathbf{O}$, we obtain an adjoint pair

$$(\mathrm{ind}_{A,\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{C}}) : \mathbf{C} \rightleftarrows A\text{-mod}(\mathbf{O}) \underset{\mathbf{O}}{\otimes} \mathbf{C} : (\mathrm{oblv}_{A,\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{C}}).$$

The functor $(\mathrm{oblv}_{A,\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{C}})$ is monadic: indeed, it commutes with all colimits and its left adjoint generates $A\text{-mod}(\mathbf{O}) \underset{\mathbf{O}}{\otimes} \mathbf{C}$.

Consider also the adjoint pair

$$\mathrm{ind}_{A,\mathbf{C}} : \mathbf{C} \rightleftarrows A\text{-mod}(\mathbf{C}) : \mathrm{oblv}_{A,\mathbf{C}}\,.$$

The functor $\mathrm{oblv}_{A,\mathbf{C}}$ is tautologically monadic.

Hence, to prove the lemma, it suffices to show that the functor $A\text{-mod}(\mathbf{O}) \underset{\mathbf{O}}{\otimes} \mathbf{C} \to A\text{-mod}(\mathbf{C})$ defines an isomorphism of the corresponding monads on $\mathbf{C}$ as plain endo-functors. However, this follows from the fact that

$$(\mathrm{oblv}_{A,\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{C}}) \circ (\mathrm{ind}_{A,\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{C}}) \simeq ((A \otimes -) \otimes \mathrm{Id}_{\mathbf{C}})$$

identifies with the action of A on $\mathbf{C}$.

□

B.1.7. *Proof of Lemma B.1.4.* We shall deduce the assertion from the Barr-Beck-Lurie theorem. Since the morhism f is quasi-compact and quasi-separated, it satisfies base change and the projection formula. and the functor f_* is continuous.

The projection formula implies that the monad $f_* \circ f^*$ is given by tensor product with $f_*(\mathcal{O}_{\mathcal{Y}_1})$. The continuity of f_* implies that it commutes with all geometric realizations. It remains to show that f_* is conservative.

By base change, the conservativity assertion reduces to the case when $\mathcal{Y}_2$ is an affine DG scheme, in which case $\mathcal{Y}_1$ is quasi-affine. In this case, the conservativity of f_* is equivalent to that of $\Gamma(\mathcal{Y}_2, -)$.

Thus, we need to show that for a quasi-affine DG scheme Y, the functor

$$\Gamma(Y, -) : \mathrm{QCoh}(Y) \to \mathrm{Vect}$$

is conservative. Let $\jmath : Y \to \overline{Y}$ be an open embedding, where $\overline{Y}$ is affine. Now, the functors $\jmath_*$ and $\Gamma(\overline{Y}, -)$ are both conservative, and hence so is $\Gamma(Y, -)$.

□

B.2. **Fiber products of passable prestacks.**

B.2.1. Recall the notion of passable prestack, see Sect. 5.1. We are going to prove:

Proposition B.2.2. *Assume that* $\mathcal{Y}_2$ *is passable and that* $\mathrm{QCoh}(\mathcal{Y}_1)$ *is dualizable as a DG category. Then* (B.1) *is an equivalence.*

The proof will use the following assertion:

Lemma B.2.3. *Let* $\mathcal{Y} \in \mathrm{PreStk}$ *be such that the diagonal morphism is representable, quasi-compact and quasi-separated, and* $\mathcal{O}_{\mathcal{Y}} \in \mathrm{QCoh}(\mathcal{Y})$ *is compact. Then the following conditions are equivalent:*

(a) *The functor* $\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}') \to \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y}')$ *is an equivalence for any* $\mathcal{Y}' \in \mathrm{PreStk}$.

(b) *The functor* $\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y})$ *is an equivalence.*

(c) *The monoidal DG category* $\mathrm{QCoh}(\mathcal{Y})$ *is rigid.*

(d) *The category* $\mathrm{QCoh}(\mathcal{Y})$ *is dualizable as a DG category.*

Proof of Proposition B.2.2. We have

$$\mathrm{QCoh}(\mathcal{Y}_1') \simeq \underset{S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2'}}{\underset{\longleftarrow}{lim}} \mathrm{QCoh}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1).$$

By Lemma B.2.3, the category $\mathrm{QCoh}(\mathcal{Y}_2)$ is rigid. Hence, by Lemma 1.4.7, the functor

$$\mathbf{C} \mapsto \mathbf{C} \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathrm{QCoh}(\mathcal{Y}_1), \quad \mathrm{QCoh}(\mathcal{Y}_2)\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}$$

commutes with limits. In particular, the functor

$$\mathrm{QCoh}(\mathcal{Y}_2') \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathrm{QCoh}(\mathcal{Y}_1) = \left(\underset{S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2'}}{\underset{\longleftarrow}{lim}} \mathrm{QCoh}(S) \right) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathrm{QCoh}(\mathcal{Y}_1) \to$$
$$\to \underset{S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2'}}{\underset{\longleftarrow}{lim}} \left(\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathrm{QCoh}(\mathcal{Y}_1) \right)$$

is an equivalence.

This reduces the assertion to the case when $\mathcal{Y}_2' = S \in \mathrm{DGSch}^{\mathrm{aff}}$. However, in the latter case, the morphism g_2 is quasi-affine, and the assertion follows from Proposition B.1.3.

□

B.2.4. *Proof of Lemma B.2.3.* The implications (a) ⇒ (b) and (c) ⇒ (d) are tautological. The implication (d) ⇒ (a) is [GL:QCoh, Proposition 1.4.4]. Thus, it remains to show that (b) implies (c).

Thus, we need to show that the right adjoint to the monoidal operation

$$\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}) \tag{B.2}$$

is continuous, and is a map of $\mathrm{QCoh}(\mathcal{Y})$-bimodule categories.

Using the assumption in (b), we identify the functor (B.2) with the functor

$$\Delta_{\mathcal{Y}}^* : \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}).$$

The required properties follow now from the assumptions on the morphism $\Delta_{\mathcal{Y}}$.

□

Appendix C. Beck-Chevalley conditions

C.1. **Monadic and co-monadic Beck-Chevalley conditions.**

C.1.1. Let $\mathbf{C}^\bullet$ be a co-simplicial category; for a map $\alpha : [j] \to [i]$ in $\mathbf{\Delta}$, let $\mathsf{T}^\alpha : \mathbf{C}^j \to \mathbf{C}^i$ denote the corresponding functor.

Definition C.1.2. *We shall say that $\mathbf{C}^\bullet$ satisfies the* monadic *Beck-Chevalley condition, if the following holds:*

- *For every i and the last face map $\partial_i : [i] \to [i+1]$, the functor T^{∂_i} admits a left adjoint.*
- *For every $\alpha : [j] \to [i]$ (and the corresponding map $\alpha+1 : [j+1] \to [i+1]$), the diagram*

$$\begin{array}{ccc} \mathbf{C}^i & \xleftarrow{(\mathsf{T}^{\partial_i})^L} & \mathbf{C}^{i+1} \\ {\scriptstyle \mathsf{T}^\alpha}\uparrow & & \uparrow{\scriptstyle \mathsf{T}^{\alpha+1}} \\ \mathbf{C}^j & \xleftarrow{(\mathsf{T}^{\partial_j})^L} & \mathbf{C}^{j+1}, \end{array}$$

that a priori commutes up to a natural transformation, commutes.

Definition C.1.3. *We say that $\mathbf{C}^\bullet$ satisfies the* co-monadic *Beck-Chevalley condition, when we replace "left adjoint" by "right adjoint" in the above definition.*

C.1.4. We also give the following definition:

Definition C.1.5. *Let $\mathbf{C}^\bullet$ be a simplicial category. We shall say that $\mathbf{C}^\bullet$ satisfies the* monadic *Beck-Chevalley condition if for every i and the last face map $\partial_i : [i] \to [i+1]$, the functor $\mathsf{T}^{\partial_i} : \mathbf{C}^{i+1} \to \mathbf{C}^i$ admits a left adjoint and for every $\alpha : [j] \to [i]$ the diagram*

$$\begin{array}{ccc} \mathbf{C}^i & \xrightarrow{(\mathsf{T}^{\partial_i})^L} & \mathbf{C}^{i+1} \\ {\scriptstyle \mathsf{T}^\alpha}\downarrow & & \downarrow{\scriptstyle \mathsf{T}^{\alpha+1}} \\ \mathbf{C}^j & \xrightarrow{(\mathsf{T}^{\partial_j})^L} & \mathbf{C}^{j+1}, \end{array}$$

that a priori commutes up to a natural transformation, commutes.

The following (tautological) observation is often useful:

Lemma C.1.6. *Let $\mathbf{C}^\bullet$ be a simplicial category, in which for every $([j] \overset{\alpha}{\to} [i]) \in \mathbf{\Delta}$, the corresponding functor $\mathsf{T}^\alpha : \mathbf{C}^i \to \mathbf{C}^j$ admits a right adjoint, and every i and the last face map $\partial_i : [i] \to [i+1]$, the functor T^{∂_i} admits a left adjoint. Then the co-simplicial category $\mathbf{C}^{\bullet,R}$, obtained by passing to the right adjoints, satisfies the monadic Beck-Chevalley condition if and only if $\mathbf{C}^\bullet$ does.*

Interchanging the words "left" and "right" in Definition C.1.5 and Lemma C.1.6 we obtain the dual definition and assertion for the co-monadic Beck-Chevalley conditions.

C.1.7. We have the following basic results (see [Lu2, Theorem 6.2.4.2]):

Lemma C.1.8. *Let $\mathbf{C}^\bullet$ satisfy the monadic Beck-Chevalley condition. Then:*

(a) *The functor of evaluation on 0-simplices*

$$\mathrm{ev}^0 : \mathrm{Tot}(\mathbf{C}^\bullet) \to \mathbf{C}^0$$

admits a left adjoint; to be denoted $(\mathrm{ev}^0)^L$.

(b) *The monad $\mathrm{ev}^0 \circ (\mathrm{ev}^0)^L$, acting on $\mathbf{C}^0$, is isomorphic, as a plain endo-functor, to*

$$(\mathsf{T}^{\mathrm{pr}_s})^L \circ \mathsf{T}^{\mathrm{pr}_t},$$

where $\mathrm{pr}_s, \mathrm{pr}_t$ are the two maps $[0] \to [1]$[13].

[13]The notation "s" is for "source" and "t" for "target."

(c) *The functor*

$$\mathrm{ev}^0 : \mathrm{Tot}(\mathbf{C}^\bullet) \to \mathbf{C}^0$$

is monadic.

Similarly, we have:

Lemma C.1.9. *Let* $\mathbf{C}^\bullet$ *satisfy the co-monadic Beck-Chevalley condition. Then:*

(a) *The functor of evaluation on* 0*-simplices*

$$\mathrm{ev}^0 : \mathrm{Tot}(\mathbf{C}^\bullet) \to \mathbf{C}^0$$

admits a right adjoint; to be denoted $(\mathrm{ev}^0)^R$.

(b) *The co-monad* $\mathrm{ev}^0 \circ (\mathrm{ev}^0)^R$*, acting on* $\mathbf{C}^0$*, is isomorphic, as a plain endo-functor, to*

$$(\mathsf{T}^{\mathrm{pr}_s})^R \circ \mathsf{T}^{\mathrm{pr}_t}.$$

(c) *The functor*

$$\mathrm{ev}^0 : \mathrm{Tot}(\mathbf{C}^\bullet) \to \mathbf{C}^0$$

is co-monadic.

C.2. **Calculating tensor products.** For future use, here are some examples, of how the Beck-Chevalley conditions can be used to calculate tensor products of categories.

C.2.1. Let $\mathbf{A}$ be a monoidal DG category, and let $\mathbf{C}^r$ and $\mathbf{C}^l$ be right and left $\mathbf{A}$-module categories respectively.

Consider the corresponding "Bar" complex, i.e., simplicial category $\mathrm{Bar}^\bullet(\mathbf{C}^r, \mathbf{A}, \mathbf{C}^l)$, so that

$$|\,\mathrm{Bar}^\bullet(\mathbf{C}^r, \mathbf{A}, \mathbf{C}^l)| = \mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l.$$

The next assertion is tautological:

Lemma C.2.2.

(i) *Assume that the action functors* $\mathrm{act}_{\mathbf{C}^r,\mathbf{A}} : \mathbf{C}^r \otimes \mathbf{A} \to \mathbf{C}^r$ *and* $\mathrm{act}_{\mathbf{A},\mathbf{C}^l} : \mathbf{A} \otimes \mathbf{C}^l \to \mathbf{C}^l$ *and the monoidal operation* $\mathrm{mult}_{\mathbf{A}} : \mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$ *all admit continuous right adjoints, and the right adjoint to* $\mathrm{act}_{\mathbf{A},\mathbf{C}^l}$ *is also a map of* $\mathbf{A}$*-module categories.*

Then the co-simplicial category $\mathrm{Bar}^{\bullet,R}(\mathbf{C}^r, \mathbf{A}, \mathbf{C}^l)$*, obtained from* $\mathrm{Bar}^\bullet(\mathbf{C}^r, \mathbf{A}, \mathbf{C}^l)$ *by passage to the right adjoint functors, satisfies the monadic Beck-Chevalley condition.*

(ii) *Assume that the action functors* $\mathrm{act}_{\mathbf{C}^r,\mathbf{A}}$ *and* $\mathrm{act}_{\mathbf{A},\mathbf{C}^l}$ *and the monoidal operation* $\mathrm{mult}_{\mathbf{A}}$ *all admit left adjoints, and that the left adjoint of the action functor* $\mathrm{act}_{\mathbf{A},\mathbf{C}^l}$ *is also a map of* $\mathbf{A}$*-module categories.*

Then the co-simplicial category $\mathrm{Bar}^{\bullet,L}(\mathbf{C}^r, \mathbf{A}, \mathbf{C}^l)$*, obtained from* $\mathrm{Bar}^\bullet(\mathbf{C}^r, \mathbf{A}, \mathbf{C}^l)$ *by passage to the left adjoint functors, satisfies the co-monadic Beck-Chevalley condition.*

(ii') *In the situation of point (ii), the simplicial category* $\mathrm{Bar}^\bullet(\mathbf{C}^r, \mathbf{A}, \mathbf{C}^l)$ *satisfies the monadic Beck-Chevalley condition.*

From here we will deduce:

Corollary C.2.3. *In the situation of either of the points of Lemma C.2.2, the right adjoint to*

$$\mathbf{C}^r \otimes \mathbf{C}^l \to \mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l$$

is monadic, and the resulting monad on $\mathbf{C}^r \otimes \mathbf{C}^l$ *is isomorphic, as a plain endo-functor to*

$$(\mathrm{act}_{\mathbf{C}^r,\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{C}^l}) \circ (\mathrm{Id}_{\mathbf{C}^r} \otimes \mathrm{act}_{\mathbf{A},\mathbf{C}^l})^R.$$

Proof. In the situation of Lemma C.2.2(i), this follows from Lemma C.1.8.

In the situation of Lemma C.2.2(ii), this follows from Lemmas C.1.8, and C.1.6, combined with [GL:DG, Lemma 1.3.3].

□

C.2.4. The next lemma implies that Corollary C.2.3 is applicable to the computation of the tensor product as along as the corresponding functors admit continuous right (resp., left) adjoints:

Assume that $\mathbf{C}^l = \mathrm{Vect}$, with the action map $\mathrm{act}_{\mathbf{A},\mathbf{C}^l}$ being given by a monoidal functor $\mathsf{F} : \mathbf{A} \to \mathrm{Vect}$. We can view F as a datum of augmentation on $\mathbf{A}$, and in this case we will write $\mathrm{Bar}^\bullet(\mathbf{C}^r, \mathbf{A})$ instead of $\mathrm{Bar}^\bullet(\mathbf{C}^r, \mathbf{A}, \mathrm{Vect})$. (When $\mathbf{C}^r$ is also Vect with the action given by F, we will simply write $\mathrm{Bar}^\bullet(\mathbf{A})$.)

We note:

Lemma C.2.5. *Assume that the functor* F *is conservative.*

(i) *If the functor* $\mathbf{A} \to \mathrm{Vect}$ *admits a continuous right adjoint, then this right adjoint is automatically a map of* $\mathbf{A}$*-module categories.*

(ii) *If the functor* $\mathbf{A} \to \mathrm{Vect}$ *admits a left adjoint, then this left adjoint is automatically a map of* $\mathbf{A}$*-module categories.*

Proof. Since F is conservative, it suffices to show that the composition

$$\mathrm{Vect} \overset{\mathsf{F}^R}{\to} \mathbf{A} \overset{\mathsf{F}}{\to} \mathrm{Vect}$$

is a map of $\mathbf{A}$-module categories. However, the latter is evident: the functor in question is given by tensor product with $\mathsf{F} \circ \mathsf{F}^R(k)$.

□

Appendix D. Rigid monoidal categories

D.1. **The notion of rigidity.**

D.1.1. Let $\mathbf{O}$ be a monoidal DG category. We shall say that $\mathbf{O}$ is *rigid* if the following conditions are satisfied:

- The right adjoint $\mathrm{mult}^R_{\mathbf{O}}$ of the monoidal operation $\mathrm{mult}_{\mathbf{O}} : \mathbf{O} \otimes \mathbf{O} \to \mathbf{O}$ is continuous;
- The functor $\mathrm{mult}^R_{\mathbf{O}} : \mathbf{O} \to \mathbf{O} \otimes \mathbf{O}$ is strictly (rather than lax) compactible with the action of $\mathbf{O} \otimes \mathbf{O}$;
- The right adjoint $\mathrm{unit}^R_{\mathbf{O}}$ of the unit functor $\mathrm{unit}_{\mathbf{O}} : \mathrm{Vect} \to \mathbf{O}$ is continuous.

D.1.2. A basic feature of rigid monoidal categories is that the monoidal structure on $\mathbf{O}$ gives rise to a canonical identification

$$\mathbf{O}^\vee \simeq \mathbf{O} \tag{D.1}$$

as plain DG categories.

Namely, we define the co-unit $\mathbf{O} \otimes \mathbf{O} \to \mathrm{Vect}$ as

$$\mathbf{O} \otimes \mathbf{O} \overset{\mathrm{mult}_{\mathbf{O}}}{\longrightarrow} \mathbf{O} \overset{\mathrm{unit}^R_{\mathbf{O}}}{\longrightarrow} \mathrm{Vect},$$

and the unit $\mathrm{Vect} \to \mathbf{O} \otimes \mathbf{O}$ as

$$\mathrm{Vect} \overset{\mathrm{unit}_{\mathbf{O}}}{\longrightarrow} \mathbf{O} \overset{\mathrm{mult}^R_{\mathbf{O}}}{\longrightarrow} \mathbf{O} \otimes \mathbf{O}.$$

The fact that the above maps indeed define a data of duality is immediate from the assumption on $\mathbf{O}$.

Under the identification (D.1), the dual of $\mathrm{mult}_{\mathbf{O}}$ is $\mathrm{mult}^R_{\mathbf{O}}$, and the dual of $\mathrm{unit}_{\mathbf{O}}$ is $\mathrm{unit}^R_{\mathbf{O}}$.

D.1.3. Assume for a moment that $\mathbf{O}$ is compactly generated. Then it is easy to show that $\mathbf{O}$ is rigid in the sense of Sect. D.1.1 if and only if every compact object of $\mathbf{O}$ admits both left and right monoidal duals.

In this case, the equialence (D.1) is given at the level of compact objects by

$$\mathbf{o} \mapsto \mathbf{o}^\vee,$$

where $\mathbf{o}^\vee$ denotes the right monoidal dual.

D.2. **Modules over a rigid category.** In this subsection we let $\mathbf{O}$ be a rigid monoidal DG category.

D.2.1. We have the following key assertion:

Proposition D.2.2. *Let $\mathbf{C}$ be an $\mathbf{O}$-module category. Then the right adjoint to the action functor*

$$\mathrm{act}_{\mathbf{C},\mathbf{O}} : \mathbf{O} \otimes \mathbf{C} \to \mathbf{C}$$

is continuous and is given by the following functor (to be denoted $\text{co-act}_{\mathbf{C},\mathbf{O}}$*):*

$$\mathbf{C} \overset{\mathrm{unit}_{\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{C}}}{\longrightarrow} \mathbf{O} \otimes \mathbf{C} \overset{\mathrm{mult}^R_{\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{C}}}{\longrightarrow} \mathbf{O} \otimes \mathbf{O} \otimes \mathbf{C} \overset{\mathrm{Id}_{\mathbf{O}} \otimes \mathrm{act}_{\mathbf{C},\mathbf{O}}}{\longrightarrow} \mathbf{O} \otimes \mathbf{C}. \tag{D.2}$$

Proof. We construct adjunction data for the functors $\mathrm{act}_{\mathbf{C},\mathbf{O}}$ and $\text{co-act}_{\mathbf{C},\mathbf{O}}$ as follows. The composition $\mathrm{act}_{\mathbf{C},\mathbf{O}} \circ \text{co-act}_{\mathbf{C},\mathbf{O}}$ is isomorphic to the composition

$$\mathbf{C} \overset{\mathrm{unit}_{\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{C}}}{\longrightarrow} \mathbf{O} \otimes \mathbf{C} \overset{\mathrm{mult}^R_{\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{C}}}{\longrightarrow} \mathbf{O} \otimes \mathbf{O} \otimes \mathbf{C} \overset{\mathrm{mult}_{\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{C}}}{\longrightarrow} \mathbf{O} \otimes \mathbf{C} \overset{\mathrm{act}_{\mathbf{C},\mathbf{O}}}{\longrightarrow} \mathbf{C}. \tag{D.3}$$

The natural transformation $\mathrm{mult}_{\mathbf{O}} \circ \mathrm{mult}^R_{\mathbf{O}} \to \mathrm{Id}_{\mathbf{O}}$ defines a natural transformation from (D.3) to

$$\mathbf{C} \overset{\mathrm{unit}_{\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{C}}}{\longrightarrow} \mathbf{O} \otimes \mathbf{C} \overset{\mathrm{act}_{\mathbf{C},\mathbf{O}}}{\longrightarrow} \mathbf{C},$$

while the latter functor is canonically isomorphic to the identity functor on $\mathbf{C}$. This defines the co-unit of the adjunction.

The composition $\text{co-act}_{\mathbf{C},\mathbf{O}} \circ \mathrm{act}_{\mathbf{C},\mathbf{O}}$ is isomorphic to the composition

$$\begin{aligned} \mathbf{O} \otimes \mathbf{C} \overset{\mathrm{unit}_{\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{C}}}{\longrightarrow} \mathbf{O} \otimes \mathbf{O} \otimes \mathbf{C} \overset{\mathrm{mult}^R_{\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{C}}}{\longrightarrow} \mathbf{O} \otimes \mathbf{O} \otimes \mathbf{O} \otimes \mathbf{C} \to \\ \overset{\mathrm{Id}_{\mathbf{O}} \otimes \mathrm{mult}_{\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{C}}}{\longrightarrow} \mathbf{O} \otimes \mathbf{O} \otimes \mathbf{C} \overset{\mathrm{Id}_{\mathbf{O}} \otimes \mathrm{act}_{\mathbf{C},\mathbf{O}}}{\longrightarrow} \mathbf{O} \otimes \mathbf{C}. \end{aligned} \tag{D.4}$$

The condition on $\mathbf{O}$ implies that the diagram

$$\begin{array}{ccc} \mathbf{O} \otimes \mathbf{O} & \xrightarrow{\mathrm{mult}^R_{\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{O}}} & \mathbf{O} \otimes \mathbf{O} \otimes \mathbf{O} \\ {\scriptstyle \mathrm{mult}_{\mathbf{O}}}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_{\mathbf{O}} \otimes \mathrm{mult}_{\mathbf{O}}} \\ \mathbf{O} & \xrightarrow{\mathrm{mult}^R_{\mathbf{O}}} & \mathbf{O} \otimes \mathbf{O} \end{array}$$

commutes.

Hence, the functor in (D.4) can be rewritten as

$$\mathbf{O} \otimes \mathbf{C} \overset{\mathrm{mult}^R_{\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{O}}}{\longrightarrow} \mathbf{O} \otimes \mathbf{O} \otimes \mathbf{C} \overset{\mathrm{Id}_{\mathbf{O}} \otimes \mathrm{act}_{\mathbf{C},\mathbf{O}}}{\longrightarrow} \mathbf{O} \otimes \mathbf{C}. \tag{D.5}$$

Now, the isomorphism

$$\mathrm{mult}_{\mathbf{O}} \circ (\mathrm{Id}_{\mathbf{O}} \otimes \mathrm{unit}_{\mathbf{O}}) \simeq \mathrm{Id}_{\mathbf{O}}$$

gives rise to a natural transformation

$$\mathrm{Id}_{\mathbf{O}} \otimes \mathrm{unit}_{\mathbf{O}} \to \mathrm{mult}^R_{\mathbf{O}}.$$

Hence, the functor in (D.5) receives a natural transformation from

$$\mathbf{O} \otimes \mathbf{C} \overset{\mathrm{Id}_{\mathbf{O}} \otimes \mathrm{unit}_{\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{O}}}{\longrightarrow} \mathbf{O} \otimes \mathbf{O} \otimes \mathbf{C} \overset{\mathrm{Id}_{\mathbf{O}} \otimes \mathrm{act}_{\mathbf{C},\mathbf{O}}}{\longrightarrow} \mathbf{O} \otimes \mathbf{C},$$

whereas the latter is the identity functor on $\mathbf{O} \otimes \mathbf{C}$.

This defines the unit for the $(\mathrm{act}_{\mathbf{C},\mathbf{O}}, \text{co-act}_{\mathbf{C},\mathbf{O}})$-adjunction The verification that the above unit and co-unit satisfy the adjunction requirements is a straightforward verification. □

D.2.3. As a formal corollary by diagram chase we obtain:

Corollary D.2.4. *For an* $\mathbf{O}$*-module* $\mathbf{C}$*, the right adjoint of the action map is a map of* $\mathbf{O}$*-module categories.*

D.3. **The dual co-monoidal category.** Let $\mathbf{O}$ be a monoidal DG category, dualizable as a plain DG category.

D.3.1. The monoidal structure on $\mathbf{O}$ gives rise to a co-monoidal structure on $\mathbf{O}^\vee$, so that we have a canonical equivalence

$$\mathbf{O}\text{-}\mathbf{mod} \simeq \mathbf{O}^\vee\text{-}\mathbf{comod}, \tag{D.6}$$

commuting with the forgetful functor to $\mathrm{DGCat}_{\mathrm{cont}}$.

D.3.2. From now in this subsection, let us assume that $\mathbf{O}$ is rigid.

The assumption on $\mathbf{O}$ implies that the *right adjoint* of the monoidal structure on $\mathbf{O}$ defines a co-monoidal structure on $\mathbf{O}$. We shall denote $\mathbf{O}$, equipped with this co-monoidal structure by $\mathbf{O}_{\mathrm{co}}$.

Furthermore, Proposition D.2.2 implies that the procedure of taking the right adjoint of the action defines a functor

$$\mathbf{O}\text{-}\mathbf{mod} \to \mathbf{O}_{\mathrm{co}}\text{-}\mathbf{comod}, \tag{D.7}$$

which commutes with the forgetful functor to $\mathrm{DGCat}_{\mathrm{cont}}$.

D.3.3. Combining (D.6) to (D.7), we obtain that there exists a canonically defined functor

$$\mathbf{O}^\vee\text{-}\mathbf{comod} \to \mathbf{O}_{\mathrm{co}}\text{-}\mathbf{comod}, \tag{D.8}$$

that commutes with the forgetful functor to $\mathrm{DGCat}_{\mathrm{cont}}$.

Hence, the functor (D.8) comes from a homomorphism of co-monoidal categories

$$\phi_{\mathbf{O}} : \mathbf{O}^\vee \to \mathbf{O}_{\mathrm{co}}. \tag{D.9}$$

Lemma D.3.4. *The homomorphism* (D.9) *is an isomorphism.*

Proof. It follows from the construction that at the level of plain DG categories, the functor (D.9) equals that of (D.1). □

D.3.5. *Homomorphisms between rigid categories.* In a similar way to the construction of (D.9) we obtain:

Proposition D.3.6. *Let* $\mathsf{F} : \mathbf{O}_1 \to \mathbf{O}_2$ *be a homomorphism between rigid monoidal categories. Then the following diagram of homomorphisms of co-monoidal categories commutes:*

$$\begin{array}{ccc} (\mathbf{O}_2)_{\mathrm{co}} & \xrightarrow{\mathsf{F}^R} & (\mathbf{O}_1)_{\mathrm{co}} \\ \downarrow & & \downarrow \\ \mathbf{O}_2^\vee & \xrightarrow{\mathsf{F}^\vee} & \mathbf{O}_1^\vee. \end{array}$$

D.3.7. *Left modules vs. right modules.* Note that if $\mathbf{O}$ is rigid, then so is the category $\mathbf{O}^{\mathrm{o}}$ with the opposite monoidal structure. Applying the construction of Sect. D.3, we obtain an isomorphism

$$\phi_{\mathbf{O}^{\mathrm{o}}} : (\mathbf{O}^{\mathrm{o}})^\vee \to (\mathbf{O}^{\mathrm{o}})_{\mathrm{co}}.$$

We obtain that there exists a canonically defined monoidal automorphism $\psi_{\mathbf{O}}$ of $\mathbf{O}$, which intertwines the isomorphisms $\phi_{\mathbf{O}^{\mathrm{o}}}$ and

$$(\mathbf{O}^{\mathrm{o}})^\vee \simeq (\mathbf{O}^\vee)^{\mathrm{o}} \xrightarrow{(\phi_{\mathbf{O}})^{\mathrm{o}}} (\mathbf{O}_{\mathrm{co}})^{\mathrm{o}} \simeq (\mathbf{O}^{\mathrm{o}})_{\mathrm{co}}.$$

We note that the automorphism $\psi_{\mathbf{O}}$ is trivial when the monoidal struture on $\mathbf{O}$ is commutative.

Remark D.3.8. When $\mathbf{O}$ is compactly generated, at the level of compact objects, the automorphism $\psi_{\mathbf{O}}$ acts as

$$\mathbf{o} \mapsto (\mathbf{o}^\vee)^\vee.$$

D.4. **Hochschild homology vs cohomology.**

D.4.1. Let $\mathbf{O}$ be a monoidal DG category, and let $\mathbf{C}^l$ and $\mathbf{C}^r$ be a left and right $\mathbf{O}$-module categories. We can form their tensor product

$$\mathbf{C}^r \underset{\mathbf{O}}{\otimes} \mathbf{C}^l \in \mathrm{DGCat}_{\mathrm{cont}},$$

which is computed as

$$|\operatorname{Bar}^\bullet(\mathbf{C}^r, \mathbf{O}, \mathbf{C}^l)|.$$

Assume that $\mathbf{O}$ is dualizable as a plain DG category, and consider $\mathbf{O}^\vee$ as a monoidal DG category. Consider the co-tensor product

$$\mathbf{C}^r \overset{\mathbf{O}}{\otimes} \mathbf{C}^l \in \mathrm{DGCat}_{\mathrm{cont}},$$

defined as

$$\operatorname{Tot}(\text{co-}\operatorname{Bar}^\bullet(\mathbf{C}^r, \mathbf{O}^\vee, \mathbf{C}^l)).$$

D.4.2. We now claim:

Proposition D.4.3. *Assume that* $\mathbf{O}$ *is rigid. Then there exists a canonical isomorphism in* $\mathrm{DGCat}_{\mathrm{cont}}$

$$\mathbf{C}^r \underset{\mathbf{O}}{\otimes} \mathbf{C}^l \simeq (\mathbf{C}^r)_{\psi_{\mathbf{O}}} \overset{\mathbf{O}}{\otimes} \mathbf{C}^l,$$

where $(\mathbf{C}^r)_{\psi_{\mathbf{O}}}$ *is the right* $\mathbf{O}$*-module category, obtained from* $\mathbf{C}^r$ *by twisting the action by the automorhism* $\psi_{\mathbf{O}}$ *of Sect. D.3.7.*

Proof. By [GL:DG, Lemma 1.3.3], the tensor product $\mathbf{C}^r \underset{\mathbf{O}}{\otimes} \mathbf{C}^l$ can be computed as the totalization of the co-simplicial category $\mathrm{Bar}^{\bullet,R}(\mathbf{C}^r, \mathbf{O}, \mathbf{C}^l)$, obtained from $\mathrm{Bar}^{\bullet}(\mathbf{C}^r, \mathbf{O}, \mathbf{C}^l)$ by passing to the right adjoint functors.

Now, by the construction of Sect. D.3, the co-simplicial categories

$$\mathrm{Bar}^{\bullet,R}(\mathbf{C}^r, \mathbf{O}, \mathbf{C}^l) \text{ and } \text{co-Bar}^{\bullet}((\mathbf{C}^r)_{\psi_{\mathbf{O}}}, \mathbf{O}^{\vee}, \mathbf{C}^l)$$

are canonically equivalent.

□

D.4.4. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two left $\mathbf{O}$-module categories, and assume that $\mathbf{C}_1$ is dualizable as a plain category. Consider $\mathbf{C}_1^{\vee}$ as a right $\mathbf{O}$-module category. Then we have

$$\underline{\mathrm{Hom}}_{\mathbf{O}}(\mathbf{C}_1, \mathbf{C}_2) \simeq \mathbf{C}_1^{\vee} \overset{\mathbf{O}}{\otimes} \mathbf{C}^l.$$

Hence, from Proposition D.4.3 we obtain:

Corollary D.4.5. *Assume that* $\mathbf{O}$ *is rigid. Then for* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *as above, there exists a canonical isomorphism*

$$\underline{\mathrm{Hom}}_{\mathbf{O}}((\mathbf{C}_1)_{\psi_{\mathbf{O}}}, \mathbf{C}_2) \simeq \mathbf{C}_1^{\vee} \underset{\mathbf{O}}{\otimes} \mathbf{C}^l.$$

D.4.6. As another corollary of Proposition D.4.3, we obtain:

Corollary D.4.7. *Let* $\mathbf{O}$ *be rigid. Then any* $\mathbf{O}$*-module category can be obtained as a totalization of a co-simplicial object, whose terms are of the form* $\mathbf{O} \otimes \mathbf{D}$ *with* $\mathbf{D} \in \mathrm{DGCat}_{\mathrm{cont}}$.

Proof. For $\mathbf{C} \in \mathbf{O}$-mod, we have

$$\mathbf{C} \simeq \mathbf{O} \underset{\mathbf{O}}{\otimes} \mathbf{C},$$

(where the right-hand side is regarded as a left $\mathbf{O}$-module category via the left action of $\mathbf{O}$ on itself). Now, by Proposition D.4.3

$$\mathbf{O} \underset{\mathbf{O}}{\otimes} \mathbf{C} \simeq \mathrm{Tot}\left(\text{co-Bar}^{\bullet}(\mathbf{O}_{\psi_{\mathbf{O}}}, \mathbf{O}, \mathbf{C})\right) \simeq \mathrm{Tot}\left(\text{co-Bar}^{\bullet}(\mathbf{O}, \mathbf{O}, (\mathbf{C})_{\psi_{\mathbf{O}}^{-1}})\right).$$

Now, the terms of $\text{co-Bar}^{\bullet}(\mathbf{O}, \mathbf{O}, (\mathbf{C})_{\psi_{\mathbf{O}}^{-1}})$, when regarded as left $\mathbf{O}$-modules, have the required form.

□

D.4.8. Finally, combining Corollaries D.2.4 and C.2.3, we obtain:

Corollary D.4.9. *Let* $\mathbf{O}$ *be rigid.*

(a) *The co-simplicial category* $\text{co-Bar}^\bullet(\mathbf{C}^r, \mathbf{O}^\vee, \mathbf{C}^l)$ *satisfies the monadic Beck-Chevalley condition.*

(b) *The functor of evaluation on* 0*-simplices*

$$\operatorname{Tot}\left(\text{co-Bar}^\bullet(\mathbf{C}^r, \mathbf{O}^\vee, \mathbf{C}^l)\right) \to \mathbf{C}^r \otimes \mathbf{C}^l$$

admits a left adjoint and is monadic. The resulting monad, viewed as a plain endo-functor of $\mathbf{C}^r \otimes \mathbf{C}^l$*, identifies with the composition*

$$\mathbf{C}^r \otimes \mathbf{C}^l \overset{\mathrm{Id}_{\mathbf{C}^r} \otimes \text{co-act}_{\mathbf{C}^l, \mathbf{O}}}{\longrightarrow} \mathbf{C}^r \otimes \mathbf{O} \otimes \mathbf{C}^l \overset{\mathrm{act}_{\mathbf{C}^r, \mathbf{O}} \otimes \mathrm{Id}_{\mathbf{C}^l}}{\longrightarrow} \mathbf{C}^r \otimes \mathbf{C}^l.$$

D.5. **Dualizability of modules over a rigid category.**

D.5.1. Let $\mathbf{O}$ be a monoidal DG category, and let $\mathbf{C}^r$ and $\mathbf{C}^l$ be a right and left $\mathbf{O}$-module categories, respectively.

Recall that a data of duality between $\mathbf{C}^r$ and $\mathbf{C}^l$ as $\mathbf{O}$-module categories consists of a unit map

$$\mathrm{Vect} \to \mathbf{C}^r \underset{\mathbf{O}}{\otimes} \mathbf{C}^l,$$

which is a map in $\mathrm{DGCat}_{\mathrm{cont}}$, and a co-unit map

$$\mathbf{C}^l \otimes \mathbf{C}^r \to \mathbf{O},$$

which is a map on $(\mathbf{O} \otimes \mathbf{O})\text{-}\mathbf{mod}$, which satisfy the usual axioms.

Equivalently, the datum of duality between $\mathbf{C}^r$ and $\mathbf{C}^l$ as $\mathbf{O}$-module categories is a functorial equivalence

$$\underline{\mathrm{Hom}}_{\mathbf{O}}(\mathbf{C}^l, \mathbf{C}) \simeq \mathbf{C}^r \underset{\mathbf{O}}{\otimes} \mathbf{C}, \quad \mathbf{C} \in \mathbf{O}\text{-}\mathbf{mod}.$$

Remark D.5.2. Assume for a moment that $\mathbf{O}$ is symmetric monoidal. Then it is easy to see that a duality data between two $\mathbf{O}$-module categories is equivalent to a duality data inside the symmetric monoidal DG category $\mathbf{O}\text{-}\mathbf{mod}$.

D.5.3. We claim:

Proposition D.5.4. *Assume that* $\mathbf{O}$ *is rigid. Then* $\mathbf{C}^l \in \mathbf{O}\text{-}\mathbf{mod}$ *is dualizable if and only if it is dualiable as a plain DG category. The DG category underlying the* $\mathbf{O}$*-module dual of* $\mathbf{O}$ *is canonically equivalent to* $(\mathbf{C}^l)^\vee$*.*

Proof. Suppose first being given a duality data between $\mathbf{C}^r$ and $\mathbf{C}^l$ as $\mathbf{O}$-module categories. We define a duality data between $\mathbf{C}^r$ and $\mathbf{C}^l$ as plain DG categories by taking the unit to be

$$\mathrm{Vect} \to \mathbf{C}^r \underset{\mathbf{O}}{\otimes} \mathbf{C}^l \to \mathbf{C}^r \otimes \mathbf{C}^l,$$

where the second arrow is the right adjoint to the tautological functor $\mathbf{C}^r \otimes \mathbf{C}^l \to \mathbf{C}^r \underset{\mathbf{O}}{\otimes} \mathbf{C}^l$ (it is continuous, e.g., by Corollary D.4.9(b)). We take the co-unit to be

$$\mathbf{C}^l \otimes \mathbf{C}^r \to \mathbf{O} \overset{\mathrm{unit}^R_{\mathbf{O}}}{\longrightarrow} \mathrm{Vect}\,.$$

The fact that the duality axioms hold is straightforward.

Vice versa, let $\mathbf{C}^l$ be dualizable as a plain DG category. Set $\mathbf{C}^r := ((\mathbf{C}^l)^\vee)_{\psi_{\mathbf{O}}}$. Now, the functorial equivalence

$$\underline{\mathrm{Hom}}_{\mathbf{O}}(\mathbf{C}^l, \mathbf{C}) \simeq \mathbf{C}^r \underset{\mathbf{O}}{\otimes} \mathbf{C}$$

follows from Corollary D.4.5.

□

Appendix E. Commutative Hopf algebras

E.1. **The setting.**

E.1.1. Let $\mathbf{O}$ be a symmetric monoidal category[14].

Consider the category co-Alg($\mathbf{O}$) of co-algebras in $\mathbf{O}$. We regard it as a symmetric monoidal category under the operation of tensor product.

By a (commutative) bi-algebra in $\mathbf{O}$ we will mean a (commutative) algebra in co-Alg($\mathbf{O}$). We shall say that a (commutative) bi-algebra is a (commutative) Hopf algebra if it is such at the level of the underlying ordinary categories (i.e., if it admits a homotopy antipode).

E.1.2. Recall that if A is an augmented co-algebra object in a monoidal category $\mathbf{O}$, we can canonically attach to it a co-simplicial object

$$\text{co-Bar}^\bullet(A).$$

If A is a bi-algebra, the object $\text{co-Bar}^\bullet(A) \in \mathbf{O}^{\mathbf{\Delta}}$ naturally lifts to one in

$$\text{Alg}(\mathbf{O}^{\mathbf{\Delta}}) \simeq (\text{Alg}(\mathbf{O}))^{\mathbf{\Delta}},$$

i.e., $\text{co-Bar}^\bullet(A)$ is a co-simplicial algebra in $\mathbf{O}$, or equivalently, a co-simplicial object of $\mathbf{O}$ endowed with a compatible family of simplex-wise monoidal structures.

E.1.3. Consider the corresponding co-simplicial category

$$\text{co-Bar}^\bullet(A)\text{-mod}$$

(where the transition functors are given by tensoring up along the maps in $\text{co-Bar}^\bullet(A)$).

Consider the totalization

$$\text{Tot}\left(\text{co-Bar}^\bullet(A)\text{-mod}\right).$$

The goal of this Appendix is to prove the following:

Proposition-Construction E.1.4. *Let A be a Hopf algebra in $\mathbf{O}$. Then there exists a canonical equivalence of categories*

$$A\text{-comod} \to \text{Tot}\left(\text{co-Bar}^\bullet(A)\text{-mod}\right).$$

E.2. **Construction of the functor.**

To construct the sought-for functor in Proposition E.1.4 we proceed as follows.

[14]In this section $\mathbf{O}$ is not necessarily stable, e.g., $\mathbf{O} = \text{DGCat}_{\text{cont}}$.

E.2.1. Let

$$\mathrm{coAlg} + \mathrm{comod}(\mathbf{O})$$

be the category of pairs

$$(A \in \text{co-Alg}^{\mathrm{aug}}(\mathbf{O}), M \in A\text{-comod}(\mathbf{O})).$$

Note that the assignment $A \rightsquigarrow \text{co-Bar}^\bullet(A)$ can be extended to a functor

$$\mathrm{coAlg} + \mathrm{comod}(\mathbf{O}) \rightsquigarrow \text{co-Bar}^\bullet(A, M') \in \mathbf{O}^{\mathbf{\Delta}}. \tag{E.1}$$

Moreover, this functor is (symmetric) monoidal, when on $\mathrm{coAlg} + \mathrm{comod}(\mathbf{O})$ we consider the (symmetric) monoidal structure

$$(A_1, M_1) \otimes (A_2, M_2) := (A_1 \otimes A_2, M_1 \otimes M_2),$$

and on $\mathbf{O}^{\mathbf{\Delta}}$ the component-wise (symmetric) monoidal structure.

E.2.2. Note that for a bi-algebra A, the pair $(A, \mathbf{1}_{\mathbf{O}})$ is naturally an algebra object in the category $\mathrm{coAlg} + \mathrm{comod}(\mathbf{O})$, and we have a canonically defined functor

$$A\text{-comod} \to (A, \mathbf{1}_{\mathbf{O}})\text{-mod}(\mathrm{coAlg} + \mathrm{comod}(\mathbf{O})). \tag{E.2}$$

We note that the value of the functor (E.1) on $(A, \mathbf{1}_{\mathbf{O}})$ is $\text{co-Bar}^\bullet(A) \in \mathrm{Alg}(\mathbf{O}^{\mathbf{\Delta}})$. Now composing the functor (E.2) and (E.1), we obtain a functor

$$A\text{-comod} \to \text{co-Bar}^\bullet(A)\text{-mod}(\mathbf{O}^{\mathbf{\Delta}}). \tag{E.3}$$

E.2.3. Now, it is easy to see that if A is a Hopf algebra, then for $M \in A$-comod and a map $[j] \to [i]$ in $\mathbf{\Delta}$, for the corresponding map of algebras and modules

$$\text{co-Bar}^j(A) \to \text{co-Bar}^i(A), \quad \text{co-Bar}^j(A, M) \to \text{co-Bar}^i(A, M),$$

the resulting map

$$\text{co-Bar}^i(A) \underset{\text{co-Bar}^j(A)}{\otimes} \text{co-Bar}^j(A, M) \to \text{co-Bar}^i(A, M)$$

is an isomorphism.

Hence, the functor (E.3) defines a functor

$$A\text{-comod} \to \mathrm{Tot}\left(\text{co-Bar}^\bullet(A)\text{-mod}\right). \tag{E.4}$$

E.3. **Proof of the equivalence.**

E.3.1. The fact that A is a Hopf algebra implies that the co-simplicial category $\text{co-Bar}^\bullet(A)$-mod satisfies the co-monadic Beck-Chevalley condition. Hence, by Lemma C.1.9, the functor ev^0 of evaluation on 0-simplices is co-monadic, and the resulting co-monad on $\mathbf{O}$ is given by tensor product with A.

E.3.2. Consider now the composition

$$A\text{-comod} \to \mathrm{Tot}\left(\text{co-Bar}^\bullet(A)\text{-mod}\right) \xrightarrow{\mathrm{ev}^0} \mathbf{O}.$$

It identifies with the forgetful functor $\mathrm{oblv}_A : A\text{-comod} \to \mathbf{O}$. In particular, it is co-monadic, and the resulting monad on $\mathbf{O}$ being the tensor product with A.

E.3.3. It remains to show that the map of co-monads, induced by the functor (E.4), is an isomorphism as plain endo-functors of $\mathbf{O}$. However, it is easy to see that the natural transformation in question is the identity map on the functor of tensor product by A.

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Department of Mathematics, Harvard University, 1 Oxford street, Cambridge MA, 02138, USA
E-mail address: gaitsgde@math.harvard.edu